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Higher order generalized Euler characteristics and generating series

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ABSTRACT

For a complex quasi-projective manifold with a finite group action, we define higher order generalized Euler characteristics with values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. We compute the generating series of generalized Euler characteristics of a fixed order of the Cartesian products of the manifold with the wreath product actions on them.

Let X be a topological space (good enough, say, a quasi-projective variety) with an action of a finite group G. For a subgroup H of G, let $X^H = \{x \in X : Hx = x\}$ be the fixed point set of H. The orbifold Euler characteristic $\chi^{orb}(X, G)$ of the G-space X is defined, e.g., in [1,2]:

$$\chi^{orb}(X, G) = \frac{1}{|G|} \sum_{\substack{(g_0, g_1)\in G\times G:\\ g_0g_1=g_1g_0}} \chi(X^{\langle g_0, g_1\rangle}) = \sum_{[g]\in G_*} \chi(X^{\langle g\rangle}/C_G(g)), \tag{1}$$

where G_* is the set of conjugacy classes of elements of G, $C_G(g) = \{h \in G : h^{-1}gh = g\}$ is the centralizer of g, and $\langle g\rangle$ and $\langle g_0, g_1\rangle$ are the subgroups generated by the corresponding elements.

The higher order Euler characteristics of (X, G) (alongside with some other generalizations) were defined in [3,4].

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Definition. The *Euler characteristic* $\chi^{(k)}(X, G)$ *of order* k of the G-space X is

$$\chi^{(k)}(X, G) = \frac{1}{|G|} \sum_{\substack{\mathbf{g}\in G^{k+1}:\\ g_i g_j = g_j g_i}} \chi(X^{\langle \mathbf{g}\rangle}) = \sum_{[g]\in G_*} \chi^{(k-1)}(X^{\langle g\rangle}, C_G(g)), \tag{2}$$

where $\mathbf{g} = (g_0, g_1, \ldots, g_k)$, $\langle \mathbf{g}\rangle$ is the subgroup generated by $g_0, g_1, \ldots, g_k$, and $\chi^{(0)}(X, G)$ is defined as $\chi(X/G)$.

The usual orbifold Euler characteristic $\chi^{orb}(X, G)$ is the Euler characteristic of order 1, $\chi^{(1)}(X, G)$.

The higher order generalized Euler characteristics take values in the Grothendieck ring of complex quasi-projective varieties extended by the rational powers of the class of the affine line. Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck ring of complex quasi-projective varieties. This is the abelian group generated by the isomorphism classes $[X]$ of quasi-projective varieties modulo the relation:

– if Y is a Zariski closed subvariety of X, then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\mathrm{Var}_{\mathbb{C}})$ is defined by the Cartesian product. The class $[X]$ of a variety X is the universal additive invariant of quasi-projective varieties and can be regarded as a generalized Euler characteristic of X. Let $\mathbb{L}$ be the class $[\mathbb{A}^1_{\mathbb{C}}]$ of the affine line and let $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ be the extension of the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ by all the rational powers of $\mathbb{L}$.

The formula for the generating series of the generalized orbifold Euler characteristics of the pairs (X^n, G_n) in [5] uses the (natural) power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ (and over $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$) defined in [6]. (See also [7] and [5] for some generalizations of this concept.) This means that for a power series $A(T) \in 1 + t \cdot R[[t]]$ ($R = K_0(\mathrm{Var}_{\mathbb{C}})$ or $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$) and for an element $m \in R$ there is defined a series $(A(T))^m \in 1 + t \cdot R[[t]]$ so that all the properties of the exponential function hold. For a quasi-projective variety M, the series $(1-t)^{-[M]}$ is the Kapranov zeta-function of M: $\zeta_{[M]}(t) := (1-t)^{-[M]} = 1 + [M]\cdot t + [\mathrm{Sym}^2 M]\cdot t^2 + [\mathrm{Sym}^3 M]\cdot t^3 + \cdots$, where $\mathrm{Sym}^k M = M^k/S_k$ is the kth symmetric power of the variety M. A geometric description of the power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is given in [6] or [5]. The (natural) power structures over $K_0(\mathrm{Var}_{\mathbb{C}})$ and over $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ possess the following properties:

(1) $(A(t^s))^m = (A(t))^m |_{t\mapsto t^s}$;
(2) $(A(\mathbb{L}^s t))^m = (A(t))^{\mathbb{L}^s m}$.

One can define a power structure over the ring $\mathbb{Z}[u_1, \ldots, u_r]$ of polynomials in r variables with integer coefficients in the following way. Let $P(u_1, \ldots, u_r) = \sum_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} p_{\underline{k}} \underline{u}^{\underline{k}} \in \mathbb{Z}[u_1, \ldots, u_r]$, where $\underline{k} = (k_1, \ldots, k_r)$, $\underline{u} = (u_1, \ldots, u_r)$, $\underline{u}^{\underline{k}} = u_1^{k_1}\cdot \ldots \cdot u_r^{k_r}$, $p_{\underline{k}} \in \mathbb{Z}$. Define

$$(1-t)^{-P(u_1,\ldots,u_r)} := \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} (1 - \underline{u}^{\underline{k}} t)^{-p_{\underline{k}}},$$

where the power (with an integer exponent $-p_{\underline{k}}$) means the usual one. This gives a λ-structure on the ring $\mathbb{Z}[u_1, \ldots, u_r]$ and therefore a power structure over it (see, e.g., [5, Proposition 1])

i.e., for polynomials $A_i(\underline{u})$, $i \geq 1$, and $M(\underline{u})$, there is defined a series $\left(1 + A_1(\underline{u})t + A_2(\underline{u})t^2 + \cdots\right)^{M(\underline{u})}$ with the coefficients from $\mathbb{Z}[u_1, \ldots, u_r]$.

Let $r = 2$, $u_1 = u$, $u_2 = v$. Let $e : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$ be the ring homomorphism which sends the class $[X]$ of a quasi-projective variety X to its Hodge–Deligne polynomial $e(X; u, v) = \sum h^{ij}_X (-u)^i (-v)^j$.

Remark. Let R_1 and R_2 be rings with power structures over them. A ring homomorphism $\varphi : R_1 \to R_2$ induces the natural homomorphism $R_1[[t]] \to R_2[[t]]$ (also denoted φ) by $\varphi\left(\sum a_i t^i\right) = \sum \varphi(a_i)\, t^i$. In [5, Proposition 2], it was shown that if a ring homomorphism $\varphi : R_1 \to R_2$ is such that $(1-t)^{-\varphi(m)} = \varphi\left((1-t)^{-m}\right)$ for any $m \in R$, then $\varphi\left((A(t))^m\right) = (\varphi(A(t)))^{\varphi(m)}$ for $A(\underline{t}) \in 1 + tR[[t]]$, $m \in R$.

There are two natural homomorphisms from the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u, v]$ of polynomials in two variables: the Euler characteristic (with compact support) $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ and the Hodge–Deligne polynomial. Both possess the following well known identities:

(1) the formula of I.G. Macdonald [8]:

$$\chi(1 + [X]t + [\mathrm{Sym}^2 X]t^2 + [\mathrm{Sym}^3 X]t^3 + \cdots) = (1-t)^{-\chi(X)},$$

(2) and the corresponding formula for the Hodge–Deligne polynomial (see [9, Proposition 1.2]):

$$e(1 + [X]t + [\mathrm{Sym}^2 X]t^2 + \cdots) = (1-T)^{-e(X;u,v)} = \prod_{p,q}\left(\frac{1}{1 - u^p v^q t}\right)^{e^{p,q}(X)}.$$

These properties and the previous remark imply that the corresponding homomorphisms respect the power structures over the corresponding rings: $K_0(\mathrm{Var}_{\mathbb{C}})$ and $\mathbb{Z}[u, v]$ respectively, see [7].

A generalization of the orbifold Euler characteristic to the orbifold (or stringly) Hodge numbers and the orbifold Hodge–Deligne polynomial (for an action of a finite group G on a non-singular quasi-projective variety X) was defined in [10].

Let X be a smooth quasi-projective variety of dimension d with an (algebraic) action of the group G. For $g \in G$, the centralizer $C_G(g)$ of g acts on the manifold $X^{\langle g\rangle}$ of fixed points of the element g. Suppose that its action on the set of connected components of $X^{\langle g\rangle}$ has N_g orbits, and let $X_1^{\langle g\rangle}, X_2^{\langle g\rangle}, \ldots, X_{N_g}^{\langle g\rangle}$ be the unions of the components of each of the orbits. At a point $x \in X_{\alpha_g}^{\langle g\rangle}$, $1 \le \alpha_g \le N_g$, the differential dg of the map g is an automorphism of finite order of the tangent space T_xX. Its action on T_xX can be represented by a diagonal matrix $\mathrm{diag}(\exp(2\pi i\theta_1), \ldots, \exp(2\pi i\theta_d))$ with $0 \le \theta_j < 1$ for $j = 1, 2, \ldots, d$ (θ_j are rational numbers). The *shift number* $F_{\alpha_g}^g$ associated with $X_{\alpha_g}^{\langle g\rangle}$ is $F_{\alpha_g}^g = \sum_{j=1}^d \theta_j \in \mathbb{Q}$. (It was introduced in [11].)

Definition. The *generalized orbifold Euler characteristic* of the pair (X, G) (see [5]) is

$$[X, G] = \sum_{[g]\in G_*} \sum_{\alpha_g=1}^{N_g} [X_{\alpha_g}^{\langle g\rangle}/C_G(g)] \cdot \mathbb{L}^{F_{\alpha_g}^g} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]. \tag{3}$$

Since the Euler characteristic and the Hodge–Deligne polynomial are additive invariants they factor through $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ and the Euler characteristic morphisms send $[X, G]$ to the orbifold Euler characteristic $\chi^{orb}(X, G)$. The Hodge–Deligne polynomial morphism sends it to the orbifold Hodge–Deligne polynomial from [12,13].

Let $G^n = G \times \cdots \times G$ be the Cartesian power of the group G. The symmetric group S_n acts on G^n by permutation of the factors: $s(g_1, \ldots, g_n) = (g_{s^{-1}(1)}, \ldots, g_{s^{-1}(n)})$. The *wreath product* $G_n = G \wr S_n$ is the semidirect product of the groups G^n and S_n defined by the described action. Namely the multiplication in the group G_n is given by the formula $(\mathbf{g}, s)(\mathbf{h}, t) = (\mathbf{g}\cdot s(\mathbf{h}), st)$, where $\mathbf{g}, \mathbf{h} \in G^n$, $s, t \in S_n$. The group G^n is a normal subgroup of the group G_n via the identification of $\mathbf{g} \in G^n$ with $(\mathbf{g}, 1) \in G_n$. For a variety X with a G-action, there is the corresponding action of the group G_n on the Cartesian power X^n given by the formula

$$((g_1, \ldots, g_n), s)(x_1, \ldots, x_n) = (g_1x_{s^{-1}(1)}, \ldots, g_nx_{s^{-1}(n)}),$$

where $x_1, \ldots, x_n \in X$, $g_1, \ldots, g_n \in G$, $s \in S_n$. One can see that the quotient X^n/G_n is naturally isomorphic to the space $\mathrm{Sym}^n(X/G) = (X/G)^n/S_n$. In particular, in the Grothendieck ring of complex quasi-projective varieties one has $[X^n/G_n] = [(X/G)^n/S_n] = [\mathrm{Sym}^n(X/G)]$.

A formula for the generating series of the kth order Euler characteristics of the pairs (X^n, G_n) in terms of the kth order Euler characteristics of the G-space X was given in [4] (see also [3]).

The generating series of the orbifold Hodge–Deligne polynomials $e(X^n, G_n; u, v)$ of the pairs (X^n, G_n) was computed in [13].

A reformulation of the result of [13] in terms of the generalized orbifold Euler characteristic with values in $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ was given in [5]. Using properties of the power structure one has ([5, Theorem 4]):

$$\sum_{n\ge0}[X^n, G_n]t^n = \left(\prod_{r=1}^{\infty}(1 - \mathbb{L}^{(r-1)d/2}t^r)\right)^{-[X,G]}. \tag{4}$$

Here we define higher order generalized Euler characteristics of a pair (X, G) (with X non-singular) and give a formula for the generating series of the kth order generalized Euler characteristic of the pairs (X^n, G_n).

Before giving the definition of the higher order generalized Euler characteristic of a pair (X, G) we discuss some versions of the definition (3) and of Eq. (4).

For a G-variety X (not necessarily non-singular) its *inertia stack* (or rather *class*) $I(X, G)$ is defined by

$$I(X, G) := \sum_{[g]\in G_*} [X^g/C_G(g)] \tag{5}$$

(see e.g. [14]). One can see that it is an analogue of the generalized orbifold Euler characteristic (3) without the shift factor $\mathbb{L}^{F_{\alpha_g}^g}$. This inspires the following version of the definition (3).

Definition. For a rational number φ_1, let

$$[X, G]_{\varphi_1} := \sum_{[g]\in G_*} \sum_{\alpha_g=1}^{N_g} [X_{\alpha_g}^{\langle g\rangle}/C_G(g)] \cdot \mathbb{L}^{\varphi_1 F_{\alpha_g}^g} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]. \tag{6}$$

That is the Zaslow shift $F_{\alpha_g}^g$ is multiplied by φ_1. For $\varphi_1 = 1$ one gets the generalized Euler characteristic $[X, G]$ from (3), for $\varphi_1 = 0$ one gets the inertia class $I(X, G)$. The arguments from [5] easily give the following version of Eq. (4).

Proposition 1.

$$\sum_{n\geq 0}[X^n, G_n]_{\varphi_1} t^n = \left(\prod_{r=1}^{\infty}(1-\mathbb{L}^{\varphi_1 (r-1)d/2}t^r)\right)^{-[X,G]}. \tag{7}$$

Thus multiplication of Zaslow's shift by a number (at least by 1 or 0) makes sense. For the corresponding definition of the higher order generalized Euler characteristic one can use factors φ_k depending on the order of the Euler characteristic.

Let X be a non-singular d-dimensional quasi-projective variety with a G action and let $\underline{\varphi} = (\varphi_1, \varphi_2, \dots)$ be a fixed sequence of rational numbers. We use the notations introduced before (3).

Definition. The *generalized orbifold Euler characteristic of order* k of the pair (X, G) is

$$[X, G]^k_{\underline{\varphi}} := \sum_{[g]\in G_*}\sum_{\alpha_g=1}^{N_g}[X^{\langle g\rangle}_{\alpha_g}, C_G(g)]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{\varphi_k F^g_{\alpha_g}} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}], \tag{8}$$

where $[X, G]^1_{\underline{\varphi}} := [X, G]_{\varphi_1}$ is the (modified) generalized orbifold Euler characteristic given by (6).

Remark. The definition (2) (as well as (1)) contains two equivalent versions. One can say that here we formulate an analogue of the second one. A formula analogous to the first one (with the factor $\frac{1}{|G|}$ in front) cannot work directly, at least without tensoring the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ by the field $\mathbb{Q}$ of rational numbers. Moreover, it seems that there is no analogue of Theorem 1 in terms of the power structure. This gives the hint that a definition of this sort makes small geometric sense (if any).

Taking the Euler characteristic, one gets $\chi([X, G]^k_{\underline{\varphi}}) = \chi^{(k)}(X, G)$.

To prove the formula for the generating series of $[X^n, G_n]^k_{\underline{\varphi}}$, we will use some technical statements.

Lemma 1.

$$[X' \times X'', G' \times G'']^k_{\underline{\varphi}} = [X', G']^k_{\underline{\varphi}} \times [X'', G'']^k_{\underline{\varphi}}. \tag{9}$$

The proof is obvious.

Let X_1 and X_2 be two G-manifolds and let $X_1^m \times X_2^{n-m}$ be embedded into $(X_1 \coprod X_2)^n$ in the natural way: a pair of elements $(x_{1,1}, \dots, x_{1,m}) \in X_1^m$ and $(x_{2,1}, \dots, x_{2,n-m}) \in X_2^{n-m}$ is identified with $(x_{1,1}, \dots, x_{1,m}, x_{2,1}, \dots, x_{2,n-m}) \in (X_1 \coprod X_2)^n$. Let $S_n(X_1^m \times X_2^{n-m})$ be the orbit of $X_1^m \times X_2^{n-m}$ under the S_n-action on $(X_1 \coprod X_2)^n$. The wreath product G_n acts on $S_n(X_1^m \times X_2^{n-m})$.

Lemma 2.

$$[S_n(X_1^m \times X_2^{n-m}), G_n]^k_{\underline{\varphi}} = [X_1^m, G_m]^k_{\underline{\varphi}} \times [X_2^{n-m}, G_{n-m}]^k_{\underline{\varphi}}. \tag{10}$$

Proof. An element $(\mathbf{g}, s) \in G_n$ has fixed points on $S_n(X_1^m \times X_2^{n-m})$ if and only if it is conjugate to an element $(\mathbf{g}', s') \in G_n$ such that $s' = (s_1, s_2) \in S_m \times S_{n-m} \subset S_n$ and the element $(\mathbf{g}', s') = ((\mathbf{g}_1, \mathbf{g}_2), (s_1, s_2))$ has fixed points on $X_1^m \times X_2^{n-m}$ (and only on it). The centralizer of the element $(\mathbf{g}', s')$ is $C_{G_m}((g_1, s_1)) \times C_{G_{n-m}}((g_2, s_2))$. The components of $(X_1^m \times X_2^{n-m})^{\langle(\mathbf{g}', s')\rangle}$ are the products $(X_1^m)^{\langle(\mathbf{g}_1, s_1)\rangle}_\alpha \times (X_2^{n-m})^{\langle(\mathbf{g}_2, s_2)\rangle}_\beta$ of the components of $(X_1^m)^{\langle(\mathbf{g}_1, s_1)\rangle}$ and $(X_2^{n-m})^{\langle(\mathbf{g}_2, s_2)\rangle}$. The shift $F^{(\mathbf{g}', s')}_{\alpha\beta}$ is equal to $F^{(\mathbf{g}_1, s_1)}_\alpha + F^{(\mathbf{g}_2, s_2)}_\beta$. Therefore

$$\begin{aligned}
[S_n(X_1^m \times X_2^{n-m}), G_n]^k_{\underline{\varphi}} &= \sum_{[(\mathbf{g}', s')]}\sum_{\alpha\beta}[(X_1^m \times X_2^{n-m})^{(\mathbf{g}', s')}_{\alpha\beta}, C_{G_m}((\mathbf{g}_1, s_1)) \times C_{G_{n-m}}((\mathbf{g}_2, s_2))]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{(F^{(\mathbf{g}_1, s_1)}_\alpha + F^{(\mathbf{g}_2, s_2)}_\beta)}\\
&= \sum_{[(\mathbf{g}_1, s_1)]}\sum_{\alpha}[(X_1^m)^{(\mathbf{g}_1, s_1)}_\alpha, C_{G_m}((\mathbf{g}_1, s_1))]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{F^{(\mathbf{g}_1, s_1)}_\alpha}\\
&\quad \times \sum_{[(\mathbf{g}_2, s_2)]}\sum_{\beta}[(X_1^{n-m})^{(\mathbf{g}_2, s_2)}_\beta, C_{G_{n-m}}((\mathbf{g}_2, s_2))]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{F^{(\mathbf{g}_2, s_2)}_\beta}\\
&= [X_1^m, G_m]^k_{\underline{\varphi}} \times [X_2^{n-m}, G_{n-m}]^k_{\underline{\varphi}}. \quad \square
\end{aligned}$$

Let X be a G-manifold and let c be an element of G acting trivially on X. Let r be a fixed positive integer. Denote by $G \cdot \langle a \rangle$ the group generated by G and the additional element a commuting with all the elements of G and such that $\langle a \rangle \cap G = \langle c \rangle$, $c = a^r$. Define the action of the group $G \cdot \langle a \rangle$ on X (an extension of the G-action) so that a acts trivially.

Lemma 3 (*Cf.* [4, *Lemma 4–1]*). *In the described situation one has*

$$[X, G \cdot \langle a \rangle]^k_{\underline{\varphi}} = r^k [X, G]^k_{\underline{\varphi}}.$$

Proof. We shall use the induction on k. For $k = 0$ this is obvious (since $[X, G]^0_{\underline{\varphi}} = [X/G]$). Each conjugacy class of elements from $G \cdot \langle a \rangle$ is of the form $[g]a^s$, where $[g] \in G_*$, $0 \le s < r$. The fixed point set of ga^s coincides with X^g, the Zaslow shift $F^{ga^s}_\alpha$ at each component of X^g coincides with F^g_α (since a acts trivially). The centralizer $C_{G\cdot\langle a\rangle}(ga^s)$ is $C_G(g) \cdot \langle a \rangle$. Therefore

$$[X, G \cdot \langle a \rangle]^k_{\underline{\varphi}} = \sum_{[g]\in G_*} r \sum_{\alpha=1}^{N_g} [X^g_\alpha, C_{G(g)} \cdot \langle a \rangle]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{F^g_\alpha} = r^k [X, G]^k_{\underline{\varphi}}. \quad \square$$

Theorem 1. *Let X be a smooth quasi-projective variety of dimension d with a G-action. Then*

$$\sum_{n\ge 0} [X^n, G_n]^k_{\underline{\varphi}} \cdot t^n = \left(\prod_{r_1,\ldots,r_k \ge 1} \left(1 - \mathbb{L}^{\Phi_k(\underline{r})d/2} \cdot t^{r_1 r_2 \cdots r_k}\right)^{r_2 r_3^2 \cdots r_k^{k-1}} \right)^{-[X,G]^k_{\underline{\varphi}}}, \tag{11}$$

where

$$\Phi_k(r_1, \ldots, r_k) = \varphi_1(r_1 - 1) + \varphi_2 r_1(r_2 - 1) + \cdots + \varphi_k r_1 r_2 \cdots r_{k-1}(r_k - 1).$$

Proof. To a big extend we shall follow the lines of the proof of Theorem A in [4]. We shall use the induction on the order k. For $k = 1$ the equation coincides with the one from Proposition 1. Assume that the statement is proved for the generalized Euler characteristic of order $k - 1$. One has

$$\sum_{n\ge 0} [X^n, G_n]^k_{\underline{\varphi}} \cdot t^n = \sum_{n \ge 0} t^n \left(\sum_{[(\mathbf{g},s)]\in G_{n*}} \sum_{comp} [(X^n)^{\langle(\mathbf{g},s)\rangle}_{comp}, C_{G_n}((\mathbf{g},s))]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{F^{(\mathbf{g},s)}_{comp}} \right),$$

where the sums are over all the conjugacy classes $[(\mathbf{g}, s)]$ of elements of G_n and over all the components of $(X^n)^{\langle(\mathbf{g},s)\rangle}$ (or rather unions of components from an orbit of the $C_{G_n}((\mathbf{g}, s))$-action on the components of it).

The conjugacy classes $[(\mathbf{g}, s)]$ of elements of G_n are characterized by their types. Let $a = (\mathbf{g}, s) \in G_n$, $\mathbf{g} = (g_1, \ldots, g_n)$. Let $z = (i_1, \ldots, i_r)$ be one of the cycles in the permutation s. The *cycle-product* of the element a corresponding to the cycle z is the product $g_{i_r} g_{i_{r-1}} \ldots g_{i_1} \in G$. The conjugacy class of the cycle-product is well-defined by the element $\mathbf{g}$ and the cycle z of the permutation s. For $[c] \in G_*$ and $r \ge 0$, let $m_r(c)$ be the number of r-cycles in the permutation s whose cycle-products lie in $[c]$. One has

$$\sum_{[c]\in G_*, r\ge 1} r m_r(c) = n.$$

The collection $\{m_r(c)\}_{r,c}$ is called the *type* of the element $a = (\mathbf{g}, s) \in G_n$. Two elements of the group G_n are conjugate to each other if and only if they are of the same type.

In [4] (see also [13]) it is shown that, for an element $(\mathbf{g}, s) \in G_n$ of type $\{m_r(c)\}$, the subspace $(X^n)^{\langle(\mathbf{g},s)\rangle}$ can be identified with

$$\prod_{[c]\in G_*} \prod_{r \ge 1} (X^{\langle c \rangle})^{m_r(c)}. \tag{12}$$

By [4, Theorem 3.5] the centralizer of the element $(\mathbf{g}, s) \in G_n$ is isomorphic to

$$\prod_{[c]\in G_*} \prod_{r\ge 1} \left\{ (C_G(c) \cdot \langle a_{r,c} \rangle) \wr S_{m_r(c)} \right\}$$

(acting on the product (12) component-wise) where $C_G(c) \cdot \langle a_{r,c} \rangle$ is the group generated by $C_G(c)$ and an element $a_{r,c}$ commuting with all the elements of $C_G(c)$ and such that $a^r_{r,c} = c$, $\langle a_{r,c} \rangle \cap C_G(c) = \langle c \rangle$, and $a_{r,c}$ acts on $(X^{\langle c \rangle})^{m_r(c)}$ trivially.

The components of $(X^{\langle c \rangle})^{m_r(c)}$ (with respect to the $C_G(c) \cdot \langle a_{r,c} \rangle$-action) are $S_{m_r(c)} \left(\prod_{\alpha=1}^{N_\alpha} (X^{\langle c \rangle}_\alpha)^{m_{r,c}(\alpha)} \right)$, where $\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha) = m_r(c)$. Here and below the sum over *comp* means the summation over all the components indicated in the summands. Therefore

$$\sum_{n\ge 0}[X^n, G_n]^k_{\underline{\varphi}} \cdot t^n = \sum_{n\ge 0} t^n \left(\sum_{[(\mathbf{g},s)]\in G_{n*}} \sum_{comp} [(X^n)^{\langle(\mathbf{g},s)\rangle}_{comp}, C_{G_n}((\mathbf{g},s))]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{F^{(\mathbf{g},s)}_{comp}} \right)$$

$$= \sum_{n\ge 0} t^n \cdot \left(\sum_{\{m_r(c)\}} \sum_{comp} \left[\prod_{[c],r} \left\{ (X^{\langle c\rangle})^{m_r(c)} \right\}_{comp}, \prod_{[c],r} \left\{ (C_G(c)\cdot\langle a_{r,c}\rangle) \wr S_{m_r(c)} \right\} \right]^{k-1}_{\underline{\varphi}} \cdot \mathbb{L}^{F^{(\mathbf{g},s)}_{comp}} \right)$$

$$= \sum_{n\ge 0} t^n \cdot \left(\sum_{\{m_{r,c}(\alpha)\}} \left\{ \prod_{[c],r} \left[S_{m_r(c)} \left(\prod_{\alpha=1}^{N_\alpha} (X^{\langle c\rangle}_\alpha)^{m_{r,c}(\alpha)} \right), (C_G(c)\cdot\langle a_{r,c}\rangle) \wr S_{m_r(c)} \right]^{k-1}_{\underline{\varphi}} \right.\right.$$

$$\left.\left. \times\ \mathbb{L}^{\phi_k(\sum_{[c],r}\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha)(F^c_\alpha + \frac{(r-1)d}{2}))} \right\} \right).$$

Iterating Lemma 2 one gets

$$= \sum_{\{m_{r,c}(\alpha)\}} t^{\sum rm_{r,c}(\alpha)} \prod_{[c],r} \left\{ \prod_{\alpha=1}^{N_\alpha} [(X^{\langle c\rangle}_\alpha)^{m_{r,c}(\alpha)}, (C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}]^{k-1}_{\underline{\varphi}} \right.$$

$$\left. \times\ \mathbb{L}^{\phi_k\left(\sum_{[c],r}\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha)(F^c_\alpha + \frac{(r-1)d}{2})\right)} \right\}$$

$$= \prod_{[c],r} \left(\prod_{\alpha=1}^{N_\alpha} \left(\sum_{\{m_{r,c}(\alpha)\}} t^{rm_{r,c}(\alpha)} [(X^{\langle c\rangle}_\alpha)^{m_{r,c}(\alpha)}, (C_G(c)\cdot\langle a_{r,c}\rangle)\wr S_{m_r(c)}]^{k-1}_{\underline{\varphi}} \right.\right.$$

$$\left.\left. \times\ \mathbb{L}^{\phi_k\left(\sum_{[c],r}\sum_{\alpha=1}^{N_\alpha} m_{r,c}(\alpha)(F^c_\alpha + \frac{(r-1)d}{2})\right)} \right) \right).$$

By induction one gets

$$= \prod_{[c],r}\prod_{\alpha=1}^{N_\alpha} \left(\prod_{r_1,\dots,r_{k-1}\ge 1} \left(1 - \mathbb{L}^{\Phi_{k-1}(r)\frac{d}{2}} (\mathbb{L}^{\varphi_k(F^c_\alpha + \frac{(r-1)d}{2})} t^r)^{r_1\cdots r_{k-1}} \right)^{r_2\cdot r_3^2\cdots r_{k-1}^{k-2}} \right)^{-[X^{\langle c\rangle}_\alpha, C_G(c)\cdot\langle a_{r,c}\rangle]^{k-1}_{\underline{\varphi}}}$$

$$= \left(\prod_{r,r_1,\dots,r_{k-1}\ge 1} \left(1 - \mathbb{L}^{\Phi_{k-1}(r)\frac{d}{2}} \mathbb{L}^{\varphi_k(r_1\cdots r_{k-1}\frac{(r-1)d}{2})} t^{r_1\cdots r_{k-1}\cdot r} \right)^{r_2\cdot r_3^2\cdots r_{k-1}^{k-2}} \right)^{-\sum_{[c],\alpha}[X^{\langle c\rangle}_\alpha, C_G(c)\cdot\langle a_{r,c}\rangle]^{k-1}_{\underline{\varphi}}\mathbb{L}^{\phi_k F^c_\alpha}}$$

(here we use the properties of the power structure)

$$= \left(\prod_{r_1,\dots,r_k\ge 1} \left(1 - \mathbb{L}^{(\Phi_{k-1}(r)+\varphi_k r_1\cdots r_{k-1}(r_k-1))\frac{d}{2}} t^{r_1\cdots r_{k-1}\cdot r_k} \right)^{r_2\cdot r_3^2\cdots r_{k-1}^{k-2}} \right)^{-r_k^{k-1}\sum_{[c],\alpha}[X^{\langle c\rangle}_\alpha, C_G(c)]^{k-1}_{\underline{\varphi}}\mathbb{L}^{\phi_k F^c_\alpha}}$$

$$= \left(\prod_{r_1,\dots,r_k\ge 1} \left(1 - \mathbb{L}^{\Phi_k(\underline{r})d/2} t^{r_1 r_2\cdots r_k} \right)^{r_2 r_3^2\cdots r_k^{k-1}} \right)^{-[X,G]^k_{\underline{\varphi}}}.$$

In the last two equations r is substituted by r_k. □

Remark. For $\underline{\varphi} = \underline{0}$, i.e. if $\varphi_i = 0$ for all i, the definition of the higher order generalized Euler characteristics does not demand X to be smooth. This way one gets the definition of a sort of higher order inertia classes and the statement of Theorem 1 holds for an arbitrary G-variety X.

Since $\chi([X^n, G_n]^k_{\underline{\varphi}}) = \chi^{(k)}(X, G)$, $\chi(\mathbb{L}) = 1$, taking the Euler characteristic of the both sides of Eq. (11) one gets Theorem A of [4]:

$$\sum_{n\ge 0}\chi^{(k)}(X^n, G_n)\cdot t^n = \left(\prod_{r_1,\dots,r_k\ge 1} \left(1 - t^{r_1 r_2\cdots r_k}\right)^{r_2 r_3^2\cdots r_k^{k-1}} \right)^{-\chi^{(k)}(X,G)}.$$

Let $e^{(k)}_{\underline{\varphi}}(X, G; u, v) := e([X, G]^k_{\underline{\varphi}}; u, v)$ be the higher order Hodge–Deligne polynomial of (X, G) (of order k). Applying the Hodge–Deligne polynomial homomorphism, one gets a generalization of the main result in [13]:

$$\sum_{n\geq 0} e^{(k)}_{\underline{\varphi}}(X^n, G_n; u, v) \cdot t^n = \left(\prod_{r_1,\ldots,r_k\geq 1} \left(1 - (uv)^{\Phi_k(\underline{r})d/2} \cdot t^{r_1 r_2 \cdots r_k}\right)^{r_2 r_3^2 \cdots r_k^{k-1}} \right)^{-e^{(k)}_{\underline{\varphi}}(X,G;u,v)}.$$

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