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On representations of the filiform Lie superalgebra $L_{m,n}$

Qi Wang [a], Hongjia Chen [b], Wende Liu [a,c,*]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin 150006, China

[b] Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

[c] School of Mathematical Sciences, Harbin Normal University, Harbin 150025, China

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ABSTRACT

In this paper, we study the representations for the filiform Lie superalgebras $L_{m,n}$, a particular class of nilpotent Lie superalgebras. We determine the minimal dimension of a faithful module over $L_{m,n}$ using the theory of linear algebra. In addition, using the method of Feingold and Frenkel (1985), we construct some finite and infinite dimensional modules over $L_{m,n}$ on the Grassmann algebra and the mixed Clifford–Weyl algebra.

0. Introduction

The concept of filiform Lie algebras (also called threadlike Lie algebras, see [1] and the AMS review of the paper [2]) was first introduced by Vergne [3] in her study of the reducibility of the varieties of nilpotent Lie algebras by using the cohomology theory of Lie algebras. She proved that the subset of filiform Lie algebras is open in the varieties of nilpotent Lie algebras which plays an important role in showing that the varieties are reducible in high dimensions. The study of filiform Lie algebras has been the subject of a number of papers, [4–7] to name a few. In particular, Lie groups of lots of filiform Lie algebras do not admit a left-invariant affine structure which are counterexamples for Milnor's conjecture [8].

As a byproduct, in [3] Vergne showed the existence of only two naturally graded filiform Lie algebras, L_m and Q_n, the second existing only in even dimension. Among them, the first algebra (is called the model filiform Lie algebra) has been a central research object for the last forty years. In particular, since its cohomology has been calculated in [9], which allowed to describe its infinitesimal deformations in a precise way, thus by studying its deformations, lots of families of characteristically nilpotent Lie algebras (that is, Lie algebras with only nilpotent derivations) have been constructed [10]. For more information on characteristically nilpotent Lie algebras, see the survey paper [11].

The filiform Lie superalgebras, a particular class of nilpotent Lie superalgebras, were introduced by Gilg in [12], which are super-analogue to the filiform Lie algebras. Analogous to what happens in the Lie case, where every filiform Lie algebra can be obtained by an infinitesimal deformation of the model filiform Lie algebra L_n, one can prove that all filiform Lie

* Corresponding author at: School of Mathematical Sciences, Harbin Normal University, Harbin 150025, China.
E-mail address: wendeliu@ustc.edu.cn (W. Liu).

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superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L_{m,n}$ [13]. The deformation of filiform Lie superalgebras were further studied recently, see [14–16]. In particular, Khakimdjanov and Navarro gave a complete classification of all the infinitesimal deformations of the model Lie superalgebra $L_{m,n}$ in [17] which play a crucial role in the theory of filiform Lie superalgebras.

However, less of work is done for the representations of filiform Lie (super)algebras. For a finite-dimensional Lie (super)algebra $\mathfrak{g}$, we define the minimal dimension of faithful $\mathfrak{g}$-modules as

$$\mu(\mathfrak{g}) = \min\{\dim V \mid V \text{ is a faithful } \mathfrak{g}\text{-module}\}.$$

In [18,19], Burde considered the faithful representations of filiform Lie algebras, in which he obtained that for any $(n+1)$-dimensional filiform Lie algebra $\mathfrak{g}$, $\mu(\mathfrak{g}) \geq n+1 = \mu(L_n)$ and also gave an upper bound for $\mu(\mathfrak{g})$. When one considers a similar problem in the super case, due to the lack of Lie's Theorem, the situation is much more complicated. Even though for the abelian Lie superalgebras with nontrivial odd part, this problem is still open. The third-named author determined the minimal dimensions of faithful modules for Heisenberg Lie superalgebras and for purely odd Lie superalgebras in [20] and [21]. To our knowledge, the minimal dimension of the faithful module for Lie superalgebra $L_{m,n}$ is still unknown. In the current paper, we determine the minimal dimensions of faithful modules for the filiform Lie superalgebras $L_{m,n}$:

Theorem 1. *For the filiform Lie superalgebra $L_{m,n}$, we have*

(1) $\mu(L_{m,0}) = m+1$;
(2) $\mu(L_{m,n}) = m+2$ *if* $m \geq n > 0$;
(3) $\mu(L_{0,n}) = n+1$ *for* $n \geq 1$; $\mu(L_{1,n}) = n+1$ *for* $n \geq 2$;
(4) $\mu(L_{m,n}) = n+2$ *if* $n > m \geq 2$.

The Clifford (or Weyl) algebras have natural representations on the exterior (or symmetric) algebras of polynomials over half of generators. Those representations are important in quantum and statistical mechanics where the generators are interpreted as operators which create or annihilate particles and satisfy Fermi (or Bose) statistics. Moreover, they have deep connections with many other important algebras, such as extended affine Lie algebras, Kac–Moody algebras and the Virasoro algebra, see [22–26]. In the last part of the paper, we use Clifford algebras, Weyl algebras and the mixed algebras to construct some infinite-dimensional representations. In addition, we also construct a finite-dimensional representation of $L_{m,n}$ on the Grassmann algebra.

Throughout this paper, we write $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{C}$ for the sets of integers, positive integers, nonnegative integers and complex numbers, respectively. Write $\mathbb{Z}_2$ for the two element field. Write e_{ij} for the standard matrix unit (1 at the spot (i,j) and 0 elsewhere). All vector spaces, algebras and Lie (super)algebras are over $\mathbb{C}$. Let $|v|$ be the $\mathbb{Z}_2$-degree of v, where v is a homogeneous element of a $\mathbb{Z}_2$-graded vector space V.

1. Preliminary remarks

For a Lie superalgebra L, we first recall the lower central series of L

$$L^0 = L, \qquad L^{i+1} = [L, L^i].$$

In order to describe the super-nilindex of a Lie superalgebra L, we inductively define two sequences:

$$L_{\bar{0}}^0 = L_{\bar{0}}, \qquad L_{\bar{0}}^{i+1} = [L_{\bar{0}}, L_{\bar{0}}^i]$$

and

$$L_{\bar{1}}^0 = L_{\bar{1}}, \qquad L_{\bar{1}}^{i+1} = [L_{\bar{0}}, L_{\bar{1}}^i].$$

Then we have the fact that L is nilpotent if and only if there exists $(m,n) \in \mathbb{N}^2$ such that $L_{\bar{0}}^m = 0$ and $L_{\bar{1}}^n = 0$, for the details see [13, Theorem 2.1].

By the above results, the super-nilindex can be defined as follows: Let L be a nilpotent Lie superalgebra, the *super-nilindex* of L is the pair (m,n) of integers such that: $L_{\bar{0}}^m = 0$, $L_{\bar{0}}^{m-1} \neq 0$ and $L_{\bar{1}}^n = 0$, $L_{\bar{1}}^{n-1} \neq 0$. It is invariant up to isomorphisms. The filiform Lie superalgebra (see [13]) is defined as follows:

Definition 1.1. Let L be a nilpotent Lie superalgebra. L is called *filiform* if its super-nilindex is $(\dim L_{\bar{0}} - 1, \dim L_{\bar{1}})$.

Denote $\mathcal{F}_{m,n}$ the set of all filiform Lie superalgebras with super-nilindex (m,n). As for the filiform Lie algebra [9], there is an adapted basis for a filiform Lie superalgebra:

For any $L = L_{\bar{0}} \oplus L_{\bar{1}} \in \mathcal{F}_{m,n}$, there exists a basis $\{X_0, X_1, \ldots, X_m, Y_1, Y_2, \ldots, Y_n\}$ of L with $X_i \in L_{\bar{0}}$ and $Y_i \in L_{\bar{1}}$ such that:

$$[X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq m-1, \qquad [X_0, X_m] = 0;$$
$$[X_1, X_2] \in \operatorname{span}_{\mathbb{C}}\{X_i \mid 4 \leq i \leq m\};$$
$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq n-1, \qquad [X_0, Y_n] = 0.$$

The above result was given by Gilg in [13].

Let $L_{m,n}$ be a filiform Lie superalgebra with a homogeneous basis

$$\{X_0, X_1, \dots, X_m \mid Y_1, \dots, Y_n\}$$

and the Lie super-brackets given by

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le m-1, \qquad [X_0, Y_j] = Y_{j+1}, \quad 1 \le j \le n-1,$$

where the other brackets vanish. We call $L_{m,n}$ the *model filiform Lie superalgebra* and call $\{X_0, X_1, \dots, X_m, Y_1, \dots, Y_n\}$ the *standard basis* of $L_{m,n}$. The even part $L_{m,n}^{\bar{0}}$ is the *model filiform Lie algebra* (always denoted by L_m) and the odd part $L_{m,n}^{\bar{1}}$ is a module of $L_{m,n}^{\bar{0}}$.

2. Minimal faithful representations

In this section, we will prove Theorem 1. We also show that $L_{m,n}$ is not the simplest filiform Lie superalgebra of $\mathcal{F}_{m,n}$ in the sense of minimal faithful representations by providing an example.

We will show the theorem case by case. The first case in Theorem 1 follows from the following result (see [29] or [19, Corollaries 2.6 and 2.8]).

Proposition 2.1. *For the model filiform Lie algebra L_m, we have $\mu(L_m) = m+1$. Moreover, the embedding $\rho : L_m \longrightarrow \mathfrak{gl}(m+1)$ given by*

$$X_0 \longmapsto \sum_{i=1}^{m-1} e_{i,i+1}, \qquad X_i \longmapsto e_{m+1-i,m+1}$$

for $1 \le i \le m$, is a Lie algebra homomorphism.

Now we have the fact:

Lemma 2.2. *If $\rho : L_m \longrightarrow \mathfrak{gl}(r,s)_{\bar{0}}$ is an injective homomorphism of Lie algebras, then $r \ge m+1$ or $s \ge m+1$, i.e., $\max(r,s) \ge m+1 = \mu(L_m)$.*

Proof. Since $\mathfrak{gl}(r,s)_{\bar{0}} = \mathfrak{gl}(r) \oplus \mathfrak{gl}(s)$, we can write $\rho = (\rho_1, \rho_2)$ such that

$$\rho(x) = \mathrm{diag}(\rho_1(x), \rho_2(x))$$

with $\rho_1(x) \in \mathfrak{gl}(r)$ and $\rho_2(x) \in \mathfrak{gl}(s)$ for any $x \in L_m$. It is clear that $\rho_1 : L_m \longrightarrow \mathfrak{gl}(r)$ and $\rho_2 : L_m \longrightarrow \mathfrak{gl}(s)$ are Lie algebra homomorphisms. By Proposition 2.1, it is enough to show that at least one of ρ_1 and ρ_2 is injective. Conversely suppose there exist nonzero elements $x, y \in L_m$ such that $\rho_1(x) = 0$ and $\rho_2(y) = 0$. Put $x = \sum_{i=0}^{m} a_i X_i$, where $\{X_0, X_1, \dots, X_m\}$ is a basis of L_m with nonzero brackets:

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le m-1.$$

If $a_0 \ne 0$, then $\rho_1(X_m) = a_0^{-1}[\rho_1(x), \rho_1(X_{m-1})] = 0$. Otherwise, take

$$k = \min\{i \mid a_i \ne 0, 1 \le i \le m\}.$$

Then we have

$$\rho_1(X_m) = a_k^{-1} \rho_1\big((\mathrm{ad}\, X_0)^{m-k}(x)\big) = 0.$$

Similarly, we have $\rho_2(X_m) = 0$. This contradicts the injectivity of ρ. $\square$

Now we can get the second case in Theorem 1:

Corollary 2.3. *$\mu(L_{m,n}) = m+2$ if $m \ge n > 0$.*

Proof. Suppose that $L_{m,n}$ can be embedded into some $\mathfrak{gl}(r,s)$, and denote the map by ρ. Then $\rho|_{L_{m,n}^{\bar{1}}} : L_{m,n}^{\bar{1}} \longrightarrow \mathfrak{gl}(r,s)_{\bar{1}}$ is an injective map of vector spaces and $\rho|_{L_{m,n}^{\bar{0}}} : L_{m,n}^{\bar{0}} \longrightarrow \mathfrak{gl}(r,s)_{\bar{0}}$ is an injective homomorphism of Lie algebras. Since $rs \ne 0$ and Lemma 2.2 implies that $\max(r,s) \ge m+1$, whence $r+s \ge m+2$. We finish the proof by constructing a Lie superalgebra homomorphism from $L_{m,n}$ to $\mathfrak{gl}(m+1,1)$ which is injective. Consider the linear map $\varphi : L_{m,n} \longrightarrow \mathfrak{gl}(m+1,1)$ given by

$$X_0 \longmapsto \sum_{i=1}^{m-1} e_{i,i+1}, \qquad X_i \longmapsto e_{m+1-i,m+1}, \qquad Y_j \longmapsto e_{n+1-j,m+2}$$

where $1 \le i \le m$, $1 \le j \le n$ and $\{X_0, X_1, \dots, X_m, Y_1, \dots, Y_n\}$ is the standard basis of $L_{m,n}$. It is easy to check that φ is an injective homomorphism of Lie superalgebras. $\square$

Proposition 2.4. *If $m = 0$ or 1 and $m < n$, then $\mu(L_{m,n}) = n + 1$.*

Proof. First, consider the case $(m, n) = (0, 1)$. It is clear that $\mu(L_{0,1}) \geq 2$, and the linear map $\varphi : L_{0,1} \longrightarrow \mathfrak{gl}(1, 1)$ given by

$$X_0 \longmapsto I_2, \qquad Y_1 \longmapsto e_{12}$$

is an injective Lie superalgebra homomorphism. Whence $\mu(L_{0,1}) = 2$.

For $n \geq 2$, the linear map $\varphi : L_{1,n} \longrightarrow \mathfrak{gl}(n, 1)$ given by

$$X_0 \longmapsto \sum_{i=1}^{n-1} e_{i,i+1}, \qquad X_1 \longmapsto I_{n+1}, \qquad Y_i \longmapsto e_{n+1-i,n+1} \quad \text{for } 1 \leq i \leq n$$

is an injective homomorphism of Lie superalgebras, thus $\mu(L_{0,n}) \leq \mu(L_{1,n}) \leq n + 1$. The first "$\leq$" follows from the fact: If L' is a subalgebra of Lie (super)algebra L, then $\mu(L') \leq \mu(L)$. Now consider $\mu(L_{0,n})$ and suppose $\rho : L_{0,n} \longrightarrow \mathfrak{gl}(r, s)$ is an injective homomorphism of Lie superalgebras with

$$X_0 \longmapsto \begin{pmatrix} J & 0 \\ 0 & K \end{pmatrix} \quad \text{and} \quad Y_1 \longmapsto \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

For a matrix T, denote T_l the operator which multiplies T on the left and T_r the operator which multiplies T on the right. Then by induction we have

$$Y_i = (\operatorname{ad} X_0)^{i-1} Y_1 \longmapsto \begin{pmatrix} 0 & (J_l - K_r)^{i-1}(A) \\ (K_l - J_r)^{i-1}(B) & 0 \end{pmatrix}.$$

Without loss of generality, assume $(J_l - K_r)^{n-1}(A) \neq 0$ and $(J_l - K_r)^n(A) = 0$. If $\sum_{i=0}^{n-1} a_i (J_l - K_r)^i(A) = 0$ for $a_i \in \mathbb{C}$ with some $a_j \neq 0$, then $(J_l - K_r)^{n-1}(A) = 0$. Whence the linear map $\varphi : L_{0,n} \longrightarrow \mathfrak{gl}(r, s)$ given by

$$\varphi(X_0) = \rho(X_0) = \begin{pmatrix} J & 0 \\ 0 & K \end{pmatrix}, \qquad \varphi(Y_1) = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(Y_i) = (\operatorname{ad} \varphi(X_0))^{i-1}(\varphi(Y_1))$$

is still an injective homomorphism of Lie superalgebras. Note that since $[Y_i, Y_j] = 0$ for $1 \leq i, j \leq n$, we can view $L_{0,n}$ as a Lie algebra if we forget its $\mathbb{Z}_2$-grading, which is isomorphic to L_n. Thus φ can induce a monomorphism $\bar{\varphi} : L_{0,n}(\cong L_n) \to \mathfrak{gl}(r+s)$ of Lie algebras such that $\bar{\varphi}(X_0) = \varphi(X_0)$ and $\bar{\varphi}(Y_i) = \varphi(Y_i)$, for $1 \leq i \leq n$. Proposition 2.1 implies that $r+s \geq \mu(L_n) = n+1$. Thus $\mu(L_{1,n}) \geq \mu(L_{0,n}) \geq n + 1$, which implies that $\mu(L_{0,n}) = \mu(L_{1,n}) = n + 1$. $\square$

Now we consider the last case.

Proposition 2.5. *$\mu(L_{m,n}) = n + 2$ if $n > m \geq 2$.*

Proof. First consider the linear map $\varphi : L_{m,n} \longrightarrow \mathfrak{gl}(n + 1, 1)$ given by

$$X_0 \longmapsto \sum_{i=1}^{n-1} e_{i,i+1}, \qquad X_i \longmapsto e_{m+1-i,n+1}, \qquad Y_j \longmapsto e_{n+1-j,n+2}.$$

It is easy to check that φ is an injective homomorphism of Lie superalgebras. Whence $\mu(L_{m,n}) \leq n + 2$. On the other hand, we have $\mu(L_{m,n}) \geq \mu(L_{0,n}) = n + 1$.

Suppose $\rho : L_{m,n} \longrightarrow \mathfrak{gl}(r, s)$ is an injective Lie superalgebra homomorphism with $r + s = n + 1$ and

$$X_0 \longmapsto \begin{pmatrix} J & 0 \\ 0 & K \end{pmatrix} \quad \text{and} \quad Y_1 \longmapsto \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

We can assume $B = 0$. Now assume the matrices J and K have the Jordan canonical forms, i.e., $J = \operatorname{diag}(J_1, \ldots, J_k)$ and $K = \operatorname{diag}(K_1, \ldots, K_l)$ with

$$J_i = \begin{pmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}, \qquad K_j = \begin{pmatrix} \eta_j & 1 & \cdots & 0 & 0 \\ 0 & \eta_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \eta_j & 1 \\ 0 & 0 & \cdots & 0 & \eta_j \end{pmatrix}.$$

Rewrite $A = (A_{ij})_{k \times l}$ in block form. Then for $1 \leq t \leq n$, we have

$$(J_l - K_r)^t(A) = \left(\left((J_i)_l - (K_j)_r \right)^t A_{ij} \right)_{k \times l}.$$

Suppose that $\left((J_i)_l - (K_j)_r\right)^{n-1} A_{ij} \neq 0$ and $\left((J_i)_l - (K_j)_r\right)^{n} A_{ij} = 0$ for some $1 \leq i \leq k$, $1 \leq j \leq l$. Since we can subtract a scalar matrix $\lambda_i I_{r+s}$ from $\varphi(X_0)$, we take $\lambda_i = 0$. Note that

$$J_i \cdot \left(\left((J_i)_l - (K_j)_r\right)^{n-1} A_{ij}\right) = \left(\left((J_i)_l - (K_j)_r\right)^{n-1} A_{ij}\right) \cdot K_j$$

implies $\eta_j = 0$ or $\left((J_i)_l - (K_j)_r\right)^{n-1} A_{ij} = 0$. Since

$$\sum_{t=0}^{n-1} (-1)^{n-t-1} \binom{n-1}{t} J_i^t A_{ij} K_j^{n-1-t} = \left((J_i)_l - (K_j)_r\right)^{n-1} A_{ij} \neq 0,$$

we have $\mathrm{rank}(J_i) + \mathrm{rank}(K_j) \geq n - 1$. As $r + s = n + 1$, we have

$$J = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{r \times r}, \qquad K = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{s \times s}$$

and

$$Y_i = (\mathrm{ad}\, X_0)^{i-1} Y_1 \longmapsto \begin{pmatrix} 0 & (J_l - K_r)^{i-1}(A) \\ 0 & 0 \end{pmatrix} \triangleq \begin{pmatrix} 0 & A_i \\ 0 & 0 \end{pmatrix}$$

where $A = (a_{ij})_{r \times s}$ and $A_i = \sum_{t=0}^{i-1} (-1)^{i-1-t} \binom{i-1}{t} J^t A K^{i-1-t}$. In particlular, we obtain

$$0 \neq (-1)^{n-r} A_n = \binom{n-1}{r-1} J^{r-1} A K^{s-1} = \binom{n-1}{r-1} e_{1r} A e_{1s} = \binom{n-1}{r-1} a_{r1} e_{1s},$$

i.e; $a_{r1} \neq 0$. Since $[X_0, X_m] = 0$ and $[X_0, X_{m-1}] = X_m$, we have

$$X_m \longmapsto \begin{pmatrix} \sum_{t=1}^{r-1} b_t J^t & 0 \\ 0 & \sum_{t=1}^{s-1} c_t K^t \end{pmatrix}.$$

Suppose $b_i = c_i = 0$ for $1 \leq i \leq \alpha - 1$. Then we have

$$\begin{aligned}
&\sum_{t=\alpha}^{r-1} b_t J^t A_{n-\alpha} - A_{n-\alpha} \sum_{t=\beta}^{s-1} c_t K^t \\
&\quad = \sum_{u=0}^{n-\alpha-1} \sum_{t=\alpha}^{r-1} (-1)^{n-\alpha-1-u} \binom{n-\alpha-1}{u} b_t J^{t+u} A K^{n-\alpha-1-u} \\
&\quad\quad - \sum_{v=0}^{n-\alpha-1} \sum_{t=\alpha}^{s-1} (-1)^{n-\alpha-1-v} \binom{n-\alpha-1}{v} c_t J^v A K^{n-\alpha-1-v+t} \\
&\quad = (-1)^{s-1} \binom{r+s-\alpha-2}{r-\alpha-1} b_\alpha a_{r1} e_{1,s} - (-1)^{s-\alpha-1} \binom{r+s-\alpha-2}{r-1} c_\alpha a_{r1} e_{1,s} + \sum_{i \neq 1 \text{ or } j \neq s} \lambda_{ij} e_{ij}.
\end{aligned}$$

For the last "=", considering the last second term, then the terms including e_{1s} are

$$\begin{aligned}
&\sum_{u=0}^{n-\alpha-1} \sum_{t=\alpha}^{r-1} (-1)^{n-\alpha-1-u} \binom{n-\alpha-1}{u} b_t e_{1,t+u+1} A e_{s-(n-\alpha-1-u),s} \\
&\quad - \sum_{v=0}^{n-\alpha-1} \sum_{t=\alpha}^{s-1} (-1)^{n-\alpha-1-v} \binom{n-\alpha-1}{v} c_t e_{1,1+v} A e_{s-(n-\alpha-1-v+t),s}.
\end{aligned}$$

Since $t + u + 1 \leq r$ and $n - \alpha - 1 - u \leq s - 1 = n - r$, we have $u + \alpha + 1 \geq r \geq t + u + 1$. On the other hand, $t \geq \alpha$. Hence, $t = \alpha$ and $u = r - \alpha - 1$. For the second one, we have $1 + v \leq r$ and $n - \alpha - 1 - v + t \leq s - 1 = n - r$. Thus $1 + v \leq r \leq 1 + v + \alpha - t$. On the other hand, $t \geq \alpha$. Hence, $t = \alpha$ and $v = r - 1$. So the coefficient of e_{1s} is

$$(-1)^{n-\alpha-1-(r-\alpha-1)} \binom{n-\alpha-1}{r-\alpha-1} b_\alpha a_{r1} - (-1)^{n-\alpha-1-(r-1)} \binom{n-\alpha-1}{r-1} c_\alpha a_{r1}.$$

Whence, $[X_m, Y_{n-\alpha}] = 0$ implies

$$\binom{r+s-\alpha-2}{r-\alpha-1} b_\alpha + (-1)^{\alpha+1} \binom{r+s-\alpha-2}{r-1} c_\alpha = 0. \tag{2.1}$$

Similarly, $[X_m, Y_{n-\alpha-1}] = 0$ implies that

$$\binom{r+s-\alpha-3}{r-\alpha-1} b_\alpha + (-1)^{\alpha+1} \binom{r+s-\alpha-3}{r-1} c_\alpha = 0. \tag{2.2}$$

Solve Eqs. (2.1) and (2.2), then we can get $b_\alpha = c_\alpha = 0$. Now by induction and noticing that $n > m$, we have $\rho(X_m) = 0$, which is a contradiction. □

Proposition 2.6. *The model filiform Lie superalgebra $L_{m,n}$ can be embedded in $\mathfrak{gl}(r, s)$ with $r + s = \mu(L_{m,n})$ and $r \geq s \geq 1$ if and only if $r \geq m + 1$.*

Proof. The "only if" part follows from Lemma 2.2.

For the "if" part, we only need to construct the embedding case by case.

(1) For $m \geq n \geq 0$, the embedding is given in Proposition 2.1 and in the proof of Corollary 2.3.
(2) For $m = 0, n = 1$, the embedding is given in Proposition 2.4.
(3) For $L_{1,n}$ with $n \geq 2$, we have that the linear map $\rho : L_{1,n} \longrightarrow \mathfrak{gl}(r, s)$ given by:

$$X_0 \longmapsto \mathrm{diag}(J_{r\times r}, J_{s\times s}), \qquad X_1 \longmapsto I_{n+1}, \qquad Y_j \longmapsto \sum_{t=0}^{j-1} (-1)^{j-t-1} \binom{j-1}{t} e_{r-t,r+j-t}$$

is an injective Lie superalgebra homomorphism (we always assume $e_{ij} = 0$ if $i < 0$ or $j < 0$ hereafter). Notice that $L_{0,n}$ is a subalgebra of $L_{1,n}$, whence we have the embedding $\rho|_{L_{0,n}}$ for $L_{0,n}$.
(4) For $n > m \geq 2$, we can define the linear map $\rho : L_{m,n} \longrightarrow \mathfrak{gl}(r, s)$ with $r + s = n + 2$ by

$$X_0 \longmapsto \mathrm{diag}(J_{r\times r}, J_{s\times s}), \qquad X_i \longmapsto e_{m+1-i,r}, \qquad Y_j \longmapsto \sum_{t=0}^{j-1} (-1)^{j-t-1} \binom{j-1}{t} e_{r-t-1,r+j-t}$$

which is an embedding. □

Remark 2.7. (1) Since we have the Lie superalgebra isomorphism $\mathfrak{gl}(r, s) \cong \mathfrak{gl}(s, r)$, Proposition 2.6 gives a description of the possible super-dimensions of the minimal faithful representations of $L_{m,n}$.
(2) For any m, n, denote $r = \max(m + 1, n + 1)$. Then we have the injective Lie superalgebra homomorphism $\rho : L_{m,n} \longrightarrow \mathfrak{gl}(r, 1)$ given by

$$X_0 \longmapsto \sum_{t=1}^{r-2} e_{t,t+1}, \qquad X_i \longmapsto e_{m+1-i,r}, \qquad Y_j \longmapsto e_{n+1-j,r+1}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In Lie algebra case, from [19, Proposition 2.5] we know that for nilpotent Lie algebra $\mathfrak{n}$ of class m, $\mu(\mathfrak{n}) \geq m+1 = \mu(L_m)$. In particular, L_m is the simplest filiform Lie algebra among all the filiform Lie algebras with dimension $m + 1$ in the sense of minimal faithful representations. But in super case, we have the following example:

Example 1. In $\mathfrak{gl}(3, 1)$, we take

$$X_0 = -e_{12} - e_{23}, \qquad X_1 = e_{12},\ X_2 = e_{13}, \qquad Y_1 = e_{34}, \qquad Y_2 = -e_{24}, \qquad Y_3 = e_{14},$$

then $[Y_i, Y_j] = 0$ for all $1 \leq i, j \leq 3$ and

$$\begin{aligned} &[X_0, X_1] = X_2, \quad [X_0, X_2] = 0, \quad [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3, \quad [X_0, Y_3] = 0, \\ &[X_1, X_2] = [X_1, Y_1] = [X_1, Y_3] = [X_2, Y_2] = [X_2, Y_3] = 0, \qquad -[X_1, Y_2] = Y_3 = [X_2, Y_1]. \end{aligned}$$

Then $L = \mathrm{span}_{\mathbb{C}}\{X_0, X_1, X_2, Y_1, Y_2, Y_3\}$ is a subalgebra of $\mathfrak{gl}(3, 1)$. On the other hand, L is a filiform Lie superalgebra with super-nilindex $(2, 3)$ which is listed in [12] ((4) in $\mathcal{F}_{2,3}$). Now we get $\mu(L_{2,3}) = 5 > 4 \geq \mu(L)$. Indeed, $\mu(L) = 4$ since $\mu(L) \geq \mu(L_{2,0}) + 1 = 4$.

Whence, in the sense of minimal faithful representations, $L_{m,n}$ is not always the simplest filiform Lie superalgebra in $\mathcal{F}_{m,n}$.

For $m, n \in \mathbb{N}$, let $L_{(m,n)}$ be the Lie algebra with a basis

$$\{X_0, X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$$

and the Lie brackets given by

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le m-1, \qquad [X_0, Y_j] = Y_{j+1}, \quad 1 \le j \le n-1,$$

where the other brackets vanish. If we forget the $\mathbb{Z}_2$-grading of $L_{m,n}$, then $L_{m,n}$ is a Lie algebra which is isomorphic to $L_{(m,n)}$. But the representations of $L_{m,n}$ and $L_{(m,n)}$ are different, such as we have that $\mu(L_{(m,1)}) = m + 1 \neq m + 2 = \mu(L_{m,1})$ for $m \ge 2$.

3. Representations of $L_{m,n}$

3.1. Lie's Theorem for $L_{m,n}$

In general, the Lie's Theorem does not hold for Lie superalgebras. But for $L_{m,n}$ we have the following result:

Proposition 3.1. *Every finite-dimensional irreducible representation of $L_{m,n}$ has dimension one.*

Proof. Suppose V is a finite-dimensional irreducible representation of $L_{m,n}$. We prove the result in the following three steps:

(1) If $m \ge 2$. Since X_m is a linear transformation on the finite-dimensional vector space V, there is $\lambda \in \mathbb{C}$ such that $V_\lambda = \{v \in V \mid X_m v = \lambda v\}$ is nonzero. As X_m is in the center of $L_{m,n}$, we have that $V_\lambda = V$. This implies that under certain basis of V, X_m becomes the scalar matrix λI_V. On the other hand, $[X_0, X_{m-1}] = X_m$ implies that the trace of X_m is zero, i.e., $\lambda = 0$ and X_m acts on V trivially. Then V is also irreducible over $L_{m-1,n} \cong L_{m,n}/\mathbb{C}X_m$. Inductively, we have that V is an irreducible representation over $L_{1,n}$.

(2) In $L_{1,n}$, X_1 is a trivial central element. This implies that if V is an irreducible representation over $L_{1,n}$ then it is also irreducible over the subalgebra $L_{0,n}$.

(3) Assume that V is a finite-dimensional irreducible representation over $L_{0,n}$ for $n \ge 2$. Since Y_n is a linear transformation on V, hence there is $\mu \in \mathbb{C}$ such that $V_\mu = \{v \in V \mid Y_n v = \mu v\}$ is nonzero. If $\mu \neq 0$, then consider the subspace $V'_\mu = V_\mu \oplus V_{-\mu}$. It is easy to check that V'_μ is a nonzero $L_{0,n}$-submodule of V. Whence $V = V'_\mu$. Suppose $\{v_1, v_2, \ldots, v_r\}$ is a basis of V_μ and $v_i = u_i + w_i$ for $u_i \in V_{\bar{0}}$ and $w_i \in V_{\bar{1}}$. Since

$$\mu u_i + \mu w_i = \mu v_i = Y_n v_i = Y_n u_i + Y_n w_i,$$

we have $Y_n u_i = \mu w_i$ and $Y_n w_i = \mu u_i$. Moreover, $Y_n(u_i - w_i) = -\mu(u_i - w_i)$ and $\{u_1, \ldots, u_r, w_1, \ldots, w_r\}$ are linearly independent. This implies that $\dim V_\mu \le \dim V_{-\mu}$. Similarly, we have $\dim V_\mu \ge \dim V_{-\mu}$, i.e., we get $\dim V_\mu = \dim V_{-\mu}$. Furthermore, we know that $\{u_1, \ldots, u_r, w_1, \ldots, w_r\}$ is a homogeneous basis of V. Under this basis

$$Y_n = \begin{pmatrix} 0 & \mu I_r \\ \mu I_r & 0 \end{pmatrix}.$$

Assume

$$X_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad Y_{n-1} = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

Then $[X_0, Y_n] = 0$ implies that $A = B$. Now

$$\begin{pmatrix} 0 & \mu I_r \\ \mu I_r & 0 \end{pmatrix} = Y_n = [X_0, Y_{n-1}] = \begin{pmatrix} 0 & AC - CA \\ AD - DA & 0 \end{pmatrix}.$$

By the fact $\mathrm{tr}(AC - CA) = 0$, we get $\mu = 0$, which is a contradiction. Whence $\mu = 0$, $V_0 = V$ and $Y_n V = 0$. Moreover, we have V is also irreducible over $L_{0,n-1} \cong L_{0,n}/\mathbb{C}Y_n$. By induction again, we have V is irreducible over the abelian Lie superalgebra $L_{0,1}$, thus V has dimension one. □

Remark 3.2. Proposition 3.1 also follows from [27, Proposition 5.2.4] given by Kac where he used a 'big' Theorem to imply the proposition. Here, we give a fundamental proof of Lie's Theorem on $L_{m,n}$ by using the theory of linear algebra as what we did in Section 2.

3.2. Some representations

If $\rho : L_{m,n} \longrightarrow \mathfrak{gl}(r, s)$ is a Lie superalgebra homomorphism, then every representation over $\mathfrak{gl}(r, s)$ becomes a representation over $L_{m,n}$ through ρ. Whence we can get lots of representations for $L_{m,n}$ by considering representations over $\mathfrak{gl}(r, s)$ such as the representations constructed in [24,28]. In this section, we would not consider all of them. We just give some interesting examples instead.

In this subsection, we fix $m, n \in \mathbb{Z}_+$ and denote $r = \max(m+1, n+1)$. Let us construct the representations for $L_{m,n}$. Since all of $L_{0,0}$, $L_{0,1}$, $L_{1,0}$ and $L_{1,1}$ are abelian Lie (super)algebras, hence we always assume $r \geq 3$.

Let $\mathcal{R}$ be an associative algebra. Let $\rho = \pm 1$. We define a ρ-bracket on $\mathcal{R}$ as follows:

$$\{a, b\}_\rho = ab + \rho ba, \quad a, b \in \mathcal{R}.$$

It is easy to see that

$$\{a, b\}_\rho = \rho\{b, a\}_\rho \quad \text{and} \quad [ab, c] = a\{b, c\}_\rho - \rho\{a, c\}_\rho b$$

for $a, b, c \in \mathcal{R}$ where $[a, b] = \{a, b\}_{-1}$ is the Lie bracket. Sometimes, we write $\{a, b\}_- = \{a, b\}_{-1}$ and $\{a, b\}_+ = \{a, b\}_{+1}$. Indeed, we have the more useful identity:

$$[ab, cd] = a\{b, c\}_\rho d - \rho ac\{b, d\}_\rho + \{a, c\}_\rho db - \rho c\{a, d\}_\rho b.$$

3.2.1. Finite-dimensional representations

Let $\Lambda(r)$ be the Grassmann algebra with r variables $\{x_1, x_2, \ldots, x_r\}$. We may view $\Lambda(r)$ as a $\mathbb{Z}_2$-graded algebra by letting $|x_i| = \bar{1}$ for $1 \leq i \leq r$. We identify x_i with the left multiplication given by x_i itself and write $\frac{\partial}{\partial x_i}$ for the partial differential operator with respect to x_i. Then, in $\mathfrak{gl}(\Lambda(r))$, we have

$$\{x_i, x_j\}_+ = \left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\}_+ = 0 \quad \text{and} \quad \left\{x_i, \frac{\partial}{\partial x_j}\right\}_+ = \delta_{ij}.$$

Now define the following linear operators on $\Lambda(r)$:

$$e_0 = \sum_{t=1}^{r-2} x_t \frac{\partial}{\partial x_{t+1}}, \qquad e_i = x_{m+1-i}\frac{\partial}{\partial x_r}, \qquad f_j = x_{n+1-j}$$

and

$$e_0' = -\sum_{t=1}^{r-2} x_{t+1}\frac{\partial}{\partial x_t}, \qquad e_i' = x_r\frac{\partial}{\partial x_{m+1-i}}, \qquad f_j' = \frac{\partial}{\partial x_{n+1-j}}.$$

Theorem 2. *$\Lambda(r)$ becomes an $L_{m,n}$-module under the actions given by*

$$X_i \longmapsto e_i, \qquad Y_j \longmapsto f_j$$

or

$$X_i \longmapsto e_i', \qquad Y_j \longmapsto f_j'$$

for $0 \leq i \leq m$ and $1 \leq j \leq n$.

Proof. Let us check the commutative relations case by case. First, we have

$$\begin{aligned}
[e_0, e_i] &= \sum_{t=1}^{r-2}\left[x_t\frac{\partial}{\partial x_{t+1}}, x_{m+1-i}\frac{\partial}{\partial x_r}\right] \\
&= \sum_{t=1}^{r-2}\left(\delta_{t+1,m+1-i}x_t\frac{\partial}{\partial x_r} - \delta_{t,r}x_{m+1-i}\frac{\partial}{\partial x_{t+1}}\right) \\
&= \begin{cases} x_{m+1-(i+1)}\dfrac{\partial}{\partial x_r} = e_{i+1} & \text{if } 1 \leq i \leq m-1 \\ 0 & \text{if } i = m \end{cases} \\
[e_i, e_j] &= \left[x_{m+1-i}\frac{\partial}{\partial x_r}, x_{m+1-j}\frac{\partial}{\partial x_r}\right] = 0 \\
[e_0, f_j] &= \sum_{t=1}^{r-2}\left[x_t\frac{\partial}{\partial x_{t+1}}, x_{n+1-j}\right] \\
&= \sum_{t=1}^{r-2}\delta_{t+1,n+1-j}x_t = \begin{cases} x_{n+1-(j+1)} = f_{j+1} & \text{if } 1 \leq j \leq n-1 \\ 0 & \text{if } j = n \end{cases} \\
[e_i, f_j] &= \left[x_{m+1-i}\frac{\partial}{\partial x_r}, x_{n+1-j}\right] = 0.
\end{aligned}$$

At last, we have

$$\{f_i, f_j\}_+ = \{x_{n+1-i}, x_{n+1-j}\}_+ = 0.$$

Similarly, we can prove the relations for e_i' and f_j'. $\square$

Remark 3.3. (1) From Section 2, we know $\mu(L_{m,n}) \le r+1$. On the other hand, we have that $\mathfrak{gl}(\Lambda(r)) \cong \mathfrak{gl}(2^{r-1}, 2^{r-1})$ and $2^{r-1} \ge r+1$. Hence we can define more actions of $L_{m,n}$ on $\Lambda(r)$ and make it as an $L_{m,n}$-module, such as the following actions:

$$X_0 \longmapsto \sum_{t=1}^{r-2} x_t \frac{\partial}{\partial x_{t+1}}, \qquad X_i \longmapsto x_1 x_{m+1-i}, \qquad Y_j \longmapsto x_{n+1-j}$$

or

$$X_0 \longmapsto \sum_{t=1}^{r-2} x_t \frac{\partial}{\partial x_{t+1}}, \qquad X_i \longmapsto \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_{m+1-i}}, \qquad Y_j \longmapsto \frac{\partial}{\partial x_{n+1-j}}$$

for $1 \le i \le m$ and $1 \le j \le n$.

(2) From Proposition 3.1, we know that in either case in Theorem 2 the $L_{m,n}$-module is reducible and has a filtration. For example, if $r=3$ we have the following filtration in the first case

$$\Lambda(3) = \Lambda(3)^0 \supseteq \Lambda(3)^1 \supseteq \Lambda(3)^2 \supseteq \cdots \supseteq \Lambda(3)^8 = 0$$

where

$$\begin{aligned}
\Lambda(3)^1 &= \mathrm{span}_{\mathbb{C}}\{x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}\\
\Lambda(3)^2 &= \mathrm{span}_{\mathbb{C}}\{x_1, x_2, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}\\
\Lambda(3)^3 &= \mathrm{span}_{\mathbb{C}}\{x_1, x_2, x_1x_2, x_1x_3, x_1x_2x_3\}\\
\Lambda(3)^4 &= \mathrm{span}_{\mathbb{C}}\{x_1, x_2, x_1x_2, x_1x_2x_3\}\\
\Lambda(3)^5 &= \mathrm{span}_{\mathbb{C}}\{x_1, x_2, x_1x_2\} = L_{m,n}\Lambda(3)\\
\Lambda(3)^6 &= \mathrm{span}_{\mathbb{C}}\{x_1, x_1x_2\}\\
\Lambda(3)^7 &= \mathrm{span}_{\mathbb{C}}\{x_1x_2\}
\end{aligned}$$

and $\Lambda(3)^i/\Lambda(3)^{i+1}$ is a trivial $L_{m,n}$-module for $i = 0, 1, \ldots, 7$. Similarly, we have filtrations in the other cases.

3.2.2. Infinite-dimensional examples

Define $\mathfrak{a}(r, \rho)$ to be the unital associative Clifford (resp. Weyl) algebra for $\rho = 1$ (resp. $\rho = -1$) with $2r$ generators a_i, a_i^*, $1 \le i \le r$, subject to relations

$$\{a_i, a_j\}_\rho = \{a_i^*, a_j^*\}_\rho = 0 \quad \text{and} \quad \{a_i, a_j^*\}_\rho = \rho\delta_{ij}.$$

Let $\mathfrak{a}_\tau(r, \rho)$ be the algebra obtained by adjoining to $\mathfrak{a}(r, \rho)$ the generator e with relations $\{e, e\}_{-\rho} = 0$ and

$$\{a_i, e\}_\tau = 0 = \{a_i^*, e\}_\tau, \qquad \text{for } \tau = \pm 1.$$

Then it is an extension of the algebra $\mathfrak{a}(r, \rho)$.

Let $\mathfrak{a}_\tau^+(r, \rho)$ be the left ideal generated by a_i^* for $1 \le i \le r$. Now we can construct our operators on the infinite-dimensional vector space

$$V_\tau(r, \rho) = \mathfrak{a}_\tau(r, \rho)/\mathfrak{a}_\tau^+(r, \rho).$$

For any $1 \le i \le m$ and $1 \le j \le n$, set

$$\tilde{e}_0 = \sum_{t=1}^{r-2} a_t a_{t+1}^*, \qquad \tilde{e}_i = a_{m+1-i} a_r^*, \qquad \tilde{f}_j = a_{n+1-j} e$$

and

$$\tilde{e}_0' = \sum_{t=1}^{r-2} a_t^* a_{t+1}, \qquad \tilde{e}_i' = a_r a_{m+1-i}^*, \qquad \tilde{f}_j' = a_{n+1-j}^* e.$$

Theorem 3. *$V_\tau(r,\rho)$ becomes a module of the Lie superalgebra $L_{m,n}$ under the action given by*

$$X_i \longmapsto \tilde{e}_i, \qquad Y_j \longmapsto \tilde{f}_j$$

or

$$X_i \longmapsto \tilde{e}'_i, \qquad Y_j \longmapsto \tilde{f}'_j$$

for $0 \le i \le m$ and $1 \le j \le n$. Moreover, the module is reducible in both cases.

Proof. We check the relations case by case:

$$\begin{aligned}
[\tilde{e}_0, \tilde{e}_i] &= \sum_{t=1}^{r-2} [a_t a_{t+1}^*, a_{m+1-i} a_r^*] \\
&= \sum_{t=1}^{r-2} \Big(\delta_{t+1,m+1-i} a_t a_r^* - \rho^2 \delta_{t,r} a_{m+1-i} a_{t+1}^* \Big) \\
&= \begin{cases} a_{m+1-(i+1)} a_r^* = \tilde{e}_{i+1} & \text{if } 1 \le i \le m-1 \\ 0 & \text{if } i = m \end{cases} \\
[\tilde{e}_i, \tilde{e}_j] &= [a_{m+1-i} a_r^*, a_{m+1-j} a_r^*] = 0 \\
[\tilde{e}_0, \tilde{f}_j] &= \sum_{t=1}^{r-2} [a_t a_{t+1}^*, a_{n+1-j} e] \\
&= \sum_{t=1}^{r-2} \delta_{t+1,n+1-j} a_t e = \begin{cases} a_{n+1-(j+1)} e = \tilde{f}_{j+1} & \text{if } 1 \le j \le n-1 \\ 0 & \text{if } j = n \end{cases} \\
[\tilde{e}_i, \tilde{f}_j] &= [a_{m+1-i} a_r^*, a_{n+1-j} e] = 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\{\tilde{f}_i, \tilde{f}_j\}_+ &= \{a_{n+1-i} e, a_{n+1-j} e\}_+ = a_{n+1-i} e a_{n+1-j} e + a_{n+1-j} e a_{n+1-i} e \\
&= -\tau a_{n+1-i} a_{n+1-j} \{e, e\}_{-\rho} = 0.
\end{aligned}$$

Similarly, we can prove the relations for $\tilde{e}'_i$ and $\tilde{f}'_j$. The irreducibility follows from that the module generated by e are proper in both cases. □

Now let us consider the associative algebra $\alpha(r,\rho)$ generated by infinite many generators

$$\{u(p) | p \in \mathbb{Z}\},$$

where $u \in \{a_i, \ a_i^* \mid 1 \le i \le r\}$ with the relations

$$\{u(p), v(q)\}_\rho = \{u, v\}_\rho \delta_{p+q,0}.$$

By the generating relations, we have

$$\begin{aligned}
&[a_i(p)a_j(q), a_k(s)] = 0, \\
&[a_i(p)a_j(q), a_k^*(s)] = -\delta_{ik}\delta_{p+s,0} a_j(q) + \rho \delta_{jk} \delta_{q+s,0} a_i(p), \\
&[a_i(p)a_j^*(q), a_k(s)] = \delta_{jk}\delta_{q+s,0} a_i(p), \\
&[a_i(p)a_j^*(q), a_k^*(s)] = -\delta_{ik}\delta_{p+s,0} a_j^*(q), \\
&[a_i^*(p)a_j^*(q), a_k(s)] = \delta_{jk}\delta_{q+s,0} a_i^*(p) - \rho \delta_{ik}\delta_{p+s,0} a_j^*(q), \\
&[a_i^*(p)a_j^*(q), a_k^*(s)] = 0
\end{aligned}$$

for $p, q, s \in \mathbb{Z}$ and $1 \le i, j, k \le r$.

Let $\alpha^+(r,\rho)$ be the subalgebra generated by $a_i(p)$, $a_j^*(q)$, $a_k^*(0)$, for $p, q > 0$, and $1 \le i, j, k \le r$. Let $\alpha^-(r,\rho)$ be the subalgebra generated by $a_i(p)$, $a_j^*(q)$, $a_k(0)$, for $p, q < 0$, and $1 \le i, j, k \le r$. Those generators in $\alpha^+(r,\rho)$ are called annihilation operators while those in $\alpha^-(r,\rho)$ are called creation operators. Let $\widetilde{V}(r,\rho)$ be a simple $\alpha(r,\rho)$-module containing an element v_0, called a"vacuum vector", and satisfying $\alpha^+(r,\rho)v_0 = 0$. So all annihilation operators kill v_0 and $\widetilde{V}(r,\rho) \cong \alpha^-(r,\rho)v_0$.

Now we are in the position to construct a class of fermions on $\widetilde{V}(r,1)$. For any $\mathbf{p}=(p_1,p_2,p_3)\in\mathbb{Z}^3$, $1\le i\le m$ and $1\le j\le n$ set

$$\begin{aligned} e_0(\mathbf{p}) &= \sum_{t=1}^{r-2}\sum_{s\in\mathbb{Z}} a_t(p_1-s)a_{t+1}^*(s),\\ e_i(\mathbf{p}) &= \sum_{s\in\mathbb{Z}} a_1\big(p_2+ip_1-s\big)a_{m+1-i}(s),\\ f_j(\mathbf{p}) &= a_{n+1-j}(p_3+jp_1). \end{aligned}$$

Theorem 4. *For any* $\mathbf{p}=(p_1,p_2,p_3)\in\mathbb{Z}^3$, $\widetilde{V}(r,1)$ *becomes a module of the Lie superalgebra* $L_{m,n}$ *under the action* $\varphi_{\mathbf{p}}: L_{m,n}\longrightarrow \mathfrak{gl}(\widetilde{V}(r,1))$ *given by*

$$X_i\longmapsto e_i(\mathbf{p}),\qquad Y_j\longmapsto f_j(\mathbf{p})$$

for $0\le i\le m$ *and* $1\le j\le n$.

Proof.

$$\begin{aligned} [e_0(\mathbf{p}),e_i(\mathbf{p})] &= \Big[\sum_{t=1}^{r-2}\sum_{s_1\in\mathbb{Z}} a_t(p_1-s_1)a_{t+1}^*(s_1),\ \sum_{s_2\in\mathbb{Z}} a_1\big(p_2+ip_1-s_2\big)a_{m+1-i}(s_2)\Big]\\ &= \sum_{t=1}^{r-2}\sum_{s\in\mathbb{Z}} \delta_{m-i+1,t+1}a_1\big(p_2+(i+1)p_1-s\big)a_t(s)\\ &= \begin{cases} \sum_{s\in\mathbb{Z}} a_1\big(p_2+(i+1)p_1-s\big)a_{m-i}(s)=e_{i+1}(\mathbf{p}) & \text{if } 1\le i\le m-1\\ 0 & \text{if } i=m. \end{cases} \end{aligned}$$

By a similar argument, we have

$$\begin{aligned} &[e_i(\mathbf{p}),e_j(\mathbf{p})]=[e_i(\mathbf{p}),f_j(\mathbf{p})]=\{f_i(\mathbf{p}),f_j(\mathbf{p})\}_+=0\\ &[e_0(\mathbf{p}),f_j(\mathbf{p})]=\begin{cases} f_{j+1}(\mathbf{p}) & \text{if } 1\le j\le n-1\\ 0 & \text{if } j=n. \end{cases} \end{aligned}$$

Whence $\widetilde{V}(r,1)$ becomes an $L_{m,n}$-module under the action $\varphi_{\mathbf{p}}$. □

Remark 3.4. (1) We can define the similar action on $\widetilde{V}(r,1)$ as what we do in Theorems 2 and 3:

$$\begin{aligned} X_0 &\longmapsto \sum_{t=1}^{r-2}\sum_{s\in\mathbb{Z}} a_t(p_1-s)a_{t+1}^*(s),\\ X_i &\longmapsto \sum_{s\in\mathbb{Z}} a_{m+1-i}\big(p_2+ip_1-s\big)a_r^*(s),\\ Y_j &\longmapsto a_{n+1-j}(p_3+jp_1). \end{aligned}$$

(2) Let us consider an extension of the algebra $\alpha(r,\rho)$. The generators

$$\{e(p)|p\in\mathbb{Z}\} \tag{3.1}$$

span an infinite-dimensional Clifford algebra or Weyl algebra with relations

$$\{e(p),e(q)\}_{-\rho}=e(p)e(q)-\rho e(q)e(p)=\rho\delta_{p+q,0}. \tag{3.2}$$

Let $\alpha_\tau(r,\rho)$ denote the algebra obtained by adjoining to $\alpha(r,\rho)$ the generators (3.1) with relations (3.2) and

$$\{a_i(p),e(q)\}_\tau=0=\{a_i^*(p),e(q)\}_\tau,\qquad \text{for } \tau=\pm1.$$

Let $\alpha_\tau^+(r,\rho)$ be the left ideal generated by $e(s)$, $a_i(p)$, $a_j^*(q)$, $a_k^*(0)$, for $p,q,s>0$, and $1\le i,j,k\le r$. We have lots of ways to define operators on the infinite-dimensional vector space

$$\widetilde{V}_\tau(r,\rho)=\alpha_\tau(r,\rho)/\alpha_\tau^+(r,\rho)$$

and make it as an $L_{m,n}$-module. The detail of constructions is left as an exercise for the reader.

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