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Classification and equivariant cohomology of circle actions on 3d manifolds

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The classification of Seifert manifolds was given in terms of numeric data by Seifert (1933), and then generalized by Raymond (1968) and Orlik and Raymond (1968) to circle actions on closed 3d manifolds. In this paper, we further generalize the classification to circle actions on 3d manifolds with boundaries by adding a numeric parameter and a graph of cycles. Then, we describe the rational equivariant cohomology of 3d manifolds with circle actions in terms of ring, module and vector-space structures. We also compute equivariant Betti numbers and Poincaré series for these manifolds and discuss the equivariant formality.

1. Introduction

The classification of closed 3d manifolds with "nice" decompositions into circles was given by Seifert [1] in terms of principal Euler number b, orientability ϵ and genus g of the underlying 2d orbifolds, and pairs of coprime integers (m_i, n_i) called Seifert invariants. Hence, these manifolds were given the name Seifert manifolds.

Later, the classification was generalized by Orlik and Raymond [2,3] to circle actions on closed 3d manifolds allowing fixed points and special exceptional orbits. Orlik and Raymond found that in their case the underlying 2d orbifolds have circle boundaries contributed by the fixed points and special exceptional orbits. Hence, besides the four types of numeric data used by Seifert, two more types of numeric data were introduced by Orlik and Raymond: the number f of fixed components and the number s of special exceptional components. Then, Orlik and Raymond proved the following:

Theorem (*Orlik-Raymond Classification of closed 3d S^1-manifolds, [2,3]*)**.** *Let S^1 act effectively and smoothly on a closed, connected smooth 3d manifold M. Then, the orbit invariants*

$$\left\{b; (\epsilon, g, f, s); (m_1,\ n_1), \ldots, (m_l,\ n_l)\right\}$$

determine M up to equivariant diffeomorphisms, subject to certain conditions. Conversely, any such set of invariants can be realized as a closed 3d manifold with an effective S^1-action.

The first goal of this paper is to further generalize the Orlik–Raymond Classification Theorem to circle actions on compact 3d manifolds, allowing boundaries. By the classification of circle actions on closed 2d manifolds, those boundaries have to be tori $\mathbb{T}$, spheres S^2, projective planes $\mathbb{R}P^2$ or Klein bottles K. Our approach relies on a careful discussion on the equivariant

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neighbourhoods of non-principal orbits and boundaries. Let t be the number of torus boundaries and $\mathcal{G}$ be a graph of cycles to keep track of the boundary types S^2, K, $\mathbb{R}P^2$, we get the following:

Theorem 3.1. *Let the circle group S^1 act effectively and smoothly on a compact, connected 3d manifold M, possibly with boundary. Then, the orbit invariants*

$$\left\{b; (\epsilon, g, f, s, t); (m_1, n_1), \dots, (m_l, n_l); \mathcal{G}\right\}$$

consisting of numeric data and a graph of cycles, determine M up to equivariant diffeomorphisms, subject to certain conditions. Conversely, any such set of invariants can be realized as a 3d manifold with an effective S^1-action.

Using the Orlik–Raymond Theorem, one can compute the fundamental groups, ordinary homology and cohomology with $\mathbb{Z}$ or $\mathbb{Z}_p$ coefficients for closed 3d S^1-manifolds, (cf.[4–7]). In this paper, we are instead interested in equivariant topological invariants.

Hence, the second goal of this paper is to describe the $\mathbb{Q}$-coefficient equivariant cohomology of compact 3d manifold M with circle action. Our main strategy is to apply the equivariant Mayer–Vietoris sequence to a decomposition of the manifold M into a fixed-point-free part and a neighbourhood of the fixed-point set. Then, we get

Theorem 4.2. *Let M be a compact connected 3d manifold(possibly with boundary) with an effective S^1-action, and F be its fixed-point set(possibly empty), then there is a short exact sequence of cohomology groups in $\mathbb{Q}$ coefficients:*

$$0 \to H^*_{S^1}(M) \to H^*(M/S^1) \oplus \Big(\mathbb{Q}[u] \otimes H^*(F)\Big) \to H^*(F) \to 0$$

Using this theorem, we can describe the ring, module and vector-space structures of the equivariant cohomology $H^*_{S^1}(M)$ in details. Furthermore, we will calculate equivariant Betti numbers and Poincaré series, and discuss a numeric condition for equivariant formality.

2. S^1-actions on 2d manifolds and closed 3d manifolds

In this section, we will recall the classification of effective S^1-actions on closed manifolds in dimensions 2 and 3, which will be crucial for our classification of effective S^1-actions on 3d manifolds with boundaries. All these results are well known, and can be found in greater details from the original papers by Orlik and Raymond [2,3] or the notes and books [4,8–10].

2.1. Some basic facts about group actions on manifolds

Throughout the paper, we always assume that a manifold M is compact, smooth and connected, and a group G is compact, unless otherwise mentioned. For convenience, we will denote a G-action on M as $G \curvearrowright M$. The quotient M/G is called the **orbit space** of the G-action on M. For any point x in M, let $G_x = \{g \in G \mid g \cdot x = x\}$ be its stabilizer. We write $M^G = \{x \in M \mid G_x = G\}$ for the set of fixed points. If $G_x = G$ for every $x \in M$, we say that the G-action on M is **trivial**. If $G_x = \{1\}$ for every $x \in M$, we say that the G-action on M is **free**. If the intersection $\cap_{x \in M} G_x = \{1\}$, we say that the G-action on M is **effective**. Throughout this paper, group actions are usually assumed to be effective, unless otherwise mentioned.

For any orbit $G \cdot x$, let V_x be an orthogonal complement of $T_x(G \cdot x)$ in T_xM. The infinitesimal action of G_x on T_xM gives a linear **isotropy representation** $G_x \curvearrowright V_x$. Then, the normal bundle of the orbit $G \cdot x$ can be written as

$$G \times_{G_x} V_x = \left\{[g, v] \mid (g, v) \sim (gh, h^{-1}v) \text{ for any } h \in G\right\}$$

with a G-action induced from the canonical G-action on the left of the first factor of $G \times V_x$.

The following theorem, proved by Koszul [11], equivariantly identifies the normal bundle with the tubular neighbourhood of an orbit $G \cdot x$.

Theorem 2.1 (*The slice theorem, [11]*)**.** *There exists an equivariant exponential map*

$$\exp : G \times_{G_x} V \longrightarrow M$$

which is an equivariant diffeomorphism from an open neighbourhood of the zero section $G \times_{G_x} \{0\}$ in $G \times_{G_x} V_x$ to an equivariant neighbourhood of $G \cdot x$ in M.

Thus, an equivariant neighbourhood of the orbit $G \cdot x$ can be specified in terms of the stabilizer G_x and the isotropy representation of G_x on the normal vector space.

Similar to the ordinary non-equivariant case, the equivariant identification between normal bundles and neighbourhoods generalizes beyond single orbit to submanifold and boundary, cf. Kankaanrinta [12].

Theorem 2.2 (*Equivariant tubular neighbourhood, [12]*)**.** *Let N be a closed G-invariant submanifold of M, and E be the normal G-vector bundle of N. There exists an equivariant exponential map*

$$\exp : E \longrightarrow M$$

which is an equivariant diffeomorphism from an open neighbourhood of the zero section in E to an equivariant tubular neighbourhood of N in M.

Theorem 2.3 (*Equivariant collaring neighbourhood, [12]*)**.** *Suppose a compact manifold M has a G-action that extends compatibly to its boundary ∂M. There exists an equivariant exponential map*

$$\exp : \partial M \times [0, \infty) \longrightarrow M$$

which is an equivariant diffeomorphism from an open neighbourhood of the boundary ∂M in $\partial M \times [0, \infty)$ to an equivariant collaring neighbourhood of ∂M in M.

Since we only consider S^1-actions, there are three types of stabilizers, namely $\{1\}$, $\mathbb{Z}/m$, S^1, whose resulting orbits will be called **principal**, **exceptional** and **singular**, respectively.

	Principal orbit	Exceptional orbit	Singular orbit
Stabilizer S^1_x	$\{1\}$	$\mathbb{Z}_m = \{e^{\frac{2\pi k i}{m}}, k = 1, 2, \dots, m\}$	S^1
Orbit $S^1 \cdot x$	S^1	$S^1/\mathbb{Z}_m$	pt

Intuitively, exceptional orbits $S^1/\mathbb{Z}_m$ are shorter than regular orbits S^1. Singular orbits $S^1/S^1 = pt$ are exactly the fixed points of the S^1-action.

Direct applications of the Slice Theorem, together with the compactness of M, leads to the following facts (cf. Audin [9] Sec I.2):

Fact 2.1. *If S^1 acts on a compact, connected manifold M, then*

- *For any subgroup H of S^1, the set $M_{(H)} = \{x \in M \mid S^1_x = H\}$ of points with stabilizer H is a submanifold of M. Moreover, S^1/H acts freely on $M_{(H)}$.*
- *There is a unique subgroup H_0 of S^1, such that the set $M_{(H_0)}$ is open and dense in M.*
- *The S^1-action on M is effective if and only if the H_0 in the previous statement is the identity group $\{1\}$.*
- *If the S^1-action on M is effective, then for every $x \in M$, the isotropy representation $S^1_x \curvearrowright V_x$ is also effective.*

Furthermore, based on the Theorem of equivariant tubular neighbourhood, the classification of effective S^1-manifolds in low dimensions can be done by listing all the possible equivariant neighbourhoods and the obstructions of patching them together to form a manifold. In dimension 1, there is only one compact effective S^1-manifold, the circle S^1 itself with the rotating action. In dimensions 2 and 3, this approach is also successful, as we will recall in the next subsections.

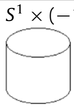
2.2. S^1-actions on 2d manifolds

We begin by listing all the possible equivariant tubular neighbourhoods of orbits, which are the same as equivariant normal bundles according to the Slice Theorem. Then, we try to patch these neighbourhoods together. The survey of this topic follows closely from Audin ([9] Sec I.3).

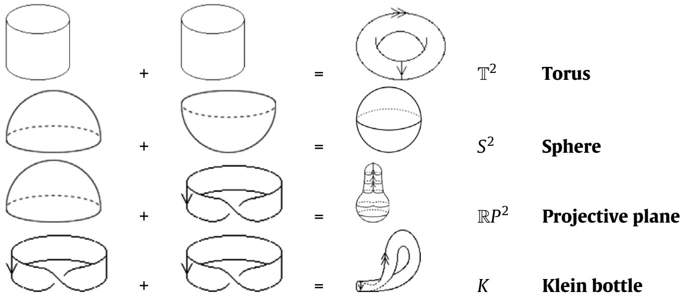
Notice that in dimension 2, for an exceptional orbit $S^1/\mathbb{Z}_m$, its isotropic representation is of dimension 1. But, there is only one such effective representation, namely the reflection $\mathbb{Z}_2 \overset{\text{reflect}}{\curvearrowright} \mathbb{R}$, which also forces the exceptional orbit to be $S^1/\mathbb{Z}_2$.

As for a singular orbit, i.e. a fixed point with stabilizer S^1, its isotropic representation is of dimension 2. The only effective S^1-representation of real dimension 2 is the rotation $S^1 \overset{\text{rotate}}{\curvearrowright} \mathbb{C}$.

So, we can summarize the list of all possible equivariant tubular neighbourhoods:

	Principal orbit	Exceptional orbit	Singular orbit
Stabilizer S^1_x	$\{1\}$	$\mathbb{Z}_2$	S^1
Orbit $S^1 \cdot x$	S^1	$S^1/\mathbb{Z}_2$	pt
Isotropic representation	$\{1\} \curvearrowright \mathbb{R}$	$\mathbb{Z}_2 \overset{\text{reflect}}{\curvearrowright} \mathbb{R}$	$S^1 \overset{\text{rotate}}{\curvearrowright} \mathbb{C}$
Equivariant neighbourhood	$S^1 \times (-1, 1)$	$S^1 \times_{\mathbb{Z}_2} (-1, 1)$	$D = \{(x, y) \mid x^2 + y^2 < 1\}$
U	 Cylinder	 Möbius band	 Disk
Orbit neighbourhood U/S^1	$(-1, 1)$	$[0, 1)$	$[0, 1)$

To form a 2-dimensional closed manifold with effective S^1-action, we now just need to patch those equivariant pieces $S^1 \times (-1, 1)$, $S^1 \times_{\mathbb{Z}_2} (-1, 1)$, D together by closing boundaries.



In the above list of 2-dimensional closed manifolds with effective S^1-action, the projective plane $\mathbb{R}P^2$ and Klein bottle K are non-orientable due to the existence of exceptional orbits $S^1/\mathbb{Z}_2$, but the torus $\mathbb{T}^2$ and the sphere S^2 are orientable.

Given a 2d compact connected effective S^1-manifold M, we can count its fixed points and exceptional orbits as f and s, respectively. If we allow M to have boundary, we can count the number of boundary components as b. Similarly, since the orbit space M/S^1 is a compact connected 1d manifold which is either a circle S^1 or an interval I, we can count the boundaries of M/S^1 as $\bar{b}$. Then, we have the classification of the 2d compact connected effective S^1-manifolds:

Theorem 2.4 (*Numeric classification of 2d S^1-manifolds*)**.** *Given a 2d compact connected effective S^1-manifold M, possibly with boundary, the integers (b, f, s) determine M up to S^1-diffeomorphism, and so do the integers $(\bar{b}, f, s)$.*

Proof. We have seen that there are three 2d effective S^1-manifolds with boundary: cylinder, disk and Möbius band, and four 2d effective S^1-manifolds without boundary: torus, sphere, projective plane and Klein bottle. The counting of boundary components as b is straightforward.

To compute (f, s), we first do this for cylinder, $f = 0$, $s = 0$; disk, $f = 1$, $s = 0$; Möbius band, $f = 0$, $s = 1$. For any one of the four closed 2d S^1-manifolds, we just add the (f, s)-vectors of its two patches.

To understand the orbit spaces, we use the standard expressions for disk, D; cylinder, $S^1 \times [-1, 1]$; sphere, S^2; torus, $S^1 \times S^1$. Their orbit spaces are $[0, 1]$, $[-1, 1]$, $[-1, 1]$ and S^1, respectively.

For the orbit spaces of the rest types of the manifolds, notice that for a compact group G and a compact subgroup H that acts on a space V, there is a relation between the G-orbit space and H-orbit space: $(G \times_H V)/G = V/H$. So, the Möbius band, projective plane and Klein bottle, written, respectively, as $S^1 \times_{\mathbb{Z}_2} [-1, 1]$, $S^2/\mathbb{Z}_2$ and $S^1 \times_{\mathbb{Z}_2} S^1$ will have S^1-orbit spaces $[0, 1]$, $[0, 1]$ and $[0, \pi]$, respectively.

Here is the complete list of the numeric data $(\bar{b}, b, f, s)$:

Manifold M	Topological expression	Orbit space M/S^1	$\#\partial(M/S^1)$ $\bar{b}$	$\#\partial M$ b	$\#M^{S^1}$ f	$\#M^{\mathbb{Z}_2}$ s
Disk	D	$[0, 1]$	2	1	1	0
Cylinder	$S^1 \times [-1, 1]$	$[-1, 1]$	2	2	0	0
Möbius band	$S^1 \times_{\mathbb{Z}_2} [-1, 1]$	$[0, 1]$	2	1	0	1
Sphere	S^2	$[-1, 1]$	2	0	2	0
Projective plane	$S^2/\mathbb{Z}_2$	$[0, 1]$	2	0	1	1
Torus	$S^1 \times S^1$	S^1	0	0	0	0
Klein bottle	$S^1 \times_{\mathbb{Z}_2} S^1$	$[0, \pi]$	2	0	0	2

From the above list, we see that different diffeomorphism types of 2d effective connected S^1-manifolds have different (b, f, s)-vectors, together with different $(\bar{b}, f, s)$-vectors, hence the claim of the theorem follows. □

Remark 2.1. Though the integer (b, f, s)-vector or $(\bar{b}, f, s)$-vector classifies all the 2d effective connected S^1-manifolds, their values are limited to the seven cases.

Remark 2.2. For 2d effective S^1-manifolds without boundary, the (f, s)-vector is enough to give the classification.

Remark 2.3. The author learned this folklore classification theorem from Audin's book ([9] Sec I.3). The numeric version here is just a simple corollary.

2.3. S^1-actions on closed 3d manifolds

The idea of classifying effective S^1-actions in dimension 3 is the same as in dimension 2 by listing all the possible equivariant tubular neighbourhoods of non-principal orbits, and then try to patch them together. But, one more dimension for the isotropic representations provides a longer list of equivariant tubular neighbourhoods.

2.3.1. Equivariant tubular neighbourhoods of principal orbits

For a point x of principal type, its isotropy group is the identity group $\{1\}$ with a trivial isotropic representation $\{1\} \curvearrowright \mathbb{R}^2$. So, an equivariant tubular neighbourhood of $S^1 \cdot x$ can be written as $S^1 \times_{\{1\}} D = S^1 \times D$, with the S^1-action concentrating entirely on the S^1-factor. So, the orbit space of this tubular neighbourhood is $(S^1 \times D)/S^1 = S^1/S^1 \times D = D$, a smooth local chart.

2.3.2. Equivariant tubular neighbourhoods of exceptional orbits

The union of exceptional orbits will be denoted as E. For an exceptional orbit $S^1/\mathbb{Z}_m$ with stabilizer $\mathbb{Z}_m = \{e^{\frac{2\pi ki}{m}}, k = 1, 2, \ldots, m\}$, its isotropic representation of $\mathbb{Z}_m$ is 2-dimensional. Such a 2-dimensional effective $\mathbb{Z}_m$-representation could preserve the orientation by rotating:

$$\mathbb{Z}_m \overset{\text{rotate}}{\curvearrowright} \mathbb{C}: \quad e^{\frac{2\pi ki}{m}} \circ z = (e^{\frac{2\pi ki}{m}})^n z$$

where the orbit invariants $(m,\ n)$, also called Seifert invariants, are coprime positive integers, and $0 < n < m$. The resulting equivariant tubular neighbourhood is $S^1 \times_{\mathbb{Z}_m} D$, whose orbit space is an orbifold disk

$$(S^1 \times_{\mathbb{Z}_m} D)/S^1 = D/\mathbb{Z}_m$$

where the central orbifold point $pt/\mathbb{Z}_m$ corresponds to the exceptional orbit $S^1/\mathbb{Z}_m$.

2.3.3. Equivariant tubular neighbourhoods of special exceptional orbits

Besides rotating, a 2-dimensional effective $\mathbb{Z}_m$-representation could also reverse the orientation by reflection:

$$\mathbb{Z}_2 \overset{\text{reflect}}{\curvearrowright} \mathbb{R}^2: \quad e^{\pi i} \circ (x, y) = (-x, y).$$

This case requires the $\mathbb{Z}_m$ to be $\mathbb{Z}_2$. Because of the reverse of orientation, we call such an orbit $S^1/\mathbb{Z}_2$ a **special exceptional orbit**. The union of all such special exceptional orbits will be denoted as SE.

If we use the open square $I \times I = \{(x, y) \mid -1 < x,\ y < 1\}$ as a neighbourhood in $\mathbb{R}^2$, an equivariant tubular neighbourhood of the special exceptional orbit $S^1/\mathbb{Z}_2$ can be written as $S^1 \times_{\mathbb{Z}_2} (I \times I)$, the orbit space by $\mathbb{Z}_2$ of the solid torus $S^1 \times (I \times I)$. Note that the reflection $\mathbb{Z}_2 \overset{\text{reflect}}{\curvearrowright} I \times I: \quad e^{\pi i} \circ (x, y) = (-x, y)$ only affects the first I-factor, so we can split the second I-factor out of the orbit space $S^1 \times_{\mathbb{Z}_2} (I \times I)$:

$$\begin{aligned} S^1 \times_{\mathbb{Z}_2} (I \times I) &= S^1 \times (I \times I)/(e^{i\theta}, x, y) \sim (-e^{i\theta}, -x, y) \\ &= \left(S^1 \times I/(e^{i\theta}, x) \sim (-e^{i\theta}, -x)\right) \times I = \text{Möb} \times I \end{aligned}$$

where we write Möb for short of the Möbius band $S^1 \times_{\mathbb{Z}_2} I$.

Because the set of points with stabilizer $\mathbb{Z}_2$ in the Möbius band $S^1 \times_{\mathbb{Z}_2} I$ is $\text{Möb}_{(\mathbb{Z}_2)} = S^1 \times_{\mathbb{Z}_2} \{0\} = S^1/\mathbb{Z}_2$ a circle, the set of points with stabilizer $\mathbb{Z}_2$ in $\text{Möb} \times I$ is $(\text{Möb} \times I)_{(\mathbb{Z}_2)} = S^1/\mathbb{Z}_2 \times I$ of dimension 2. Thus, if a 3d S^1-manifold M has a special exceptional orbit $S^1/\mathbb{Z}_2$, then the connected component of $M_{(\mathbb{Z}_2)}$ that contains this orbit will be of dimension 2 and is acted freely by $S^1/\mathbb{Z}_2$, hence has to be $S^1/\mathbb{Z}_2 \times S^1$ according to the list of 2d S^1-manifolds.

Now, an equivariant tubular neighbourhood of this torus $S^1/\mathbb{Z}_2 \times S^1$ will be a bundle of Möbius band over S^1, which is actually a product bundle $\text{Möb} \times S^1$, cf. Raymond [2].

Notice that the S^1-action concentrates entirely on the factor of Möbius band, so the orbit space is $(\text{Möb} \times I)/S^1 = \text{Möb}/S^1 \times I = [0, 1) \times I$ with a boundary circle $\{0\} \times S^1$.

2.3.4. Equivariant tubular neighbourhoods of fixed points

The set of fixed points will be denoted as F. For a fixed point x with stabilizer S^1, its isotropic representation is of dimension 3. There is only one such effective 3-dimensional S^1-representation $S^1 \curvearrowright \mathbb{C} \oplus \mathbb{R}$ by acting on the $\mathbb{C}$-factor rotationally and acting on the $\mathbb{R}$-factor trivially.

So, an equivariant tubular neighbourhood of x can be written as $D \times I$, with fixed point set $\{0\} \times I$, an interval. We can continue to glue along this fixed interval to form S^1, a connected component of the fixed point set. Now, an enlarged equivariant tubular neighbourhood of the fixed circle S^1 is going to be a disk bundle over the S^1, which is actually a product bundle $D \times S^1$, cf. Raymond [2].

Notice that the S^1-action concentrates entirely on the D-factor, so the orbit space is $(D \times S^1)/S^1 = D/S^1 \times S^1 = [0, 1) \times S^1$ with a boundary circle $\{0\} \times S^1$.

2.3.5. Patching: from local to global

First, we can summarize all the local pictures into a list

	Principal	Exceptional	Special exceptional	Singular
Stabilizer S^1_x	$\{1\}$	$\mathbb{Z}_m$	$\mathbb{Z}_2$	S^1
Isotropic representation	$\{1\} \curvearrowright \mathbb{C}$	$\mathbb{Z}_m \overset{\text{rotate}}{\curvearrowright} \mathbb{C}$	$\mathbb{Z}_2 \overset{\text{reflect}}{\curvearrowright} \mathbb{R}^2$	$S^1 \overset{\text{rotate}}{\curvearrowright} \mathbb{C} \oplus \mathbb{R}$
Orbit $S^1 \cdot x$	S^1	$S^1/\mathbb{Z}_m$	$S^1/\mathbb{Z}_2$	pt
Equivariant neighbourhood	$S^1 \times D$	$S^1 \times_{\mathbb{Z}_m} D$	Möb $\times I$	$D \times I$
Orbit neighbourhood	D	$D/\mathbb{Z}_m$	$[0, 1) \times I$	$[0, 1) \times I$
Component of orbits of same type		$S^1/\mathbb{Z}_m$	$S^1/\mathbb{Z}_2 \times S^1$	$pt \times S^1$
Enlarged equivariant neighbourhood		$S^1 \times_{\mathbb{Z}_m} D$	Möb $\times S^1$	$D \times S^1$
Enlarged orbit neighbourhood		$D/\mathbb{Z}_m$	$[0, 1) \times S^1$	$[0, 1) \times S^1$

From the above list, we see that, passing to the orbit space, the local neighbourhood of an exceptional orbit $S^1/\mathbb{Z}_m$ contributes to an orbifold neighbourhood $D/\mathbb{Z}_m$. Both the local neighbourhoods of special exceptional orbits and the local neighbourhoods of fixed circles give rise to half closed, half open annuli $[0, 1) \times S^1$ with circle boundaries $\{0\} \times S^1$.

Theorem 2.5 (*Orbit space of closed 3d S^1-manifold, [2,3]*)**.** *For a compact closed 3d effective S^1-manifold M, the orbit space $M^* = M/S^1$ is a 2d orbifold surface, possibly with boundaries. The orbifold surface M^* has finite number of interior orbifold points with Seifert invariants $\{(m_1, n_1), \dots, (m_l, n_l)\}$, and boundary $\partial M^* = F \cup SE/S^1$ coming from the fixed circles and special exceptional orbits.*

To express $M^* = M/S^1$ and its orbifold points into numeric data, let us denote ϵ as the orientability of the 2d orbit space $M^* = M/S^1$, g the genus, (f, s) the numbers of circles formed from fixed circles and components of special exceptional orbits, respectively.

As for the total space M, after specifying the neighbourhoods of non-principal orbits, there is an obstruction integer b of finding a cross section over the principal part of the orbit space. The theorem by Orlik and Raymond says that, these invariants completely classify the 3d S^1-manifolds, after adding some constraints within these invariants. The following version is taken from Orlik's lecture notes [8].

Theorem 2.6 (*Equivariant classification of closed 3d S^1-manifolds, [2,3]*)**.** *Let S^1 act effectively and smoothly on a closed, connected smooth 3d manifold M. Then, the orbit invariants*

$$\left\{b; (\epsilon, g, f, s); (m_1, n_1), \dots, (m_l, n_l)\right\}$$

determine M up to equivariant diffeomorphisms, subject to the following conditions:

(1) $b = 0$, *if* $f + s > 0$
$b \in \mathbb{Z}$, *if* $f + s = 0$ *and* $\epsilon = o$, *orientable*
$b \in \mathbb{Z}_2$, *if* $f + s = 0$ *and* $\epsilon = n$, *non-orientable*
$b = 0$, *if* $f + s = 0$, $\epsilon = n$ *and* $m_i = 2$ *for some* i
(2) $0 < n_i < m_i$, $(m_i, n_i) = 1$ *if* $\epsilon = o$
$0 < n_i \leqslant \frac{m_i}{2}$, $(m_i, n_i) = 1$ *if* $\epsilon = n$.

Conversely, any such set of invariants can be realized as a closed 3d manifold with an effective S^1-action.

Remark 2.4. Raymond's idea of proving this classification theorem is as follows: given any two closed 3d S^1-manifolds $M, \bar{M}$ with the same orbit invariants, firstly we can establish an orbifold diffeomorphism between M/S^1 and $\bar{M}/S^1$. Secondly, we can lift this orbifold diffeomorphism to $E \cup F \cup SE \to \bar{E} \cup \bar{F} \cup \bar{SE}$ between the three types of non-principal orbits and extend this map to a tubular neighbourhood of the non-principal orbits. Finally, we can extend this map to all the principal orbits using local cross sections, which actually gives a global S^1-diffeomorphism if the principal Euler numbers $b, \bar{b}$ are the same.

Remark 2.5. When M has neither fixed point nor special exceptional orbit, i.e. $f = s = 0$, then this is the case of Seifert manifolds.

Remark 2.6. The invariants in $M = \left\{b; (\epsilon, g, f, s); (m_1, n_1), \dots, (m_l, n_l)\right\}$ mostly come from the orbit space $M^* = M/S^1$ except the invariant b. Therefore, the constraint ($b = 0$, if $f + s > 0$) says that if the orbifold M^* has boundaries, then $M = \left\{b = 0; (\epsilon, g, f, s); (m_1, n_1), \dots, (m_l, n_l)\right\}$ is determined by the orbifold M/S^1 and the assignment of its boundary circles being either from fixed components or special exceptional components.

Remark 2.7. The above classification is up to equivariant diffeomorphisms. But, Orlik and Raymond also discussed in certain conditions, more than one S^1-actions can appear on the same 3d manifold.

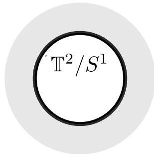


Fig. 1. Cylinder $S^1 \times [0, 1)$ flattened as annulus.

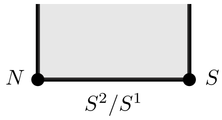


Fig. 2. One-side-open rectangle $[-1, 1] \times [0, 1)$.

For an orientable S^1-manifold M, the orbit space $M^* = M/S^1$ will be orientable, i.e. $\epsilon = o$, and there will be no special exceptional orbits, i.e. $s = 0$.

Corollary 2.1 (*Classification of closed orientable 3d S^1-manifolds, [2,3]*)**.** *If a closed 3d S^1-manifold is oriented and the S^1-action preserves the orientation. Then, the orbit invariants*

$$\left\{b; (\epsilon = o, g, f, s = 0); (m_1,\ n_1), \ldots, (m_l,\ n_l)\right\}$$

determine M up to equivariant diffeomorphisms, subject to the following conditions:

(1) $b = 0$, *if* $f > 0$
$b \in \mathbb{Z}$, *if* $f = 0$
(2) $0 < n_i < m_i,\ (m_i,\ n_i) = 1$

3. S^1-actions on 3d manifolds with boundaries

Let M^3 be a compact connected 3d manifold with an effective S^1-action that extends compatibly to its non-empty boundary ∂M. Combining the classification of S^1-actions on closed 2d and 3d manifolds, we can generalize the Orlik–Raymond classification theorem to S^1-actions on 3d manifolds with boundaries.

3.1. Equivariant collaring and tubular neighbourhoods

Similar to the case of 3d S^1-manifolds without boundary, we will first give a complete description of the equivariant neighbourhoods of boundaries and non-principal orbits.

3.1.1. Collaring neighbourhoods of boundaries and their orbit spaces

Theorem 2.3 of equivariant collaring neighbourhood says that for any S^1-invariant boundary component B in ∂M, an equivariant collaring neighbourhood of B in M looks like $B \times [0, 1)$, whose orbit space is $B/S^1 \times [0, 1)$.

We have seen in the discussion of 2d closed S^1-manifolds that there are four of them up to equivariant diffeomorphisms: $\mathbb{T}^2,\ S^2,\ K,\ \mathbb{R}P^2$ which will appear as boundaries of 3d S^1-manifolds.

The non-principal orbits appearing in $\mathbb{T}^2,\ S^2,\ K,\ \mathbb{R}P^2$ are either fixed points or $S^1/\mathbb{Z}_2$ with isotropy representation being a reflection, hence a special exceptional orbit. Therefore, among the union of the non-principal orbits $E \cup F \cup SE$, the boundary ∂M is separated from the exceptional orbits E, but could possibly have common points with the fixed points and special exceptional orbits $F \cup SE$.

More explicitly, each boundary component $\mathbb{T}^2$ has an equivariant collaring neighbourhood $\mathbb{T}^2 \times [0, 1)$ consisting of only principal orbits. The orbit space $\mathbb{T}^2/S^1 \times [0, 1) = S^1 \times [0, 1)$ is a half closed, half open annulus with a circle boundary (see Fig. 1): where the boundary circle is the orbit space $\mathbb{T}^2/S^1 = S^1$.

Each boundary component S^2 has an equivariant collaring neighbourhood $S^2 \times [0, 1)$ with the two fixed poles $N,\ S$ attached to the two fixed intervals $N \times [0, 1),\ S \times [0, 1)$, respectively. The orbit space $S^2/S^1 \times [0, 1) = [-1, 1] \times [0, 1)$ is an open manifold with 3 boundaries and 2 corners (see Fig. 2):

where the bottom interval is the orbit space S^2/S^1, the left and right intervals come from the two fixed intervals $N \times [0, 1),\ S \times [0, 1)$, the two corner points are the two poles $N,\ S$.

Each boundary component $\mathbb{R}P^2$ has an equivariant collaring neighbourhood $\mathbb{R}P^2 \times [0, 1)$ with a fixed point p and the orbit $S^1/\mathbb{Z}_2$ attached to a fixed interval $p \times [0, 1)$ and a special exceptional component $S^1/\mathbb{Z}_2 \times [0, 1)$ respectively. The orbit space

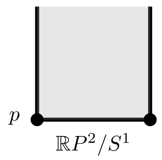


Fig. 3. One-side-open rectangle $[0, 1] \times [0, 1)$.

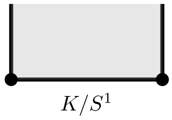


Fig. 4. One-side-open rectangle $[0, \pi] \times [0, 1)$.

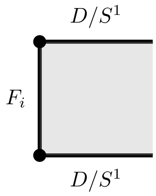


Fig. 5. One-side-open rectangle $[0, 1) \times [0, 1]$.

$\mathbb{R}P^2/S^1 \times [0, 1) = [0, 1] \times [0, 1)$ is an open manifold with 3 boundaries and 2 corners (see Fig. 3): where the bottom interval is the orbit space $\mathbb{R}P^2/S^1$, the left interval is the fixed interval $p \times [0, 1)$ with the corner point p, and the right interval comes from the orbit space $(S^1/\mathbb{Z}_2 \times [0, 1))/S^1$ with the other corner point.

Each boundary component K has an equivariant collaring neighbourhood $K \times [0, 1)$ with the two $S^1/\mathbb{Z}_2$-orbits attached to special exceptional components $S^1/\mathbb{Z}_2 \times [0, 1)$, respectively. The orbit space $K/S^1 \times [0, 1) = [0, \pi] \times [0, 1)$ is an open manifold with 3 boundaries and 2 corners (see Fig. 4): where the bottom interval is the orbit space K/S^1, the left and right intervals come from the orbit spaces $(S^1/\mathbb{Z}_2 \times [0, 1))/S^1$ with corner points.

3.1.2. Tubular neighbourhoods of non-principal orbits and their orbit spaces

Equivariant tubular neighbourhoods of an exceptional orbit $S^1/\mathbb{Z}_m$, a fixed circle S^1 or a special exceptional component $S^1/\mathbb{Z}_2 \times S^1$ will still be $S^1 \times_{\mathbb{Z}_m} D$, $D \times S^1$ or Möb $\times S^1$ respectively, the same as we see in the case of 3d S^1-manifolds without boundary. The orbit spaces of these neighbourhoods provide orbifold chart $D/\mathbb{Z}_m$ and annulus charts $[0, 1) \times S^1$ for M/S^1.

Suppose a fixed component F_i has common points with the boundary ∂M. As a 1d compact manifold with boundary, F_i has to be an interval, denoted as $[0, 1]$. Its equivariant tubular neighbourhood will be $D \times [0, 1]$, with boundary $(D \times [0, 1]) \cap \partial M = D \times \{0\} \cup D \times \{1\}$. The orbit space

$$\big(D \times [0, 1]\big)/S^1 = D/S^1 \times [0, 1] = [0, 1) \times [0, 1]$$

is an open manifold with 3 boundaries and 2 corners (see Fig. 5): where the left interval is the fixed interval F_i, the bottom and top intervals come from the orbit space of the boundary $D \times \{0\} \cup D \times \{1\}$.

Similarly, suppose a special exceptional connected component SE_j has common points with the boundary ∂M. As a 2d compact principal $S^1/\mathbb{Z}_2$-manifold with boundary, SE_j has to be a cylinder $S^1/\mathbb{Z}_2 \times [0, 1]$, according to the classification theorem in dimension 2. Its equivariant tubular neighbourhood will be Möb $\times [0, 1]$, with boundary (Möb $\times [0, 1]) \cap \partial M =$ Möb $\times \{0\} \cup$ Möb $\times \{1\}$. The orbit space

$$\big(\text{Möb} \times [0, 1]\big)/S^1 = \text{Möb}/S^1 \times [0, 1] = [0, 1) \times [0, 1]$$

is an open manifold with 3 boundaries and 2 corners (see Fig. 6): where the left interval comes from the orbit space SE_j/S^1, the bottom and top intervals come from the orbit space of the boundary Möb $\times \{0\} \cup$ Möb $\times \{1\}$.

3.2. Orbit spaces and classifications of 3d s^1-manifolds with boundaries

Using the discussion of local orbit spaces, we have the following theorem about orbit space of 3d S^1-manifolds with boundaries:

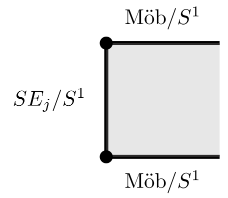


Fig. 6. One-side-open rectangle $[0, 1) \times [0, 1]$.

Proposition 3.1 (*Orbit space of 3d S^1-manifold*)**.** *For a compact connected 3d effective S^1-manifold M possibly with boundary, the orbit space $M^* = M/S^1$ is a 2d orbifold surface, possibly with boundary and corners. The orbifold surface M^* has a finite number of interior orbifold points E/S^1 with Seifert invariants $\{(m_1, n_1), \ldots, (m_l, n_l)\}$. Moreover, it has boundary $\partial M^* = F \cup SE/S^1 \cup (\partial M)/S^1$ and corner points $\partial^2 M^* = (F \cup SE/S^1) \cap (\partial M)/S^1$.*

Proof. This follows from the previous local discussions. □

Remark 3.1. The notation $\partial^2 M^*$ for corners is because in dimension 2, corners are boundary of the boundary.

If we trace the boundaries of the orbit space $M^* = M/S^1$, the circle boundaries of M^* are formed from either $\mathbb{T}^2$ boundaries of M, or fixed components and special exceptional components in M that do not meet ∂M. While the interval boundaries of M^* are formed from either S^2, K, $\mathbb{R}P^2$ boundaries of M, or fixed components and special exceptional components in M that meet ∂M. At the meeting points, corners of M^* are formed and connect intervals into cycles.

The orbit invariants of an S^1-manifold M with non-empty boundary can be given as follows:

- ϵ: o if $M^* = M/S^1$ is orientable, or n otherwise
- g: the genus of the quotient orbifold surface M^*
- f: the number of fixed components in M not touching ∂M
- s: the number of special exceptional components in M not touching ∂M
- t: the number of $\mathbb{T}^2$ boundaries of M
- $\mathcal{G}$: the graph of cycles whose edges and vertices are, respectively, the interval boundaries and corner points of M^*. Along with the edges, we will also record the types of these edges formed from S^2, K, $\mathbb{R}P^2$ boundaries of M, or fixed components and special exceptional components in M that meet ∂M.
- $(m_1, n_1), \ldots, (m_l, n_l)$: the Seifert invariants.

Remark 3.2. When an S^1-manifold M^3 is closed, there is the invariant b denoting the Euler number of the principal part of M. As we will see shortly, we can include the invariant b back by setting $b = 0$ for S^1-manifolds with boundaries.

Condition 3.1. Taking into account the types of edges, every cycle in the graph $\mathcal{G}$ must satisfy the following conditions of type distributions (for example, see Fig. 7):

(1) Every type-F edge connects edges of type-S^2 or type-$\mathbb{R}P^2$
(2) Every type-SE edge connects edges of types-K or type-$\mathbb{R}P^2$
(3) Every type-S^2 edge connects type-F edges
(4) Every type-K edge connects type-SE edges
(5) Every type-$\mathbb{R}P^2$ edge connects one edge of type-F and one edge of type-SE.

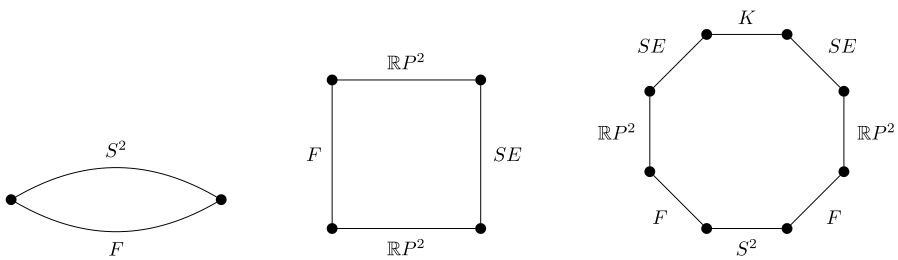


Fig. 7. Examples of cycles satisfying the condition of type distributions.

A simple counting of the numbers of vertices in type F and SE, respectively, gives the following:

Proposition 3.2. *Given a graph $\mathcal{G}$ constructed as above from an S^1-manifold M^3 with non-empty boundary, let s_p, k, r be the numbers of edges formed from $S^2,\ K,\ \mathbb{R}P^2$ boundaries of M, and let f', s' be the numbers of edges formed from fixed components and special exceptional components of M, respectively. Then, there are numeric relations:*

$$2f' = 2s_p + r$$
$$2s' = 2k + r.$$

In particular, the number r of $\mathbb{R}P^2$ boundaries of M is even.

Remark 3.3. The above counting is valid not only for the entire graph $\mathcal{G}$, but also for each cycle in $\mathcal{G}$.

Now, we can give the classification of 3d S^1-manifolds with boundary

Theorem 3.1 (*Classification of 3d S^1-manifolds with boundary*)**.** *Let S^1 act effectively and smoothly on a compact, connected 3d manifold M with boundary. Then, the orbit invariants*

$$\left\{(\epsilon, g, f, s, t); (m_1,\ n_1), \ldots, (m_l,\ n_l); \mathcal{G}\right\}$$

consisting of numeric data and a graph of cycles, determine M up to equivariant diffeomorphisms, subject to the following conditions:

(1) $0 < n_i < m_i,\ (m_i,\ n_i) = 1$ *if* $\epsilon = o$*, orientable*
$0 < n_i \leqslant \frac{m_i}{2},\ (m_i,\ n_i) = 1$ *if* $\epsilon = n$*, non-orientable*
(2) *The graph $\mathcal{G}$ satisfies the Condition 3.1 of distribution of edge types*

Conversely, any such set of invariants can be realized by an effective S^1-action on a compact 3d manifold with boundary.

Proof. The proof of this non-closed case is similar to the closed case in Raymond's paper [2]. Given orbit invariants $\left\{(\epsilon, g, f, s, t); (m_1,\ n_1), \ldots, (m_l,\ n_l); \mathcal{G}\right\}$, we firstly recover the boundary from the data $\{t, \mathcal{G}\}$ and the non-principal orbits from the data $\{(f, s); (m_1,\ n_1), \ldots, (m_l,\ n_l); \mathcal{G}\}$, where t is the number of $\mathbb{T}^2$ boundaries and $\mathcal{G}$ records the distributions of $S^2,\ K,\ \mathbb{R}P^2$ boundaries and their nearby fixed components and special exceptional components. Secondly, we enlarge the boundary and non-principal orbits to their neighbourhoods using the local description given in the previous subsection. Finally, we take a surface N with orientability ϵ, genus g and ($f + s + t +$ number of cycles in $\mathcal{G}$) disks deleted. The circle bundle over ∂N is inherited from our second step, and can be extended over the entire N uniquely up to equivariant diffeomorphisms because the obstruction of doing so is $H^2(N, \mathbb{Z}) = 0$. These three steps realize the given orbit invariants $\left\{(\epsilon, g, f, s, t); (m_1,\ n_1), \ldots, (m_l,\ n_l); \mathcal{G}\right\}$ uniquely up to equivariant diffeomorphisms. □

Remark 3.4. If we add the $b = 0$ to the case of 3d S^1-manifolds with boundary, we can synthesize the cases with or without boundary into one single theorem.

4. Equivariant cohomology of 3d s^1-manifolds

The classification of 3d S^1-manifolds (possibly with boundaries) in terms of numeric invariants and graphs gives us an S^1-equivariant stratification of every such manifold and enables us to calculate all kinds of topological data. For example, the fundamental groups, ordinary homology and cohomology with $\mathbb{Z}$ or $\mathbb{Z}_p$ coefficients have been computed extensively for closed 3d S^1-manifolds in literature [4–7], and now can be generalized to 3d S^1-manifolds with boundaries, using the classification Theorems 2.6 and 3.1. But, not much has been discussed for S^1-equivariant cohomology, which is the goal of the current section.

In the following subsections, we will first prove our core Theorem 4.2 in full generality. When we explore more delicate computational invariants, we will try to keep the presentation of results in a manageable way.

4.1. Some basic facts about equivariant cohomology

In the following discussion, the coefficient of cohomology will always be $\mathbb{Q}$ unless otherwise mentioned. For a group action of G on M, the equivariant cohomology ring is defined using the Borel construction $H^*_G(M) = H^*(EG{\times}_G M)$, where $H^*(-)$ is the ordinary simplicial cohomology theory, EG is the universal principal G-bundle and $EG{\times}_G M$ is the associated bundle with fibre M. The pull-back $\pi^* : H^*_G(pt) \longrightarrow H^*_G(M)$ of the trivial map $\pi : M \longrightarrow pt$ gives $H^*_G(M)$ a module structure of the ring $H^*_G(pt)$.

In general, the equivariant cohomology $H^*_G(M)$ is not the same as the ordinary cohomology $H^*(M/G)$ of the orbit space M/G. If we choose any fibre inclusion $\iota : M \to EG \times M$ and pass to the orbit spaces $\bar{\iota} : M/G \to EG{\times}_G M$, then the pull-back $\bar{\iota}^* : H^*_G(M) = H^*(EG{\times}_G M) \to H^*(M/G)$ gives a natural map between $H^*_G(M)$ and $H^*(M/G)$.

We will need some basic facts to compute equivariant cohomology, see any of the expository surveys [13,14] for details.

The first set of facts is about equivariant cohomology of homogeneous space, i.e. space with one single orbit:

Fact 4.1. *Let G be a compact Lie group, and H a closed Lie subgroup. Denote $BG = EG/G$ and $BH = EH/H$ for the classifying space of G-bundles and H-bundles, respectively. Then,*

- $H^*_G(pt) = H^*(EG/G) = H^*(BG)$
- $H^*_G(G/H) \cong H^*_H(pt) = H^*(BH)$

The second set of facts is about equivariant cohomology of extremal types of group actions:

Fact 4.2. *Let a compact Lie group G act on a compact manifold M.*

- *If the action $G \curvearrowright M$ is free, then $H^*_G(M) \cong H^*(M/G)$.*
- *If the action $G \curvearrowright M$ is trivial, then $H^*_G(M) \cong H^*(M) \otimes H^*_G(pt)$.*

In particular, when $G = S^1$, there are three types of orbits: S^1, $S^1/\mathbb{Z}_m$, S^1/S^1. For a principal orbit, $H^*_{S^1}(S^1) \cong H^*(pt)$. For an exceptional orbit $S^1/\mathbb{Z}_m$, the classifying space $B\mathbb{Z}_m = S^\infty/\mathbb{Z}_m$ is the infinite Lens space with cohomology in $\mathbb{Q}$-coefficient the same as $H^*(pt)$. For a fixed point S^1/S^1, the classifying space $BS^1 = \mathbb{C}P^\infty$ is the infinite projective space with cohomology $\mathbb{Q}[u]$ a polynomial ring, where the parameter u is the generator of $H^2(\mathbb{C}P^1)$ in degree 2.

	Principal orbit	Exceptional orbit	Singular orbit
Orbit $\mathcal{O}$	S^1	$S^1/\mathbb{Z}_m$	S^1/S^1
$H^*_{S^1}(\mathcal{O}, \mathbb{Q})$	$H^*(pt, \mathbb{Q})$	$H^*(pt, \mathbb{Q})$	$\mathbb{Q}[u]$

The third set of facts enables us to compute equivariant cohomology by deforming, cutting and pasting, similar to the computation in ordinary cohomology:

Fact 4.3. *Let U_1, U_2 be two G-spaces, and A, B be two G-subspaces of a G-space X.*

Homotopy invariance *If $\varphi : U_1 \xrightarrow{\simeq} U_2$ is a G-homotopic equivalence, then $\varphi^* : H^*_G(U_2) \xrightarrow{\cong} H^*_G(U_1)$ is an isomorphism.*
Mayer–Vietoris sequence *If $X = A^\circ \cup B^\circ$ is the union of interiors of A and B, then there is a long exact sequence:*

$$\cdots \longrightarrow H^i_G(X) \longrightarrow H^i_G(A) \oplus H^i_G(B) \longrightarrow H^i_G(A \cap B) \xrightarrow{\delta} H^{i+1}_G(X) \longrightarrow \cdots$$

Remark 4.1. Besides the Borel model of equivariant cohomology, there are also Cartan model and Weil model (cf. Guillemin–Sternberg [15]) using equivariant de Rham theory. In this paper, we prefer the Borel model because the homotopy invariance and Mayer–Vietoris sequence are more natural for the Borel model, from the topological rather than the differential point of view.

The fourth set of facts deals with equivariant cohomology of product spaces:

Fact 4.4. *Let $G \curvearrowright M$ and $H \curvearrowright N$ be two group actions on manifolds. Then, for the product action $G \times H \curvearrowright M \times N$, we get*

$$H^*_{G\times H}(M \times N) \cong H^*_G(M) \otimes H^*_H(N).$$

Especially, for the product action $G \curvearrowright M \times N$ where G acts on N trivially, we get

$$H^*_G(M \times N) \cong H^*_G(M) \otimes H^*(N).$$

4.2. A short exact sequence

Let S^1 act effectively on a compact connected 3d manifold M, possibly with boundary. We will compute the equivariant cohomology group $H^*_{S^1}(M, \mathbb{Q})$ by cutting and pasting, with the help of the classification theorem from previous sections.

As we have seen from the previous computation of $H^*_{S^1}(\mathcal{O})$ for each S^1-orbit $\mathcal{O}$, the S^1-equivariant cohomology in $\mathbb{Q}$ coefficient does not distinguish principal orbit S^1 from exceptional orbit $S^1/\mathbb{Z}_m$ or special exceptional orbit $S^1/\mathbb{Z}_2$. However, there is big difference between the S^1-equivariant cohomology of fixed point and non-fixed orbit.

If a 3d S^1-manifold M does not have fixed points, we would hope that its S^1-equivariant cohomology is the ordinary cohomology of the orbit space M/S^1. Actually, a more general statement is true due to Satake [16]. The version here is taken from Duistermaat's lecture notes [17].

Definition 4.1. An action of a compact Lie group G on a manifold M is **locally free**, if for every $x \in M$, the isotropy group G_x is finite.

Theorem 4.1 (*Satake [16]*)**.** *If a compact Lie group G acts locally freely on a compact manifold M, then M/G is an orbifold, and $H^*_G(M, \mathbb{R}) \cong H^*(M/G, \mathbb{R})$.*

We can certainly apply the Theorem of Satake to our special case of S^1-actions. However, there is a subtlety in Satake's definition of $H^*(M/S^1, \mathbb{R})$ for the orbifold M/S^1 in terms of *orbifold* differential forms (cf. [16,17]). Moreover, because of the use of differential forms, the above theorem is originally stated for $\mathbb{R}$-coefficients not for $\mathbb{Q}$-coefficients.

In our definition of $H^*(M/S^1, \mathbb{Q})$, we will simply use the ordinary simplicial cohomology for the topological space M/S^1 by forgetting its orbifold structure.

Our method of calculating equivariant cohomology is based on the equivariant Mayer–Vietoris sequence and induction on the number of non-principal components which is finite because of the compactness of M.

Proposition 4.1. *Let S^1 act effectively on a compact connected 3d manifold M, possibly with boundary. If M does not have fixed points, then $H^*_{S^1}(M, \mathbb{Q}) \cong H^*(M/S^1, \mathbb{Q})$.*

Proof. We will proceed by induction on the number of non-principal components.

To begin with, suppose M does not have non-principal component. Since we assume there is no fixed point, then S^1 acts on M freely and hence $H^*_{S^1}(M) \cong H^*(M/S^1)$.

Now, suppose the proposition is true for any 3d fixed-point-free S^1-manifold with $k \geqslant 0$ non-principal components, and suppose M has $k+1$ non-principal components. Let C be a non-principal component together with an equivariant tubular neighbourhood N, then the complement $M' = M \setminus N$ has k non-principal components and $H^*_{S^1}(M') \cong H^*(M'/S^1)$ according to our assumption. Let us also denote $L = M' \cap N$.

The equivariant Mayer–Vietoris sequence for the union $M = M' \cup N$ and the ordinary Mayer–Vietoris sequence for the union $M/S^1 = M'/S^1 \cup N/S^1$ gives

$$\begin{array}{ccccccccc}
H^{*-1}_{S^1}(M') \oplus H^{*-1}_{S^1}(N) & \longrightarrow & H^{*-1}_{S^1}(L) & \longrightarrow & H^*_{S^1}(M) & \longrightarrow & H^*_{S^1}(M') \oplus H^*_{S^1}(N) & \longrightarrow & H^*_{S^1}(L) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{*-1}(M'/S^1) \oplus H^{*-1}(N/S^1) & \longrightarrow & H^{*-1}(L/S^1) & \longrightarrow & H^*(M/S^1) & \longrightarrow & H^*(M'/S^1) \oplus H^*(N/S^1) & \longrightarrow & H^*(L/S^1)
\end{array}$$

where the second and the fifth vertical maps are isomorphisms, because the intersection $L = M' \cap N$ does not touch non-principal orbits and consists of only principal orbits.

According to the Five Lemma in homological algebra, in order to prove that the middle vertical map is an isomorphism, we now need to prove the first and the fourth maps are isomorphisms. But, we already have the isomorphism $H^*_{S^1}(M') \cong H^*(M'/S^1)$. So, we only need to prove $H^*_{S^1}(N) \cong H^*(N/S^1)$.

In the 3d fixed-point-free S^1-manifold M, according to our detailed discussion in Section 3.1, there are three cases for a non-fixed, non-principal component C, its equivariant neighbourhood N and orbit space N/S^1. Note that, for each case, there is an equivariant deformation retraction $N \simeq C$, so we have $H^*_{S^1}(N) \cong H^*_{S^1}(C)$. Also, recall that we have calculated $H^*_{S^1}(S^1/\mathbb{Z}_m, \mathbb{Q}) \cong H^*(pt, \mathbb{Q})$.

C	$S^1/\mathbb{Z}_m$	$S^1/\mathbb{Z}_2 \times S^1$	$S^1/\mathbb{Z}_2 \times I$
N	$S^1 \times_{\mathbb{Z}_m} D^2$	Möb $\times S^1$	Möb $\times I$
N/S^1	$D^2/\mathbb{Z}_m$	$I \times S^1$	$I \times I$
$H^*_{S^1}(N) \cong H^*_{S^1}(C)$	$H^*(pt)$	$H^*(S^1)$	$H^*(I)$
$H^*(N/S^1)$	$H^*(D^2/\mathbb{Z}_m)$	$H^*(S^1)$	$H^*(I)$

For the second and the third case, it is clear that $H^*_{S^1}(N) \cong H^*(N/S^1)$. For the first case, the orbit space $D^2/\mathbb{Z}_m$, viewed as an ice-cream cone, has a deformation retract to the cone's tip pt, so $H^*_{S^1}(N) \cong H^*(pt) \cong H^*(D^2/\mathbb{Z}_m) = H^*(N/S^1)$. □

If a 3d S^1-manifold M has fixed points, then the number of fixed components will be finite due to the compactness of M, and every fixed component is either a circle S^1 or an interval I according to our discussion in the previous Sections 2 and 3. The calculation of S^1 equivariant cohomology of a general 3d S^1-manifold M will be carried out by doing induction on the number of connected components of these fixed points. The beginning case of no fixed points is just the previous Proposition 4.1.

Suppose now that an S^1-manifold M has $k > 0$ connected components of fixed points. Let us choose any such connected component F, with its equivariant neighbourhood N. If $F = S^1$, then $N = D \times S^1$; if $F = I$, then $N = D \times I$. In both cases, $N = D \times F$. If we set the complement $M' = M \setminus N$, then M is attached equivariantly by M' and $N = D \times F$ along $S^1 \times F$. The Mayer–Vietoris sequence of equivariant cohomology groups then gives

$$\to H^*_{S^1}(M, \mathbb{Q}) \to H^*_{S^1}(M', \mathbb{Q}) \oplus H^*_{S^1}(D \times F, \mathbb{Q}) \to H^*_{S^1}(S^1 \times F, \mathbb{Q}) \to H^{*+1}_{S^1}(M, \mathbb{Q}) \to .$$

However, since the S^1-action on $D \times F$ and $S^1 \times F$ concentrates on their first components, respectively, we have

$$\begin{array}{ccc} H^*_{S^1}(D\times F) & \longrightarrow & H^*_{S^1}(S^1\times F) \\ \| & & \| \\ H^*_{S^1}(D)\otimes H^*(F) & \longrightarrow & H^*_{S^1}(S^1)\otimes H^*(F) \\ \| & & \| \\ \mathbb{Q}[u]\otimes H^*(F) & \longrightarrow & H^*(F): \end{array} \qquad f(u)\otimes\alpha \longmapsto f(0)\cdot\alpha$$

where the upper 2 vertical isomorphisms are because of the cohomology of product spaces, the lower left vertical isomorphism is because of homotopy between D and pt, and the lower right vertical isomorphism is because the S^1 is a principal orbit.

The bottom map is obviously surjective, so is the top map $H^*_{S^1}(D\times F)\to H^*_{S^1}(S^1\times F)$. This means that the long exact sequence actually stops at $H^*_{S^1}(M')\oplus H^*_{S^1}(D\times F)\to H^*_{S^1}(S^1\times F)\to 0$. We then conclude that the long exact sequence reduces into the following short exact sequence:

$$0\to H^*_{S^1}(M)\to H^*_{S^1}(M')\oplus\Big(\mathbb{Q}[u]\otimes H^*(F)\Big)\to H^*(F)\to 0$$

where we have replaced the $H^*_{S^1}(D\times F)$ and $H^*_{S^1}(S^1\times F)$ by $\mathbb{Q}[u]\otimes H^*(F)$ and $H^*(F)$, respectively.

We can now consider all the k components of fixed points $F_1, F_2, \dots, F_k$, together with their equivariant tubular neighbourhood $N_1, N_2, \dots, N_k$. If we set the complement $M_\circ = M\smallsetminus\cup_i N_i$, an S^1-manifold without fixed points, then there is a short exact sequence of cohomology groups:

$$0\to H^*_{S^1}(M)\to H^*_{S^1}(M_\circ)\oplus\oplus_i\Big(\mathbb{Q}[u]\otimes H^*(F_i)\Big)\to\oplus_i H^*(F_i)\to 0 \qquad (\dagger)$$

Since $M_\circ$ is fixed-point-free, $H^*_{S^1}(M_\circ,\mathbb{Q})\cong H^*(M_\circ/S^1,\mathbb{Q})$ by Proposition 4.1. To understand the orbit space $M_\circ/S^1$, we can compare it with the orbit space M/S^1.

Lemma 4.1. *Following the above notation, the two orbit spaces $M_\circ/S^1$ and M/S^1 are topologically homotopic. Especially, $H^*(M_\circ/S^1,\mathbb{Q})\cong H^*(M/S^1,\mathbb{Q})$.*

Proof. Since the majority of $M_\circ/S^1$ and M/S^1 is isomorphic, we only need to check what happens in an equivariant neighbourhood N near an S^1-fixed component F of M.

Let N' be an equivariant neighbourhood slightly larger than N. If we choose local S^1-equivariant coordinates properly, we can write $N' = D_1\times F$ and $N = D_{\frac{1}{2}}\times F$, where D_1 and $D_{\frac{1}{2}}$ are 2-dimensional disks of radii 1 and $\frac{1}{2}$, such that S^1 acts on the disks by standard rotation.

Now $N'\smallsetminus N = (D_1\smallsetminus D_{\frac{1}{2}})\times F$ and $N' = D_1\times F$ are equivariant neighbourhoods of $M_\circ = M\smallsetminus N$ and M, respectively. Their orbit spaces by the S^1-action give neighbourhoods $(N'\smallsetminus N)/S^1$ and N'/S^1 of $M_\circ/S^1$ and M/S^1, respectively.

However,

$$(N'\smallsetminus N)/S^1 = \Big((D_1\smallsetminus D_{\frac{1}{2}})/S^1\Big)\times F = [\frac{1}{2},1)\times F$$

and

$$N'/S^1 = \Big(D_1/S^1\Big)\times F = [0,1)\times F$$

are homotopic. Thus, $M_\circ/S^1$ and M/S^1 are homotopic. □

Finally, we can combine all the above discussions and get the following:

Theorem 4.2. *Let M be a compact connected 3d effective S^1-manifold(possibly with boundary), and F be its fixed-point set(possibly empty), then there is a short exact sequence of cohomology groups in $\mathbb{Q}$ coefficients:*

$$0\to H^*_{S^1}(M)\to H^*(M/S^1)\oplus\Big(\mathbb{Q}[u]\otimes H^*(F)\Big)\to H^*(F)\to 0 \qquad (\ddagger)$$

Proof. If the fixed-point set F is not empty, then we can use the short exact sequenceEq. (†), and the replacement $H^*_{S^1}(M_\circ)\cong H^*(M_\circ/S^1)\cong H^*(M/S^1)$ because of the Lemma 4.1. If the fixed-point set $F=\varnothing$ is empty, then $H^*(F)=0$. We just use the Proposition 4.1 which says $H^*_{S^1}(M)\cong H^*(M/S^1)$. □

Remark 4.2. To be more specific about the maps involved in the above short exact sequence (‡):

(1) $H^*_{S^1}(M)\to H^*(M/S^1)$ is the natural map between equivariant cohomology of M and ordinary cohomology of M/S^1

(2) $H^*_{S^1}(M) \to \mathbb{Q}[u] \otimes H^*(F)$ is the equivariant restriction map from M to its fixed-point set F
(3) $H^*(M/S^1) \to H^*(F)$ is the restriction map from M/S^1 to its boundary formed by F
(4) $\mathbb{Q}[u] \otimes H^*(F) \to H^*(F)$ is the evaluation map given by $f(u) \otimes \alpha \mapsto f(0)\alpha$.

4.3. The ring and module structure

By the short exact sequence (‡) of Theorem 4.2, we have the inclusion of cohomology groups: $H^*_{S^1}(M) \hookrightarrow H^*(M/S^1) \oplus \big(\mathbb{Q}[u] \otimes H^*(F)\big)$. Since this inclusion is the direct sum of two restriction maps of cohomology rings, it preserves ring structure. Therefore, we can describe the ring structure of $H^*_{S^1}(M)$ explicitly in terms of elements and constraints in $H^*(M/S^1)$ and $\mathbb{Q}[u] \otimes H^*(F)$.

For simplicity, we will focus on closed 3d S^1-manifolds. If M does not have fixed points, then the Proposition 4.1 says that its equivariant cohomology ring is the cohomology ring of the orbit space.

Thus, we will only be interested in the case where M has non-empty set of fixed points. According to the classification theorem, we can write $M = \big\{b = 0; (\epsilon, g, f, s); (m_1, n_1), \ldots, (m_l, n_l)\big\}$ with $f > 0$. Topologically, M/S^1 is a 2d surface of genus g, with $f + s > 0$ boundary circles.

Let us first give a description of the involved cohomologies $H^*(M/S^1)$ and $\mathbb{Q}[u] \otimes H^*(F_i)$.

The orbit space M/S^1 as a topological 2d surface of genus g, has f boundary circles $\cup_{i=1}^{f} F_i$ from fixed components and s boundary circles $\cup_{j=1}^{s} SE_j$ from the orbit spaces of special exceptional components. For a fixed circle $F_i = S^1$, $1 \leqslant i \leqslant f$, we write $H^*(F_i, \mathbb{Q}) = \mathbb{Q}\delta_i \oplus \mathbb{Q}\theta_i$, where δ_i and θ_i are generators of $H^0(F_i, \mathbb{Z})$ and $H^1(F_i, \mathbb{Z})$, respectively. Similarly, for $SE_j = S^1$, $1 \leqslant j \leqslant s$, we write $H^*(SE_j, \mathbb{Q}) = \mathbb{Q}\delta_{f+j} \oplus \mathbb{Q}\theta_{f+j}$. If the orbit space M/S^1 is orientable, i.e. $\epsilon = o$, though $\pm\theta_i$ are both generators for $H^1(F_i, \mathbb{Z})$, we only choose θ_i compatible with the boundary orientation on F_i. The same rule of choice also applies to θ_{f+j}. Moreover, we can write $\mathbb{Q}[u] \otimes H^*(F_i) = \mathbb{Q}[u]\delta_i \oplus \mathbb{Q}[u]\theta_i$ such that every element of $\mathbb{Q}[u] \otimes H^*(F_i)$ can be expressed as $p_i(u)\delta_i + q_i(u)\theta_i$ for polynomials $p_i(u), p_i(u) \in \mathbb{Q}[u]$.

Using the classic calculation of cohomology of 2d surfaces with boundaries, the cohomology $H^*(M/S^1)$ has two different descriptions according to whether M/S^1 is orientable or not.

If M/S^1 is an orientable surface of genus g with $f+s > 0$ boundary circles, then it is homotopic to a wedge of $2g+f+s-1$ circles. Let us denote α_k, β_k, $1 \leqslant k \leqslant g$ for the generators of $H^1(-)$ of the $2g$ circles used in the polygon presentation of the surface M/S^1. Then, we can write $H^*(M/S^1)$ as a sub-ring of $\mathbb{Q}\delta_0 \oplus \oplus_{k=1}^{g} \big(\mathbb{Q}\alpha_k \oplus \mathbb{Q}\beta_k\big) \oplus \big(\oplus_{i=1}^{f} \mathbb{Q}\theta_i\big) \oplus \big(\oplus_{j=1}^{s} \mathbb{Q}\theta_{f+j}\big)$, such that every element of $H^*(M/S^1)$ can be expressed as $D\delta_0 + \sum_k (A_k\alpha_k + B_k\beta_k) + \sum_i C_i\theta_i + \sum_j C_{f+j}\theta_{f+j}$ for $D, A_k, B_k, C_i, C_{f+j} \in \mathbb{Q}$, under the constraint that $\sum_k (A_k + B_k) + \sum_i C_i + \sum_j C_{f+j} = 0$.

Moreover, we have the restriction maps to each fixed circle F_i:

$$\mathbb{Q}[u] \otimes H^*(F_i) \to H^*(F_i): \quad p_i(u)\delta_i + q_i(u)\theta_i \mapsto p_i(0)\delta_i + q_i(0)\theta_i$$

and

$$H^*(M/S^1) \to H^*(F_i): \quad D\delta_0 + \sum_{k=1}^{g} (A_k\alpha_k + B_k\beta_k) + \sum_{i=1}^{f} C_i\theta_i + \sum_{j=1}^{s} C_{f+j}\theta_{f+j} \mapsto D\delta_i + C_i\theta_i.$$

If M/S^1 is a non-orientable surface of genus g with $f+s > 0$ boundary circles, then it is homotopic to a wedge of $g+f+s-1$ circles. We can denote α_k, $1 \leqslant k \leqslant g$ for the generators of $H^1(-)$ of the g circles used in the polygon presentation of the surface M/S^1. The description of the cohomology $H^*(M/S^1)$ together with the restriction maps is similar to the orientable case, with the only difference that there is no β_k, B_k for the non-orientable case.

Following the above notations, we get the following:

Theorem 4.3. *For a closed 3d S^1-manifold $M = \big\{b = 0; (\epsilon = o, g, f, s); (m_1, n_1), \ldots, (m_l, n_l)\big\}$ with $f > 0$ and an orientable orbit space M/S^1, an element of its equivariant cohomology $H^*_{S^1}(M)$ can be written as*

$$\left(D\delta_0 + \sum_{k=1}^{g} (A_k\alpha_k + B_k\beta_k) + \sum_{i=1}^{f} C_i\theta_i + \sum_{j=1}^{s} C_{f+j}\theta_{f+j}\,;\, \sum_{i=1}^{f} (p_i(u)\delta_i + q_i(u)\theta_i)\right) \qquad (*)$$

in $H^*(M/S^1) \oplus \oplus_i \Big(\mathbb{Q}[u] \otimes H^*(F_i)\Big)$, under the relations

(1) $\sum_{k=1}^{g} (A_k + B_k) + \sum_{i=1}^{f} C_i + \sum_{j=1}^{s} C_{f+j} = 0$
(2) $p_1(0) = p_2(0) = \cdots = p_f(0) = D$
(3) $q_i(0) = C_i$ *for each* i

Breaking the equivariant cohomology $H^*_{S^1}(M)$ into different degrees, we have

- $H^0_{S^1}(M) = \mathbb{Q}$

- $H^1_{S^1}(M)$ *is a subgroup of* $H^1(M/S^1) \oplus \oplus_i H^1(F_i)$ *consisting of elements*

$$\Big(\sum_{k=1}^{g}(A_k\alpha_k + B_k\beta_k) + \sum_{i=1}^{f} C_i\theta_i + \sum_{j=1}^{s} C_{f+j}\theta_{f+j}\,;\, \sum_{i=1}^{f} C_i\theta_i\Big)$$

under the constraint $\sum_{k=1}^{g}(A_k+B_k) + \sum_{i=1}^{f} C_i + \sum_{j=1}^{s} C_{f+j} = 0$.
- $H^{\geqslant 2}_{S^1}(M) = \oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$ *where* $\mathbb{Q}[u]_+$ *consists of polynomials without constant terms.*

Proof. The expression (∗) of elements of $H^*_{S^1}(M)$ comes from the description of cohomologies $H^*(M/S^1)$ and $\mathbb{Q}[u] \otimes H^*(F_i)$. The relations (1)(2)(3) are due to the Theorem 4.2 that $H^*_{S^1}(M)$ is the kernel of the restriction map $H^*(M/S^1) \oplus \oplus_i\Big(\mathbb{Q}[u] \otimes H^*(F_i)\Big) \to \oplus_i H^*(F_i)$. Thus, the images of restrictions are the same: $p_1(0) = p_2(0) = \cdots = p_f(0) = D$, and $q_i(0) = C_i$. Since the relations (1)(2)(3) only live in degree less than 2, we get the description of $H^*_{S^1}(M)$ in different degrees. □

Remark 4.3. For a closed 3d S^1-manifold $M = \{b = 0; (\epsilon = n, g, f, s); (m_1, n_1), \ldots, (m_l, n_l)\}$ with $f > 0$ and a non-orientable orbit space M/S^1, the explicit expression of elements of $H^*_{S^1}(M)$ is almost the same as the oriented case, with the only modification that there is no β_k, B_k term.

Theorem 4.4. *For a closed 3d* S^1*-manifold* $M = \{b = 0; (\epsilon, g, f, s); (m_1, n_1), \ldots, (m_l, n_l)\}$ *with* $f > 0$*, the graded ring structure of* $H^*_{S^1}(M)$ *is as follows:*

(1) $H^0_{S^1}(M) \otimes H^*_{S^1}(M) \xrightarrow{\cup} H^*_{S^1}(M)$ *and* $H^*_{S^1}(M) \otimes H^0_{S^1}(M) \xrightarrow{\cup} H^*_{S^1}(M)$ *are just scalar multiplication.*
(2) $H^1_{S^1}(M) \otimes H^1_{S^1}(M) \xrightarrow{\cup} H^2_{S^1}(M)$ *is a zero map*
(3) $H^1_{S^1}(M) \otimes H^{\geqslant 2}_{S^1}(M) \xrightarrow{\cup} H^{\geqslant 3}_{S^1}(M)$ *fits into a commutative diagram:*

$$\begin{tikzcd} H^1_{S^1}(M) \otimes H^{\geq 2}_{S^1}(M) \arrow[r] \arrow[d] & H^{\geq 3}_{S^1}(M) \arrow[d, equal] \\ \Big(\oplus_i H^1(F_i)\Big) \otimes \Big(\oplus_i \big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\big)\Big) \arrow[r] & \oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big) \end{tikzcd}$$

where the left map is the restriction map $H^1_{S^1}(M) \to \oplus_i H^1_{S^1}(F_i) = \oplus_i H^1(F_i)$ *tensored with the identification* $H^{\geqslant 2}_{S^1}(M) \cong \oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$*, and the bottom map is the component-wise multiplication in* $\oplus_i\Big(\mathbb{Q}[u] \otimes H^*(F_i)\Big)$.
(4) $H^{\geqslant 2}_{S^1}(M) \otimes H^{\geqslant 2}_{S^1}(M) \xrightarrow{\cup} H^{\geqslant 2}_{S^1}(M)$ *is just the component-wise multiplication of* $\oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$.

Proof. We will explain the above breakdown one by one for the case when M/S^1 is orientable.

(1) This is clear.
(2) From Theorem 4.3, $H^1_{S^1}(M)$ is generated by the basis $\alpha_j,\ \beta_j,\ \theta_i$, which have zero cup product among them.
(3) Similar to the above remark, the $H^1(M/S^1)$ component of $H^1_{S^1}(M) \subset H^1(M/S^1) \oplus \oplus_i H^1(F_i)$ has zero cup-product. So, only the cup product involving $\oplus_i H^1(F_i)$ will survive.
(4) Since $H^{\geqslant 2}_{S^1}(M) \cong \oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$, the cup product among $H^{\geqslant 2}_{S^1}(M)$ is inherited from $\oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$.

The argument is exactly the same for the case when M/S^1 is non-orientable, because of the Remark 4.3. □

Using the cup product of Theorem 4.4, we can now describe the $H^*_{S^1}(pt)$-module structure of $H^*_{S^1}(M)$.

Theorem 4.5. *Following the notations of Theorem 4.4, for a closed 3d* S^1*-manifold* M *with non-empty set of fixed points , the forgetful map* $\pi : M \to pt$ *induces the map* $\pi^* : H^*_{S^1}(pt) = \mathbb{Q}[u] \to H^*_{S^1}(M)$*, with the image of the generator* u *being* $\pi^*(u) = \sum_i u\delta_i$*. The generator* u *acts on* $H^*_{S^1}(M)$ *by multiplying with* $\pi^*(u) = \sum_i u\delta_i$ *using the cup product of* $H^*_{S^1}(M)$.

Proof. $u \in \mathbb{Q}[u]$ is of degree 2, so is $\pi^*(u) \in H^{\geqslant 2}_{S^1}(M) \cong \oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$. Hence, we only need to know the restriction of $\pi^*(u)$ from $H^*_{S^1}(M)$ to $H^*_{S^1}(F_i)$ for each fixed circle F_i. The commutative diagram of forgetful maps

$$\begin{tikzcd} F_i \arrow[r, hook] \arrow[dr, "\pi_i"'] & M \arrow[d, "\pi"] \\ & pt \end{tikzcd}$$

induces the commutative diagram of maps between equivariant cohomologies

$$\begin{tikzcd} H^*_{S^1}(F_i) & H^*_{S^1}(M) \arrow[l] \\ & H^*_{S^1}(pt) \arrow[ul, "\pi_i^*"] \arrow[u, "\pi^*"'] \end{tikzcd}$$

Thus, the restriction of $\pi^*(u)$ from $H^*_{S^1}(M)$ to $H^*_{S^1}(F_i)$ is the image $\pi_i^*(u)$ via the map $\pi_i^* : H^*_{S^1}(pt) = \mathbb{Q}[u] \to H^*_{S^1}(F_i) = \mathbb{Q}[u]\delta_i \oplus \mathbb{Q}[u]\theta_i$. Since F_i is a fixed component of the S^1-action on M, $u \in \mathbb{Q}[u]$ acts trivially on $H^*_{S^1}(F_i)$ with $\pi_i^*(u) = u\delta_i$.

In conclusion, if we combine the contribution from all the fixed components F_i, we get $\pi^*(u) = \sum_i u\delta_i$. □

If a closed 3d S^1-manifold M does not have fixed point, then the image $\pi^*(u)$ is in $H^2_{S^1}(M) \cong H^2(M/S^1)$ by the Proposition 4.1. In this case, a condition for $\pi^*(u) = 0$ is to make sure that $H^2(M/S^1) = 0$.

Proposition 4.2. *For a closed 3d fixed-point-free S^1-manifold $M = \{b; (\epsilon, g, f = 0, s); (m_1, n_1), \ldots, (m_l, n_l)\}$, if $\epsilon = n$ or $s > 0$, then $H^2_{S^1}(M) \cong H^2(M/S^1) = 0$, hence $\pi^*(u) = 0$.*

Proof. By the classic calculation of cohomology of surfaces. A sufficient condition for $H^2(M/S^1) = 0$ is that M/S^1 is non-orientable or has non-empty boundary, which corresponds to the condition: $\epsilon = n$ or $s > 0$. □

If $\epsilon = o$ and $s = 0$, then this is exactly the case of oriented Seifert manifold. The image $\pi^*(u) \in H^2_{S^1}(M) = H^2(M/S^1)$ is calculated by Niederkrüger in his thesis (cf. [10] Theorem III.13).

Theorem 4.6 (Niederkrüger, [10]). *Given an oriented Seifert manifold $M = \{b; (\epsilon = o, g, f = 0, s = 0); (m_1, n_1), \ldots, (m_l, n_l)\}$, let l_i be the unique solution of $l_i n_i \equiv 1 \mod m_i$, $0 < l_i < m_i$ for each coprime pair (m_i, n_i). Then,*

$$\pi^*(u) = b + \sum_{i=1}^{r} \frac{l_i}{m_i} \in H^2(M/S^1) = \mathbb{Q}.$$

Remark 4.4. The rational number $b + \sum_{i=1}^r l_i/m_i$ is exactly the orbifold Euler characteristic of the oriented Seifert manifold, with integer b contributed by the principal orbits and fraction $\sum_{i=1}^r l_i/m_i$ contributed by the exceptional orbits.

4.4. The vector-space structure

Since we are working in $\mathbb{Q}$-coefficient, the group structure of the equivariant cohomology $H^*_{S^1}(M)$ is simply the $\mathbb{Q}$-vector-space structure. In the short exact sequence (‡), we note that the surjective map $\mathbb{Q}[u] \otimes H^*(F) \longrightarrow H^*(F)$ by sending a polynomial $f(u) \in \mathbb{Q}[u]$ to its constant term $f(0)$, has a kernel $\mathbb{Q}[u]_+ \otimes H^*(F)$, where $\mathbb{Q}[u]_+$ consists of polynomials without constant terms.

Proposition 4.3. *Let M be a compact connected 3d effective S^1-manifold(possibly with boundary), and F be its fixed-point set(possibly empty), we get*

$$H^*_{S^1}(M) \cong H^*(M/S^1) \oplus \Big(\mathbb{Q}[u]_+ \otimes H^*(F)\Big) \quad \text{as graded vector spaces}$$

where $\mathbb{Q}[u]_+$ consists of polynomials without constant terms.

Proof. For a graded vector space, its isomorphism type is determined by the dimension at each grading. In order to prove the proposition, we only need to show the dimension of $H^*_{S^1}(M)$ is the same as the dimension of $H^*(M/S^1) \oplus \big(\mathbb{Q}[u]_+ \otimes H^*(F)\big)$ at each grading. From Theorem 4.2, we know that $H^*_{S^1}(M)$ is the kernel of the surjective map $H^*(M/S^1) \oplus \big(\mathbb{Q}[u] \otimes H^*(F)\big) \to H^*(F)$. Write $\mathbb{Q}[u] = \mathbb{Q}[u]_+ \oplus \mathbb{Q}$, then $H^*(M/S^1) \oplus \big(\mathbb{Q}[u] \otimes H^*(F)\big) = H^*(M/S^1) \oplus \big(\mathbb{Q}[u]_+ \otimes H^*(F)\big) \oplus \big(\mathbb{Q} \otimes H^*(F)\big)$ where the third summand $\mathbb{Q} \otimes H^*(F)$ is isomorphic to $H^*(F)$, and hence the direct sum of the first two summands $H^*(M/S^1)$, $\mathbb{Q}[u]_+ \otimes H^*(F)$ is isomorphic to $H^*_{S^1}(M)$ as graded vector spaces. □

Remark 4.5. The above expression of $H^*_{S^1}(M)$ as a direct sum usually does not preserve the ring structure, unless $F = \varnothing$, i.e. M is fixed-point-free.

Remark 4.6. If the fixed-point set $M^{S^1} = F = \cup_i F_i$ is non-empty, then the orbit space M/S^1 has boundaries, so $H^{*\geqslant 2}(M/S^1) = 0$. Also, note $\mathbb{Q}[u]_+ \otimes H^*(F_i)$ has degrees at least 2. So, the above theorem says that when $M^{S^1} \neq \varnothing$, we have

(1) $H^{*\leqslant 1}_{S^1}(M) \cong H^*(M/S^1)$ is determined by the orbit space and $H^{*\geqslant 2}_{S^1}(M) \cong \oplus_i\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big)$ is determined by the fixed-point set.
(2) Since $H^*(S^1)$ contributes to both even and odd degrees, but $H^*(I)$ only contributes to even degrees. We have $\#\{F_i = S^1\} = \dim H^3_{S^1}(M)$ and $\#\{F_i = I\} = \dim H^2_{S^1}(M) - \dim H^3_{S^1}(M)$.

4.5. Equivariant Betti numbers and Poincaré series

Given an S^1-manifold M, we can calculate its equivariant Betti numbers $b^k_{S^1} = \dim H^k_{S^1}(M)$ and the equivariant Poincaré series $P^M_{S^1}(x) = \sum_{k=0}^{\infty} b^k_{S^1} x^k$.

When a closed 3d S^1-manifold M has neither fixed points nor special exceptional orbits, i.e. $f = s = 0$, also called Seifert manifold, its orbit space M/S^1 is a closed 2d orbifold of genus g. By Proposition 4.1, $H^*_{S^1}(M, \mathbb{Q}) \cong H^*(M/S^1, \mathbb{Q})$ and the classic calculation of cohomology of closed surfaces, we have

Proposition 4.4. *For a closed 3d S^1-manifold M without fixed points nor special exceptional orbits, i.e. $M = \big\{b; (\epsilon, g, f = 0, s = 0); (m_1,\ n_1), \ldots, (m_l,\ n_l)\big\}$, the equivariant Poincaré series are $1 + 2gx + x^2$ if M is orientable, or $1 + gx$ if M is non-orientable.*

When the set of fixed points or special exceptional orbits is non-empty, we will get the following:

Theorem 4.7. *For a closed 3d S^1-manifold $M = \big\{b; (\epsilon, g, f, s); (m_1,\ n_1), \ldots, (m_l,\ n_l)\big\}$ with $f + s > 0$(hence $b = 0$), its equivariant Betti numbers are*

$$\begin{aligned} b^0_{S^1} &= 1 \\ b^1_{S^1} &= \begin{cases} 2g + f + s - 1 & \text{if } \epsilon = o \\ g + f + s - 1 & \text{if } \epsilon = n \end{cases} \\ b^{2k}_{S^1} &= f \quad \text{for } k \geqslant 1 \\ b^{2k+1}_{S^1} &= f \quad \text{for } k \geqslant 1 \end{aligned}$$

with the equivariant Poincaré series

$$P^M_{S^1}(x) = \sum_{k=0}^{\infty} b^k_{S^1} x^k = \begin{cases} 1 + (2g + f + s - 1)x + f \cdot \dfrac{x^2}{1 - x} & \text{if } \epsilon = o \\[2ex] 1 + (g + f + s - 1)x + f \cdot \dfrac{x^2}{1 - x} & \text{if } \epsilon = n. \end{cases}$$

Proof. By Proposition 4.3, the equivariant cohomology of M is

$$H^*_{S^1}(M) \cong H^*(M/S^1) \oplus \oplus_{i=1}^{f}\Big(\mathbb{Q}[u]_+ \otimes H^*(F_i)\Big) \quad \text{as graded vector spaces}$$

where $\mathbb{Q}[u]_+$ is the set of polynomials without constant terms and $F = \cup_i^f F_i$ is the union of fixed circles.

Note that, M/S^1 is a 2d surface of genus g with $f + s > 0$ boundaries. Its Poincaré series are $1 + (2g + f + s - 1)x$ if $\epsilon = o$, or $1 + (g + f + s - 1)x$ if $\epsilon = n$, using the classic result on the cohomology of 2d surface with boundary. For each $\mathbb{Q}[u]_+ \otimes H^*(F_i),\ 1 \leqslant i \leqslant f$, it is easy to see that the Poincaré series are $\frac{x^2}{1-x^2} \cdot (1 + x) = \frac{x^2}{1-x}$.

Then, we can calculate the equivariant Poincaré series $P^M_{S^1}(x)$ and equivariant Betti numbers $b^*_{S^1}$ of M additively from those of M/S^1 and F_i. $\square$

4.6. Equivariant formality

Using the explicit description of the ring and module structures, we can determine when a closed 3d S^1-manifold is equivariantly formal in the following sense.

Definition 4.2. A G-action on a manifold M is **equivariantly formal**, if the equivariant cohomology $H^*_G(M)$ is a free $H^*_G(pt)$-module.

When talking about equivariant formality, we will only be interested in the case of closed manifolds in this paper.

Theorem 4.8. *A closed 3d S^1-manifold $M = \{b; (\epsilon, g, f, s); (m_1, n_1), \dots, (m_l, n_l)\}$ is S^1-equivariantly formal if and only if $f > 0$, $b = 0$ and*

$$\begin{cases} g = s = 0 \text{ or } g = 0,\ s = 1 & \text{if } \epsilon = o \\ g = 1,\ s = 0 & \text{if } \epsilon = n. \end{cases}$$

Proof. For the necessity, when M is S^1-equivariantly formal, $H^*_{S^1}(M)$ is a free $H^*_{S^1}(pt)$-module. Since the polynomial ring $H^*_{S^1}(pt) = \mathbb{Q}[u]$ is infinite dimensional, so is $H^*_{S^1}(M)$. Therefore, it must have non-empty fixed-point set to generate elements of degree to the infinity, so $f > 0$, and hence $b = 0$.

The polynomial ring $\mathbb{Q}[u]$, with u of degree 2, has non-decreasing Betti numbers in odd degrees and even degrees, respectively. Hence, so does any free $H^*_{S^1}(pt) = \mathbb{Q}[u]$-module.

$$\begin{aligned} b^{2k}_{S^1} &\leqslant b^{2k+2}_{S^1} \quad \text{for } k \geqslant 0 \\ b^{2k+1}_{S^1} &\leqslant b^{2k+3}_{S^1} \quad \text{for } k \geqslant 0. \end{aligned}$$

Especially, we will verify $b^1_{S^1} \leqslant b^3_{S^1}$ by substituting our calculation of the Betti numbers $b^*_{S^1}$ from Theorem 4.7.

When $\epsilon = o$, we get $2g+f+s-1 \leqslant f$, or equivalently, $2g+s \leqslant 1$. Here, s as the number of special exceptional components in M, is non-negative; g as the genus of an orientable surface, is also non-negative. These constraints force $g = s = 0$ or $g = 0,\ s = 1$.

When $\epsilon = n$, we get $g + f + s - 1 \leqslant f$, or equivalently, $g + s \leqslant 1$. Here s again is non-negative. But g as the genus of a non-orientable surface, is strictly positive. These constraints force $g = 1,\ s = 0$.

For the sufficiency, let us first assume $f > 0,\ b = 0$.

When $\epsilon = o,\ g = 0,\ s = 0$, there are no $\alpha_k, \beta_k, \theta_{f+j}$ terms, by Theorem 4.3. Also, note that $D\delta_0 + \sum_{i=1}^f C_i\theta_i$ can be absorbed into $\sum_i (p_i(u)\delta_i + q_i(u)\theta_i)$ because of the relations(2)(3) in that theorem. Hence, there is a much nicer expression of an element of the equivariant cohomology $H^*_{S^1}(M)$:

$$\sum_{i=0}^f \big(p_i(u)\delta_i + q_i(u)\theta_i\big) \in \mathbb{Q}[u] \otimes H^*(F)$$

under the relations:

$$p_1(0) = p_2(0) = \cdots = p_f(0) \text{ and } \sum_{i=0}^f q_i(0) = 0.$$

This is indeed a free $\mathbb{Q}[u]$-module, since we can find its $\mathbb{Q}[u]$-module generators without extra relations:

$$\begin{aligned} &\sum_{i=0}^f \delta_i && (1 \text{ term in deg } 0) \\ &\theta_1 - \theta_2, \dots, \theta_1 - \theta_f && (f-1 \text{ terms in deg } 1) \\ &u(\delta_1 - \delta_2), \dots, u(\delta_1 - \delta_f) && (f-1 \text{ terms in deg } 2) \\ &u\sum_{i=0}^f \theta_i && (1 \text{ term in deg } 3). \end{aligned}$$

When $\epsilon = o,\ g = 0,\ s = 1$, there are no α_k, β_k terms and only one θ_{f+1} term among the θ_{f+j} terms, by Theorem 4.3. Again, we can absorb $D\delta_0 + \sum_{i=1}^f C_i\theta_i$ into $\sum_i (p_i(u)\delta_i + q_i(u)\theta_i)$. Moreover, the condition (1) in Theorem 4.3 says $C_{f+1} + \sum_{i=0}^f q_i(0) = 0$, so we can absorb $C_{f+1}\theta_{f+j}$ into $\sum_i q_i(u)\theta_i$. Hence, every element of the equivariant cohomology $H^*_{S^1}(M)$ can be expressed as follows:

$$\sum_{i=0}^f \big(p_i(u)\delta_i + q_i(u)\theta_i\big) \in \mathbb{Q}[u] \otimes H^*(F)$$

under the relations:

$$p_1(0) = p_2(0) = \cdots = p_f(0).$$

This is indeed a free $\mathbb{Q}[u]$-module, since we can find its $\mathbb{Q}[u]$-module generators without extra relations:

$$\begin{array}{ll} \sum_{i=0}^{f} \delta_i & (1 \text{ term in deg } 0) \\ \theta_1, \ldots, \theta_f & (f \text{ terms in deg } 1) \\ u(\delta_1 - \delta_2), \ldots, u(\delta_1 - \delta_f) & (f-1 \text{ terms in deg } 2). \end{array}$$

When $\epsilon = n$, $g = 1$, $s = 0$, there is only one α_1 term among the α_k's, but no β_k, θ_{f+j} terms, by Theorem 4.3 and the remark next to it. Again, we can absorb $D\delta_0 + \sum_{i=1}^{f} C_i\theta_i$ into $\sum_i (p_i(u)\delta_i + q_i(u)\theta_i)$. Moreover, the condition (1) in Theorem 4.3 says $A_1 + \sum_{i=0}^{f} q_i(0) = 0$, so we can absorb $A_1\alpha_1$ into $\sum_i q_i(u)\theta_i$. Hence, every element of the equivariant cohomology $H^*_{S^1}(M)$ can be expressed as follows:

$$\sum_{i=0}^{f} \big(p_i(u)\delta_i + q_i(u)\theta_i\big) \in \mathbb{Q}[u] \otimes H^*(F)$$

under the relations:

$$p_1(0) = p_2(0) = \cdots = p_f(0).$$

This is indeed a free $\mathbb{Q}[u]$-module, since we can find its $\mathbb{Q}[u]$-module generators without extra relations:

$$\begin{array}{ll} \sum_{i=0}^{f} \delta_i & (1 \text{ term in deg } 0) \\ \theta_1, \ldots, \theta_f & (f \text{ terms in deg } 1) \\ u(\delta_1 - \delta_2), \ldots, u(\delta_1 - \delta_f) & (f-1 \text{ terms in deg } 2). \quad \square \end{array}$$

If we focus on the oriented case with $\epsilon = o$, $s = 0$, then

Corollary 4.1. *A closed oriented 3d S^1-manifold $M = \{b; (\epsilon = o, g, f, s = 0); (m_1, n_1), \ldots, (m_l, n_l)\}$ is S^1-equivariantly formal if and only if $f > 0$, $b = 0$, $g = s = 0$.*

When a closed 3d S^1-manifold M satisfies $\{\epsilon = o, f > 0, b = 0, g = s = 0\}$, we get its Poincaré series using Theorem 4.7:

$$P^M_{S^1}(x) = 1 + (f-1)x + f \cdot \frac{x^2}{1-x}.$$

On the other hand, the enumeration of $\mathbb{Q}[u]$-module generators in the above proof of Theorem 4.8 gives the Poincaré series

$$\begin{aligned} P^M_{S^1}(x) &= \big(1 + (f-1)x + (f-1)x^2 + x^3\big) \cdot P^{pt}_{S^1}(x) \\ &= \big(1 + (f-1)x + (f-1)x^2 + x^3\big) \cdot (1 + x^2 + x^4 + \cdots) \\ &= \frac{1 + (f-1)x + (f-1)x^2 + x^3}{1 - x^2}. \end{aligned}$$

However, one can easily check that these two expressions are the same.

Similarly, when a closed 3d S^1-manifold M satisfies $\{\epsilon = o, f > 0, b = 0, g = 0, s = 1\}$ or $\{\epsilon = n, f > 0, b = 0, g = 1, s = 0\}$, we get its Poincaré series using Theorem 4.7:

$$P^M_{S^1}(x) = 1 + fx + f \cdot \frac{x^2}{1-x}.$$

On the other hand, the enumeration of $\mathbb{Q}[u]$-module generators in the above proof of Theorem 4.8 gives the Poincaré series

$$\begin{aligned} P^M_{S^1}(x) &= \big(1 + fx + (f-1)x^2\big) \cdot P^{pt}_{S^1}(x) \\ &= \big(1 + fx + (f-1)x^2\big) \cdot (1 + x^2 + x^4 + \cdots) \\ &= \frac{1 + fx + (f-1)x^2}{1 - x^2}. \end{aligned}$$

One can also easily check that these two expressions are the same.

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