



# Gauge transformations for categorical bundles

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## ABSTRACT

A gauge transformation of categorical principal bundles arises from a functorial isomorphism between such bundles. We determine the geometric nature of such gauge transformations. For a twisted-product categorical principal bundle whose structure group is given by a pair of Lie groups  $G$  and  $H$  we show that a pair consisting of a traditional gauge transformation  $\theta$ , given by a  $G$ -valued function, and an  $L(H)$ -valued 1-form  $\Lambda^H$  determine a categorical gauge transformation. More general gauge transformations are also studied.

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## 1. Introduction

The purpose of this paper is to develop a counterpart of the classical gauge transformation in the setting of categorical bundles. Briefly put, a categorical bundle is a structure, formulated in the language of category theory, that encodes a classical principal bundle equipped with connection and some additional structure. (Here and always we use the terms ‘classical principal bundle’ to mean a principal bundle, in the usual sense from topology and differential geometry, as distinct from a categorical principal bundle.) Just as a classical principal bundle has a structure group, a categorical principal bundle involves two structure groups. Our framework for categorical bundles is motivated by the geometric and physical background and is distinct from more category-theory motivated frameworks.

A gauge transformation, in its most basic form, is given by a smooth function

$$\theta : U \rightarrow G,$$

where  $U$  is an open subset of a manifold and  $G$  is a Lie group that describes the symmetries of a system. In terms of principal bundles, the function  $\theta$  corresponds to the bundle automorphism

$$U \times G \rightarrow U \times G : (b, g) \mapsto (b, \theta(b)g),$$

where we think of  $U \times G$  as the product bundle over  $U$ . A connection form can, in this context, be described by a smooth 1-form  $A_1$  on  $U$  with values in  $L(G)$ , the Lie algebra of  $G$ ; the effect of the gauge transformation  $\theta$  on  $A_1$  is to transform it into

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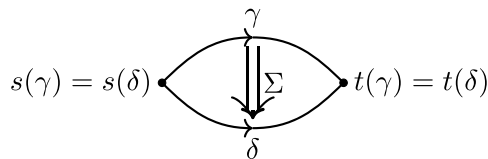


Fig. 1. A morphism of  $\mathcal{P}_2(M)$ .

the connection form  $A_2$  given by

$$A_2 = \theta A_1 \theta^{-1} - (d\theta) \theta^{-1}. \quad (1.1)$$

In this paper we determine the counterpart of this for categorical principal bundles. Such a structure is given by a functor

$$\pi : \mathbf{P} \rightarrow \mathbf{M},$$

along with a categorical group  $\mathbf{G}$  that acts functorially on the right on  $\mathbf{P}$ . We will explain these notions in Section 2, but for now let us note that a categorical group  $\mathbf{G}$ , when unraveled into non-categorical language, involves two Lie groups  $G$  and  $H$ , intertwined in a special structure.

A gauge transformation corresponds, in the categorical context, to a functorial bundle automorphism  $\mathbf{P} \rightarrow \mathbf{P}$ . We focus on the case where the categorical principal bundle is “trivial” in a certain special sense, with  $\mathbf{P} = \mathbf{U} \times_{\eta} \mathbf{G}$  (this structure is described below in Eq. (3.8)), which contains geometric information beyond a simple product bundle structure. Theorem 4.1.1, which is one of our main results, provides an explicit determination of such a functor in the setting where the morphisms of  $\mathbf{U}$  are given by paths on  $U = \text{Obj}(\mathbf{U})$ , which is a manifold. Roughly stated, such a functor is specified by two ‘gauge transformations’: a  $G$ -valued function

$$\theta : U \rightarrow G$$

and an  $L(H)$ -valued 1-form  $\Lambda^H$  on  $U$ .

### 1.1. Other works and approaches

There is a considerable literature on category-theoretic approaches to gauge theories. A brief sample of this includes the many works of Baez et al. [1,2], Martins et al. [3–5], Parzygnat [6,7], Sati et al. [8], Schreiber et al. [9,10], Soncini and Zucchini [11], Waldorf [12–14], Wang [15–17].

Much of the literature mentioned above approaches the theory with a category-theoretic motivation. (The ‘box category’ structure used in Martins and Picken [5] is closer to our framework than is the standard 2-bundle theory.) The physics literature closest to our approach includes the works of Girelli and Pfeiffer [18,19]. Abbaspour and Wagemann [20] provide a brief comparison between some of the different approaches to higher gauge theory.

### 1.2. Comparison with other approaches

Our approach to categorical principal bundles, following the framework developed in our earlier papers [21,22], has a more geometric motivation and setting but uses category-theoretic structures to formulate the theory. We have developed this theory in several directions, including the construction of categorical bundles from local data [23], and in the study of twisted actions of categorical groups [24].

There are some basic differences between our framework and that of the 2-bundle approach. Fundamentally, our framework is a general one, that can be used to understand classical principal bundles as well as “higher” bundles over path spaces.

Let us first look at the situation for base spaces/categories. In the 2-category framework, the “higher path category” for the base manifold  $M$  is  $\mathcal{P}_2(M)$ , with objects corresponding to paths  $\gamma$  on  $M$  and morphisms  $\Sigma : \gamma \rightarrow \delta$  running only between  $\gamma$  and  $\delta$  that have a common source and a common target as shown in Fig. 1. In our framework, a higher morphism  $\Gamma : \gamma_1 \rightarrow \gamma_2$  can run, in principle, between any two ‘paths’  $\gamma_1$  and  $\gamma_2$  on  $M$ , as shown in Fig. 2. More generally, in our framework,  $\text{Obj}(\mathbf{M}_1) = \text{Mor}(\mathbf{M})$ , as we pass from a ‘lower category’  $\mathbf{M}$  to a higher category  $\mathbf{M}_1$ . Our approach is closer to the framework of double categories [25].

In our framework of *categorical principal bundles* there is a classical principal  $G$ -bundle that serves as ‘object bundle’, whereas such a structure does not directly appear in the 2-bundle approach. In other approaches the traditional cocycle defining a  $G$ -bundle is replaced by a weaker, functorial, notion, which also appears in our approach but in a different way [23].

Overall, our motivation is more differential geometric than category theoretic, and the central motivating examples, that of the decorated bundle (Section 2.14) and twisted-product bundles (Section 3), appear to be unique to our approach. At the bundle level, in the case of most interest in our framework, a morphism of the bundle category  $\mathbf{P}$  is not simply a path on

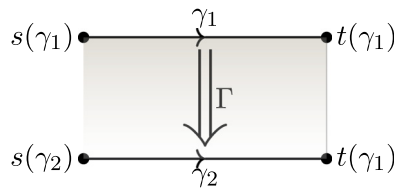


Fig. 2. Objects of  $\mathbf{M}_1$  are morphisms of  $\mathbf{M}$ .

the bundle space  $P$  but a pair  $(\bar{\gamma}, h)$ , where  $\bar{\gamma}$  is a path on  $P$  horizontal with respect to a given connection  $\bar{A}$  on the bundle  $P \rightarrow M$  and  $h$  is an element of the group  $H$  that is associated to a categorical group  $\mathbf{G}$  as explained below in Section 2.2.

A *categorical connection*  $\mathbf{A}$  on a categorical principal bundle  $\mathbf{P} \rightarrow \mathbf{M}$ , in our framework, prescribes a horizontal lift  $\bar{\gamma}_u^{\mathbf{A}}$ , or parallel-transport, for every morphism  $\gamma$  on the base category and every object  $u$  lying above the source  $s(\gamma)$ . The lift is required to satisfy certain natural rigidity conditions (Section 2.11).

Moving up to the case  $\mathbf{P}_1 \rightarrow \mathbf{M}_1$ , where  $\text{Obj}(\mathbf{M}_1) = \text{Mor}(\mathbf{M})$  and  $\text{Obj}(\mathbf{P}_1) = \text{Mor}(\mathbf{P})$ , a categorical connection specifies how to parallel transport an initial morphism of  $\mathbf{P}$  over a ‘higher path’  $\Gamma$  on the space of paths on  $M$ . Thus, unlike in the 2-bundle theory where a connection is required by definition to satisfy some functorial properties and is therefore a highly constrained object specified by a connection form  $A$  and an  $L(H)$ -valued 2-form  $B$  satisfying a special zero-fake-curvature condition, a general *categorical connection* (Section 2.11) in our framework involves several distinct 1-forms and 2-forms that may be used to prescribe parallel-transport.

Our results concerning gauge transformations are similar to those obtained in more category theory oriented approaches, but, since there are basic differences between our framework and those studied elsewhere in the literature, the results are not identical.

### 1.3. Results and organization

After summarizing the basic notions of categorical groups and bundles in Section 2, we review the concept of *twisted-product bundles* in Section 3. These are the categorical counterparts of the traditional product bundles. Following this we come to some new results in Section 4. Here we determine the nature of functorial isomorphisms of twisted-product bundles. Theorem 4.1.1 states one of our main results, specifying the structure of functorial isomorphisms of twisted-product bundles. In essence, a local gauge transformation is specified by two entities:

- (i) a function  $\theta : U \rightarrow G$ , and
- (ii) a function  $\theta_H$  that associates to each path  $\gamma$  on  $U$  an element  $\theta_H(\gamma) \in H$ , satisfying certain consistency properties.

In Section 5 (Theorem 5.2.1) we show that in the generic case  $\theta_H$  can be expressed in terms of a path-ordered integral of an  $L(H)$ -valued 1-form  $\Lambda^H$ . Thus, gauge transformations of interest are specified by a ‘classical’ gauge transformation function  $\theta : U \rightarrow G$  and an  $L(H)$ -valued 1-form  $\Lambda^H$ . The traditional gauge transformation formula (1.1) is replaced by the formula (5.19):

$$\bar{A}_2 = \theta \bar{A}_1 \theta^{-1} - (d\theta)\theta^{-1} + \tau \Lambda^H. \quad (1.2)$$

This result is derived as a ‘differential’ form of a more general result on transformations of connections that is proved in Theorem 4.1.1.

The very special case of (1.2) where  $\Lambda^H$  is 0 gives the classical gauge transformation formula. Just as the classical gauge transformation formula (1.1) for a gauge potential allows us to view the gauge potential as a connection on a principal bundle, a field whose transformation law is given by (1.2) (in the sense that physical quantities obtained from it are invariant under (1.2)) can be viewed as a categorical connection on a principal categorical bundle.

Traditional gauge transformations may be studied either at the local level in terms of  $G$ -equivariant fiber-preserving diffeomorphisms of  $U \times G$  or they can be studied globally in terms of diffeomorphisms of the type

$$P \rightarrow P : p \mapsto pg_p.$$

Analogously we can also study categorical bundle automorphisms at a global level in the framework of *decorated categorical bundles*. This is carried out in Section 6. Finally, in Section 7, we specialize to the case where  $G$  and  $H$  are abelian and determine the effect of gauge transformations on higher parallel-transport.

## 2. Categorical groups and categorical bundles

We summarize here the basic notions and notation we will use. The objects and morphisms of the categories we work with form sets, and, indeed, the object sets of interest to us are smooth manifolds and functors of interest are given by means of smooth functions.

## 2.1. Categorical bundles: a quick overview

Our general definition of categorical principal bundles in Section 2.8 includes the special case of *twisted-product bundles*, which form the categorical counterpart of product bundles, and *decorated categorical bundles* (Section 2.14), which form the main examples of interest for us. Section 3 is devoted to a closer study of twisted-product categorical bundles. Roughly put, a decorated categorical bundle contains within its structure a classical principal bundle equipped with a connection, while a twisted-product categorical bundle contains a classical principal *product bundle* equipped with a connection. Let us note that twisted-product categorical bundles are a type of decorated categorical bundles, and the latter form a type of principal categorical bundles.

## 2.2. Categorical groups

A *categorical group*  $\mathbf{G}$  is a small category along with a functor

$$\mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G} \quad (2.1)$$

that makes both the object set  $\text{Obj}(\mathbf{G})$  and the morphism set  $\text{Mor}(\mathbf{G})$  groups. The source and target maps

$$s, t : \text{Mor}(\mathbf{G}) \rightarrow \text{Obj}(\mathbf{G})$$

are homomorphisms. (There are other definitions of this notion; see, for example, [26, Def 2.1].)

We say that the categorical group  $\mathbf{G}$  is a *categorical Lie group* if  $\text{Obj}(\mathbf{G})$  and  $\text{Mor}(\mathbf{G})$  are Lie groups and  $s$  and  $t$  are smooth mappings and so is  $x \mapsto 1_x$ .

For any object  $a$  we denote by  $1_a$  the identity morphism  $a \rightarrow a$ . A functor carries identity morphisms to identity morphisms; thus, functoriality of the group operation implies that  $1_a 1_b = 1_{ab}$ . From this we see that  $1_e$  is the identity element in the group  $\text{Mor}(\mathbf{G})$ , if  $e$  is the identity element in  $\text{Obj}(\mathbf{G})$ .

The group structures on  $\text{Obj}(\mathbf{G})$  and  $\text{Mor}(\mathbf{G})$  give rise to maps

$$\text{Obj}(\mathbf{G}) \rightarrow \text{Obj}(\mathbf{G}) : g \mapsto g^{-1} \quad \text{Mor}(\mathbf{G}) \rightarrow \text{Mor}(\mathbf{G}) : \phi \mapsto \phi^{\text{inv}}, \quad (2.2)$$

where  $\phi^{\text{inv}}$  is the inverse of  $\phi \in \text{Mor}(\mathbf{G})$  relative to the group structure (not the compositional inverse). Since  $s$  and  $t$  are homomorphisms, it follows that  $s(\phi^{\text{inv}}) = s(\phi)^{-1}$  and  $t(\phi^{\text{inv}}) = t(\phi)^{-1}$ . Moreover, since the group operation is functorial, for any morphisms  $\phi : a \rightarrow b$  and  $\psi : b \rightarrow c$  in  $\mathbf{G}$ , we have

$$(\psi^{\text{inv}} \circ \phi^{\text{inv}})(\psi \circ \phi) = (\psi^{\text{inv}} \psi) \circ (\phi^{\text{inv}} \phi) = 1_e \circ 1_e = 1_e,$$

and so

$$(\psi \circ \phi)^{\text{inv}} = \psi^{\text{inv}} \circ \phi^{\text{inv}}. \quad (2.3)$$

Taking the special case where  $\psi$  is  $1_b$ , it follows that  $\phi^{\text{inv}} = 1_b^{\text{inv}} \circ \phi^{\text{inv}}$ ; now taking  $\phi = 1_{b^{-1}}$  we have

$$1_{b^{-1}} = 1_b^{\text{inv}} \circ 1_{b^{-1}} = 1_b^{\text{inv}}. \quad (2.4)$$

Thus  $\phi \mapsto \phi^{\text{inv}}$  preserves compositions and carries identity morphisms to identity morphisms. Thus (2.2) specifies a functor

$$\text{Inv} : \mathbf{G} \rightarrow \mathbf{G}. \quad (2.5)$$

Associated to a categorical group  $\mathbf{G}$  is a *crossed module*  $(G, H, \alpha, \tau)$ , where  $G$  and  $H$  are groups, and

$$\tau : H \rightarrow G \quad \text{and} \quad \alpha : G \rightarrow \text{Aut}(H) : g \mapsto \alpha_g \quad (2.6)$$

are homomorphisms satisfying the Peiffer identities [27]:

$$\begin{aligned} \tau(\alpha_g(h)) &= g\tau(h)g^{-1} \\ \alpha_{\tau(h)}(h') &= hh'h^{-1} \end{aligned} \quad (2.7)$$

for all  $g \in G$  and  $h \in H$ . (This relationship is attributed to George Janelidze by Mac Lane [28, page 285]; see also [29] and references therein.) As a consequence of the first Peiffer identity the image  $\tau(H)$  is a normal subgroup of  $G$ :

$$g\tau(h)g^{-1} = \tau(\alpha_g(h)) \in \tau(H) \text{ for all } h \in H \text{ and } g \in G. \quad (2.8)$$

The crossed module is given by

$$G = \text{Obj}(\mathbf{G}) \quad \text{and} \quad H = \ker s \subset \text{Mor}(\mathbf{G}).$$

The morphism group  $\text{Mor}(\mathbf{G})$  is then isomorphic to the semidirect product  $H \rtimes_{\alpha} G$ :

$$\text{Mor}(\mathbf{G}) \simeq H \rtimes_{\alpha} G, \quad (2.9)$$

with  $(h, g) \in H \rtimes_{\alpha} G$  having source  $g$  and target  $\tau(h)g$ :

$$s(h, g) = g \quad \text{and} \quad t(h, g) = \tau(h)g. \quad (2.10)$$

Composition of morphisms in  $\text{Mor}(\mathbf{G})$  corresponds, when  $\text{Mor}(\mathbf{G})$  is identified with  $H \rtimes_{\alpha} G$ , to the law

$$(h_2, g_2) \circ (h_1, g_1) = (h_2 h_1, g_1). \quad (2.11)$$

Here  $(h_1, g_1)$  is a morphism from  $g_1$  to  $g_2 = \tau(h_1)g_1$ , and  $(h_2, g_2)$  is a morphism from  $g_2$  to  $\tau(h_2)g_2 = \tau(h_2 h_1)g_1$ .

In contrast, the product operation in  $\text{Mor}(\mathbf{G})$  is given by the semidirect product operation

$$(h_2, g_2)(h_1, g_1) = (h_2 \alpha_{g_2}(h_1), g_2 g_1). \quad (2.12)$$

The categorical group  $\mathbf{G}$  is a categorical Lie group if and only if  $G$  and  $H$  are Lie groups and the mappings  $\tau : h \mapsto \tau(h)$  and  $(h, g) \mapsto \alpha_g(h)$  are smooth.

### 2.3. The semidirect product $H \rtimes_{\alpha} G$

We will often identify  $H$  and  $G$  with the subgroups  $H \times \{e\}$  and  $\{e\} \times G$  in  $H \rtimes_{\alpha} G$ . Thus,  $(h, g) \in H \rtimes_{\alpha} G$  can be written simply as a product:

$$hg = (h, g). \quad (2.13)$$

With this notation, conjugation by  $g$  (that is,  $(e, g)$ ) is simply  $\alpha_g$ :

$$ghg^{-1} = \alpha_g(h). \quad (2.14)$$

On the left here we have used  $g$  to denote the element  $(e, g) \in H \rtimes_{\alpha} G$  and  $h$  to be the element  $(h, e) \in H \rtimes_{\alpha} G$ . This confusion of notation that makes the identity (2.14) work proves to be very convenient in computations. We can view the target homomorphism

$$t : H \rtimes_{\alpha} G \rightarrow G : hg \mapsto \tau(h)g \quad (2.15)$$

as extending the homomorphism  $\tau : H \rightarrow G$  to the domain  $H \rtimes_{\alpha} G \simeq \text{Mor}(\mathbf{G})$ . We then have

$$t(gh) = t(ghg^{-1} \cdot g) = \tau(ghg^{-1})g = g\tau(h)g^{-1} \cdot g = g\tau(h) \quad (2.16)$$

wherein we have used the first of the Peiffer identities (2.7).

### 2.4. Categorical groups $\mathbf{G}_1$ and $\mathbf{G}_0$ from a classical group $G$

Let  $G$  be a group. Then we can form a categorical group  $\mathbf{G}_0$  whose object group is  $G$  and whose only morphisms are the identity morphisms  $1_g : g \rightarrow g$ . For a more useful example, let  $\mathbf{G}_1$  be the categorical group whose object group is again  $G$  but for which there is a unique morphism  $g_0 \rightarrow g_1$  for every  $g_0, g_1 \in G$ . Thus,  $\text{Mor}(\mathbf{G}_1) \simeq G \times G$ , with source of  $g_0 \rightarrow g_1$  being  $g_0$  and target  $g_1$ , and composition is defined in the unique way possible:  $(g_1 \rightarrow g_2) \circ (g_0 \rightarrow g_1) = (g_1 \rightarrow g_2)$ . We note that the identity morphism at  $g \in G$  is

$$1_g = g \rightarrow g. \quad (2.17)$$

The associated crossed module is  $(G, G, \alpha, \tau)$ , where  $\tau(g) = g$  and  $\alpha_g(g') = gg'g^{-1}$  for all  $g, g' \in G$ . The ordered pair  $(g_1, g_2) \in G \rtimes_{\alpha} G$  corresponds to the morphism  $g_1 \rightarrow g_2 g_1$ .

### 2.5. Categorical group actions

Let  $\mathbf{C}$  be a category,  $\mathbf{G}$  a categorical group. By a categorical *right action*

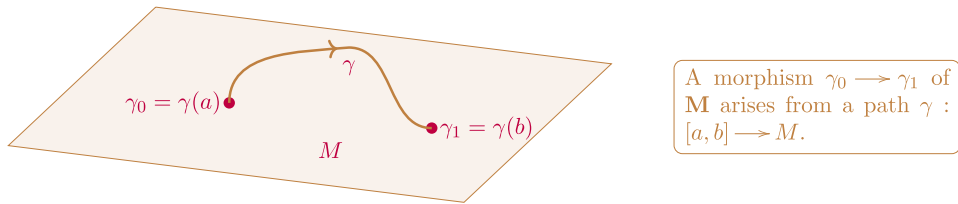
$$R : \mathbf{C} \times \mathbf{G} \rightarrow \mathbf{C}$$

we mean a functor that is a right action both at the object level and at the morphism level. Thus

$$\begin{aligned} a(gg') &= (ag)g' & \text{for all } a \in \text{Obj}(\mathbf{C}), g, g' \in \text{Obj}(\mathbf{G}) \\ f(\phi\phi') &= (f\phi)\phi' & \text{for all } f \in \text{Mor}(\mathbf{C}), \phi, \phi' \in \text{Mor}(\mathbf{G}), \end{aligned} \quad (2.18)$$

and the identity elements in  $\text{Obj}(\mathbf{G})$  and  $\text{Mor}(\mathbf{G})$  act by the identity maps on  $\text{Obj}(\mathbf{C})$  and  $\text{Mor}(\mathbf{C})$ . The functoriality of  $R$  imposes conditions that are not present in traditional actions of groups on spaces; for instance, if  $f_1, f_2 \in \text{Mor}(\mathbf{C})$  and  $\phi_1, \phi_2 \in \text{Mor}(\mathbf{G})$ , with the compositions  $f_2 \circ f_1$  and  $\phi_2 \circ \phi_1$  both defined, then

$$(f_2 \circ f_1)(\phi_2 \circ \phi_1) = (f_2 \phi_2) \circ (f_1 \phi_1). \quad (2.19)$$

Fig. 3. The category  $\mathbf{M}$ .

Moreover, for any  $f \in \text{Mor}(\mathbf{C})$  and  $\phi \in \text{Mor}(\mathbf{G})$ , corresponding to  $hg \in H \rtimes_{\alpha} G \simeq \text{Mor}(\mathbf{G})$ , we have the following sources and targets:

$$s(fhg) = s(f)g \quad \text{and} \quad t(fhg) = t(f)\tau(h)g. \quad (2.20)$$

## 2.6. Categories from spaces

Let  $M$  be a manifold. Then we can form a category  $\mathbf{M}$  whose object set is  $M$  and whose morphisms arise from paths on  $M$ . By a 'path' here we mean a  $C^{\infty}$  mapping  $\gamma : [a, b] \rightarrow M$  that is constant near  $a$  and near  $b$ ; moreover, we regard  $\gamma$  as being equivalent to any time-translate  $\gamma_{+c} : [a - c, b - c] \rightarrow M : t \mapsto \gamma(t + c)$  for all  $c \in \mathbb{R}$ , and then  $\text{Mor}(\mathbf{M})$  consists of all equivalence classes of such paths. Composition of morphisms corresponds to composition of paths.

## 2.7. Smooth structures

We will not need to explicitly work with the smooth structures on the object and morphism spaces, but we will summarize here in very compact form the essential notions. We refer to the Appendix in [6] for more information and references. By a smooth space we mean a nonempty set  $X$  along with a set  $C^{\infty}(U, X)$  of functions  $U \rightarrow X$  for every open subset  $U$  of every  $\mathbb{R}^n$  such that: (i) every constant map  $U \rightarrow X$  is in  $C^{\infty}(U, X)$ ; (ii) if  $U = \cup_{i \in I} U_i$ , where each  $U_i$  is any open subset in  $\mathbb{R}^n$ , then a mapping  $\phi : U \rightarrow X$  is in  $C^{\infty}(U, X)$  if each restriction  $\phi|_{U_i}$  is in  $C^{\infty}(U_i, X)$ ; (iii) if  $\phi \in C^{\infty}(U, X)$  and if  $g : V \rightarrow U$  is a  $C^{\infty}$  function, where  $U$  and  $V$  are open subsets of Euclidean spaces, then  $\phi \circ g \in C^{\infty}(V, X)$ . The function  $f \in C^{\infty}(U, X)$  are called *plots*. Every smooth manifold is a smooth space in the natural way. If  $X$  and  $Y$  are smooth spaces then any nonempty subset of  $X$  is a smooth space in the obvious way and so is  $X \times Y$ . Moreover, by  $C^{\infty}(X, Y)$  we mean the set of all mappings  $h : X \rightarrow Y$  such that  $h \circ \phi \in C^{\infty}(Y)$  for any plot  $\phi \in C^{\infty}(U, X)$  with  $U$  being any open subset of any Euclidean space. The set  $C^{\infty}(X, Y)$  becomes a smooth space, with  $C^{\infty}(U, C^{\infty}(X, Y))$  consisting of all maps  $\Phi : U \rightarrow C^{\infty}(X, Y)$  for which the function  $U \times X \rightarrow Y : (u, x) \mapsto \Phi(u, x)$  is in  $C^{\infty}(U \times X, Y)$ . Let  $\phi \in C^{\infty}(U, X)$  be a plot,  $p \in U$ , and  $v \in T_p U$ , and let  $(\psi, q, w)$  be another such triple; we say that these two triples are tangent to each other if  $\phi(p) = \psi(q)$  and  $D(f \circ \phi)|_p v = D(f \circ \psi)|_q w$  for all  $f \in C^{\infty}(X, W)$  where  $W$  is any open subset of any Euclidean space. Equivalence classes of such triples form the tangent space  $T_{\phi(p)} X$ , but this is not necessarily a vector space.

## 2.8. Categorical principal bundles

A categorical principal bundle with structure categorical group  $\mathbf{G}$  is a functor

$$\pi : \mathbf{P} \rightarrow \mathbf{M}$$

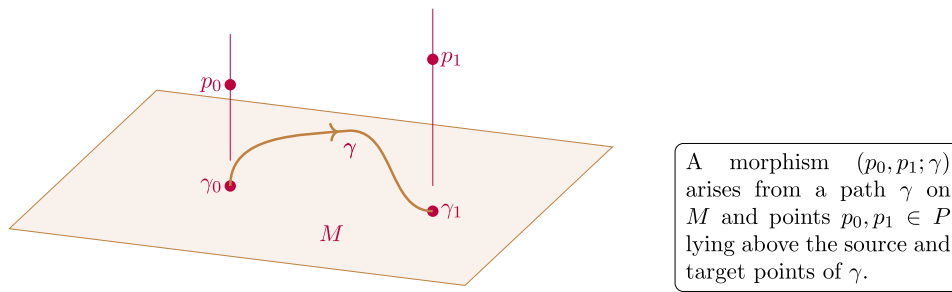
that is surjective both on the level of objects and on the level of morphisms, along with a functor

$$\mathbf{P} \times \mathbf{G} \rightarrow \mathbf{P} \quad (2.21)$$

that is a free right action both on objects and on morphisms, such that  $\pi(pg) = \pi(p)$  for all objects/morphisms  $p$  of  $\mathbf{P}$  and all objects/morphisms  $g$  of  $\mathbf{G}$ , the action is transitive on each fiber. (Categorical principal bundles are developed in more detail in [21].) In practice we are only concerned with the case where  $\mathbf{G}$  is a categorical Lie group,  $\text{Obj}(\mathbf{P})$  and  $\text{Obj}(\mathbf{M})$  are smooth manifolds, and the object bundle

$$\text{Obj}(\mathbf{P}) \rightarrow \text{Obj}(\mathbf{M})$$

is a principal  $G$ -bundle, where  $G = \text{Obj}(\mathbf{G})$ . The main example of interest to us is the case of *decorated bundles*, which we describe below in 2.14. In all examples of interest the object and morphism sets are smooth spaces,  $\mathbf{G}$  is a categorical Lie group, the action (2.21) is smooth both at the object level and the morphism level, and  $\pi$  is  $C^{\infty}$  both on objects and morphisms.

Fig. 4. The categorical bundle  $\mathbf{P}_1$ .

## 2.9. Categorical principal bundles from classical ones

(This is Example P2 in [21].) Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle. We can view this as a categorical principal bundle in the following way. Let  $\mathbf{G}_1$  be the categorical group described in Section 2.4. Let  $\mathbf{M}$  be the category whose object set is  $M$  and morphisms are paths on  $M$ , as discussed before (see Fig. 3). Let  $\mathbf{P}_1$  be the category whose objects are the points of  $P$ , and whose morphisms are of the following form:

$$(p, q; \gamma),$$

where  $p, q \in P$  and  $\gamma \in \text{Mor}(\mathbf{M})$  runs from  $\pi(p)$  to  $\pi(q)$  (see Fig. 4). The functor  $\pi : \mathbf{P}_1 \rightarrow \mathbf{M}$  is just  $\pi : P \rightarrow M$  on objects and on morphisms it is given by  $\pi(p, q; \gamma) = \gamma$ . The action of  $\mathbf{G}_1$  on  $\mathbf{P}_1$  is given at the object-level by the action of  $G$  on  $P$  and at the morphism level by

$$(p, q; \gamma)(g_0 \rightarrow g_1) = (pg_0, qg_1; \gamma). \quad (2.22)$$

Let us note the special case where the morphism  $(g_0 \rightarrow g_1) \in \text{Mor}(\mathbf{G}_1)$  is the identity morphism at  $g \in G$ ; that is, for  $(g_0 \rightarrow g_1)$  we take  $1_g = (g \rightarrow g)$  (as in (2.17)) and obtain:

$$(p, q; \gamma)1_g = (pg, qg; \gamma). \quad (2.23)$$

## 2.10. Functorial isomorphisms

Let  $\pi : \mathbf{P} \rightarrow \mathbf{M}$  and  $\pi' : \mathbf{P}' \rightarrow \mathbf{M}$  be categorical principal  $\mathbf{G}$ -bundles. A functorial isomorphism  $\Phi : \pi \rightarrow \pi'$  is a functor  $\Phi : \mathbf{P} \rightarrow \mathbf{P}'$  that commutes with  $\pi$  and is  $\mathbf{G}$ -equivariant in the sense that

$$\begin{aligned} \Phi(pg) &= \Phi(p)g & \text{for all } (p, g) \in \text{Obj}(\mathbf{P}) \times \text{Obj}(\mathbf{G}) \\ \Phi(\overline{\gamma}\phi) &= \Phi(\overline{\gamma})\phi & \text{for all } (\overline{\gamma}, \phi) \in \text{Mor}(\mathbf{P}) \times \text{Mor}(\mathbf{G}). \end{aligned} \quad (2.24)$$

The definition of a categorical principal bundle implies that  $\Phi$  is, in fact, a bijection both on the object spaces and on the morphism spaces.

## 2.11. Categorical connections

The classical notion of a connection generalizes readily to the setting of categorical bundles. This notion of a *categorical connection* is developed in [21, sec. 7]. The essential idea of a connection on a classical principal  $G$ -bundle  $\pi : P \rightarrow M$  is a prescription of how to *parallel transport*: given a path  $\gamma$  on  $M$  and a point  $u$  on the fiber above the initial point  $\gamma_0$  of  $\gamma$ , the connection provides a path  $\overline{\gamma}_u$ , called the *horizontal lift* of  $\gamma$  starting at  $u$ , in  $P$  that lies above  $\gamma$  and initiates at  $u$ . Moreover, the classical parallel transport has a ‘rigidity’ property: if the initial point  $u$  is replaced by  $ug$ , where  $g \in G$ , then the horizontal lift initiating at  $ug$  is given by  $\overline{\gamma}_{ug}$ , the right-translate of the entire path by  $g$ :

$$\overline{\gamma}_{ug} = \overline{\gamma}_u g. \quad (2.25)$$

Furthermore, the composition of horizontal paths is horizontal, reflecting the idea that the velocity of parallel-transport is determined by the velocity of the base path  $\gamma$  rather than the entire ‘history’ of the path.

These ideas extend in a straightforward way to categorical bundles. Let  $\mathbf{G}$  be a categorical group, and

$$\pi : \mathbf{P} \rightarrow \mathbf{M}$$

be categorical principal  $\mathbf{G}$ -bundle. In particular,  $G = \text{Obj}(\mathbf{G})$  acts on the right on  $\text{Obj}(\mathbf{P})$  and  $\text{Mor}(\mathbf{G})$  acts on the right on  $\text{Mor}(\mathbf{P})$ . A *categorical connection*  $\mathbf{A}$  on this bundle is a specification associating to each  $\gamma \in \text{Mor}(\mathbf{M})$  and object  $u \in \text{Obj}(\mathbf{P})$



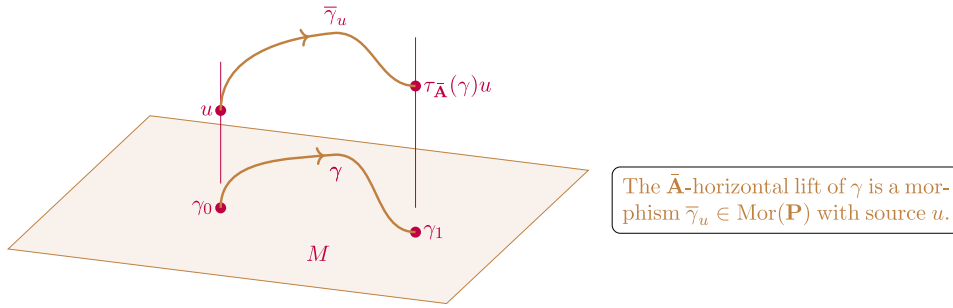


Fig. 5. Horizontal lifts.

on the fiber over  $s(\gamma)$ , a morphism  $\bar{\gamma}_u \in \text{Mor}(\mathbf{P})$ , called the  $\bar{A}$ -horizontal lift of  $\gamma$  initiating at  $u$ , satisfying the following conditions:

- $s(\bar{\gamma}_u) = u$ ;
- $\bar{\gamma}_u 1_g = \bar{\gamma}_{ug}$  for all  $g \in G$ , and all  $\gamma$  and  $u$  as above;
- the composition of  $\bar{A}$ -horizontal morphisms is  $\bar{A}$ -horizontal.

The target or terminal point of  $\bar{\gamma}_u$  is the *parallel-translate* of  $u$  along  $\gamma$  by  $\bar{A}$ , and we denote it by  $\tau_{\bar{A}}(\gamma)u$ . These notions are illustrated in Fig. 5.

We have not discussed smoothness issues, but when all the object and morphism spaces are equipped with smooth-space structures then we also require  $(u, \gamma) \mapsto \bar{\gamma}_u$  to be smooth.

### 2.12. Categorical connections on $\mathbf{P}_1$

Let us take a quick look at categorical connections in the context of the categorical principal  $\mathbf{G}_1$ -bundle  $\mathbf{P}_1 \rightarrow \mathbf{M}$  we discussed above in Section 2.9. A categorical connection  $\bar{A}$  on this bundle is, by definition, a prescription of *horizontal lifts*: thus for each  $\gamma \in \text{Mor}(\mathbf{M})$  and every  $p \in \text{Obj}(\mathbf{P})$  above  $s(\gamma)$  there should be a morphism  $\bar{\gamma}_p \in \text{Mor}(\mathbf{P})$ , subject to some natural consistency conditions described in detail in Section 2.11; these conditions include the requirement that

$$\bar{\gamma}_{pg} = \bar{\gamma}_p 1_g. \quad (2.26)$$

Clearly the morphism  $\bar{\gamma}_p = (p; q; \gamma)$  is uniquely specified by the terminal point  $q$ . Thus, every classical connection  $\bar{A}$  on the bundle  $\pi : P \rightarrow M$  gives rise to a categorical connection  $\bar{A}$  on  $\mathbf{P} \rightarrow \mathbf{M}$ . Moreover, at least conceptually, every categorical connection on  $\mathbf{P} \rightarrow \mathbf{M}$  arises from a classical connection on  $\pi : P \rightarrow M$ . Let us note the meaning of the condition (2.26) more explicitly (writing  $\bar{\gamma}_p 1_g$  on the left side this time):

$$(p, q; \gamma) 1_g = (pg, qg; \gamma) \quad (2.27)$$

where we have used (2.23). This makes sense, in that the  $\bar{A}$ -horizontal lift  $\bar{\gamma}_{pg}$  of  $\gamma$  with initial point  $pg$  is obtained by right-translating the entire path  $\bar{\gamma}_p$  by  $g$ :

$$\bar{\gamma}_{pg} = \bar{\gamma}_p g, \quad (2.28)$$

and the terminal point of  $\bar{\gamma}_{pg}$  is indeed  $qg$ .

### 2.13. Reduction to the horizontal bundle

Let  $\bar{A}$  be a categorical connection on a categorical principal bundle  $\mathbf{P} \rightarrow \mathbf{M}$  with some structure categorical group  $\mathbf{G}$ . (We explain the general notion of categorical connections in Section 2.11.) Let us focus now on the  $\bar{A}$ -horizontal morphisms of  $\mathbf{P}$ . Let  $\mathbf{P}^{\bar{A}}$  be the category with object space  $\text{Obj}(\mathbf{P})$  and morphisms being just the  $\bar{A}$ -horizontal morphisms of  $\mathbf{P}$ . By (2.26) it is not the full categorical group  $\mathbf{G}$  that acts on  $\mathbf{P}$ , but the smaller group  $\mathbf{G}_0$  whose morphisms are only the identity morphisms. Thus we have a categorical principal  $\mathbf{G}_0$ -bundle  $\mathbf{P}^{\bar{A}} \rightarrow \mathbf{M}$ . Let us note that the structure group for this bundle is the discrete categorical group  $\mathbf{G}_0$ , and the information about  $\text{Mor}(\mathbf{G})$  is erased. In terms of the notation in Section 2.9, a morphism of  $\mathbf{P}^{\bar{A}}$  is of the form

$$(p, q; \gamma),$$

where  $q$  is now uniquely determined by the initial point  $p$  and the path  $\gamma$ ; we think of  $q$  as being obtained by parallel-transport of  $p$  along  $\gamma$ . (See Fig. 6.)



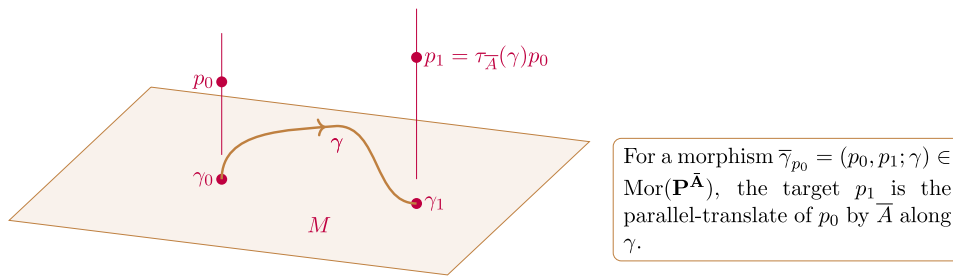
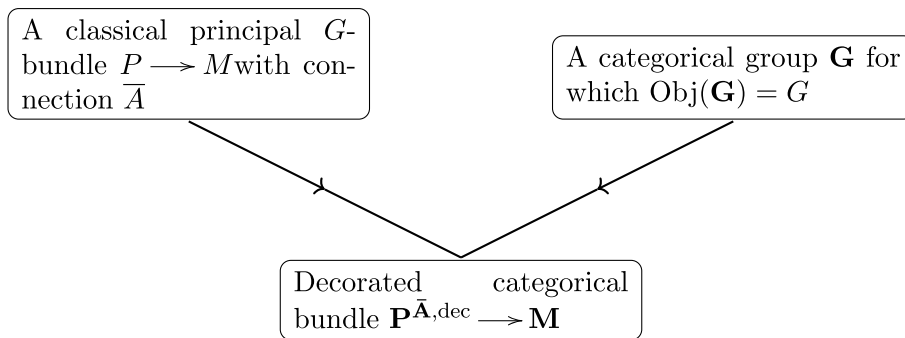
Fig. 6. The categorical bundle  $\mathbf{P}^{\bar{A}}$ .

Fig. 7. The decorated bundle construction.

#### 2.14. The decorated categorical principal bundle $\mathbf{P}^{\bar{A},\text{dec}}$

The main examples of categorical bundles of interest for us are *decorated categorical bundles*. We review the construction of such categorical principal bundles from [21]. Consider a classical principal  $G$ -bundle  $\pi : P \rightarrow M$ , equipped with a connection form  $\bar{A}$ . The base category  $\mathbf{M}$  has object set  $M$ , while morphisms arise from smooth paths on  $M$ , as described in Section 2.6. The decorated bundle category  $\mathbf{P}^{\bar{A},\text{dec}}$  has object set  $P$  but the morphisms of  $\mathbf{P}^{\bar{A},\text{dec}}$  are pairs  $(\bar{\gamma}, h)$ , where  $\bar{\gamma}$  is any  $\bar{A}$ -horizontal path on  $P$  and  $h$  any element of  $H$ . We denote by  $\bar{\gamma}g$  the path on  $P$  obtained by acting with  $g \in G$  on the right pointwise; specifically,  $(\gamma g)(u) = \gamma(u)g$  for all  $u$  in the domain of the path  $\gamma$ . The source and target maps are

$$s(\bar{\gamma}, h) = s(\bar{\gamma}) \quad \text{and} \quad t(\bar{\gamma}, h) = t(\bar{\gamma})\tau(h). \quad (2.29)$$

Composition of morphisms is defined by

$$(\bar{\gamma}_2, h_2) \circ (\bar{\gamma}_1, h_1) = (\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1, h_1 h_2) \quad (2.30)$$

when the source of  $(\bar{\gamma}_2, h_2)$  is the target of  $(\bar{\gamma}_1, h_1)$ . The categorical group  $\mathbf{G}$  acts on  $\mathbf{P}^{\bar{A},\text{dec}}$  on the right as follows: at the object level it is just the right action of  $G$  on  $P$ , while for morphisms we define the action by

$$(\bar{\gamma}, h)h_1g_1 = (\bar{\gamma}g_1, g_1^{-1}hh_1g_1), \quad (2.31)$$

with obvious notation. The projection functor  $\pi : \mathbf{P}^{\bar{A},\text{dec}} \rightarrow \mathbf{M}$  is just the bundle projection  $\pi : P \rightarrow M$  at the object level, and for morphisms is given by  $\pi(\bar{\gamma}, h) = \pi \circ \bar{\gamma}$ . We call  $\mathbf{P}^{\bar{A},\text{dec}} \rightarrow \mathbf{M}$  the *decorated categorical principal bundle* arising from the bundle  $P \rightarrow M$  and the connection  $\bar{A}$ .

Let us note again that a decorated categorical principal bundle is an example of a categorical principal bundle. It is the main example of categorical principal bundles of interest to us, and is constructed from a classical principal  $G$ -bundle equipped with a connection. (See Fig. 7.)

#### 2.15. Decorated categorical principal bundle: general construction

In the preceding discussion we began with a classical connection  $\bar{A}$  on a classical principal bundle. However, the construction works just as described when applied to the horizontal bundle  $\mathbf{P}^{\bar{A}} \rightarrow \mathbf{M}$  for any categorical connection  $\bar{A}$  on a principal categorical  $\mathbf{G}'$ -bundle  $\mathbf{P} \rightarrow \mathbf{M}$  and categorical group  $\mathbf{G}$  whose object group is the same as the object group  $\mathbf{G}'$ . The result is a categorical principal  $\mathbf{G}$ -bundle  $\mathbf{P}^{\bar{A},\text{dec}} \rightarrow \mathbf{M}$ .

### 3. Twisted-product bundles

In this section we will study a special type of categorical principal bundles called twisted-product categorical bundles. For traditional bundles there is the notion of triviality: a bundle is trivial if it is isomorphic to a product bundle. In the case of principal  $G$ -bundles, the product bundle over a space  $U$  is given simply by the projection

$$U \times G \rightarrow U : (u, g) \mapsto u.$$

A classical principal bundle is locally trivial, isomorphic locally to a product bundle.

A categorical bundle encodes more geometrical information than a traditional bundle. In particular, the notion of local triviality is richer for categorical bundles, for which the relevant notion is that of *twisted-product bundles*, developed in [30]. We will review these ideas in this section. We begin with some fairly standard notation that is useful in the context of parallel transport.

#### 3.1. Path ordered exponentials

Suppose  $[a, b] \rightarrow L(K) : u \mapsto X_u$  is a smooth path on the Lie algebra  $L(K)$  of a Lie group  $K$ . Let  $[a, b] \rightarrow K : t \mapsto k_t$  be the solution of the differential equation

$$\dot{k}_u k_u^{-1} = X_u \quad \text{for all } u \in [a, b] \quad (3.1)$$

with initial condition  $k_a = e$ . Then we use the *path ordered exponential* notation to denote  $k_u$ :

$$Pe^{\int_a^u X_v dv} \stackrel{\text{def}}{=} k_u. \quad (3.2)$$

Next suppose  $A$  is an  $L(K)$ -valued smooth 1-form on a manifold  $M$  and  $\gamma : [a, b] \rightarrow M$  a piecewise smooth path on  $M$ . Then we use the notation

$$Pe^{\int_\gamma A} = Pe^{\int_a^b A(\gamma'(u)) du}. \quad (3.3)$$

The path ordered exponential has the following useful composition property:

$$Pe^{\int_{\gamma_2 \circ \gamma_1} A} = Pe^{\int_{\gamma_2} A} Pe^{\int_{\gamma_1} A} \quad (3.4)$$

whenever the two factors on the right are defined and the composition  $\gamma_2 \circ \gamma_1$  is also defined. This is verified by noting that if  $u \mapsto k_u$  satisfies (3.1) then so does  $u \mapsto k_u c$  for any fixed  $c \in K$ .

#### 3.2. Twisted-product bundles

**Proposition 3.2.1** (which is from [30, Prop 8]) below describes the construction of a categorical principal bundle  $\pi_\eta : \mathbf{U} \times_\eta \mathbf{G} \rightarrow \mathbf{U}$ , where  $\mathbf{U}$  is any category (of course, we are primarily interested in categories that arise from spaces such as manifolds). The full structure of this bundle is set out in the statement of **Proposition 3.2.1**. We call this categorical principal bundle a *twisted-product bundle*. The structure involves a function  $\eta$  that associates to each path  $\gamma$  on  $U$  an element of  $G$  that we could think of as describing parallel-transport along  $\gamma$  by some connection. Thus, instead of the classical product bundle  $U \times G$  we have here, in addition, also a connection form on this bundle.

**Proposition 3.2.1.** Let  $\mathbf{U}$  be a category, and  $\mathbf{G}$  a categorical group with associated crossed module  $(G, H, \alpha, \tau)$ . Let

$$\eta : \text{Mor}(\mathbf{U}) \rightarrow G = \text{Obj}(\mathbf{G})$$

be a map satisfying

$$\eta(\gamma_2 \circ \gamma_1) = \eta(\gamma_2)\eta(\gamma_1) \quad (3.5)$$

for all  $\gamma_1, \gamma_2 \in \text{Mor}(\mathbf{U})$  for which the composition  $\gamma_2 \circ \gamma_1$  is defined. Then there is a category  $\mathbf{U} \times_\eta \mathbf{G}$  for which:

- the object set is  $\text{Obj}(\mathbf{U}) \times \text{Obj}(\mathbf{G})$ ,
- the morphism set is  $\text{Mor}(\mathbf{U}) \times \text{Mor}(\mathbf{G})$ ,
- the source and target maps are given by

$$\begin{aligned} s_\eta(\gamma, hg) &= (s(\gamma), g) \\ t_\eta(\gamma, hg) &= (t(\gamma), \eta(\gamma)\tau(h)g), \end{aligned} \quad (3.6)$$

and

- composition is given by

$$(\gamma_2, h_2 g_2) \circ_\eta (\gamma_1, h_1 g_1) = (\gamma_2 \circ \gamma_1, g^{-1} h_2 g h_1 g_1), \quad (3.7)$$

where  $g = \eta(\gamma_1)$ , and the source  $s_\eta(\gamma_2, h_2 g_2)$  is the target  $t_\eta(\gamma_1, h_1 g_1)$ .

Moreover, the projection on the first factor

$$\pi_\eta : \mathbf{U} \times_\eta \mathbf{G} \rightarrow \mathbf{U}, \quad (3.8)$$

along with the usual right action of  $\mathbf{G}$  on  $\mathbf{U} \times_\eta \mathbf{G}$ , makes  $\pi_\eta$  a categorical principal bundle.

### 3.3. Twisted-product bundles and decorated product bundles

We can understand twisted-product bundles in terms of decorated bundles. To this end let us consider a smooth 1-form  $\bar{A}_0$  on  $U$ , with values in the Lie algebra  $L(G)$ . This corresponds to a connection form  $\bar{A}$  on the product bundle  $U \times G$ , given by

$$\bar{A}|_{(u,g)}(X, Z) = \bar{A}_0(X) + g^{-1}Z, \quad (3.9)$$

for all  $X \in T_u U$  and  $Z \in T_g G$ , and all  $(u, g) \in U \times G$ . Now let  $\eta : \text{Mor}(\mathbf{U}) \rightarrow G$  be given by

$$\eta(\gamma) = \eta_{\bar{A}_0}(\gamma) \stackrel{\text{def}}{=} P e^{-\int_\gamma \bar{A}_0}. \quad (3.10)$$

The condition (3.5) is satisfied, because of the composition law behavior (3.4). We denote by  $\bar{\gamma}_g$  the  $\bar{A}$ -horizontal lift of a path  $\gamma : [a, b] \rightarrow U$  with initial point  $(\gamma(a), g)$ :

$$\bar{\gamma}_g(u) = (\gamma(u), \eta(\gamma|_{[a,u]})). \quad (3.11)$$

A typical morphism of the twisted-product bundle  $\mathbf{U} \times_\eta \mathbf{G}$  is  $(\gamma, hg)$ , where  $g \in G$ ,  $h \in H$ , and  $\gamma \in \text{Mor}(\mathbf{U})$ , with source and target given by

$$s_\eta(\gamma, hg) = (s(\gamma), g) \quad \text{and} \quad t_\eta(\gamma, hg) = (t(\gamma), \eta(\gamma)\tau(h)g), \quad (3.12)$$

as in (3.6), and composition law given by

$$(\gamma_2, h_2 g_2) \circ_\eta (\gamma_1, h_1 g_1) = (\gamma_2 \circ \gamma_1, g^{-1} h_2 g h_1 g_1), \quad (3.13)$$

as before in (3.7). On the other hand, for the decorated categorical bundle  $(\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}}$ , the typical morphism is of the form  $(\bar{\gamma}_g, h)$ , with notation as in (3.11). Then we have the mappings

$$\begin{aligned} \text{Obj}((\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}}) &\rightarrow \text{Obj}(\mathbf{U} \times_\eta \mathbf{G}) : (u, g) \mapsto (u, g) \\ \text{Mor}((\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}}) &\rightarrow \text{Mor}(\mathbf{U} \times_\eta \mathbf{G}) : (\bar{\gamma}_g, h) \mapsto (\gamma, gh), \end{aligned} \quad (3.14)$$

where, recall,  $gh = (ghg^{-1})g \in H \rtimes_\alpha G \simeq \text{Mor}(\mathbf{G})$ . These specify an isomorphism of categorical principal bundles:

$$(\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}} \simeq \mathbf{U} \times_\eta \mathbf{G}. \quad (3.15)$$

## 4. Isomorphisms of twisted-product bundles

In this section we present our main result, [Theorem 4.1.1](#), which provides an explicit description of the categorical counterpart of bundle automorphisms  $U \times G \rightarrow U \times G$ .

We work with the framework of the preceding section. In particular, we work with the categorical group  $\mathbf{G}$ , with corresponding morphism group  $\text{Mor}(\mathbf{G}) \simeq H \rtimes_\alpha G$ , and we use the convention of identifying  $h \in H$  with  $(h, e) \in H \rtimes_\alpha G$  and  $g \in G$  with  $(e, g) \in H \rtimes_\alpha G$ , so that

$$(h, g) \in H \rtimes_\alpha G \text{ is written as } hg.$$

Recall also from (2.14) that then

$$\alpha_g(h) = ghg^{-1},$$

Our goal is to determine all functors

$$\Theta : \mathbf{U} \times_{\eta_1} \mathbf{G} \rightarrow \mathbf{U} \times_{\eta_2} \mathbf{G} \quad (4.1)$$

that are isomorphisms of these categorical bundles. The latter condition means (as in Section 2.10) that  $\Theta$  is a bijection on objects and morphisms, commutes with the projection onto the base category  $\mathbf{U}$ , and preserves the right action of  $\mathbf{G}$  on the two categorical bundles.

The condition that  $\Theta$  maps fibers to fibers (commuting with the projections to the base categories) and preserves the action of  $\mathbf{G}$  is equivalent to the condition that  $\Theta$  is given

$$\text{on objects by: } \Theta(u, g) = (u, \theta(u)g) \quad (4.2)$$

and

$$\text{on morphisms by: } \Theta(\gamma, hg) = (\gamma, \theta(\gamma)hg), \quad (4.3)$$

where  $\theta(u) \in G$  for all objects  $u \in \text{Obj}(\mathbf{U})$  and

$$\theta(\gamma) = \theta_H(\gamma)\theta_G(\gamma) \in H \rtimes_\alpha G \quad \text{for all morphisms } \gamma \in \text{Mor}(\mathbf{U}). \quad (4.4)$$

Let us note that we use  $\theta$  to denote both a mapping  $U \rightarrow G$  and a mapping  $\text{Mor}(\mathbf{U}) \rightarrow \text{Mor}(G)$ .

#### 4.1. Determination of the functor $\Theta$

Our main result gives an explicit description of the functor  $\Psi$ . In the following result we use the notation  $\theta, \theta_G, \theta_H, \eta_i$ , and additional framework used above.

**Theorem 4.1.1.** *The functor  $\Theta$  in (4.1) is a functorial isomorphism between the twisted-product bundles  $\mathbf{U} \times_{\eta_i} \mathbf{G}$  if and only if:*

$$\theta_G(\gamma) = \theta(\gamma_0) \quad (4.5)$$

$$\eta_2(\gamma) = \theta(\gamma_1)\eta_1(\gamma)\theta(\gamma_0)^{-1}\tau(\theta_H(\gamma))^{-1} \quad (4.6)$$

$$\theta_H(\gamma' \circ \gamma) = [\eta_2(\gamma)^{-1}\theta_H(\gamma')\eta_2(\gamma)]\theta_H(\gamma) \quad (4.7)$$

for all morphisms  $\gamma \in \text{Mor}(\mathbf{U})$ , with  $\gamma_0 = s(\gamma)$  being the source of  $\gamma$  and  $\gamma_1 = t(\gamma)$  being the target, and all  $\gamma' \in \text{Mor}(\mathbf{U})$  for which the composite  $\gamma' \circ \gamma$  is defined.

The rest of this subsection is devoted to proving this result.

One major difference between traditional gauge transformations and categorical gauge transformations in our framework is expressed in the formula (4.6): in the classical case the multiplier  $\tau(\theta_H(\gamma))^{-1}$  is absent. It is this term that will modify the classical gauge transformation law for connection forms by introducing an additional term as will see later in Theorem 5.2.1.

**Proof.** For  $\Theta$  to be a functor, it must match sources and targets. The source condition is:

$$s\Theta = \Theta s.$$

This means that

$$s(\gamma, \theta(\gamma)hg) = \Theta(\gamma_0, g) \quad \text{for all } \gamma \in \text{Mor}(\mathbf{U}) \text{ and } hg \in H \rtimes_\alpha G. \quad (4.8)$$

Recalling the definition of the source and target functions  $s$  and  $t$  as given in (2.29), we see that the source condition is

$$(\gamma_0, \theta_G(\gamma)g) = (\gamma_0, \theta(\gamma_0)g), \quad (4.9)$$

which, in turn, is equivalent to

$$\theta_G(\gamma) = \theta(\gamma_0) \quad (4.10)$$

Next we turn to targets. The functoriality of  $\Theta$  requires also that it matches targets to targets:

$$t\Theta(\gamma, hg) = \Theta(t(\gamma), hg), \quad (4.11)$$

which is equivalent to

$$t(\gamma, \theta(\gamma)hg) = \Theta(\gamma_1, \eta_1(\gamma)\tau(h)g), \quad (4.12)$$

which, in turn, means

$$(\gamma_1, \eta_2(\gamma)\tau\theta_H(\gamma) \cdot \theta_G(\gamma)\tau(h)g) = (\gamma_1, \theta(\gamma_1)\eta_1(\gamma)\tau(h)g). \quad (4.13)$$

Since  $\theta_G(\gamma) = \theta(\gamma_0)$  we see that the target condition is equivalent to

$$\eta_2(\gamma)\tau\theta_H(\gamma)\theta(\gamma_0) = \theta(\gamma_1)\eta_1(\gamma) \quad (4.14)$$

for all  $\gamma \in \text{Mor}(\mathbf{U})$ . Here, and elsewhere, we write  $\tau\theta_H$  to mean the composite  $\tau \circ \theta_H$ , so that  $\tau\theta_H(\gamma)$  is  $\tau(\theta_H(\gamma))$ . The relation (4.6) then follows immediately.

Now we examine compositions. Functoriality of  $\Theta$  means that it carries composition of morphisms to composition of morphisms. Working with composable morphisms, we have, on the one hand,

$$\begin{aligned} \Theta(\gamma', h'g') \circ_{\eta_2} \Theta(\gamma, hg) \\ &= (\gamma', \theta(\gamma')h'g') \circ_{\eta_2} (\gamma, \theta(\gamma)hg) \\ &= (\gamma' \circ \gamma, x) \end{aligned} \quad (4.15)$$

where

$$x = \eta_2(\gamma)^{-1} [\theta_H(\gamma')\theta(\gamma'_0)h'\theta(\gamma'_0)^{-1}] \eta_2(\gamma) \cdot \theta_H(\gamma)\theta(\gamma_0)h\theta(\gamma_0)^{-1} \cdot \theta(\gamma_0)g; \quad (4.16)$$

here we have used

$$\theta(\gamma') = \theta_H(\gamma')\theta_G(\gamma') = \theta_H(\gamma')\theta(\gamma'_0)$$

and in order to do the composition properly we have written, for example,  $(\gamma', \theta(\gamma')h'g')$  as  $(\gamma', \theta_H(\gamma') \cdot \theta(\gamma'_0)h'\theta(\gamma'_0)^{-1} \cdot \theta(\gamma'_0)g')$ .

On the other hand we have

$$\begin{aligned} \Theta((h', g') \circ_{\eta_1} (h, g)) &= \Theta(\gamma' \circ \gamma, \eta_1(\gamma)^{-1}h'\eta_1(\gamma)h \cdot g) \\ &= (\gamma' \circ \gamma, \theta(\gamma' \circ \gamma)\eta_1(\gamma)^{-1}h'\eta_1(\gamma)h \cdot g) \\ &= (\gamma' \circ \gamma, \theta_H(\gamma' \circ \gamma)\theta(\gamma_0) \cdot \eta_1(\gamma)^{-1}h'\eta_1(\gamma)h \cdot g), \end{aligned} \quad (4.17)$$

where we have used

$$\theta(\gamma' \circ \gamma) = \theta_H(\gamma' \circ \gamma)\theta_G(\gamma' \circ \gamma) \quad (4.18)$$

along with

$$\theta_G(\gamma' \circ \gamma) = \theta((\gamma' \circ \gamma)_0) = \theta(\gamma_0). \quad (4.19)$$

Comparing (4.15) and (4.17) we see that  $\Theta$  respects composition of morphisms if and only if

$$\begin{aligned} \theta_H(\gamma' \circ \gamma)\theta(\gamma_0)\eta_1(\gamma)^{-1}h'\eta_1(\gamma)h \cdot g \\ &= \eta_2(\gamma)^{-1} [\theta_H(\gamma')\theta(\gamma_1)h'\theta(\gamma_1)^{-1}] \eta_2(\gamma) \theta_H(\gamma) [\theta(\gamma_0)h\theta(\gamma_0)^{-1}] \cdot \theta(\gamma_0)g \end{aligned} \quad (4.20)$$

To properly compare the  $H$  and  $G$  components we write the left hand side in the form

$$\theta_H(\gamma' \circ \gamma)\theta(\gamma_0)[\eta_1(\gamma)^{-1}h'\eta_1(\gamma)]h\theta(\gamma_0)^{-1} \cdot \theta(\gamma_0)g. \quad (4.21)$$

Thus the condition is

$$\begin{aligned} \theta_H(\gamma' \circ \gamma)\theta(\gamma_0)[\eta_1(\gamma)^{-1}h'\eta_1(\gamma)]h\theta(\gamma_0)^{-1} \cdot \theta(\gamma_0)g \\ &= \eta_2(\gamma)^{-1} [\theta_H(\gamma')\theta(\gamma_1)h'\theta(\gamma_1)^{-1}] \eta_2(\gamma) \theta_H(\gamma) [\theta(\gamma_0)h\theta(\gamma_0)^{-1}] \cdot \theta(\gamma_0)g \end{aligned} \quad (4.22)$$

The  $G$ -components manifestly agree. Focusing on the  $H$ -components the (necessary and sufficient) condition becomes

$$\begin{aligned} \theta_H(\gamma' \circ \gamma)\theta(\gamma_0)[\eta_1(\gamma)^{-1}h'\eta_1(\gamma)]h\theta(\gamma_0)^{-1} \\ &= \eta_2(\gamma)^{-1} [\theta_H(\gamma')\theta(\gamma_1)h'\theta(\gamma_1)^{-1}] \eta_2(\gamma) \theta_H(\gamma) [\theta(\gamma_0)h\theta(\gamma_0)^{-1}] \end{aligned} \quad (4.23)$$

Right-multiplying by  $\theta(\gamma_0)h^{-1}\theta(\gamma_0)^{-1}$ , we obtain:

$$\begin{aligned} \theta_H(\gamma' \circ \gamma)\theta(\gamma_0)[\eta_1(\gamma)^{-1}h'\eta_1(\gamma)]\theta(\gamma_0)^{-1} \\ &= \eta_2(\gamma)^{-1} [\theta_H(\gamma')\theta(\gamma_1)h'\theta(\gamma_1)^{-1}] \eta_2(\gamma) \theta_H(\gamma) \\ &= [\eta_2(\gamma)^{-1}\theta_H(\gamma')\eta_2(\gamma)] \tau\theta_H(\gamma)\theta(\gamma_0)\eta_1(\gamma)^{-1}h'\eta_1(\gamma) \cdot \\ &\quad \cdot \theta(\gamma_0)^{-1} \tau\theta_H(\gamma)^{-1} \eta_2(\gamma)^{-1} \eta_2(\gamma) \theta_H(\gamma) \\ &= [\eta_2(\gamma)^{-1}\theta_H(\gamma')\eta_2(\gamma)] \theta_H(\gamma)\theta(\gamma_0)\eta_1(\gamma)^{-1}h'\eta_1(\gamma)\theta(\gamma_0)^{-1} \cdot \\ &\quad \cdot \theta_H(\gamma)^{-1} \eta_2(\gamma)^{-1} \eta_2(\gamma) \theta_H(\gamma) \end{aligned} \quad (4.24)$$

where in the second last equality we used (4.6) and in the last equality we used the second Peiffer identity (2.7). Simplifying a bit further we obtain

$$\begin{aligned}\theta_H(\gamma' \circ \gamma) &= [\theta(\gamma_0) \eta_1(\gamma)^{-1} h' \eta_1(\gamma) \theta(\gamma_0)^{-1}] \\ &= [\eta_2(\gamma)^{-1} \theta_H(\gamma') \eta_2(\gamma)] \theta_H(\gamma) [\theta(\gamma_0) \eta_1(\gamma)^{-1} h' \eta_1(\gamma) \theta(\gamma_0)^{-1}].\end{aligned}\quad (4.25)$$

So at last we are down to the simple relation

$$\theta_H(\gamma' \circ \gamma) = [\eta_2(\gamma)^{-1} \theta_H(\gamma') \eta_2(\gamma)] \theta_H(\gamma). \quad (4.26)$$

This completes the proof of Theorem 4.1.1.  $\square$

The condition (4.26) can also be written as

$$\eta_2(\gamma' \circ \gamma) \theta_H(\gamma' \circ \gamma) = [\eta_2(\gamma') \theta_H(\gamma')] [\eta_2(\gamma) \theta_H(\gamma)]. \quad (4.27)$$

This relation can also be obtained immediately, as a necessary condition, from (4.24) by taking the special case  $h' = e$  and left-multiplying both sides by  $\eta_2(\gamma' \circ \gamma) = \eta_2(\gamma') \eta_2(\gamma)$ .

#### 4.2. Specification of functorial gauge transformations

The relation (4.27) shows that the product  $\eta_2(\gamma) \theta_H(\gamma)$  behaves the same way as a parallel-transport specifier along  $\gamma$ . In more detail, for any smooth  $L(H \rtimes_\alpha G)$ -valued 1-form  $\Lambda$  on  $U$ , the path ordered exponential

$$\Pi(\gamma) = Pe^{-\int_\gamma \Lambda}$$

satisfies the relation

$$\Pi(\gamma' \circ \gamma) = \Pi(\gamma') \Pi(\gamma). \quad (4.28)$$

for any smooth composition of paths  $\gamma' \circ \gamma$ . (We have noted this earlier in (3.4).)

Thus for us the functorial gauge transformations of most interest are those for which there is an  $L(H \rtimes_\alpha G)$ -valued 1-form  $\Lambda$  on  $U$  such that

$$\eta_2(\gamma) \theta_H(\gamma) = Pe^{-\int_\gamma \Lambda} \quad (4.29)$$

for all smooth paths  $\gamma \in \text{Mor}(\mathbf{U})$ . This Ansatz actually implies that  $\eta_2$  arises from a 1-form  $\bar{A}_2$ , as we see in Proposition 5.1.1.

### 5. Gauge transformation of the background connection form

A traditional gauge transformation is specified by a function  $\theta : U \rightarrow G$ . In this section we shall see that a functorial gauge transform arises from a pair  $(\theta, \Lambda^H)$ , where  $\theta : U \rightarrow G$  is a smooth function and  $\Lambda^H$  is an  $L(H)$ -valued smooth 1-form over  $U$ .

We continue with the framework from the previous section, specifically with the functorial isomorphism

$$\Theta : \mathbf{U} \times_{\eta_1} \mathbf{G} \rightarrow \mathbf{U} \times_{\eta_2} \mathbf{G}$$

given on objects and morphisms by

$$\Theta(u, g) = (u, \theta(u)g), \quad \text{and} \quad \Theta(\gamma, hg) = (\gamma, \theta(\gamma)hg), \quad (5.1)$$

for all  $u \in \text{Obj}(\mathbf{U})$ ,  $\gamma \in \text{Mor}(\mathbf{U})$ . The elements  $\theta(u) \in G$  and  $\theta(\gamma) = \theta_H(\gamma) \theta_G(\gamma) \in H \rtimes_\alpha G$  satisfy the relations (4.5)–(4.7) discussed in Theorem 4.1.1.

We specialize the discussion of the preceding section by assuming that there is a smooth  $L(G)$ -valued 1-form  $\bar{A}_1$  on  $U$  such that for any smooth path  $\gamma : [a, b] \rightarrow U$  the function  $u \mapsto \eta_1(\gamma|[a, u]) \in G$  satisfies the differential equation

$$\frac{d\eta_1(\gamma|[a, u])}{du} \eta_1(\gamma|[a, u])^{-1} = -\bar{A}_1(\gamma'(u)) \quad (5.2)$$

for  $u \in [a, b]$ , with initial value  $\eta_1(\gamma|[a, a]) = e$ . This definition is compressed into the path-ordered exponential notation

$$\eta_1(\gamma) = Pe^{-\int_\gamma \bar{A}_1}. \quad (5.3)$$

Moreover, we assume that (4.29) holds for some 1-form  $\Lambda$  on  $U$ .

### 5.1. $\theta_H$ as a path-ordered integral

We have then the following result.

**Proposition 5.1.1.** Suppose  $\Lambda$  is a smooth  $L(H \rtimes_\alpha G)$ -valued 1-form on  $U$  for which (4.29) holds. Then  $\eta_2$  arises from an  $L(G)$ -valued 1-form  $\bar{A}_2$  on  $U$  in the sense that  $\eta_2(\gamma) = Pe^{-\int_\gamma \bar{A}_2}$  for all morphisms/paths  $\gamma_2$  on  $U$ , where  $\bar{A}_2$  is the  $L(G)$ -component of  $\Lambda$ :

$$\bar{A}_2 = \Lambda^G. \quad (5.4)$$

Moreover, for any smooth path  $\gamma : [a, b] \rightarrow U$ , we have

$$\theta_H(\gamma) = Pe^{-\int_\gamma g_\gamma^{-1} \Lambda^H g_\gamma}, \quad (5.5)$$

where  $\Lambda^H$  is the  $L(H)$ -component of the  $L(H \rtimes_\alpha G)$ -valued 1-form  $\Lambda$ , and  $g_\gamma$  is the function along the path  $\gamma$  given by  $g_\gamma(u) = Pe^{-\int_{\gamma|_{[a,u]} \bar{A}_2}$  for all  $u \in [a, b]$ .

Let  $\eta_2(\gamma|_u)$  be defined, as with  $\eta_1(\gamma)$  in (5.3), by

$$\eta_2(\gamma|_u) = Pe^{-\int_{\gamma|_u} \bar{A}_2}. \quad (5.6)$$

Then, in view of (5.3), the element  $g_\gamma(u)$  in (5.5) is  $\eta_2(\gamma|_u)$ , where  $\gamma|_u = \gamma|_{[a,u]}$ . Thus

$$\frac{d\theta_H(\gamma|_u)}{du} \theta_H(\gamma|_u)^{-1} = -\eta_2(\gamma|_u)^{-1} \Lambda^H(\dot{\gamma}(u)) \eta_2(\gamma|_u), \quad (5.7)$$

**Proof.** Let us recall that the group  $H$  is the kernel of the source morphism  $\text{Mor}(\mathbf{G}) \rightarrow G$ , and so it is a normal subgroup of  $H \rtimes_\alpha G$ ; alternatively,  $ghg^{-1} = \alpha_g(h)$ , as seen earlier in (2.14), is an element of  $H$  for all  $h \in H$  and  $g \in G$ . Writing an element  $gh$ , where  $g \in G$  and  $h \in H$ , as  $(ghg^{-1})g$  shows that the source of  $gh$  is  $g$ . The proof follows then from Lemma 5.1.1 below.  $\square$

The following observation describes the  $H$ -component and the  $G$ -component of a path ordered exponential in  $H \rtimes_\alpha G$ .

**Lemma 5.1.1.** Suppose

$$[a, b] \rightarrow L(H \rtimes_\alpha G) : u \mapsto X_u + Y_u \quad (5.8)$$

is a smooth function, where  $X_u \in L(H)$  and  $Y_u \in L(G)$ . Then

$$Pe^{\int_a^b (X_u + Y_u) du} = Pe^{\int_a^b Y_u du} Pe^{\int_a^b g_u^{-1} X_u g_u du}, \quad (5.9)$$

where  $g_u = Pe^{\int_a^u Y_v dv}$ . In this formula,  $g_u^{-1} X_u g_u$  lies in  $L(H)$ , so that  $Pe^{\int_a^b g_u^{-1} X_u g_u du}$  lies in  $H$ , and the first term  $Pe^{\int_a^b Y_u du}$  lies in  $G$ .

**Proof.** Analogously to  $g_u$ , let  $h_u = Pe^{\int_a^u g_v^{-1} X_v g_v dv} \in H$ . Then for the path

$$[a, b] \rightarrow H \rtimes_\alpha G : u \mapsto g_u h_u \quad (5.10)$$

we have

$$\begin{aligned} \frac{d(g_u h_u)}{du} (g_u h_u)^{-1} &= \dot{g}_u g_u^{-1} + g_u \dot{h}_u h_u^{-1} g_u^{-1} = Y_u + g_u (g_u^{-1} X_u g_u) g_u^{-1} \\ &= Y_u + X_u. \end{aligned} \quad (5.11)$$

Hence

$$Pe^{\int_a^b (X_u + Y_u) du} = g_b h_b,$$

which proves (5.9). As noted in the proof of the Proposition,  $H$  is a normal subgroup in  $H \rtimes_\alpha G$ , and so  $g_u^{-1} X_u g_u$  lies in  $L(H)$ .  $\square$

### 5.2. Gauge transforming with $\theta$ and $\Lambda$

Recall that  $\text{Mor}(\mathbf{G})$  can be identified with the semidirect product  $H \rtimes_\alpha G$ , and then the element  $(h, g)$ , which we also write as  $hg$ , runs from the source  $g$  to the target  $\tau(h)g$ :

$$s(hg) = g, \quad \text{and} \quad t(hg) = \tau(h)g, \quad (5.12)$$

where now the target and source maps should be viewed as

$$s : H \rtimes_\alpha G \rightarrow G, \quad \text{and} \quad t : H \rtimes G \rightarrow G. \quad (5.13)$$



The path ordered exponential  $Pe^{-\int_{\gamma} \Lambda}$  is obtained as the terminal value  $x_b$  of the solution of the  $H \rtimes_{\alpha} G$ -valued differential equation

$$\dot{x}_u x_u^{-1} = -\Lambda(\gamma'(u)) \quad \text{for all } u \in [a, b], \text{ with } x_a = e. \quad (5.14)$$

(We have discussed equations of this type earlier in (3.1) and (3.3).) Here  $x_u \in H \rtimes_{\alpha} G$ . Applying  $t$  to both sides we obtain:

$$\dot{g}_u g_u^{-1} = -t\Lambda(\gamma'(u)) \quad \text{for all } u \in [a, b], \text{ with } g_a = e, \quad (5.15)$$

where now  $g_u = t(x_u)$ , and where  $t\Lambda$  is the  $L(G)$ -valued 1-form obtained by composing the  $L(H \rtimes_{\alpha} G)$ -valued 1-form  $\Lambda$  with

$$dt|_e : L(H \rtimes_{\alpha} G) \rightarrow L(G) : (X, Y) \mapsto d\tau|_e X + Y. \quad (5.16)$$

Thus,

$$t\left(Pe^{-\int_{\gamma} \Lambda}\right) = Pe^{-\int_{\gamma} t\Lambda}. \quad (5.17)$$

Then from (4.29) we have:

$$\eta_2(\gamma)\tau(\theta_H(\gamma)) = Pe^{-\int_{\gamma} t\Lambda} \quad (5.18)$$

**Theorem 5.2.1.** With framework as in Proposition 5.1.1 and Eqs. (5.2), (5.3), the connection forms  $\bar{A}_1$  and  $\bar{A}_2$  are related by the following transformation:

$$\bar{A}_2 = \theta \bar{A}_1 \theta^{-1} - (d\theta)\theta^{-1} - \tau \Lambda^H. \quad (5.19)$$

Here  $\Lambda^H$  is the  $L(H)$ -component of the  $L(H \rtimes_{\alpha} G)$ -valued 1-form  $\Lambda$ .

This result is superficially similar to the gauge transformation law in higher gauge theories (for example, [17, equation (1.2)]). However, here our context and interpretation are quite different. Our result arises from an isomorphism between a pair of twisted-product bundles as described in Theorem 4.1.1. Since gauge transformations on such bundles have not been explored before, our result is in fact new, and indeed it is interesting that we arrive at this kind of transformation law using a completely different approach.

**Proof.** Let  $\gamma : [a, b] \rightarrow U$  be a smooth path, and for all  $u \in [a, b]$ , let

$$\gamma|_u = \gamma|[a, u]. \quad (5.20)$$

From (4.6) we have

$$\eta_2(\gamma|_u) = \delta(u)\beta(u)^{-1} \quad (5.21)$$

where

$$\begin{aligned} \delta(u) &= \theta(\gamma(u))\eta_1(\gamma|_u)\theta(\gamma(a))^{-1} \\ \beta(u) &= \tau(\theta_H(\gamma|_u)). \end{aligned} \quad (5.22)$$

Then

$$\frac{d\eta_2(\gamma|_u)}{du} \eta_2(\gamma|_u)^{-1} = \dot{\delta}(u)\delta(u)^{-1} - \eta_2(\gamma|_u)\dot{\beta}(u)\beta(u)^{-1}\eta_2(\gamma|_u)^{-1}. \quad (5.23)$$

Writing  $V$  for the tangent vector  $\gamma'(u)$ , we have

$$\begin{aligned} -\bar{A}_2(V) &= \frac{d\eta_2(\gamma|_u)}{du} \eta_2(\gamma|_u)^{-1} \\ &= \dot{\delta}(u)\delta(u)^{-1} - \eta_2(\gamma|_u)\dot{\beta}(u)\beta(u)^{-1}\eta_2(\gamma|_u)^{-1} \\ &= (d\theta|_{\gamma(u)}V)\theta(\gamma(u))^{-1} + \theta(\gamma(u))(-\bar{A}_1(V))\theta(\gamma(u))^{-1} \\ &\quad - \eta_2(\gamma|_u)[- \eta_2(\gamma|_u)^{-1}\tau\Lambda^H(V)\eta_2(\gamma|_u)]\eta_2(\gamma|_u)^{-1} \end{aligned} \quad (5.24)$$

where in the last step we used the relation (5.7). Thus,

$$-\bar{A}_2 = (d\theta)\theta^{-1} - \theta\bar{A}_1\theta^{-1} + \tau\Lambda^H. \quad (5.25)$$

Thus we obtain the desired result (5.19).  $\square$

## 6. Functorial automorphisms of decorated bundles

In this section we work at the global level and determine the structure of automorphisms of categorical bundles. We focus on categorical bundles that arise as *decorated bundles*, wherein the morphisms arise from paths that are horizontal with respect to a given connection on a principal bundle.

In this section we will often use the notion of a categorical connection defined in Section 2.11.

### 6.1. Decorated categorical bundles

We turn to decorated bundles at an abstract categorical level. We have discussed this briefly earlier in Section 2.15. Let  $\mathbf{G}_0$  be a categorical group, and

$$\mathbf{P} \rightarrow \mathbf{M}$$

be a categorical principal  $\mathbf{G}_0$ -bundle. Let  $\bar{\mathbf{A}}$  be a categorical connection on this bundle. Let  $\mathbf{P}^{\bar{\mathbf{A}}}$  be the subcategory of  $\mathbf{P}$  with the same object set but with morphisms being the  $\bar{\mathbf{A}}$ -horizontal morphisms. Then  $\pi$  restricts to a categorical principal bundle

$$\mathbf{P}^{\bar{\mathbf{A}}} \rightarrow \mathbf{M}$$

where the structure categorical group is a discrete category with the object group being  $\text{Obj}(\mathbf{G}_0)$ . (The morphism group  $\text{Mor}(\mathbf{G}_0)$  will play no role in the discussion below.)

Now consider a categorical group  $\mathbf{G}$  for which

$$\text{Obj}(\mathbf{G}) = \text{Obj}(\mathbf{G}_0), \quad (6.1)$$

as groups. Associated to  $\mathbf{G}$  is the crossed module  $H \rtimes_{\alpha} G$ , where  $G = \text{Obj}(\mathbf{G})$  and  $H$  is the kernel of the source map  $\text{Mor}(\mathbf{G}) \rightarrow \text{Obj}(\mathbf{G})$ .

From this data we can form a category, denoted  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$ , whose object set is the same as  $\text{Obj}(\mathbf{P})$  and whose morphisms are of the form  $(\bar{\gamma}, h)$ , with  $\bar{\gamma}$  being any  $\bar{\mathbf{A}}$ -horizontal morphism in  $\mathbf{P}$  and  $h \in H$ . Thus:

$$\text{Obj}(\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}) = \text{Obj}(\mathbf{P}) \quad (6.2)$$

and

$$\text{Mor}(\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}) = \text{Mor}(\mathbf{P}^{\bar{\mathbf{A}}}) \times H, \quad (6.3)$$

with source and target maps given by (see Fig. 8)

$$\begin{aligned} s(\bar{\gamma}, h) &= s(\bar{\gamma}) \\ t(\bar{\gamma}, h) &= t(\bar{\gamma})\tau(h), \end{aligned} \quad (6.4)$$

and composition given by

$$(\bar{\gamma}_2, h_2) \circ (\bar{\gamma}_1, h_1) = (\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1, h_1 h_2), \quad (6.5)$$

when  $t(\bar{\gamma}_1, h_1) = s(\bar{\gamma}_2, h_2)$ . We have a right action of  $\mathbf{G}$  on  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$  given on objects by the right action of  $\text{Obj}(\mathbf{G})$  on  $\mathbf{P}$  and on morphisms by

$$(\bar{\gamma}, h)h_1g_1 = (\bar{\gamma}g_1, g_1^{-1}hh_1g_1). \quad (6.6)$$

The decorated categorical bundle

$$\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}} \rightarrow \mathbf{M} \quad (6.7)$$

is the functor given on objects by the original projection  $\text{Obj}(\mathbf{P}) \rightarrow \text{Obj}(\mathbf{M})$  and on morphism by  $(\bar{\gamma}, h) \mapsto \pi(\bar{\gamma})$ , along with the right action of  $\mathbf{G}$  on  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$  described above.

It will be convenient to write  $(\bar{\gamma}, h)$  as  $\bar{\gamma}h$ :

$$\bar{\gamma}h \stackrel{\text{def}}{=} (\bar{\gamma}, h). \quad (6.8)$$

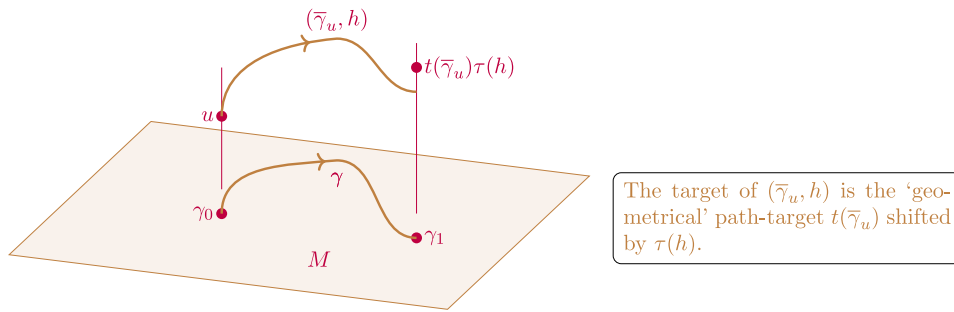
For  $\bar{\gamma} \in \text{Mor}(\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}})$  and  $(h, g) \in H \rtimes_{\alpha} G$ , the notation

$$\bar{\gamma}gh$$

means the pair

$$(\bar{\gamma}g, h),$$

where  $\bar{\gamma}g \in \text{Mor}(\mathbf{P})$  is obtained from  $\bar{\gamma} \in \text{Mor}(\mathbf{P})$  by the right action of  $1_g \in \text{Mor}(\mathbf{G})$ .



**Fig. 8.** The decorated categorical bundle  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$ .

## 6.2. Functorial automorphisms

Consider now a functorial bundle automorphism

$$\theta : \mathbf{P}^{\bar{\mathbf{A}}, \text{dec}} \rightarrow \mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}, \quad (6.9)$$

that is fiber-preserving. Then on objects  $\theta$  is given by

$$\theta(p) = pg_p \quad (6.10)$$

with  $g_p \in G = \text{Obj}(\mathbf{G})$  for all  $p \in P = \text{Obj}(\mathbf{P})$ , and on morphisms it is given by

$$\theta(\bar{\gamma}, h) = \theta(\bar{\gamma}, e)h = ((\bar{\gamma}, e)g_{\bar{\gamma}}h_{\bar{\gamma}})h, \quad (6.11)$$

where  $g_{\bar{\gamma}} \in G$  and  $h_{\bar{\gamma}} \in H$ . We can also write this as

$$\theta(\bar{\gamma}, h) = (\bar{\gamma}g_{\bar{\gamma}}, h_{\bar{\gamma}}h). \quad (6.12)$$

## 6.3. Conditions for $\theta$ to be an automorphism

For  $\theta$  in (6.9) to be an automorphism of the categorical bundle  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$  means, in particular, that it preserves the categorical right action of  $\mathbf{G}$  on  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$ .

For  $\theta$  to be a functor it must match sources and targets appropriately. Comparing sources we have

$$\begin{aligned} \theta(s(\bar{\gamma}, h)) &= \theta(s(\bar{\gamma})) = s(\bar{\gamma})g_{s(\bar{\gamma})} \\ s(\theta(\bar{\gamma}, h)) &= s(\bar{\gamma}g_{\bar{\gamma}}) = s(\bar{\gamma})g_{\bar{\gamma}}. \end{aligned} \quad (6.13)$$

Thus, for the element  $g_{\bar{\gamma}}h_{\bar{\gamma}} \in H \rtimes_{\alpha} G$  that specifies  $\theta(\bar{\gamma}, e)$ , we have the condition

$$g_{\bar{\gamma}} = g_{s(\bar{\gamma})}, \quad (6.14)$$

where  $\bar{\gamma}$  is any  $\bar{\mathbf{A}}$ -horizontal morphism. This is the counterpart for decorated bundles of the condition (4.10) for gauge transformations of twisted-product bundles.

Since  $\theta$  is equivariant under the right action of  $\mathbf{G}$  we have

$$\begin{aligned} \theta(pg) &= \theta(p)g \\ \theta((\bar{\gamma}, h)g_1h_1) &= \theta(\bar{\gamma}, h)g_1h_1, \end{aligned} \quad (6.15)$$

for all  $g, g_1 \in G$  and  $h, h_1 \in H$ . From these equations we have, upon using (6.14) and some algebraic simplification, the conditions

$$\begin{aligned} g_{pg} &= g^{-1}g_pg \\ h_{\bar{\gamma}g_1} &= g_1^{-1}h_{\bar{\gamma}}g_1. \end{aligned} \quad (6.16)$$

(The first equation is the same as for traditional automorphisms on principal bundles.)

Next, comparing targets, we have

$$\begin{aligned} \theta(t(\bar{\gamma}, h)) &= \theta(t(\bar{\gamma})\tau(h)) = t(\bar{\gamma})g_{t(\bar{\gamma})}\tau(h) \\ t(\theta(\bar{\gamma}, h)) &= t(\bar{\gamma}g_{\bar{\gamma}}, h_{\bar{\gamma}}h) = t(\bar{\gamma})g_{\bar{\gamma}}\tau(h_{\bar{\gamma}}h). \end{aligned} \quad (6.17)$$

Thus

$$g_{t(\bar{\gamma})} = g_{\bar{\gamma}} \tau(h_{\bar{\gamma}}). \quad (6.18)$$

Using this along with (6.14) we have the condition

$$\tau(h_{\bar{\gamma}}) = g_{s(\bar{\gamma})}^{-1} g_{t(\bar{\gamma})}. \quad (6.19)$$

Finally, let us examine the consequences of functoriality in terms of the effect on composition of morphisms:

$$\theta((\bar{\gamma}_2, h_2) \circ (\bar{\gamma}_1, h_1)) = \theta(\bar{\gamma}_2, h_2) \circ \theta(\bar{\gamma}_1, h_1). \quad (6.20)$$

Using (6.12) we have then

$$\begin{aligned} & ((\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1) g_{\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1}, h_{\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1} h_1 h_2) \\ &= (\bar{\gamma}_2 g_{\bar{\gamma}_2}, h_{\bar{\gamma}_2} h_2) \circ (\bar{\gamma}_1 g_{\bar{\gamma}_1}, h_{\bar{\gamma}_1} h_1) \\ &= (\bar{\gamma}_2 g_{\bar{\gamma}_2} \tau(h_{\bar{\gamma}_1} h_1)^{-1} \circ \bar{\gamma}_1 g_{\bar{\gamma}_1}, h_{\bar{\gamma}_1} h_1 h_{\bar{\gamma}_2} h_2). \end{aligned} \quad (6.21)$$

From (6.14) we have

$$g_{\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1} = g_{s(\bar{\gamma}_1)} \quad (6.22)$$

and using (6.19) we have

$$\begin{aligned} g_{\bar{\gamma}_2} \tau(h_{\bar{\gamma}_1} h_1)^{-1} &= g_{t(\bar{\gamma}_1) \tau(h_1)} \tau(h_1)^{-1} (g_{s(\bar{\gamma}_1)}^{-1} g_{t(\bar{\gamma}_1)})^{-1} \\ &= g_{s(\bar{\gamma}_1)}. \end{aligned} \quad (6.23)$$

Substituting in (6.21) we obtain

$$\begin{aligned} & ((\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1) g_{s(\bar{\gamma}_1)}, h_{\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1} h_1 h_2) \\ &= ((\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1) g_{s(\bar{\gamma}_1)}, h_{\bar{\gamma}_1} h_1 h_{\bar{\gamma}_2} h_2). \end{aligned} \quad (6.24)$$

Thus the condition is

$$h_{\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1} h_1 = h_{\bar{\gamma}_1} h_1 h_{\bar{\gamma}_2}. \quad (6.25)$$

From this we have

$$\begin{aligned} h_{\bar{\gamma}_2 \tau(h_1)^{-1} \circ \bar{\gamma}_1} &= h_{\bar{\gamma}_1} h_1 h_{\bar{\gamma}_2} h_1^{-1} \\ &= h_{\bar{\gamma}_1} \tau(h_1) h_{\bar{\gamma}_2} \tau(h_1)^{-1} \text{ (using the second Peiffer identity (2.7))} \\ &= h_{\bar{\gamma}_1} h_{\bar{\gamma}_2 \tau(h_1)^{-1}} \text{ (using (6.16)).} \end{aligned} \quad (6.26)$$

Thus the condition for functoriality of composition is

$$h_{\bar{\gamma}_2 \circ \bar{\gamma}_1} = h_{\bar{\gamma}_1} h_{\bar{\gamma}_2} \quad (6.27)$$

whenever the composition  $\bar{\gamma}_2 \circ \bar{\gamma}_1$  is defined.

#### 6.4. Comparison with twisted-product bundles

Let us now denote by  $\bar{\gamma}$  the  $\bar{A}$ -horizontal lift of  $\gamma \in \text{Mor}(\mathbf{U})$  to a path on  $U \times G$  initiating at  $(s(\gamma), e)$ . Comparing with the gauge transformation formula (4.3) for twisted-product bundles it follows that:

$$g_{\bar{\gamma}} h_{\bar{\gamma}} \text{ corresponds to } \theta(\gamma) = \theta_H(\gamma) \theta_G(\gamma);$$

which means that

- (i)  $g_{\bar{\gamma}}$  corresponds to  $\theta_G(\gamma)$ ;
- (ii)  $h_{\bar{\gamma}}$  corresponds to  $\theta_G(\gamma)^{-1} \theta_H(\gamma) \theta_G(\gamma)$ .

In more detail, suppose we identify the decorated bundle  $(\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}}$  with  $\mathbf{U} \times_{\eta} \mathbf{G}$  as in (3.15). We use the notation  $\bar{\gamma}_g$  from (3.11). Then, by (3.14), the morphism  $(\bar{\gamma}_e, h)$  in the category  $(\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}}$  corresponds to  $(\gamma, h) \in \text{Mor}(\mathbf{U} \times_{\eta} \mathbf{G})$ , and this is gauge-transformed to  $(\gamma, \theta(\gamma)h)$ , which, in turn, corresponds to

$$(\bar{\gamma}_e \theta_G(\gamma), \theta_G(\gamma)^{-1} \theta_H(\gamma) \theta_G(\gamma) h) \in \text{Mor}((\mathbf{U} \times \mathbf{G})^{\bar{A}, \text{dec}}).$$

Then, looking back at (6.12), we see that (i) and (ii) follow, with  $\bar{\gamma} = \bar{\gamma}_e$ .

### 6.5. Functorial isomorphisms of decorated bundles

So far in this section we have studied *automorphisms* of the decorated categorical bundle  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}}$ . We examine briefly how one might proceed to study the more general case of isomorphisms of decorated categorical bundles. To this end, let  $\bar{\mathbf{A}}$  and  $\tilde{\mathbf{A}}$  be categorical connections on a categorical principal  $\mathbf{G}$ -bundle  $\mathbf{P} \rightarrow \mathbf{M}$ . As in Section 2.10, a *functorial isomorphism*

$$\theta : \mathbf{P}^{\bar{\mathbf{A}}, \text{dec}} \rightarrow \mathbf{P}^{\tilde{\mathbf{A}}, \text{dec}}, \quad (6.28)$$

is a fiber-preserving  $\mathbf{G}$ -equivariant functor. As before,  $\theta$  is given on objects by

$$\theta(p) = pg_p \quad (6.29)$$

with  $g_p \in G = \text{Obj}(\mathbf{G})$  for all  $p \in P = \text{Obj}(\mathbf{P})$ , and on morphisms it is given by

$$\theta(\bar{\gamma}, h) = \theta(\bar{\gamma}, e)h = ((\tilde{\gamma}, e)g_{\tilde{\gamma}}h_{\tilde{\gamma}})h, \quad (6.30)$$

where  $g_{\tilde{\gamma}} \in G$  and  $h_{\tilde{\gamma}} \in H$ , and  $\tilde{\gamma}$  is the  $\tilde{\mathbf{A}}$ -horizontal morphism with source  $s(\tilde{\gamma})$  that projects by  $\pi$  to  $\pi(\tilde{\gamma}) \in \text{Mor}(\mathbf{M})$ .

We can also write (6.30) as

$$\theta(\bar{\gamma}, h) = (\tilde{\gamma}g_{\tilde{\gamma}}, h_{\tilde{\gamma}}h). \quad (6.31)$$

We shall not pursue this further, but an analysis analogous to that in Section 6.3 could be carried out.

## 7. Transformation of higher parallel-transport

In this section we will study ‘higher parallel-transport’, by which we mean essentially a connection on the bundle whose base space is comprised of paths on  $M$  and whose bundle space consists of the pairs  $(\bar{\gamma}, h)$ , with  $\bar{\gamma}$  being any horizontal path on the principal  $G$ -bundle, relative to some fixed connection  $\bar{A}$ , and  $h$  any ‘decoration’ of  $\bar{\gamma}$  drawn from  $H$ . After discussing the general case we will focus on the simplest situation: the case in which the groups  $G$  and  $H$  are abelian and we work with a product bundle  $U \times G \rightarrow U$ .

### 7.1. Categorical connections on higher bundles

Let us consider the case where  $\mathbf{P} \rightarrow \mathbf{M}$  is a decorated categorical  $\mathbf{G}$ -bundle, arising from a classical principal  $G$ -bundle  $\pi : P \rightarrow M$  equipped with a classical connection for  $\bar{A}$ , and  $\mathbf{G}$  is associated with a crossed module  $(G, H, \alpha, \tau)$ . Let us then consider the ‘morphism bundle’ of this categorical bundle; its base space is  $M_1$ , comprised of paths on  $M$ , and its bundle space is  $P_1^{\bar{A}, \text{dec}}$ , comprised of pairs  $(\bar{\gamma}, h)$ , where  $\bar{\gamma}$  is any  $\bar{A}$ -horizontal paths on  $P$  and  $h \in H$ . Then

$$P_1^{\bar{A}, \text{dec}} \rightarrow M \quad (7.1)$$

is itself a principal  $H \rtimes_{\alpha} G$ -bundle. This is just the ‘morphism bundle’ of the categorical principal bundle  $\mathbf{P}^{\bar{\mathbf{A}}, \text{dec}} \rightarrow \mathbf{M}$ .

Let  $\bar{A}_1$  be a (classical) connection on the bundle (7.1). Suppose  $\mathbf{G}_1$  is a categorical group whose object group is  $\text{Mor}(\mathbf{G})$ . Let  $(H, K, \alpha_1, \tau_1)$  be the associated crossed module, so that

$$\text{Mor}(\mathbf{G}_1) \simeq K \rtimes_{\alpha_1} H.$$

Then we can form a new ‘higher’ categorical principal bundle

$$\mathbf{P}_1^{\bar{\mathbf{A}}, \text{dec}} \rightarrow \mathbf{M}_1,$$

whose object bundle is  $P_1^{\bar{A}_1, \text{dec}} \rightarrow M_1$ . Here  $\mathbf{M}_1$  is the category whose objects are the morphisms of  $\mathbf{M}$ , i.e. paths on  $M$ , and whose morphisms are paths of paths on  $M$  (this is described in more detail below in (7.2)). Objects of  $\mathbf{P}_1^{\bar{\mathbf{A}}, \text{dec}}$  are the elements  $(\bar{\gamma}, h) \in P_1^{\bar{A}, \text{dec}}$  and morphisms of  $\mathbf{P}_1^{\bar{\mathbf{A}}, \text{dec}}$  are of the form

$$(\bar{\Gamma}, h, k)$$

where  $(\bar{\Gamma}, h)$  is a path of paths on  $P_1^{\bar{A}, \text{dec}}$ , horizontal with respect to  $\bar{A}_1$ , and  $k$  is a decorating element drawn from  $K$ . Finally, a *categorical connection*  $\bar{\mathbf{A}}_1$  on this categorical bundle would be a specification of horizontal lifts:

$$\Gamma \mapsto (\bar{\Gamma}, h, k),$$

for any given  $\Gamma$ , path of paths on  $M$ , and initial path  $(\bar{\gamma}, h)$  that projects down to the initial path  $\Gamma_0$ .

## 7.2. Parallel-transport of decorated paths

We specialize to the local situation, where the base manifold is  $U$  and the bundle  $P$  over  $U$  is trivialized as a product  $U \times G$ . We specify a higher parallel-transport process by means of the following data:

- (a) two  $L(G)$ -valued 1-forms  $A$  and  $\bar{A}$  on  $U$ ;
- (b) an  $L(H)$ -valued 2-form  $B$  on the base manifold  $U$ ;
- (c) an  $L(H)$ -valued 1-form  $C$  on  $U$ .

The forms  $A$  and  $\bar{A}$  should be viewed as providing connections on the product bundle  $U \times G \rightarrow U$ . We use the categories  $\mathbf{U}$  and  $\mathbf{U}_1$  analogously to  $\mathbf{M}$  and  $\mathbf{M}_1$ . Thus an object of  $\mathbf{U}_1$  arises from a path on  $U$ , and a morphism of  $\mathbf{U}_1$  arises from a path of paths on  $U$  as we now describe.

Consider a  $C^\infty$  mapping

$$\Gamma : [u_0, u_1] \times [v_0, v_1] \rightarrow M : (u, v) \mapsto \Gamma(u, v) = \Gamma_u(v) = \Gamma^v(u), \quad (7.2)$$

such that there is an  $\epsilon > 0$  for which  $\Gamma_u$  is constant on  $[v_0, v_0 + \epsilon]$  and on  $[v_1 - \epsilon, v_1]$  for all  $u \in [u_0, u_1]$  and  $\Gamma^v$  is constant on  $[u_0, u_0 + \epsilon]$  and on  $[u_1 - \epsilon, u_1]$  for all  $v \in [v_0, v_1]$ . We view

$$u \mapsto \Gamma_u$$

as a path of paths on  $U$ , and thus a morphism of  $\mathbf{U}_1$ .

We now describe a connection on the bundle

$$P_1^{\bar{A}, \text{dec}} \rightarrow M_1.$$

This is a prescription for parallel-transporting an initial object  $(\bar{\gamma}, h)$  along a path  $u \mapsto \Gamma_u$  on  $M_1$ . Here  $\bar{\gamma}$  is an  $\bar{A}$ -horizontal path on  $U \times G$  lying above  $\Gamma_{u_0}$ , and hence is of the form

$$[v_0, v_1] \rightarrow U \times G : v \mapsto (\Gamma_{u_0}(v), g_{u_0}(v)),$$

where

$$g_{u_0}(v) = Pe^{-\int_{\Gamma_{u_0}|_{[v_0, v]}} \bar{A}}, \quad (7.3)$$

the path-ordered exponential  $Pe^{\dots}$  being as defined in (3.3). The parallel-transport process would be a path

$$u \mapsto \bar{\Gamma}_u = (\Gamma_u(\cdot), g_u(\cdot)),$$

where each path

$$v \mapsto \bar{\Gamma}_u(v) = (\Gamma_u(v), g_u(v)) \in U \times G$$

is  $\bar{A}$ -horizontal. We generate a path of such horizontal paths:

$$[u_0, u_1] \times [v_0, v_1] \rightarrow U \times G : (u, v) \mapsto (\Gamma_u(v), g_u(v)),$$

by taking

$$g_u(v) = Pe^{-\int_{\Gamma_u|_{[v_0, v]}} \bar{A}} g_{u_0}(v_0), \quad (7.4)$$

where the initial value  $g_u(v_0)$  is specified by a second  $L(G)$ -valued 1-form  $A$ :

$$g_u(v_0) = Pe^{-\int_{\Gamma^{v_0}|_{[u_0, u]}} A}. \quad (7.5)$$

Now we describe how an initial decoration  $h_{u_0}$  evolves:

$$h_u = c_u x_u c_{u_0}^{-1} h_{u_0}, \quad (7.6)$$

where

$$c_u = Pe^{-\int_{v_0}^{v_1} C(\partial_v \Gamma(u, v)) dv} \quad (7.7)$$

and  $u \mapsto x_u \in H$  solves

$$\begin{aligned} \frac{dx_u}{du} x_u^{-1} = & - \left[ \int_{v_0}^{v_1} g_u(v)^{-1} B(\partial_1 \Gamma(u, v), \partial_2 \Gamma(u, v)) g_u(v) dv \right. \\ & \left. + C(\partial_1 \Gamma(u, v_0)) - C(\partial_1 \Gamma(u, v_1)) \right] \end{aligned} \quad (7.8)$$

with initial value  $x_{u_0} = e$ .

The effect of a gauge transformation  $(\theta, \Lambda^H)$  on this is as follows:

$$\begin{aligned} g_u(v_0) &\mapsto \theta(\Gamma_u(v_0)) g_u(v_0) \\ h_u &\mapsto \tilde{h}_u = \theta_H(\Gamma_u) \theta(\Gamma_u(v_0)) h_u \theta(\Gamma_u(v_0))^{-1} \end{aligned} \quad (7.9)$$

where we have used the formulas in (4.2) and (4.3); an explicit expression for  $\theta(\Gamma_u)$  in terms of  $\Lambda^H$  is given in (5.5).

### 7.3. The abelian case

The transformation law (7.9) is quite intricate. In the case of abelian structure groups the formulas simplify greatly and become more transparent. First let us observe from (7.6) that in the abelian case we have

$$h_u = \exp(-X_u) h_{u_0}, \quad (7.10)$$

where

$$X_u = \int_{\Gamma[[u_0, u] \times [v_0, v_1]]} B + \int_{\Gamma^{v_0}[[u_0, u]]} C - \int_{\Gamma^{v_1}[[u_0, u]]} C. \quad (7.11)$$

Then using the second formula in (7.9) we have:

$$\tilde{h}_u = \tilde{h}_{u_0} \exp(-\tilde{X}_u), \quad (7.12)$$

where  $X_u$  is given by

$$\tilde{X}_u = \int_{\Gamma[[u_0, u] \times [v_0, v_1]]} B + \int_{\Gamma^{v_0}[[u_0, u]]} C - \int_{\Gamma^{v_1}[[u_0, u]]} C + \int_{\Gamma_u} \Lambda^H - \int_{\Gamma_{u_0}} \Lambda^H. \quad (7.13)$$

Using Stokes' theorem we have:

$$\tilde{X}_u = \int_{\Gamma[[u_0, u] \times [v_0, v_1]]} (B + d\Lambda^H) - \int_{\Gamma^{v_0}[[u_0, u]] \cup \Gamma^{v_1}[[u_0, u]]} (C - \Lambda^H). \quad (7.14)$$

Thus the effect of a gauge transformation  $(\theta, \Lambda^H)$  on the  $B$ -field is the translation

$$B \mapsto B + d\Lambda^H \quad (7.15)$$

and the effect on  $C$  is to transform it to the 1-form  $C - \Lambda^H$ . The effect on the  $B$ -field, in this abelian case, happens to be the same as in a similar transformation in 2-gauge theory; however, even in the abelian case our structure here has the additional term  $C$  whose transformation is part of the description of the gauge transformation.

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