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Central configurations of nested regular polyhedra for the spatial $2n$-body problem

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ABSTRACT

We consider $2n$ masses located at the vertices of two nested regular polyhedra with the same number of vertices. Assuming that the masses in each polyhedron are equal, we prove that for each ratio of the masses of the inner and the outer polyhedra there exists a unique ratio of the length of the edges of the inner and the outer polyhedra such that the configuration is central.

1. Introduction

We consider the N-body problem in the ℓ-dimensional space with $\ell = 2, 3$,

$$m_i \ddot{\mathbf{q}}_i = -\sum_{j=1, j\neq i}^{N} G m_i m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3}, \quad i = 1, \dots, N,$$

where $\mathbf{q}_i \in \mathbb{R}^\ell$ is the position vector of the punctual mass m_i in an inertial coordinate system and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. We fix the center of mass $\sum_{i=1}^{N} m_i \mathbf{q}_i / \sum_{i=1}^{N} m_i$ of the system at the origin of $\mathbb{R}^{\ell N}$. The *configuration space* of the N-body problem in $\mathbb{R}^\ell$ is

$$\mathcal{E} = \left\{ (\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathbb{R}^{\ell N} : \sum_{i=1}^{N} m_i \mathbf{q}_i = 0, \mathbf{q}_i \neq \mathbf{q}_j, \text{ for } i \neq j \right\}.$$

Given $m_1, \dots, m_N$, a configuration $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_i = -\lambda \mathbf{q}_i, \quad i = 1, \dots, N, \tag{1}$$

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that is if the acceleration $\ddot{\mathbf{q}}_i$ of each point mass m_i is proportional to its position $\mathbf{q}_i$ relative to the center of mass of the system and is directed towards the center of mass.

The central configurations of the N-body problem are important because they allow the computation of all the homographic solutions; every motion starting and ending in a total collision is asymptotic to a central configuration, and every parabolic motion of the N bodies (i.e. the N bodies tend to infinity as the time tends to infinity with zero radial velocity) is asymptotic to a central configuration (see [9,2]); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see [10]); etc.

Two central configurations in $\mathbb{R}^\ell$ are in the same *class* if there exist a rotation and a homothecy of $\mathbb{R}^\ell$ which transform one into the other.

The first known central configurations are the three classes of collinear central configurations for the 3-body problem found in 1767 by Euler [3]. In 1772 Lagrange [6] proved that when $N = 3$, for each values of the masses m_1, m_2 and m_3, there are two classes of central configurations with the masses located at the vertices of an equilateral triangle. Those five classes are all the classes of central configurations of the 3-body problem. Only partial results on central configurations are known for $N > 3$.

A central configuration of $\mathbb{R}^\ell$ is called *planar* if the configuration of the N bodies is contained in a plane, and it is called *spatial* if there does not exist a plane containing the configuration of the N bodies.

The simplest known planar central configuration of the N-body problem for $N \geqslant 2$ is obtained by taking N equal masses at the vertices of a regular N-gon. We cannot find in the literature who was the first in discovering such planar central configurations. If we take N equal masses at the vertices of a regular polyhedron with N vertices, then we obtain a spatial central configuration of the N-body problem (see [1]).

A *homographic solution* is a solution of the N-body problem such that at every time the configuration of the N bodies is central. If the central configuration is planar, then there exist three types of homographic solutions, the homothetic, the relative equilibrium and the composition of both. Let $(\mathbf{q}_1, \dots \mathbf{q}_N) \in \mathbb{R}^{2N}$ be a planar central configuration. A *homothetic solution* is of the form $(\varrho(t)\mathbf{q}_1, \dots, \varrho(t)\mathbf{q}_N)$, and a *relative equilibrium solution* is of the form $(A(t)\mathbf{q}_1, \dots, A(t)\mathbf{q}_N)$ where $A(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$. For the spatial central configurations the unique possible homographic solutions are the homothetic ones. For more details on homographic motions see Witner [11].

It is also known the existence of planar central configurations for the $2n$-body problem where the masses are at the vertices of two nested regular n-gons with a common center. In such configurations all the masses on the same n-gon are equal but masses on different n-gons could be different. It seems that the first in studying these nested planar central configurations was Longley [8] in 1907, later on in 1927 and 1929 Bilimovitch (see [4]), and in 1967 Klemplerer [5] also studied them. More recently they have been also studied in [12,13].

We say that two regular polyhedra are *nested* if they have the same number of vertices n, the same center and the positions of the vertices of the inner polyhedron $\mathbf{r}_i$ and the ones of the outer polyhedron $\mathbf{R}_i$ satisfy the relation $\mathbf{R}_i = \rho\mathbf{r}_i$ for some *scale factor* $\rho > 1$ and for all $i = 1, \dots, n$.

In this paper we shall prove that for convenient masses at the vertices of two nested regular polyhedra (see Fig. 1) we get spatial central configurations for the $2n$-body problem in $\mathbb{R}^3$. As in the planar case all the masses located at the vertices of the same polyhedron must be equal, but masses on different polyhedra could be different. There are five regular polyhedra: the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron with 4, 6, 8, 12 and 20 vertices, respectively. Some preliminary results in this direction restricted to the tetrahedron and octahedron can be found in [14,7]. Here we give an unified analytic proof for all five regular polyhedra.

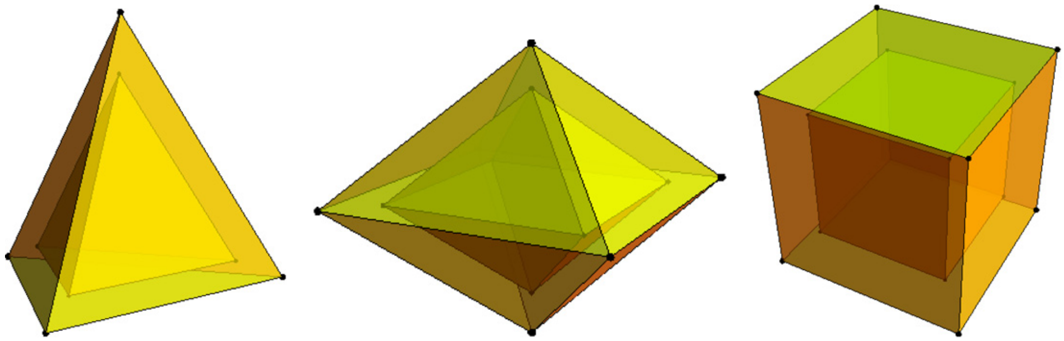
The nested regular tetrahedra (octahedra, cube, icosahedra and dodecahedra) central configurations are characterized in Section 2 (3–6, respectively). The main results of these sections are summarized in the following theorem.

Theorem 1. *We consider* $2n$ *masses at the vertices of two nested regular polyhedra of* n *vertices, where* n *can be either* 4, 6, 8, 12 *or* 20. *Assume that the masses of the inner polyhedron are equal to* m_1 *and the masses of the outer polyhedron are equal to* m_2. *Then given two arbitrary positive values of* m_1 *and* m_2 *there exists a unique value of the scale factor* ρ *of the nested polyhedra for which this configuration is central.*

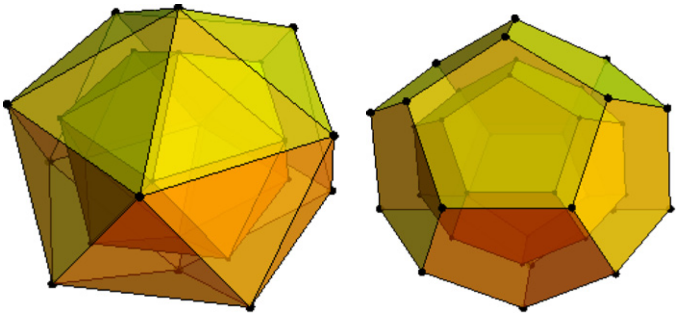
Assume that $\mathbf{q}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, then the equations of the spatial central configurations given by (1) can be written as

$$
\begin{aligned}
ex_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j(x_i - x_j)}{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{3/2}} - \lambda x_i = 0, \\
ey_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j(y_i - y_j)}{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{3/2}} - \lambda y_i = 0, \\
ez_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j(z_i - z_j)}{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)^{3/2}} - \lambda z_i = 0,
\end{aligned}
\tag{2}
$$

for $i = 1, \dots, N$.



(a) Nested regular tetrahedra. (b) Nested regular octahedra. (c) Nested regular cube.



(d) Nested regular icosahedra. (e) Nested regular dodecahedra.

Fig. 1. Nested regular polyhedra.

2. Nested tetrahedra

In this section we study the spatial central configurations of the 8-body problem when the masses are located at the vertices of two nested tetrahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner tetrahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner tetrahedron have length 2. Recall that the set of central configurations is invariant under homothecies.

Proposition 2. *Consider four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ located at the vertices of a regular tetrahedron with edge length 2 having positions $(x_1, y_1, z_1) = (-1, -1/\sqrt{3}, -1/\sqrt{6})$, $(x_2, y_2, z_2) = (1, -1/\sqrt{3}, -1/\sqrt{6})$, $(x_3, y_3, z_3) = (0, 2/\sqrt{3}, -1/\sqrt{6})$, and $(x_4, y_4, z_4) = (0, 0, \sqrt{3/2})$. Consider four additional equal masses $m_5 = m_6 = m_7 = m_8 = m$ at the vertices of a second nested regular tetrahedron having positions $(x_{i+4}, y_{i+4}, z_{i+4}) = \rho(x_i, y_i, z_i)$ for $i = 1, \ldots, 4$ and $\rho > 1$ (see Fig. 1(a)). Then the following statements hold.*

(a) *Such configuration is central for the spatial 8-body problem when*

$$m = f_8(\rho) = \frac{\frac{(2/3)^{3/2}}{(\rho-1)^2} - \frac{\rho}{2} + \frac{2\sqrt{2}(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}}}{-\frac{1/2}{\rho^2} - \frac{(2/3)^{3/2}\rho}{(\rho-1)^2} + \frac{2\sqrt{2}\rho(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}}},$$

and $\rho > \alpha = 1.8899915758445007\ldots$, where α is the unique real solution of $f_8(\rho) = 0$ for $\rho > 1$.

(b) *For a fixed value of $m > 0$ there exists a unique $\rho > \alpha$ for which the nested regular tetrahedra is a central configuration.*

Proof. It is easy to check that in the statement of Proposition 2 the positions and the values of the masses have been taken so that the center of mass of the configuration is located at the origin.

We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_3 = ex_4 = ex_7 = ex_8 = ey_4 = ey_8 = 0$, $ex_1 = -ex_2$, $ex_5 = -ex_6$, $ey_1 = ey_2 = -ex_2/\sqrt{3}$, $ey_3 = 2ex_2/\sqrt{3}$, $ey_5 = ey_6 = -ex_6/\sqrt{3}$, $ey_7 = 2ex_6/\sqrt{3}$, $ez_1 = ez_2 = ez_3 = -ex_2/\sqrt{6}$, $ez_4 = \sqrt{3/2}ex_2$, $ez_5 = ez_6 = ez_7 = -ex_6/\sqrt{6}$ and $ez_8 = \sqrt{3/2}ex_6$. Therefore system (2) is equivalent to system

$$\begin{aligned} ex_2 &= -\lambda + \frac{1}{2} - \frac{(2/3)^{3/2}m}{(\rho-1)^2} + \frac{2\sqrt{2}m(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}} = 0, \\ ex_6 &= -\lambda\rho + \frac{m}{2\rho^2} + \frac{(2/3)^{3/2}}{(\rho-1)^2} + \frac{2\sqrt{2}(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}} = 0. \end{aligned} \tag{3}$$

Solving system (3) with respect to the variables λ and m we get $\lambda = a(\rho)/f(\rho)$ and $m = b(\rho)/f(\rho)$ where

$$a(\rho) = -\frac{8/27}{(\rho-1)^4} - \frac{1/4}{\rho^2} + \frac{8(3\rho^2+10\rho+3)}{(3\rho^2+2\rho+3)^3} - \frac{16/(3\sqrt{3})}{(\rho-1)(3\rho^2+2\rho+3)^{3/2}},$$
$$b(\rho) = \frac{(2/3)^{3/2}}{(\rho-1)^2} - \frac{\rho}{2} + \frac{2\sqrt{2}(3\rho+1)}{(3\rho^2+2\rho+3)^{3/2}},$$
$$f(\rho) = -\frac{1/2}{\rho^2} - \frac{(2/3)^{3/2}\rho}{(\rho-1)^2} + \frac{2\sqrt{2}\rho(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}}.$$

Since $\rho > 1$ and equation $3\rho^2+2\rho+3=0$ has no real solutions, $a(\rho)$, $b(\rho)$ and $f(\rho)$ are well defined for $\rho > 1$. The function $f(\rho)$ has no real zeros when $\rho > 1$. Indeed $f(\rho)$ can be written as

$$\frac{36\sqrt{2}(\rho-1)^2\rho^3(\rho+3) - \left(3\rho^2+2\rho+3\right)^{3/2}\left(4\sqrt{6}\rho^3+9\rho^2-18\rho+9\right)}{18(\rho-1)^2\rho^2\left(3\rho^2+2\rho+3\right)^{3/2}}.$$

So $f(\rho)=0$ if and only if

$$36\sqrt{2}(\rho-1)^2\rho^3(\rho+3) = \left(3\rho^2+2\rho+3\right)^{3/2}\left(4\sqrt{6}\rho^3+9\rho^2-18\rho+9\right).$$

We transform this equation into a polynomial one by squaring both sides of the equality. Solving numerically the resulting polynomial equation we see that there are no real solutions of $f(\rho)=0$ with $\rho > 1$, in particular $f(\rho) < 0$ for $\rho > 1$. Therefore λ and m are well defined for $\rho > 1$.

The solution of (3) gives a central configuration of the 8-body problem if and only if $\lambda > 0$ and $m > 0$. Next we analyze the sign of λ and m for $\rho > 1$. The function $a(\rho)$ can be written as

$$a(\rho) = -\frac{a_1(\rho)+a_2(\rho)}{108(\rho-1)^4\rho^2\left(3\rho^2+2\rho+3\right)^{9/2}},$$

where

$$a_1(\rho) = 192\sqrt{3}(\rho-1)^3\rho^2\left(3\rho^2+2\rho+3\right)^3,$$
$$\begin{aligned} a_2(\rho) \ = \ & \left(3\rho^2+2\rho+3\right)^{3/2}\left(729\rho^{10}-1458\rho^9-27\rho^8-216\rho^7+24\,642\rho^6\right. \\ & \left. -\ 30\,956\rho^5+24\,642\rho^4-216\rho^3-27\rho^2-1458\rho+729\right). \end{aligned}$$

We solve numerically the polynomial equation $(a_1(\rho))^2 = (-a_2(\rho))^2$ and we see that it has no real solutions. Therefore λ is always different from zero. In particular it is positive for $\rho > 1$.

Finally $b(\rho)$ can be written as

$$\frac{36\sqrt{2}(\rho-1)^2(3\rho+1) - \left(3\rho^2+2\rho+3\right)^{3/2}\left(9\rho^3-18\rho^2+9\rho-4\sqrt{6}\right)}{18(\rho-1)^2\left(3\rho^2+2\rho+3\right)^{3/2}}.$$

Solving numerically the equation

$$\left(36\sqrt{2}(\rho-1)^2(3\rho+1)\right)^2 = \left(3\rho^2+2\rho+3\right)^3\left(9\rho^3-18\rho^2+9\rho-4\sqrt{6}\right)^2,$$

we see that it has only two real solutions with $\rho > 1$, these solutions are $\rho = 1.6903479049860676\ldots$ and $\rho = \alpha = 1.8899915758445007\ldots$, but $\rho = \alpha$ is the unique one that satisfies equation $b(\rho) = 0$. Furthermore, $b(\rho) > 0$ for $1 < \rho < \alpha$ and $b(\rho) < 0$ for $\rho > \alpha$, so $m < 0$ for $1 < \rho < \alpha$, and $m > 0$ for $\rho > \alpha$. This proves statement (a).

In order to prove statement (b) it is sufficient to prove that m is an increasing function of ρ for $\rho > \alpha$. The derivative of m with respect to ρ is

$$\frac{dm}{d\rho} = \frac{1}{f(\rho)^2}\left(\frac{db}{d\rho}(\rho)f(\rho) - b(\rho)\frac{df}{d\rho}(\rho)\right) \tag{4}$$

where

$$\frac{db}{d\rho}(\rho) = -\frac{12\sqrt{2}\left(3\rho^2+2\rho-1\right)}{\left(3\rho^2+2\rho+3\right)^{5/2}} - \frac{2(2/3)^{3/2}}{(\rho-1)^3} - \frac{1}{2},$$
$$\frac{df}{d\rho}(\rho) = \frac{(2/3)^{3/2}(\rho+1)}{(\rho-1)^3} - \frac{2\sqrt{2}\left(3\rho^3+17\rho^2-3\rho-9\right)}{\left(3\rho^2+2\rho+3\right)^{5/2}} + \frac{1}{\rho^3}.$$

We have seen that $f(\rho) < 0$ and $b(\rho) < 0$ for $\rho > \alpha$. Since $3\rho^2 + 2\rho - 1 > 0$ for $\rho > 1$, $\mathrm{d}b/\mathrm{d}\rho$ is negative for $\rho > 1$. Finally, we solve equation $\mathrm{d}f/\mathrm{d}\rho(\rho) = 0$ by proceeding in a similar way as for the resolution of equation $f(\rho) = 0$ and we see that this equation has no real zeros for $\rho > 1$. In particular, $\mathrm{d}f/\mathrm{d}\rho(\rho) > 0$ for $\rho > 1$. Therefore $\mathrm{d}m/\mathrm{d}\rho > 0$ for $\rho > \alpha$. This proves statement (b). □

3. Nested octahedra

In this section we study the spatial central configurations of the 12-body problem when the masses are located at the vertices of two nested octahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner octahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner octahedron have length 2.

Proposition 3. *Consider six equal masses $m_i = 1$ for $i = 1, \dots, 6$ at the vertices of a regular octahedron with edge length 2 having positions $(x_1, y_1, z_1) = (1, 0, 0)$, $(x_2, y_2, z_2) = (-1, 0, 0)$, $(x_3, y_3, z_3) = (0, 1, 0)$, $(x_4, y_4, z_4) = (0, -1, 0)$, $(x_5, y_5, z_5) = (0, 0, 1)$, $(x_6, y_6, z_6) = (0, 0, -1)$. Consider six additional equal masses $m_i = m$ for $i = 7, \dots, 12$ at the vertices of a second nested regular octahedron having positions $(x_{i+6}, y_{i+6}, z_{i+6}) = \rho(x_i, y_i, z_i)$ for $i = 1, \dots, 6$ and $\rho > 1$ (see Fig. 1(b)). Then the following statements hold.*

(a) *Such configuration is central for the spatial 12-body problem when*

$$m = f_{12}(\rho) = \frac{\frac{4\rho}{(\rho^2+1)^{3/2}} - \frac{(1+4\sqrt{2})\rho}{4} + \frac{2(\rho^2+1)}{(\rho^2-1)^2}}{-\frac{4\rho^2}{(\rho^2-1)^2} + \frac{4\rho}{(\rho^2+1)^{3/2}} - \frac{1+4\sqrt{2}}{4\rho^2}},$$

and $\rho > \alpha = 1.7298565115043054\dots$, where α is the unique real solution of $f_{12}(\rho) = 0$ for $\rho > 1$.

(b) *For a fixed value of $m > 0$ there exists a unique $\rho > \alpha$ for which the nested regular octahedra is a central configuration.*

Proof. It is easy to check that the center of mass of the configuration defined in Proposition 3 is at the origin. We substitute the positions and the values of the masses into (2). After some computations we get that $ex_3 = ex_4 = ex_5 = ex_6 = ex_9 = ex_{10} = ex_{11} = ex_{12} = ey_1 = ey_2 = ey_5 = ey_6 = ey_7 = ey_8 = ey_{11} = ey_{12} = ez_1 = ez_2 = ez_3 = ez_4 = ez_7 = ez_8 = ez_9 = ez_{10} = 0$, $ey_3 = ez_5 = ex_1$, $ex_2 = ey_4 = ez_6 = -ex_1$, $ey_9 = ez_{11} = ex_7$, and $ex_8 = ey_{10} = ez_{12} = -ex_7$. Therefore system (2) is equivalent to system

$$\begin{aligned} ex_1 &= \frac{4m}{\left(\rho^2+1\right)^{3/2}} - \frac{4\rho m}{\left(\rho^2-1\right)^2} - \lambda + \sqrt{2} + \frac{1}{4} = 0, \\ ex_7 &= \frac{\left(1/4+\sqrt{2}\right)m}{\rho^2} - \lambda\rho + \frac{2\left(\rho^2+1\right)}{\left(\rho^2-1\right)^2} + \frac{4\rho}{\left(\rho^2+1\right)^{3/2}} = 0. \end{aligned} \tag{5}$$

Solving system (5) with respect to the variables λ and m we get $\lambda = a(\rho)/f(\rho)$ and $m = b(\rho)/f(\rho)$ where

$$\begin{aligned} a(\rho) &= \frac{16\rho}{\left(\rho^2+1\right)^3} - \frac{8\left(\rho^3+\rho\right)}{\left(\rho^2-1\right)^4} - \frac{8}{\left(\rho^2-1\right)\left(\rho^2+1\right)^{3/2}} - \frac{33+8\sqrt{2}}{16\rho^2}, \\ b(\rho) &= \frac{4\rho}{\left(\rho^2+1\right)^{3/2}} - \frac{\left(1+4\sqrt{2}\right)\rho}{4} + \frac{2\left(\rho^2+1\right)}{\left(\rho^2-1\right)^2}, \\ f(\rho) &= -\frac{4\rho^2}{\left(\rho^2-1\right)^2} + \frac{4\rho}{\left(\rho^2+1\right)^{3/2}} - \frac{1+4\sqrt{2}}{4\rho^2}. \end{aligned}$$

Since $\rho > 1$, $a(\rho)$, $b(\rho)$ and $f(\rho)$ are well defined for $\rho > 1$. The function $f(\rho)$ can be written as

$$f(\rho) = \frac{f_1(\rho) - f_2(\rho)}{4(\rho-1)^2\rho^2(\rho+1)^2\left(\rho^2+1\right)^{3/2}},$$

where

$$\begin{aligned} f_1(\rho) &= 16\rho^3\left(\rho^2-1\right)^2, \\ f_2(\rho) &= \left(\rho^2+1\right)^{3/2}\left(\left(17+4\sqrt{2}\right)\rho^4 - 2\left(1+4\sqrt{2}\right)\rho^2 + 1 + 4\sqrt{2}\right). \end{aligned}$$

Solving numerically the polynomial equation $(f_1(\rho))^2 = (f_2(\rho))^2$ we see that $f(\rho) = 0$ has no real solutions when $\rho > 1$, in particular $f(\rho) < 0$ for $\rho > 1$. Therefore λ and m are well defined for $\rho > 1$.

We can write

$$a(\rho)=-\frac{a_1(\rho)+a_2(\rho)}{16(\rho-1)^4\rho^2(\rho+1)^4\left(\rho^2+1\right)^{9/2}},$$

where

$$\begin{aligned}a_1(\rho)&=128\rho^2\left(\rho^4-1\right)^3,\\ a_2(\rho)&=\left(\rho^2+1\right)^{3/2}\left(\left(33+8\sqrt{2}\right)\rho^{14}-\left(33+8\sqrt{2}\right)\rho^{12}-128\rho^{11}\right.\\ &\quad-3\left(33+8\sqrt{2}\right)\rho^{10}+1536\rho^9+3\left(33+8\sqrt{2}\right)\rho^8-768\rho^7\\ &\quad\left.+3\left(33+8\sqrt{2}\right)\rho^6+1536\rho^5-3\left(33+8\sqrt{2}\right)\rho^4-128\rho^3-\left(33+8\sqrt{2}\right)\rho^2+33+8\sqrt{2}\right).\end{aligned}$$

Solving numerically the polynomial equation $(a_1(\rho))^2=(-a_2(\rho))^2$ we see that $a(\rho)$ has no real solutions for $\rho>1$. Therefore λ is different from zero for $\rho>1$. In particular it is positive for $\rho>1$.

Finally we have

$$b(\rho)=\frac{b_1(\rho)-b_2(\rho)}{4(\rho-1)^2(\rho+1)^2\left(\rho^2+1\right)^{3/2}}$$

where

$$\begin{aligned}b_1(\rho)&=16\rho\left(\rho^2-1\right)^2,\\ b_2(\rho)&=\left(\rho^2+1\right)^{3/2}\left(\left(1+4\sqrt{2}\right)\rho^5-2\left(1+4\sqrt{2}\right)\rho^3-8\rho^2+4\sqrt{2}\rho+\rho-8\right).\end{aligned}$$

We solve numerically equation $(b_1(\rho))^2=(b_2(\rho))^2$ and we see that it has only two real roots with $\rho>1$, they are $\rho=1.5419308914910530\ldots$ and $\rho=\alpha=1.7298565115043054\ldots$, but $\rho=\alpha$ is the unique one that satisfies equation $b(\rho)=0$. Furthermore, $b(\rho)>0$ for $1<\rho<\alpha$ and $b(\rho)<0$ for $\rho>\alpha$, so $m<0$ for $1<\rho<\alpha$, and $m>0$ for $\rho>\alpha$. This proves statement (a).

In order to prove statement (b) we proceed as in Section 2. The derivative $\mathrm{d}m/\mathrm{d}\rho$ is given by (4) where

$$\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}\rho}(\rho)&=-\frac{4\rho\left(\rho^2+3\right)}{\left(\rho^2-1\right)^3}-\frac{8\rho^2-4}{\left(\rho^2+1\right)^{5/2}}-\sqrt{2}-\frac{1}{4},\\ \frac{\mathrm{d}f}{\mathrm{d}\rho}(\rho)&=\frac{4-8\rho^2}{\left(\rho^2+1\right)^{5/2}}+\frac{8\left(\rho^3+\rho\right)}{\left(\rho^2-1\right)^3}+\frac{1+4\sqrt{2}}{2\rho^3}.\end{aligned}$$

As above $f(\rho)<0$ and $b(\rho)<0$ for $\rho>\alpha$. Since $8\rho^2-4>0$ for $\rho>1$, $\mathrm{d}b/\mathrm{d}\rho$ is negative for $\rho>1$. We solve equation $\mathrm{d}f/\mathrm{d}\rho(\rho)=0$ by proceeding in a similar way as for the resolution of equation $f(\rho)=0$ and we get that it has no real solutions for $\rho>1$, in particular $\mathrm{d}f/\mathrm{d}\rho(\rho)>0$ for $\rho>1$. Therefore m is increasing for $\rho>\alpha$. This proves statement (b). □

4. Nested cube

In this section we study the spatial central configurations of the 16-body problem when the masses are located at the vertices of two nested cubes. Taking conveniently the unit of masses we can assume that all the masses of the inner cube are equal to one. We also choose the unit of length in such a way that the edges of the inner cube have length 2.

Proposition 4. *Consider eight equal masses $m_i=1$ for $i=1,\ldots,8$ at the vertices of a regular cube with edge length 2 having positions $(x_1,y_1,z_1)=(1,1,1)$, $(x_2,y_2,z_2)=(1,1,-1)$, $(x_3,y_3,z_3)=(1,-1,1)$, $(x_4,y_4,z_4)=(-1,1,1)$, $(x_5,y_5,z_5)=(1,-1,-1)$, $(x_6,y_6,z_6)=(-1,1,-1)$, $(x_7,y_7,z_7)=(-1,-1,1)$, and $(x_8,y_8,z_8)=(-1,-1,-1)$. Consider eight additional equal masses $m_i=m$ for $i=9,\ldots,16$ at the vertices of a second nested regular cube having positions $(x_{i+8},y_{i+8},z_{i+8})=\rho(x_i,y_i,z_i)$ for $i=1,\ldots,8$ and $\rho>1$ (see Fig. 1(c)). Then the following statements hold.*

(a) *Such configuration is central for the spatial 16-body problem when $m=b(\rho)/f(\rho)$ where*

$$b(\rho)=-\frac{1}{72}\left(18+9\sqrt{2}+2\sqrt{3}\right)\rho+\frac{2\left(\rho^2+1\right)}{3\sqrt{3}\left(\rho^2-1\right)^2}+\frac{3\rho-1}{\left(3\rho^2-2\rho+3\right)^{3/2}}+\frac{3\rho+1}{\left(3\rho^2+2\rho+3\right)^{3/2}},$$

$$f(\rho)=-\frac{18+9\sqrt{2}+2\sqrt{3}}{72\rho^2}-\frac{4\rho^2}{3\sqrt{3}\left(\rho^2-1\right)^2}-\frac{(\rho-3)\rho}{\left(3\rho^2-2\rho+3\right)^{3/2}}+\frac{(\rho+3)\rho}{\left(3\rho^2+2\rho+3\right)^{3/2}},$$

and $\rho>\alpha=1.643646762940176\ldots$ where α is the unique real solution of $b(\rho)=0$ for $\rho>1$.

(b) *For a fixed value of $m > 0$ there exists a unique $\rho > \alpha$ for which the nested regular cube is a central configuration.*

Proof. It is easy to check that the center of mass of the configuration defined in Proposition 4 is at the origin. We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_2 = ex_3 = ex_5 = ey_1 = ey_2 = ey_4 = ey_6 = ez_1 = ez_3 = ez_4 = ez_7 = ex_1$, $ex_4 = ex_6 = ex_7 = ex_8 = ey_3 = ey_5 = ey_7 = ey_8 = ez_2 = ez_5 = ez_6 = ez_8 = -ex_1$, $ex_{10} = ex_{11} = ex_{13} = ey_9 = ey_{10} = ey_{12} = ey_{14} = ez_9 = ez_{11} = ez_{12} = ez_{15} = ex_9$, and $ex_{12} = ex_{14} = ex_{15} = ex_{16} = ey_{11} = ey_{13} = ey_{15} = ey_{16} = ez_{10} = ez_{13} = ez_{14} = ez_{16} = -ex_9$. Therefore system (2) is equivalent to system

$$\begin{aligned} ex_1 &= \frac{1}{72}\left(18+9\sqrt{2}+2\sqrt{3}\right) - \lambda - \frac{4m\rho}{3\sqrt{3}\left(\rho^2-1\right)^2} - \frac{m(\rho-3)}{\left(3\rho^2-2\rho+3\right)^{3/2}} + \frac{m(\rho+3)}{\left(3\rho^2+2\rho+3\right)^{3/2}} = 0, \\ ex_9 &= \frac{\left(18+9\sqrt{2}+2\sqrt{3}\right)m}{72\rho^2} - \lambda\rho + \frac{2\left(\rho^2+1\right)}{3\sqrt{3}\left(\rho^2-1\right)^2} + \frac{3\rho-1}{\left(3\rho^2-2\rho+3\right)^{3/2}} + \frac{3\rho+1}{\left(3\rho^2+2\rho+3\right)^{3/2}} = 0. \end{aligned} \tag{6}$$

Solving system (6) with respect to the variables λ and m we get $\lambda = a(\rho)/f(\rho)$ and $m = b(\rho)/f(\rho)$ where

$$\begin{aligned} a(\rho) = &-\frac{83+54\sqrt{2}+12\sqrt{3}+6\sqrt{6}}{864\rho^2} - \frac{8\left(\rho^3+\rho\right)}{27\left(\rho^2-1\right)^4} + \frac{3\rho^2+10\rho+3}{\left(3\rho^2+2\rho+3\right)^3} - \frac{3\rho^2-10\rho+3}{\left(3\rho^2-2\rho+3\right)^3} \\ &- \frac{2(\rho+3)}{3\sqrt{3}\left(\rho^2-1\right)\left(3\rho^2-2\rho+3\right)^{3/2}} + \frac{2(\rho-3)}{3\sqrt{3}\left(\rho^2-1\right)\left(3\rho^2+2\rho+3\right)^{3/2}} \\ &+ \frac{16\rho}{\left(3\rho^2-2\rho+3\right)^{3/2}\left(3\rho^2+2\rho+3\right)^{3/2}}. \end{aligned}$$

Since $\rho > 1$ and equations $3\rho^2+2\rho+3 = 0$ and $3\rho^2-2\rho+3 = 0$ have no real solutions, $a(\rho)$, $b(\rho)$ and $f(\rho)$ are well defined for $\rho > 1$. Next we find the real zeros of $f(\rho)$ when $\rho > 1$. The function $f(\rho)$ can be written as

$$f_1(\rho)/\left(72(\rho-1)^2\rho^2(\rho+1)^2\left(3\rho^2-2\rho+3\right)^{3/2}\left(3\rho^2+2\rho+3\right)^{3/2}\right)$$

where $f_1(\rho)$ is given by

$$\begin{aligned} &72\left(\rho^2-1\right)^2\rho^3\left[(\rho+3)\left(3\rho^2-2\rho+3\right)^{3/2} - (\rho-3)\left(3\rho^2+2\rho+3\right)^{3/2}\right] \\ &\quad - \left(3\rho^2-2\rho+3\right)^{3/2}\left(3\rho^2+2\rho+3\right)^{3/2}\left(\left(18+9\sqrt{2}+34\sqrt{3}\right)\rho^4\right. \\ &\quad \left. - 2\left(18+9\sqrt{2}+2\sqrt{3}\right)\rho^2 + 18+9\sqrt{2}+2\sqrt{3}\right). \end{aligned}$$

Then $f(\rho) = 0$ if and only $f_1(\rho) = 0$. Notice that equation $f_1(\rho) = 0$ can be written as

$$g_1(\rho)\sqrt{G_1(\rho)} + g_2(\rho)\sqrt{G_2(\rho)} = g_3(\rho)\sqrt{G_1(\rho)G_2(\rho)}.$$

Squaring both sides of this equation we get

$$g_1(\rho)^2G_1(\rho) + g_2(\rho)^2G_2(\rho) - g_3(\rho)^2G_1(\rho)G_2(\rho) = -2g_1(\rho)g_2(\rho)\sqrt{G_1(\rho)G_2(\rho)}.$$

Squaring again both sides of the last equation we get a polynomial equation. We solve it numerically and we see that it has a unique real solution $\rho = 4.26968682884071\ldots$ with $\rho > 1$ which is not a solution of the initial equation $f_1(\rho) = 0$. Therefore $f(\rho) = 0$ has no real solutions with $\rho > 1$ and consequently λ and m are well defined for $\rho > 1$. Moreover $f(\rho) < 0$ for $\rho > 1$.

Repeating the same arguments for $b(\rho)$ we see that $b(\rho)$ has a unique real zero when $\rho > 1$ which is given by $\rho = \alpha = 1.643646762940176\ldots$. Furthermore, $b(\rho) > 0$ for $1 < \rho < \alpha$ and $b(\rho) < 0$ for $\rho > \alpha$, so $m < 0$ for $1 < \rho < \alpha$, and $m > 0$ for $\rho > \alpha$.

Let $a_1(\rho)$ be the numerator of $a(\rho)$. It is easy to check that $a_1(\rho) = 0$ can be written as

$$g_1(\rho)\sqrt{G_1(\rho)} + g_2(\rho)\sqrt{G_2(\rho)} = g_3(\rho)\sqrt{G_1(\rho)G_2(\rho)} + g_4(\rho).$$

Squaring both sides of this equation we get

$$\begin{aligned} &g_1(\rho)^2G_1(\rho) + g_2(\rho)^2G_2(\rho) - g_3(\rho)^2G_1(\rho)G_2(\rho) - g_4(\rho)^2 \\ &\quad = -(2g_1(\rho)g_2(\rho) + 2g_3(\rho)g_4(\rho))\sqrt{G_1(\rho)G_2(\rho)}. \end{aligned}$$

Squaring again both sides of the last equation we get a polynomial equation. We solve it numerically and we see that it has no real solutions with $\rho > 1$. Therefore $a(\rho)$ is different from zero for $\rho > 1$. This proves statement (a).

Now we prove statement (b). The derivative $\mathrm{d}m/\mathrm{d}\rho$ is given by (4) where

$$\frac{\mathrm{d}b}{\mathrm{d}\rho}(\rho)=-\frac{4\rho\left(\rho^2+3\right)}{3\sqrt{3}\left(\rho^2-1\right)^3}-\frac{18\rho^2-12\rho-6}{\left(3\rho^2-2\rho+3\right)^{5/2}}-\frac{6\left(3\rho^2+2\rho-1\right)}{\left(3\rho^2+2\rho+3\right)^{5/2}}-\frac{1}{72}\left(18+9\sqrt{2}+2\sqrt{3}\right),$$
$$\frac{\mathrm{d}f}{\mathrm{d}\rho}(\rho)=\frac{8\rho\left(\rho^2+1\right)}{3\sqrt{3}\left(\rho^2-1\right)^3}-\frac{3\rho^3+17\rho^2-3\rho-9}{\left(3\rho^2+2\rho+3\right)^{5/2}}+\frac{3\rho^3-17\rho^2-3\rho+9}{\left(3\rho^2-2\rho+3\right)^{5/2}}+\frac{18+9\sqrt{2}+2\sqrt{3}}{36\rho^3}.$$

We have seen that $f(\rho)<0$ and $b(\rho)<0$ for $\rho>\alpha$. Since $18\rho^2-12\rho-6>0$ and $3\rho^2+2\rho-1>0$ for $\rho>1$, $\mathrm{d}b/\mathrm{d}\rho$ is negative for $\rho>1$. Finally, we solve equation $\mathrm{d}f/\mathrm{d}\rho(\rho)=0$ by proceeding in a similar way as for the resolution of equation $f(\rho)=0$ and we get that it has no real zeros for $\rho>1$, in particular $\mathrm{d}f/\mathrm{d}\rho(\rho)>0$ for $\rho>1$. Therefore $\mathrm{d}m/\mathrm{d}\rho>0$ for $\rho>\alpha$. This proves statement (b). □

5. Nested icosahedra

In this section we study the spatial central configurations of the 24-body problem when the masses are located at the vertices of two nested icosahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner icosahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner icosahedron have length 2.

Proposition 5. *Consider twelve equal masses $m_i=1$ for $i=1,\dots,12$ located at the vertices of a regular icosahedron with edge length 2 having positions $(x_1,y_1,z_1)=(0,1,\phi)$, $(x_2,y_2,z_2)=(0,1,-\phi)$, $(x_3,y_3,z_3)=(0,-1,\phi)$, $(x_4,y_4,z_4)=(0,-1,-\phi)$, $(x_5,y_5,z_5)=(1,\phi,0)$, $(x_6,y_6,z_6)=(1,-\phi,0)$, $(x_7,y_7,z_7)=(-1,\phi,0)$, $(x_8,y_8,z_8)=(-1,-\phi,0)$, $(x_9,y_9,z_9)=(\phi,0,1)$, $(x_{10},y_{10},z_{10})=(\phi,0,-1)$, $(x_{11},y_{11},z_{11})=(-\phi,0,1)$, and $(x_{12},y_{12},z_{12})=(-\phi,0,-1)$, where $\phi=(1+\sqrt{5})/2$ is the golden ratio. Consider twelve additional equal masses $m_i=m$ for $i=12,\dots,24$ at the vertices of a second nested regular icosahedron having positions $(x_{i+12},y_{i+12},z_{i+12})=\rho(x_i,y_i,z_i)$ for $i=1,\dots,12$ and $\rho>1$ (see Fig. 1(d)). Then the following statements hold.*

(a) *Such configuration is central for the spatial 24-body problem when $m=b(\rho)/f(\rho)$ where*

$$b(\rho)=\frac{2\sqrt{5-2\sqrt{5}}\left(\rho^2+1\right)}{5\left(\rho^2-1\right)^2}-\frac{2\sqrt{2}\left(\sqrt{5}-5\rho\right)}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{3/2}}+\frac{2\sqrt{2}\left(5\rho+\sqrt{5}\right)}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{3/2}}-\frac{1}{20}\left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right)\rho,$$
$$f(\rho)=-\frac{4\sqrt{5-2\sqrt{5}}\rho^2}{5\left(\rho^2-1\right)^2}-\frac{2\sqrt{2}\left(\sqrt{5}\rho-5\right)\rho}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{3/2}}+\frac{2\sqrt{2}\left(\sqrt{5}\rho+5\right)\rho}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{3/2}}-\frac{5\sqrt{5}+\sqrt{5-2\sqrt{5}}}{20\rho^2},$$

$\varphi=5+\sqrt{5}$ and $\rho>\alpha=1.549351115672993\dots$ where α is the unique real solution of $b(\rho)=0$.

(b) *For a fixed value of $m>0$ there exists a unique $\rho>\alpha$ for which the nested regular icosahedra is a central configuration.*

Proof. It is easy to check that the center of mass of the configuration defined in Proposition 5 is at the origin. We substitute the positions and the values of the masses into (2). After some computations we get that $ex_1=ex_2=ex_3=ex_4=ex_{13}=ex_{14}=ex_{15}=ex_{16}=ey_9=ey_{10}=ey_{11}=ey_{12}=ey_{21}=ey_{22}=ey_{23}=ey_{24}=ez_5=ez_6=ez_7=ez_8=ez_{17}=ez_{18}=ez_{19}=ez_{20}=0$, $ex_6=ey_1=ey_2=ez_9=ez_{11}=ex_5$, $ex_7=ex_8=ey_3=ey_4=ez_{10}=ez_{12}=-ex_5$, $ex_9=ex_{10}=ey_5=ey_7=ez_1=ez_3=\phi ex_5$, $ex_{11}=ex_{12}=ey_6=ey_8=ez_2=ez_4=-\phi ex_5$, $ex_{18}=ey_{13}=ey_{14}=ez_{21}=ez_{23}=ex_{17}$, $ex_{19}=ex_{20}=ey_{15}=ey_{16}=ez_{22}=ez_{24}=-ex_{17}$, $ex_{21}=ex_{22}=ey_{17}=ey_{19}=ez_{13}=ez_{15}=\phi ex_{17}$, and $ex_{23}=ex_{24}=ey_{18}=ey_{20}=ez_{14}=ez_{16}=-\phi ex_{17}$. Therefore system (2) is equivalent to system

$$\begin{aligned}ex_5&=-\lambda-\frac{2\sqrt{2}m\left(\sqrt{5}\rho-5\right)}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{3/2}}+\frac{2\sqrt{2}m\left(\sqrt{5}\rho+5\right)}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{3/2}}-\frac{4\sqrt{5-2\sqrt{5}}m\rho}{5\left(\rho^2-1\right)^2}+\frac{1}{20}\left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right),\\ ex_{17}&=\frac{\left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right)m}{20\rho^2}-\lambda\rho+\frac{2\sqrt{5-2\sqrt{5}}\left(\rho^2+1\right)}{5\left(\rho^2-1\right)^2}-\frac{2\sqrt{2}\left(\sqrt{5}-5\rho\right)}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{3/2}}+\frac{2\sqrt{2}\left(5\rho+\sqrt{5}\right)}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{3/2}}.\end{aligned}\tag{7}$$

Solving system (7) with respect to the variables λ and m we get $\lambda=a(\rho)/f(\rho)$ and $m=b(\rho)/f(\rho)$ where $a(\rho)$ is given by

$$\frac{8\left(2\sqrt{5}-5\right)\left(\rho^2+1\right)\rho}{25\left(\rho^2-1\right)^4}-\frac{65-\sqrt{5}+5\sqrt{5\left(5-2\sqrt{5}\right)}}{200\rho^2}-\frac{4\sqrt{10-4\sqrt{5}}\left(\sqrt{5}\rho+5\right)}{5\left(\rho^2-1\right)\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{3/2}}$$

$$+\frac{4\sqrt{10-4\sqrt{5}}\left(\sqrt{5}\rho-5\right)}{5\left(\rho^2-1\right)\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{3/2}}-\frac{40\left(\sqrt{5}\rho^2-6\rho+\sqrt{5}\right)}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^3}+\frac{40\left(\sqrt{5}\rho^2+6\rho+\sqrt{5}\right)}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^3}$$
$$+\frac{320\rho}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{3/2}\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{3/2}}.$$

Since $\rho > 1$ and equations $\varphi\rho^2+4\phi\rho+\varphi = 0$, and $\varphi\rho^2-4\phi\rho+\varphi = 0 = 0$ have no real solutions, $a(\rho)$, $b(\rho)$ and $f(\rho)$ are well defined for $\rho > 1$. We proceed as in Section 4 and after doing a lot of tedious computations we prove that $f(\rho) < 0$ for $\rho > 1$, so λ and m are well defined for $\rho > 1$. We also prove that $a(\rho)$ has no real zeros when $\rho > 1$ and that $b(\rho)$ has a unique real zero when $\rho > 1$ which is given by $\rho = \alpha = 1.549351115672993\ldots$. Furthermore, $b(\rho) > 0$ for $1 < \rho < \alpha$ and $b(\rho) < 0$ for $\rho > \alpha$, so $m < 0$ for $1 < \rho < \alpha$, and $m > 0$ for $\rho > \alpha$. This proves statement (a).

Now we prove statement (b). The derivative $\mathrm{d}m/\mathrm{d}\rho$ is given by (4) where

$$\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}\rho}(\rho) &= -\frac{4\sqrt{5-2\sqrt{5}}\rho\left(\rho^2+3\right)}{5\left(\rho^2-1\right)^3}-\frac{4\sqrt{2}\left(5\varphi\rho^2-20\phi\rho-\varphi\right)}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{5/2}}-\frac{4\sqrt{2}\left(5\varphi\rho^2+20\phi\rho-\varphi\right)}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{5/2}}\\ &\quad -\frac{1}{20}\left(5\sqrt{5}+\sqrt{5-2\sqrt{5}}\right),\\ \frac{\mathrm{d}f}{\mathrm{d}\rho}(\rho) &= \frac{8\sqrt{5-2\sqrt{5}}\rho\left(\rho^2+1\right)}{5\left(\rho^2-1\right)^3}+\frac{2\sqrt{2}\left(10\phi\rho^3-9\varphi\rho^2-10\phi\rho+5\varphi\right)}{\left(\varphi\rho^2-4\phi\rho+\varphi\right)^{5/2}}\\ &\quad -\frac{2\sqrt{2}\left(10\phi\rho^3+9\varphi\rho^2-10\phi\rho-5\varphi\right)}{\left(\varphi\rho^2+4\phi\rho+\varphi\right)^{5/2}}+\frac{5\sqrt{5}+\sqrt{5-2\sqrt{5}}}{10\rho^3}.\end{aligned}$$

We have seen that $f(\rho) < 0$ and $b(\rho) < 0$ for $\rho > \alpha$. We solve equations $\mathrm{d}b/\mathrm{d}\rho(\rho) = 0$ and $\mathrm{d}f/\mathrm{d}\rho(\rho) = 0$ by proceeding in a similar way as for the resolution of equation $f(\rho) = 0$ and we get that they have no real solutions with $\rho > 1$. In particular, $\mathrm{d}b/\mathrm{d}\rho(\rho) < 0$ and $\mathrm{d}f/\mathrm{d}\rho(\rho) > 0$ for $\rho > 1$. Therefore $\mathrm{d}m/\mathrm{d}\rho > 0$ for $\rho > \alpha$. This proves statement (b). □

6. Nested dodecahedra

In this section we study the spatial central configurations of the 40-body problem when the masses are located at the vertices of two nested dodecahedra. Taking conveniently the unit of masses we can assume that all the masses of the inner dodecahedron are equal to one. We also choose the unit of length in such a way that the edges of the inner dodecahedron have length 2.

Proposition 6. *Consider twenty equal masses $m_i = 1$ for $i = 1, \ldots, 20$ located at the vertices of a regular dodecahedron with edge length 2 having positions $(x_1, y_1, z_1) = (1, 1, 1)$, $(x_2, y_2, z_2) = (-1, 1, 1)$, $(x_3, y_3, z_3) = (1, -1, 1)$, $(x_4, y_4, z_4) = (1, 1, -1)$, $(x_5, y_5, z_5) = (-1, -1, 1)$, $(x_6, y_6, z_6) = (-1, 1, -1)$, $(x_7, y_7, z_7) = (1, -1, -1)$, $(x_8, y_8, z_8) = (-1, -1, -1)$, $(x_9, y_9, z_9) = (0, 1/\phi, \phi)$, $(x_{10}, y_{10}, z_{10}) = (0, -1/\phi, \phi)$, $(x_{11}, y_{11}, z_{11}) = (0, 1/\phi, -\phi)$, $(x_{12}, y_{12}, z_{12}) = (0, -1/\phi, -\phi)$, $(x_{13}, y_{13}, z_{13}) = (1/\phi, \phi, 0)$, $(x_{14}, y_{14}, z_{14}) = (-1/\phi, \phi, 0)$, $(x_{15}, y_{15}, z_{15}) = (1/\phi, -\phi, 0)$, $(x_{16}, y_{16}, z_{16}) = (-1/\phi, -\phi, 0)$, $(x_{17}, y_{17}, z_{17}) = (\phi, 0, 1/\phi)$, $(x_{18}, y_{18}, z_{18}) = (-\phi, 0, 1/\phi)$, $(x_{19}, y_{19}, z_{19}) = (\phi, 0, -1/\phi)$, and $(x_{20}, y_{20}, z_{20}) = (-\phi, 0, -1/\phi)$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Consider twenty additional equal masses $m_i = m$ for $i = 21, \ldots, 40$ at the vertices of a second nested regular dodecahedron having positions $(x_{i+20}, y_{i+20}, z_{i+20}) = \rho(x_i, y_i, z_i)$ for $i = 1, \ldots, 20$ and $\rho > 1$ (see Fig. 1(e)). Then the following statements hold.*

(a) *Such configuration is central for the spatial 40-body problem when $m = b(\rho)/f(\rho)$ where*

$$\begin{aligned}b(\rho) &= -\frac{1}{36}\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right)\rho+\frac{2\left(\rho^2+1\right)}{3\sqrt{3}\left(\rho^2-1\right)^2}+\frac{6\rho-2}{\left(3\rho^2-2\rho+3\right)^{3/2}}+\frac{6\rho+2}{\left(3\rho^2+2\rho+3\right)^{3/2}}\\ &\quad +\frac{6\phi\rho-\sqrt{5}-5}{2\phi\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}}+\frac{3\rho+\sqrt{5}}{\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}},\\ f(\rho) &= -\frac{4\rho^2}{3\sqrt{3}\left(\rho^2-1\right)^2}-\frac{2(\rho-3)\rho}{\left(3\rho^2-2\rho+3\right)^{3/2}}+\frac{2(\rho+3)\rho}{\left(3\rho^2+2\rho+3\right)^{3/2}}\\ &\quad -\frac{\left(\left(5+\sqrt{5}\right)\rho-6\phi\right)\rho}{2\phi\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}}+\frac{\left(\left(5+\sqrt{5}\right)\rho+6\phi\right)\rho}{2\phi\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}}-\frac{18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}}{36\rho^2}\end{aligned}$$

and $\rho > \alpha = 1.462226054217616\ldots$ where α is the unique real solution of $b(\rho) = 0$.

(b) *For a fixed value of $m > 0$ there exists a unique $\rho > \alpha$ for which the nested regular dodecahedra is a central configuration.*

Proof. It is easy to check that the center of mass of the configuration defined in Proposition 6 is at the origin. We substitute the positions and the values of the masses into (2). After some computations we obtain that $ex_9 = ex_{10} = ex_{11} = ex_{12} = ex_{29} = ex_{30} = ex_{31} = ex_{32} = ey_{17} = ey_{18} = ey_{19} = ey_{20} = ey_{37} = ey_{38} = ey_{39} = ey_{40} = ez_{13} = ez_{14} = ez_{15} = ez_{16} = ez_{33} = ez_{34} = ez_{35} = ez_{36} = 0$, $ex_3 = ex_4 = ex_7 = ey_1 = ey_2 = ey_4 = ey_6 = ez_1 = ez_2 = ez_3 = ez_5 = ex_1$, $ex_2 = ex_5 = ex_6 = ex_8 = ey_3 = ey_5 = ey_7 = ey_8 = ez_4 = ez_6 = ez_7 = ez_8 = -ex_1$, $ex_{13} = ex_{15} = ey_9 = ey_{11} = ez_{17} = ez_{18} = ex_1/\phi$, $ex_{14} = ex_{16} = ey_{10} = ey_{12} = ez_{19} = ez_{20} = -ex_1/\phi$, $ex_{17} = ex_{19} = ey_{13} = ey_{14} = ez_9 = ez_{10} = \phi ex_1$, $ex_{18} = ex_{20} = ey_{15} = ey_{16} = ez_{11} = ez_{12} = -\phi ex_1$, $ex_{23} = ex_{24} = ex_{27} = ey_{21} = ey_{22} = ey_{24} = ey_{26} = ez_{21} = ez_{22} = ez_{23} = ez_{25} = ex_{21}$, $ex_{22} = ex_{25} = ex_{26} = ex_{28} = ey_{23} = ey_{25} = ey_{27} = ey_{28} = ez_{24} = ez_{26} = ez_{27} = ez_{28} = -ex_{21}$, $ex_{33} = ex_{35} = ey_{29} = ey_{31} = ez_{37} = ez_{38} = ex_{21}/\phi$, $ex_{34} = ex_{36} = ey_{30} = ey_{32} = ez_{39} = ez_{40} = -ex_{21}/\phi$, $ex_{37} = ex_{39} = ey_{33} = ey_{34} = ez_{29} = ez_{30} = \phi ex_{21}$ and $ex_{38} = ex_{40} = ey_{35} = ey_{36} = ez_{31} = ez_{32} = -\phi ex_{21}$. Therefore system (2) is equivalent to system

$$\begin{aligned}
ex_1 &= -\lambda - \frac{m\left(\left(5+\sqrt{5}\right)\rho - 6\phi\right)}{2\phi\left(3\rho^2 - 2\sqrt{5}\rho + 3\right)^{3/2}} + \frac{m\left(\left(5+\sqrt{5}\right)\rho + 6\phi\right)}{2\phi\left(3\rho^2 + 2\sqrt{5}\rho + 3\right)^{3/2}} \\
&\quad - \frac{2m(\rho-3)}{(3\rho^2-2\rho+3)^{3/2}} + \frac{2m(\rho+3)}{(3\rho^2+2\rho+3)^{3/2}} - \frac{4m\rho}{3\sqrt{3}\left(\rho^2-1\right)^2} + \frac{1}{36}\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right), \\
ex_{21} &= -\lambda\rho + \frac{2\left(\rho^2+1\right)}{3\sqrt{3}\left(\rho^2-1\right)^2} + \frac{6\phi\rho - \sqrt{5} - 5}{2\phi\left(3\rho^2 - 2\sqrt{5}\rho + 3\right)^{3/2}} \\
&\quad + \frac{3\rho+\sqrt{5}}{\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}} + \frac{6\rho-2}{(3\rho^2-2\rho+3)^{3/2}} + \frac{6\rho+2}{(3\rho^2+2\rho+3)^{3/2}} + \frac{\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right)m}{36\rho^2}.
\end{aligned} \tag{8}$$

Solving system (8) with respect to the variables λ and m we get $\lambda = a(\rho)/f(\rho)$ and $m = b(\rho)/f(\rho)$ where

$$\begin{aligned}
a(\rho) &= \frac{4(\rho-3)}{3\sqrt{3}\left(\rho^2-1\right)\left(3\rho^2+2\rho+3\right)^{3/2}} + \frac{4\left(3\rho^2+10\rho+3\right)}{\left(3\rho^2+2\rho+3\right)^3} - \frac{4\left(3\rho^2-2\left(1+2\sqrt{5}\right)\rho+3\right)}{\phi\left(3\rho^2+2\rho+3\right)^{3/2}\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}} \\
&\quad + \frac{4\left(3\rho^2+\left(2+4\sqrt{5}\right)\rho+3\right)}{\phi\left(3\rho^2-2\rho+3\right)^{3/2}\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}} - \frac{2\left(3\left(3+\sqrt{5}\right)\rho^2-2\left(7+5\sqrt{5}\right)\rho+3\left(3+\sqrt{5}\right)\right)}{\phi\left(3\rho^2-2\rho+3\right)^{3/2}\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}} \\
&\quad + \frac{2\left(3\left(3+\sqrt{5}\right)\rho^2+2\left(7+5\sqrt{5}\right)\rho+3\left(3+\sqrt{5}\right)\right)}{\phi\left(3\rho^2+2\rho+3\right)^{3/2}\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}} + \frac{3\left(5+\sqrt{5}\right)\rho^2+28\phi\rho+3\left(5+\sqrt{5}\right)}{2\phi\left(3\rho^2+2\sqrt{5}\rho+3\right)^3} \\
&\quad - \frac{3\left(5+3\sqrt{5}\right)\rho^2-14\left(3+\sqrt{5}\right)\rho+9\sqrt{5}+15}{\left(3+\sqrt{5}\right)\left(3\rho^2-2\sqrt{5}\rho+3\right)^3} - \frac{8\left(\rho^3+\rho\right)}{27\left(\rho^2-1\right)^4} - \frac{4(\rho+3)}{3\sqrt{3}\left(\rho^2-1\right)\left(3\rho^2-2\rho+3\right)^{3/2}} \\
&\quad - \frac{\left(5+\sqrt{5}\right)\rho+6\phi}{3\sqrt{3}\phi\left(\rho^2-1\right)\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}} + \frac{\left(5+\sqrt{5}\right)\rho-6\phi}{3\sqrt{3}\phi\left(\rho^2-1\right)\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}} \\
&\quad + \frac{8\rho}{\left(3\rho^2-2\sqrt{5}\rho+3\right)^{3/2}\left(3\rho^2+2\sqrt{5}\rho+3\right)^{3/2}} + \frac{64\rho}{\left(3\rho^2-2\rho+3\right)^{3/2}\left(3\rho^2+2\rho+3\right)^{3/2}} \\
&\quad - \frac{149+54\sqrt{2}+6\sqrt{3}+54\sqrt{5}+3\sqrt{6}+27\sqrt{10}+3\sqrt{15}}{216\rho^2} - \frac{4\left(3\rho^2-10\rho+3\right)}{\left(3\rho^2-2\rho+3\right)^3}.
\end{aligned}$$

Since $\rho > 1$ and equations $3\rho^2 + 2\sqrt{5}\rho + 3 = 0$, $3\rho^2 - 2\sqrt{5}\rho + 3 = 0$, $3\rho^2 + 2\rho + 3 = 0$, and $3\rho^2 - 2\rho + 3 = 0$ have no real solutions, $a(\rho)$, $b(\rho)$ and $f(\rho)$ are well defined for $\rho > 1$. We solve equations $a(\rho) = 0$, $b(\rho) = 0$ and $f(\rho) = 0$ in a similar way as in the previous sections (see the Appendix) and we see that $f(\rho)$ and $a(\rho)$ are negative for $\rho > 1$, and that there exists $\alpha = 1.462226054217616\ldots$ such that $b(\rho)$ is negative for $\rho > \alpha$, and positive for $1 < \rho < \alpha$. Therefore $\lambda > 0$ for $\rho > 1$, $m < 0$ for $1 < \rho < \alpha$, and $m > 0$ for $\rho > \alpha$. This proves statement (a).

Now we prove statement (b). The derivative $\mathrm{d}m/\mathrm{d}\rho$ is given by (4) where

$$\begin{aligned}\frac{\mathrm{d}b}{\mathrm{d}\rho}(\rho) \;=\; & -\frac{4\rho\left(\rho^2+3\right)}{3\sqrt{3}\left(\rho^2-1\right)^3}-\frac{36\rho^2-24\rho-12}{\left(3\rho^2-2\rho+3\right)^{5/2}}-\frac{12\left(3\rho^2+2\rho-1\right)}{\left(3\rho^2+2\rho+3\right)^{5/2}}\\ & -\frac{3\left(6\phi\rho^2-2\left(5+\sqrt{5}\right)\rho+2\phi\right)}{\phi\left(3\rho^2-2\sqrt{5}\rho+3\right)^{5/2}}-\frac{6\left(3\rho^2+2\sqrt{5}\rho+1\right)}{\left(3\rho^2+2\sqrt{5}\rho+3\right)^{5/2}}-\frac{1}{36}\left(18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}\right),\\ \frac{\mathrm{d}f}{\mathrm{d}\rho}(\rho) \;=\; & \frac{8\rho\left(\rho^2+1\right)}{3\sqrt{3}\left(\rho^2-1\right)^3}-\frac{2\left(3\rho^3+17\rho^2-3\rho-9\right)}{\left(3\rho^2+2\rho+3\right)^{5/2}}+\frac{2\left(3\rho^3-17\rho^2-3\rho+9\right)}{\left(3\rho^2-2\rho+3\right)^{5/2}}+\frac{18+9\sqrt{2}+\sqrt{3}+9\sqrt{5}}{18\rho^3}\\ & -\frac{3\left(5+\sqrt{5}\right)\rho^3+26\phi\rho^2-3\left(5+\sqrt{5}\right)\rho-18\phi}{2\phi\left(3\rho^2+2\sqrt{5}\rho+3\right)^{5/2}}\\ & +\frac{3\left(5+\sqrt{5}\right)\rho^3-26\phi\rho^2-3\left(5+\sqrt{5}\right)\rho+18\phi}{2\phi\left(3\rho^2-2\sqrt{5}\rho+3\right)^{5/2}}.\end{aligned}$$

We have seen that $f(\rho) < 0$ and $b(\rho) < 0$ for $\rho > \alpha$. We solve numerically equations $\mathrm{d}b/\mathrm{d}\rho(\rho) = 0$ and $df/d\rho(\rho) = 0$ (see the Appendix) and we get that they have no real solutions with $\rho > 1$. In particular, $\mathrm{d}b/\mathrm{d}\rho(\rho) < 0$ and $\mathrm{d}f/\mathrm{d}\rho(\rho) > 0$ for $\rho > 1$. Therefore $\mathrm{d}m/\mathrm{d}\rho > 0$ for $\rho > \alpha$. This proves statement (b). □

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Appendix

In this appendix we analyze the resolution of equations of the form $F(\rho) = 0$ when F is a rational function containing radicals. These type of equations are solved by the following steps.

(1) We eliminate the fractions by multiplying equation $F(\rho) = 0$ by the least common denominator of $F(\rho)$.
(2) We eliminate the radicals of the resulting equation by isolating in a convenient way one or more radicals on one side of the equation and squaring both sides of the equation. If the resulting equation still contains radicals, then we repeat the process again. In the end we obtain a polynomial equation.
(3) We find numerically all the solutions of the polynomial equation obtained in step (2).
(4) We check which of these solutions are really solutions of the initial equation $F(\rho) = 0$.

Next we detail how to group the radicals in step (2) for each type of equation that appears in this work.

(a) Equations with one radical: $\alpha_1\sqrt{a} + \alpha_2 = 0$. We eliminate the radicals by applying step (2) in the following way

$$(\alpha_1\sqrt{a})^2 = (-\alpha_2)^2.$$

(b) Equations of the form $\alpha_1\sqrt{a} + \alpha_2\sqrt{b} + \alpha_3\sqrt{a}\sqrt{b} = 0$. Applying step (2) in the following way we obtain an equation with one radical

$$(\alpha_1\sqrt{a} + \alpha_2\sqrt{b})^2 = (-\alpha_3\sqrt{a}\sqrt{b})^2,$$

we obtain an equation with one radical of the form $\beta_1\sqrt{a}\sqrt{b} + \beta_2 = 0$.

(c) Equations of the form $\alpha_1\sqrt{a} + \alpha_2\sqrt{b} + \alpha_3\sqrt{a}\sqrt{b} + \alpha_4 = 0$ can be reduced directly to an equation with one radical by applying step (2) in the following way

$$(\alpha_1\sqrt{a} + \alpha_2\sqrt{b})^2 = (-\alpha_3\sqrt{a}\sqrt{b} - \alpha_4)^2.$$

(d) Equations of the form $\alpha_1\sqrt{b}\sqrt{c}\sqrt{d} + \alpha_2\sqrt{a}\sqrt{c}\sqrt{d} + \alpha_3\sqrt{a}\sqrt{b}\sqrt{d} + \alpha_4\sqrt{a}\sqrt{b}\sqrt{c} + \alpha_5\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d} = 0$. Applying step (2) by grouping the terms in the following way

$$(\alpha_1\sqrt{b}\sqrt{c}\sqrt{d} + \alpha_2\sqrt{a}\sqrt{c}\sqrt{d})^2 = (-\alpha_3\sqrt{a}\sqrt{b}\sqrt{d} - \alpha_4\sqrt{a}\sqrt{b}\sqrt{c} - \alpha_5\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d})^2,$$

we obtain an equation of the form $\beta_1\sqrt{c}+\beta_2\sqrt{d}+\beta_3\sqrt{a}\sqrt{b}+\beta_4\sqrt{c}\sqrt{d}+\beta_5=0$. Applying step (2) to this equation in the following way

$$(\beta_1\sqrt{c}+\beta_2\sqrt{d}+\beta_4\sqrt{c}\sqrt{d}+\beta_5)^2=(-\beta_3\sqrt{a}\sqrt{b})^2,$$

we obtain an equation with three radicals of the form $\gamma_1\sqrt{c}+\gamma_2\sqrt{d}+\gamma_3\sqrt{c}\sqrt{d}+\gamma_4=0$. These types of equations have been analyzed in item (c).

(f) Equations of the form $\alpha_1\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d}+\alpha_2\sqrt{b}\sqrt{c}\sqrt{d}+\alpha_3\sqrt{a}\sqrt{c}\sqrt{d}+\alpha_4\sqrt{a}\sqrt{b}\sqrt{d}+\alpha_5\sqrt{a}\sqrt{b}\sqrt{c}+\alpha_6\sqrt{c}\sqrt{d}+\alpha_7\sqrt{b}\sqrt{d}+\alpha_8\sqrt{b}\sqrt{c}+\alpha_9\sqrt{a}\sqrt{d}+\alpha_{10}\sqrt{a}\sqrt{c}+\alpha_{11}\sqrt{a}\sqrt{b}=0$. We apply step (2) by grouping the terms in the following way

$$\begin{aligned}&(\alpha_2\sqrt{b}\sqrt{c}\sqrt{d}+\alpha_3\sqrt{a}\sqrt{c}\sqrt{d}+\alpha_4\sqrt{a}\sqrt{b}\sqrt{d}+\alpha_5\sqrt{a}\sqrt{b}\sqrt{c})^2\\&\quad=(-\alpha_6\sqrt{c}\sqrt{d}-\alpha_7\sqrt{b}\sqrt{d}-\alpha_8\sqrt{b}\sqrt{c}-\alpha_9\sqrt{a}\sqrt{d}-\alpha_{10}\sqrt{a}\sqrt{c}-\alpha_{11}\sqrt{a}\sqrt{b}-\alpha_1\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d})^2.\end{aligned}$$

The resulting equation is of the form $\beta_1\sqrt{a}\sqrt{b}+\beta_2\sqrt{a}\sqrt{c}+\beta_3\sqrt{a}\sqrt{d}+\beta_4\sqrt{b}\sqrt{c}+\beta_5\sqrt{b}\sqrt{d}+\beta_6\sqrt{c}\sqrt{d}+\beta_7\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d}+\beta_8=0$. We apply step (2) by grouping the terms in the following way

$$(\beta_1\sqrt{a}\sqrt{b}+\beta_2\sqrt{a}\sqrt{c}+\beta_3\sqrt{a}\sqrt{d}+\beta_7\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d})^2=(-\beta_4\sqrt{b}\sqrt{c}-\beta_5\sqrt{b}\sqrt{d}-\beta_6\sqrt{c}\sqrt{d}-\beta_8)^2.$$

We obtain an equation of the form $\gamma_1\sqrt{b}\sqrt{c}+\gamma_2\sqrt{b}\sqrt{d}+\gamma_3\sqrt{c}\sqrt{d}+\gamma_4=0$. Finally, we apply step (2) by grouping the terms in the following way

$$(\gamma_1\sqrt{b}\sqrt{c}+\gamma_3\sqrt{c}\sqrt{d})^2=(-\gamma_2\sqrt{b}\sqrt{d}+\gamma_4)^2.$$

and we obtain an equation of the form $\delta_1\sqrt{b}\sqrt{d}+\delta_2=0$.

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