



The existence of homogeneous geodesics in homogeneous pseudo-Riemannian and affine manifolds

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ABSTRACT

It is well known that, in any homogeneous Riemannian manifold, there is at least one homogeneous geodesic through each point. For the pseudo-Riemannian case, even if we assume reductivity, this existence problem is still open. The standard way to deal with homogeneous geodesics in the pseudo-Riemannian case is to use the so-called “Geodesic Lemma”, which is a formula involving the inner product. We shall use a different approach: namely, we imbed the class of all homogeneous pseudo-Riemannian manifolds into the broader class of all homogeneous *affine* manifolds (possibly with torsion) and we apply a new, purely affine method to the existence problem. In dimension 2, it was solved positively in a previous article by three authors. Our main result says that any homogeneous affine manifold admits at least one homogeneous geodesic through each point. As an immediate corollary, we prove the same result for the subclass of all homogeneous pseudo-Riemannian manifolds.

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1. Introduction

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold*. Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p , then M can be identified with the *homogeneous space* G/H . In general, there may exist more than one such group $G \subset I_0(M)$. For any fixed choice $M = G/H$, G acts effectively on G/H from the left. The pseudo-Riemannian metric g on M can be considered as a G -invariant metric on G/H . The pair $(G/H, g)$ is then called a *pseudo-Riemannian homogeneous space*.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive homogeneous space*. We denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of G and H , respectively, and consider the adjoint representation $\text{Ad}: H \times \mathfrak{g} \rightarrow \mathfrak{g}$ of H on \mathfrak{g} . There exists a direct sum decomposition (the *reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If the metric g is indefinite, the reductive decomposition may not exist (see for instance [1] or [2] for examples of nonreductive pseudo-Riemannian homogeneous spaces). For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_e G$ with the tangent space $T_p M$ via the projection $\pi: G \rightarrow G/H = M$. Using this natural identification and the scalar product g_p on $T_p M$, we obtain a scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{m} . This scalar product is obviously $\text{Ad}(H)$ -invariant.

We start with the definition of a homogeneous geodesic in the pseudo-Riemannian case.

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Definition 1. A geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be *homogeneous* if there exist (1) a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J ; and (2) a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$. The vector X is then called a *geodesic vector*.

For results on homogeneous geodesics in homogeneous Riemannian manifolds we refer the reader to, for example, [3–6]. A homogeneous Riemannian manifold all of whose geodesics are homogeneous is called a *Riemannian g.o. manifold*. For many results and further references on Riemannian g.o. manifolds see, for example, [7–11]. In pseudo-Riemannian geometry, null homogeneous geodesics are of particular interest. In [2,12], the authors study plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics. In these papers, each geodesic vector X is characterized by the formula (1) below. See also [13–16]. A rigorous mathematical proof of this characterization is given in [14].

Lemma 2. Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if

$$([X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}) = k(X_{\mathfrak{m}}, Z) \quad \text{for all } Z \in \mathfrak{m}, \text{ where } k \in \mathbb{R} \text{ is some constant.} \quad (1)$$

Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

For the study of homogeneous geodesics in homogeneous affine manifolds, we cannot use the algebraic tools as in pseudo-Riemannian geometry. In [17], the present author, together with Kowalski and Vlášek, described a new, elementary method for studying homogeneous geodesics in homogeneous affine manifolds.

Definition 3. Let ∇ be an affine connection on a manifold M . Then ∇ , or also (M, ∇) , is said to be *homogeneous* if for each two points $x, y \in M$ there exists an affine transformation $\varphi : M \rightarrow M$ such that $\varphi(x) = y$. This means that φ is a diffeomorphism such that

$$\nabla_{\varphi_* X} \varphi_* Y = \varphi_*(\nabla_X Y)$$

holds for every vector fields X, Y defined on M .

Definition 4. In a homogeneous affine manifold (M, ∇) , by a *homogeneous geodesic* we mean a geodesic which is an orbit of a one-parameter group of affine diffeomorphisms. (Here the canonical parameter of the group need not be the affine parameter of the geodesic.)

The following proposition is well known.

Proposition 5. Let (M, ∇) be a homogeneous affine manifold and $p \in M$. There exist $n = \dim(M)$ affine Killing vector fields which are linearly independent at each point of some neighbourhood \mathcal{U} of p .

The homogeneous affine connections in dimension 2 are completely classified (see [18–20]). In [17], the authors studied these homogeneous affine connections in detail; they found classes of affine g.o. manifolds and they proved that these connections always admit a homogeneous geodesic. In this paper, we are going to prove the existence of a homogeneous geodesic in any homogeneous affine manifold.

2. The main result

First, let us restate the first part of Proposition 2.4. from [17] and recall the proof, which uses the idea of the proof of the generalized Geodesic Lemma for the pseudo-Riemannian situation given in [14].

Proposition 6. Let Z be a Killing vector field without null values defined in a neighbourhood \mathcal{U} of some point $p \in M = (G/H, \nabla)$ and $\gamma(t)$ an integral curve of Z with $\gamma(0) = p$. If the formula

$$\nabla_{Z_{\gamma(t)}} Z = k_{\gamma} \cdot Z_{\gamma(t)} \quad (2)$$

holds for $t = 0$, then it holds with the same k_{γ} for all sufficiently small values of the parameter t , and $\gamma(t)$ is a local geodesic.

Proof. Let g_t be the local one-parameter group of affine diffeomorphisms corresponding to the local Killing vector field Z ; hence

$$Z_x = \left. \frac{d}{dt} \right|_{t=0} g_t(x), \quad \forall x \in M. \quad (3)$$

Clearly, $\gamma(t) = g_t(p)$. Let us choose a small value u of the parameter t . (In the following lines, t and u are always supposed to be sufficiently small.) We obtain directly

$$(dg_u)_p Z_p = (dg_u)_p \left. \frac{d}{dt} \right|_0 g_t(p) = \left. \frac{d}{dt} \right|_0 g_t(g_u(p)) = \left. \frac{d}{dt} \right|_0 g_{u+t}(p) = Z_{g_u(p)}, \quad (4)$$

which means that $Z_{\gamma(u)} = \frac{d\gamma(t+u)}{dt}|_{t=0}$ and hence $Z_{\gamma(t)} = \frac{d\gamma(t)}{dt}$ for all values of t . It is well known that the covariant derivative $\nabla_{Z_{\gamma(u)}}Z$ depends, for each u , only on the values of the vector field Z along the curve $\gamma(t)$. Since g_u is an affine diffeomorphism, we easily get

$$(dg_u)|_{g_t(p)}(\nabla_Z Z|_{g_t(p)}) = \nabla_Z Z|_{g_{t+u}(p)} \quad \text{for arbitrary } t. \quad (5)$$

The formula (2) for arbitrary t follows from the equalities (3)–(5) and the formula (2) for $t = 0$. If k_γ is nonzero, direct calculation easily shows that the curve $\tilde{\gamma}(s)$ defined by $\tilde{\gamma}(\exp(k_\gamma t)) = \gamma(t)$ satisfies the condition $\nabla_{\tilde{\gamma}'(s)}\tilde{\gamma}'(s) = 0$. \square

Theorem 7. Let $M = (G/H, \nabla)$ be a homogeneous affine manifold and $p \in M$. Then there exists a homogeneous geodesic through p .

Proof. According to Proposition 5, let us consider the Killing vector fields K_1, \dots, K_n which are linearly independent at each point of some neighbourhood \mathcal{U} of p and denote by B the basis $\{K_1(p), \dots, K_n(p)\}$ of $T_p M$. Any tangent vector $X \in T_p M$ has coordinates (x_1, \dots, x_n) with respect to the basis B , and it determines the Killing vector field $X^* = x_1 K_1 + \dots + x_n K_n$ and an integral curve γ of X^* through p . We are going to show that there exists a vector $\tilde{X} \in T_p M$ such that the corresponding integral curve is geodesic. Let us consider the sphere S^{n-1} of vectors $X \in T_p M$ whose coordinates (x_1, \dots, x_n) have norm equal to 1 with respect to the Euclidean scalar product. For each $X \in S^{n-1}$, denote by $v(X)$ the covariant derivative $\nabla_{X^*} X^*|_{t=0}$. Obviously, the components of $v(X)$ are quadratic forms in the variables x_1, \dots, x_n with constant coefficients. Further, denote by $t(X)$ the vector $v(X) - \langle v(X), X \rangle X$. Then $t(X) \perp X$ for each $X \in S^{n-1}$. Clearly, the map $X \mapsto t(X)$ defines a smooth tangent vector field on the sphere S^{n-1} .

Assume now that $t(X) \neq 0$ everywhere. Putting $f(X) = t(X)/\|t(X)\|$, we obtain a smooth map $f: S^{n-1} \rightarrow S^{n-1}$ without fixed points. According to a well-known statement from differential topology, the degree of f is $\deg(f) = (-1)^n$. On the other hand, we have $v(X) = v(-X)$, and hence $f(X) = f(-X)$ for each X . If Y is a regular value of f , then the inverse image $f^{-1}(Y)$ consists of an even number of elements. Hence $\deg(f)$ is an even number, which is a contradiction. It follows that there is a vector $\tilde{X} \in T_p M$ such that $t(\tilde{X}) = 0$, and hence $v(\tilde{X}) = k_\gamma \tilde{X}$, where $k_\gamma = \langle v(\tilde{X}), \tilde{X} \rangle$ is a constant. We see immediately that $\nabla_{\tilde{X}^*} \tilde{X}^* = k_\gamma \tilde{X}^*$, and the corresponding integral curve γ is a (local) homogeneous geodesic. It can be uniquely prolonged to a global homogeneous geodesic in the sense of Definition 4. \square

Corollary 8. Let $M = (G/H, g)$ be a homogeneous pseudo-Riemannian manifold (not necessarily reductive) and $p \in M$. Then M admits a homogeneous geodesic through p .

Proof. G is a transitive group of affine diffeomorphisms of M with respect to the Levi-Civita connection ∇ . Then we can use Theorem 7 immediately. \square

Let us mention that the relation between the full isometry group of (M, g) and the full group of affine diffeomorphisms of (M, ∇) is not relevant for the proof.

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