Journal of Geometry and Physics 61 (2011) 1932–1949

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Hamilton–Jacobi diffieties

Luca Vitagliano *

DipMat, Università degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy
Istituto Nazionale di Fisica Nucleare, GC Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy
Istituto Tullio Levi-Civita, via Colacurcio 54, 83050 Santo Stefano del Sole (AV), Italy

ARTICLE INFO

Article history:
Received 1 April 2011
Received in revised form 4 May 2011
Accepted 8 May 2011
Available online 13 May 2011

MSC:
53B15
58A20
70S05

Keywords:
Diffieties
Hamilton–Jacobi theory
Geometry of PDEs
Lagrangian formalism
Hamiltonian formalism
Fiber bundles

ABSTRACT

Diffieties formalize geometrically the concept of differential equations. We introduce and study Hamilton–Jacobi diffieties. They are finite dimensional subdiffieties of a given diffiety and appear to play a special role in the field theoretic version of the geometric Hamilton–Jacobi theory.

0. Introduction

Diffieties [1,2] are geometric objects formalizing, in a coordinate free manner, the concept of (systems of) differential equation(s) (much as varieties formalize, in a coordinate free manner, the concept of algebraic equations). Roughly speaking, a diffiety is a manifold $\mathscr{E}$, of countable dimension, endowed with an involutive distribution $\mathscr{C}$ of finite dimension. Let $(\mathscr{E}, \mathscr{C})$ be a diffiety representing a certain system of PDEs $\mathscr{E}_0$. Then, (locally) maximal integral submanifolds of $(\mathscr{E}, \mathscr{C})$ represent (local) solutions of $\mathscr{E}_0$. Notice that the Frobenius theorem fails for generic infinite dimensional diffieties. In fact, a PDE need not possess solutions with given (admissible) jet or may possess many solutions with the same (admissible) jet. A particularly simple class of PDEs is made up of compatible PDEs of the form

$$\frac{\partial \boldsymbol{y}}{\partial x^i} = f_i(\boldsymbol{x}, \boldsymbol{y}). \tag{1}$$

They are represented by a particularly simple class of diffieties, namely, finite dimensional ones. For such diffieties the Frobenius theorem holds and, in fact, integrable PDEs of the form (1) possess exactly one (germ of a) solution for any (admissible) jet. Given any PDE $\mathscr{E}_0$, one can search for another compatible PDE $\mathscr{Y}$ of the simple form (1) that implies $\mathscr{E}_0$. One may then obtain some solutions of $\mathscr{E}_0$ by integrating such a $\mathscr{Y}$. Geometrically, one can search for finite dimensional

* Corresponding address: DipMat, Università degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy. Tel.: +39 089963329; fax: +39 089963303.

E-mail address: lvitagliano@unisa.it.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2011.05.003

subdiffieties of the diffiety $(\mathscr{E},\mathscr{C})$ representing $\mathscr{E}_0$. In this paper we name as *Hamilton–Jacobi diffieties* such subdiffieties and study their main properties. The choice of this specific name is motivated by the following example.

Consider a system of ordinary (not necessarily autonomous) Euler–Lagrange equations

$$\frac{\partial L}{\partial x^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i}=0,\quad i=1,\ldots,n. \tag{2}$$

Suppose, for simplicity, that the Legendre transform

$$p_i=\frac{\partial L}{\partial \dot{x}^i}(t,x,\dot{x})$$

is invertible and let

$$\dot{x}^i=v^i(t,x,p)$$

be its inverse. Eq. (2) is represented by a suitable diffiety $\mathscr{E}_{EL}$. Let

$$\frac{\mathrm{d}}{\mathrm{d}t}x^i=X^i(t,x) \tag{3}$$

be a first-order equation. It is of the form (1) and, since it has just one independent variable, it is automatically compatible. Therefore, it is represented by a finite dimensional diffiety $\mathscr{Y}$. Moreover, $\mathscr{Y}\subset\mathscr{E}_{EL}$, i.e., solutions of Eq. (3) are solutions of Eq. (2), iff

$$\frac{\partial L}{\partial x^i}(t,x,X)-\frac{\partial^2 L}{\partial \dot{x}^i\partial x^j}(t,x,X)X^j-\frac{\partial^2 L}{\partial \dot{x}^i\partial \dot{x}^j}(t,x,X)\left(\frac{\partial X^j}{\partial t}+X^k\frac{\partial X^j}{\partial x^k}\right)=0. \tag{4}$$

Notice that if $S=S(t,x)$ is a solution of the standard Hamilton–Jacobi (HJ) equation

$$\frac{\partial S}{\partial t}+H(t,x,\partial S/\partial x)=f(t),\qquad H=v^i\frac{\partial L}{\partial \dot{x}^i}(t,x,v)-L(t,x,v), \tag{5}$$

then

$$X^i=v^i(t,x,\partial S/\partial x)$$

is a solution of Eq. (4). On the other hand, let $X^i=X^i(t,x)$ be a solution of Eq. (4). Put

$$T_i=\frac{\partial L}{\partial \dot{x}^i}(t,x,X).$$

If

$$\frac{\partial T_j}{\partial x^i}-\frac{\partial T_i}{\partial x^j}=0, \tag{6}$$

then $T_i=\partial S/\partial x^i$ for a solution $S=S(t,x)$ of the standard HJ Eq. (5). We conclude that the standard HJ equation is basically equivalent to Eq. (4) plus Eq. (6). Therefore, we name Eq. (4) the *generalized HJ equation* (see below; see also [3,4]). According to the above definition, its solutions correspond to *HJ subdiffieties* of $\mathscr{E}_{EL}$. This motivates the choice of the name for these specific diffieties.

Similarly, under suitable integrability conditions, solutions of the field theoretic HJ equation [5] may be interpreted as finite dimensional subdiffieties of the diffiety of field equations. Hence, both the ordinary and the field theoretic HJ theories fit well within the theory of HJ diffieties. In fact, we show below that the geometric HJ theory presented in [3,4] (see also [6] for the nonholonomic case) and generalized to the case of regular field theories in [7], can be naturally generalized to the case of singular (i.e., gauge), higher derivative, field theories. In this generalization, HJ diffieties play a central role.

The paper is divided into seven sections. In Section 1 we review the basic differential geometric constructions used throughout the paper, and collect notation and conventions. In Section 2 we recall the concept of diffiety and briefly review the geometric theory of PDEs and its application to the calculus of variations. In Section 3 we introduce the concept of HJ diffieties and elementary HJ diffieties, and study their relation. In Section 4 we illustrate the general theory by presenting some simple examples of (elementary) HJ subdiffieties of noteworthy diffieties. In Section 5 we review the geometric formulation of higher derivative, Lagrangian and Hamiltonian field theories as defined in [8]. In Section 6 we propose a field theoretic version of the geometric HJ theory of [3,4] and show that it is naturally linked to the theory of (elementary) HJ subdiffieties of the field equations. In Section 7 we present one final example of (elementary) HJ subdiffieties of an Euler–Lagrange equation. The reviews in Sections 1, 2 and 5 are included to make the paper as self-consistent as possible.

1. The differential geometric background

In this section we collect notation, conventions, and the main geometric constructions needed in the paper. Let N be a smooth manifold. If $L\subset N$ is a submanifold, we denote the inclusion by $i_L:L\hookrightarrow N$. We denote by $C^\infty(N)$ the $\mathbb{R}$-algebra of smooth, $\mathbb{R}$-valued functions on N. We will always understand a vector field X on N as a derivation $X : C^\infty(N) \longrightarrow C^\infty(N)$. We denote by $D(N)$ the $C^\infty(N)$-module of vector fields over N, by $\Lambda(M) = \bigoplus_k \Lambda^k(N)$ the graded $\mathbb{R}$-algebra of differential forms over N and by $d : \Lambda(N) \longrightarrow \Lambda(N)$ the de Rham differential. If $F : N_1 \longrightarrow N$ is a smooth map of manifolds, we denote by $F^* : \Lambda(N) \longrightarrow \Lambda(N_1)$ the pull-back via F.

Let $\alpha : A \longrightarrow N$ be an affine bundle (for instance, a vector bundle) and $F : N_1 \longrightarrow N$ a smooth map of manifolds. Let $\mathscr{A}$ be the affine space of smooth sections of α. The affine bundle on N_1 induced by α via F will be denoted by $F^\circ(\alpha) : F^\circ(A) \longrightarrow N$:

$$\begin{tikzcd} F^\circ(A) \arrow[r] \arrow[d, "F^\circ(\alpha)"'] & A \arrow[d, "\alpha"] \\ N_1 \arrow[r, "F"] & N \end{tikzcd},$$

and the space of its section by $F^\circ(\mathscr{A})$. For any section a of α there exists a unique section of $F^\circ(\alpha)$, which we denote by $F^\circ(a)$, such that the diagram

$$\begin{tikzcd} F^\circ(A) \arrow[r] & A \\ N_1 \arrow[r, "F"] \arrow[u, "F^\circ(a)"] & N \arrow[u, "a"'] \end{tikzcd}$$

commutes. If $F : N_1 \longrightarrow N$ is the embedding of a submanifold, we also write $\bullet\,|_F$ for $F^\circ(\bullet)$.

We will often understand the sum over repeated upper–lower indices and multi-indices. Our notation as regards multi-indices is as follows. We will use the capital letters I, J, K for multi-indices. Let n be a positive integer. A multi-index of length k is a k-tuple of indices $I = (i_1, \ldots, i_k)$, $i_1, \ldots, i_k \leq n$. We identify multi-indices differing only by the order of the entries. If I is a multi-index of length k, we put $|I| := k$. Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_h)$ be multi-indices, and i be an index. We denote by IJ (resp. Ii) the multi-index $(i_1, \ldots, i_k, j_1, \ldots, j_h)$ (resp. $(i_1, \ldots, i_k, i)$). We write $\partial^{|I|}/\partial x^I$ for $\partial/\partial x^{i_1} \circ \cdots \circ \partial/\partial x^{i_k}$.

Let $\xi : P \longrightarrow M$ be a fiber bundle. For $k \leq \infty$, we denote by $\xi_k : J^k\xi \longrightarrow M$ the bundle of k-jets of local sections of ξ. For any (local) section $s : M \longrightarrow P$ of π, we denote by $j_k s : M \longrightarrow J^k\xi$ its kth jet prolongation. Let $\ldots, x^i, \ldots$ be coordinates on M and $\ldots, x^i, \ldots, y^a, \ldots$ bundle coordinates on E. We denote by $\ldots, x^i, \ldots, y^a_I, \ldots$ the associated jet coordinates on $J^k\xi$, $|I| \leq k$. For $0 \leq h \leq k \leq \infty$, we denote by $\xi_{k,h} : J^k\xi \longrightarrow J^h\xi$ the canonical projection. We will always understand the monomorphisms $\pi^*_{k,h} : \Lambda(J^h\xi) \longrightarrow \Lambda(J^k\xi)$. For all $k \geq 0$, $\xi_{k+1,k} : J^{k+1}\xi \longrightarrow J^k\xi$ is an affine subbundle of $(\pi_k)_{1,0} : J^1\pi_k \longrightarrow J^k\pi$ and the inclusion $i : J^{k+1}\pi \subset J^1\pi_k$ is locally defined by $i^*(u^\alpha_I)_i = u^\alpha_{Ii}$, $|I| \leq k$.

Let ξ be as above. We denote by $\Lambda_1(P, \xi) = \bigoplus_k \Lambda^k_1(P, \xi) \subset \Lambda(P)$ the differential (graded) ideal in $\Lambda(P)$ made of differential forms on P vanishing when pulled back to fibers of ξ, by $\Lambda_q(P, \xi) = \bigoplus_k \Lambda^k_q(P, \xi)$ its qth exterior power, $q \geq 0$, and by $V\Lambda(P, \xi) = \bigoplus_k V\Lambda^k(P, \xi)$ the quotient differential algebra $\Lambda(P)/\Lambda_1(P, \xi)$, $d^V : V\Lambda(P, \xi) \longrightarrow V\Lambda(P, \xi)$ being its (quotient) differential. By abusing the notation, we also denote by d^V the (quotient) differential in $\Lambda_q(P, \xi)/\Lambda_{q+1}(P, \xi) \simeq V\Lambda(P, \xi) \otimes \Lambda^q_q(P, \xi)$. We denote by $J^\dagger\xi \longrightarrow P$ the *reduced multimomentum bundle* of ξ (see, for instance, [9]). It is the vector bundle over P whose module of sections is $V\Lambda^1(P, \xi) \otimes \Lambda^{n-1}_{n-1}(P, \xi)$. Equivalently, it may be defined as the bundle of affine morphisms from $J^1\xi$ to $\Lambda^n T^*M$.

A connection ∇ in ξ is a section of the first jet bundle $\xi_{1,0} : J^1\xi \longrightarrow P$. We will also interpret ∇ as an element in $\Lambda^1(P) \otimes VD(P, \xi)$, where $VD(P, \xi)$ is the module of ξ-vertical vector fields on P. Put $\ldots, \nabla^a_i := \nabla^*(y^a_i), \ldots$, where $\ldots, y^a_i, \ldots$ are jet coordinates in $J^1\xi$. Then, locally

$$\nabla = (dy^a - \nabla^a_i dx^i) \otimes \frac{\partial}{\partial y^a}.$$

Recall that a (local) section $\sigma : M \longrightarrow P$ is ∇-constant for a connection ∇ iff, by definition, $\nabla \circ \sigma = j_1\sigma$. A connection ∇ in P determines splittings of the exact sequence

$$0 \longrightarrow VD(P, \xi) \longrightarrow D(P) \longrightarrow \xi^\circ(D(M)) \longrightarrow 0, \tag{7}$$

and its dual

$$0 \longleftarrow V\Lambda^1(P, \xi) \longleftarrow \Lambda^1(P) \longleftarrow \Lambda^1_1(P, \xi) \longleftarrow 0. \tag{8}$$

Thus, using ∇ one can lift a vector field X on M to a vector field X^∇ on P transversal to fibers of ξ. Moreover, ∇ determines an isomorphism

$$\Lambda(P) \simeq \bigoplus_{p,q} V\Lambda^p(P, \xi) \otimes \Lambda^q_q(P, \xi),$$

and, in particular, for any p, q, a projection

$$i^{p,q}(\nabla) : \Lambda^{p+q}(P) \longrightarrow V\Lambda^p(P, \xi) \otimes \Lambda^q_q(P, \xi),$$

and an embedding

$$\mathrm{e}^{p,q}(\nabla) : V\Lambda^p(P,\xi) \otimes \Lambda^q_q(P,\xi) \longrightarrow \Lambda^{p+q}(P)$$

taking its values in $\Lambda^{p+q}_q(P,\xi)$. Notice that the "insertions" $i^{p,q}(\nabla)$ are actually pointwise and, therefore, can be restricted to maps. Namely, if $F : P_1 \longrightarrow P$ is a smooth map, and Δ a section of the pull-back $F^\circ(\xi_{1,0}) : F^\circ(J^1\xi) \longrightarrow P_1$, then the element

$$i^{p,q}(\Delta)F^\circ(\omega) \in F^\circ(V\Lambda^p(P,\xi) \otimes_A \Lambda^q_q(P,\xi))$$

is well-defined for every $\omega \in \Lambda^{p+q}(P)$.

Every connection ∇ defines a vector-valued differential 2-form, $R^\nabla \in \Lambda^2_2(P,\xi) \otimes VD(P,\xi)$, called the curvature, via

$$R^\nabla(X,Y) := [X^\nabla, Y^\nabla] - [X,Y]^\nabla, \quad X, Y \in D(M).$$

Locally,

$$R^\nabla = R^a_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \frac{\partial}{\partial y^a}, \qquad R^a_{ij} = \frac{1}{2}(D_i\nabla^a_j - D_j\nabla^a_i) \circ \nabla$$

where $D_i := \partial/\partial x^i + y^a_i\partial/\partial y^a$, $i = 1, \ldots, n$. A connection ∇ is flat iff, by definition, $R^\nabla = 0$. If ∇ is a flat connection in ξ, then P is locally foliated by (local) ∇-constant sections of ξ.

Example 1. Let $\xi : P \longrightarrow M$ be as above and $\sigma : M \longrightarrow P$ a (local) section of ξ. It is sometimes useful to understand $j_1\sigma : M \longrightarrow J^1\xi$ as a section of the pull-back bundle $\xi_{1,0}|_\sigma : J^1\xi|_\sigma \longrightarrow M$. For instance, if $\omega \in \Lambda^{n+1}_{n-1}(P,\xi)$ is a PD Hamiltonian system in ξ in the sense of [10], the *PD Hamilton equations* for σ read

$$i^{1,n}(j_1\sigma)\omega|_\sigma = 0.$$

Let ∇ be a connection in ξ. If ∇ is flat, the de Rham complex of P, $(\Lambda(P), d)$, splits into a bicomplex

$$(V\Lambda(P,\xi) \otimes \Lambda^\bullet_\bullet(P,\xi), \overline{d}_\nabla, d^V), \tag{9}$$

where

$$\overline{d}_\nabla(\omega \otimes \sigma) := (i^{p,q+1}(\nabla) \circ d \circ \mathrm{e}^{p,q}(\nabla))(\omega \otimes \sigma),$$

$\omega \in V\Lambda^p(P,\xi)$ and $\sigma \in \Lambda^q_q(P,\xi)$, $p, q \geq 0$. For every fixed p, the complex $(V\Lambda^p(P,\xi) \otimes \Lambda^\bullet_\bullet(P,\xi), \overline{d}_\nabla)$ is locally acyclic in positive degree.

2. Geometry of PDEs and the calculus of variations

In this section we recall basic facts about the geometric theory of partial differential equations (PDEs). For more details see [11].

Let $\pi : E \longrightarrow M$ be a fiber bundle, with $\dim M = n$, $\dim E = n + m$, and $\ldots, u^\alpha, \ldots$ fiber coordinates in E. Recall that, for all $k \leq \infty$, $J^k\pi$ is endowed with the *Cartan distribution*

$$\mathscr{C}_k : J^k\pi \ni \theta \longmapsto \mathscr{C}_k(\theta) \subset T_\theta J^k\pi,$$

where $\mathscr{C}_k(\theta)$ is defined as follows. Suppose that $\theta = (j_ks)(x)$ for some $x \in M$ and s a local section of π around x. The image of $d_xj_ks : T_xM \longrightarrow T_\theta J^k\pi$ is said to be an *R-plane* at θ. Put

$$\mathscr{C}_k(\theta) := \mathrm{span}\{R\text{-planes at } \theta\}.$$

Now, suppose that $k < \infty$. $\mathscr{C}_k$ is locally spanned by vector fields

$$\ldots, \frac{\partial}{\partial x^i} + \sum_{|I|<k} u^\alpha_{Ii}\frac{\partial}{\partial u^\alpha_I}, \ldots, \frac{\partial}{\partial u^\alpha_J}, \ldots, \quad |J| = k.$$

Local infinitesimal symmetries of $\mathscr{C}_k$ are called *Lie fields*. Every Lie field $Z \in D(J^k\pi)$ can be uniquely lifted to a Lie field $Z_r \in D(J^{k+r}\pi)$. Moreover, according to the Lie Bäcklund Theorem, every Lie field $Z \in D(J^k\pi)$ is the lift of

1. a vector field $Y \in D(E)$ if $m > 1$,
2. a Lie field $Y' \in D(J^1\pi)$ if $m = 1$.

If $Y \in D(E)$ is locally given by $Y = X^i\partial/\partial x^i + Y^\alpha\partial/\partial u^\alpha$, then the Lie field $Y_r \in D(J^r\pi)$ is locally given by

$$Y_r = X^i\left(\frac{\partial}{\partial x^i} + \sum_{|I|<r} u^\alpha_{Ii}\frac{\partial}{\partial u^\alpha_I}\right) + \sum_{|I|<r} D_I(Y^\alpha - u^\alpha_i X^i)\frac{\partial}{\partial u^\alpha_I}, \tag{10}$$

where $D_{j_1\cdots j_s} := D_{j_1} \circ \cdots \circ D_{j_s}$, $D_j := \partial/\partial x^j + u^\alpha_{Ij}\partial/\partial u^\alpha_I$ being the *jth total derivative*, and $j, j_1, \ldots, j_s = 1, \ldots, n$.

Remark 1. The local expression (10) shows, in particular, that, for any $\xi \in T_\theta J^r\pi$, there exists $Y \in D(E)$ such that $\xi = (Y_r)_\theta$.

A kth-order (system of ℓ) PDE(s) on sections of π is an ℓ-codimensional closed submanifold $\mathscr{E}_0 \subset J^k\pi$. For $0 \leq r \leq \infty$ one can define the *rth prolongation of E* as

$$\mathscr{E}_r := \{j_{k+r}(s)(x) \in J^{k+r}\pi \, : \, \mathrm{im} j_k(s) \text{ is tangent to } \mathscr{E}_0 \text{at } j_k(s)(x) \text{ up to the order } r, x \in M\} \subset J^{k+r}\pi.$$

If $\mathscr{E}$ is locally given by

$$\Phi^a(\ldots, x^i, \ldots, u^\alpha_I, \ldots) = 0, \quad a = 1, \ldots, \ell, \quad |I| \leq k,$$

then $\mathscr{E}_r$ is locally given by

$$(D_J \Phi^a)(\ldots, x^i, \ldots, u^\alpha_I, \ldots) = 0, \quad |J| \leq r. \tag{11}$$

In the following we will always assume that

1. $\mathscr{E}_r \longrightarrow M$ is a smooth (closed) subbundle of π_{k+r}, $r \leq \infty$,
2. the (possibly non-surjective) maps $\pi_{k+r+1,k+r} : \mathscr{E}_{r+1} \longrightarrow \mathscr{E}_r$, $r < \infty$, have constant rank.

A local section s of π is a *(local) solution of* $\mathscr{E}_0$ iff, by definition, $\mathrm{im} j_k s \subset \mathscr{E}_0$ or, which is the same, $\mathrm{im} j_{k+r} s \subset \mathscr{E}_r$ for some $r \leq \infty$. Lie fields in $D(J^k\pi)$ preserving $\mathscr{E}_0$ are called *Lie symmetries* of $\mathscr{E}_0$. The flow of a Lie symmetry maps (images of kth prolongations of) solutions to (images of kth prolongations of) solutions.

The Cartan distribution $\mathscr{C} := \mathscr{C}_\infty \subset TJ^\infty\pi$ is locally spanned by total derivatives, D_i, $i = 1, \ldots, n$, and restricts to any submanifold $\mathscr{E} \subset J^\infty\pi$ of the form $\mathscr{E} = \mathscr{E}_\infty$ for some PDE $\mathscr{E}_0$. Denote again by $\mathscr{C}$ the restricted distribution. It is a flat connection in $\mathscr{E} \longrightarrow M$ sometimes called the *Cartan connection*. $\mathscr{C}$-constant sections are of the form $j_\infty s$, with s a (local) solution of $\mathscr{E}_0$ and vice versa. Therefore, we can identify the space of $\mathscr{C}$-constant sections and the space of solutions of $\mathscr{E}_0$. The pair $(\mathscr{E}, \mathscr{C})$ is called a(n elementary) *diffiety* [1,2] and contains all the relevant information about the original PDE $\mathscr{E}_0$. We will often identify $(\mathscr{E}, \mathscr{C})$ and $\mathscr{E}_0$.

Remark 2. A diffiety $(\mathscr{E}, \mathscr{C})$ can be generically embedded in many ways in an infinite jet space. Informally speaking, any such embedding corresponds to a choice of dependent variables in the original equation. Properties of a diffiety that do not depend on its embedding in an infinite jet space are referred to as *intrinsic*.

In the following we will indicate $\mathscr{C}^p\Lambda^p := V\Lambda^p(J^\infty\pi, \pi_\infty)$ and $\overline{\Lambda}^q := \Lambda^q_q(J^\infty\pi, \pi_\infty)$. We also put $\mathscr{C}^\bullet\Lambda := \bigoplus_p \mathscr{C}^p\Lambda^p$ and $\overline{\Lambda} := \bigoplus_q \overline{\Lambda}^q$. The Cartan connection endows the de Rham complex $(\Lambda(J^\infty\pi), d)$ of $J^\infty\pi$ with a bicomplex structure $(\mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}, \overline{d}, d^V)$, where $\overline{d} := d_{\mathscr{C}}$, called the *variational bicomplex*. The variational bicomplex allows a cohomological formulation of the calculus of variations [11–13] (see below). Similarly, the de Rham complex $(\Lambda(\mathscr{E}), d)$ of a diffiety $\mathscr{E}$ is naturally endowed with a bicomplex structure denoted by $(\mathscr{C}^\bullet\Lambda(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}, d^V)$.

In the following we will understand the isomorphism $\Lambda(J^\infty) \simeq \mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}$. The complex

$$0 \longrightarrow C^\infty(J^\infty) \xrightarrow{\overline{d}} \overline{\Lambda}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \overline{\Lambda}^q \xrightarrow{\overline{d}} \overline{\Lambda}^{q+1} \xrightarrow{\overline{d}} \cdots$$

is called the *horizontal de Rham complex*. An element $\mathscr{L} \in \overline{\Lambda}^n$ is naturally interpreted as a *Lagrangian density* and its cohomology class $[\mathscr{L}] \in H^n(\overline{\Lambda}, \overline{d})$ as an *action functional* on sections of π. The associated Euler–Lagrange equations can then be obtained as follows.

Consider the complex

$$0 \longrightarrow \mathscr{C}\Lambda^1 \xrightarrow{\overline{d}} \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q \xrightarrow{\overline{d}} \cdots, \tag{12}$$

and the $C^\infty(J^\infty\pi)$-submodule $\varkappa^\dagger \subset \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ generated by elements in $\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n \cap \Lambda^{n+1}(J^1\pi)$. $\varkappa^\dagger$ is locally spanned by elements $d^V u^\alpha \otimes d^n x$, $d^n x := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

Theorem 1 (*[12]*). *Complex (12) is acyclic in the qth term, for $q \neq n$. Moreover, for any $\omega \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ there exists a unique element $\boldsymbol{E}_\omega \in \varkappa^\dagger \subset \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q$ such that $\boldsymbol{E}_\omega - \omega = \overline{d}\vartheta$ for some $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ and the correspondence $H^n(\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}, \overline{d}) \ni [\omega] \longmapsto \boldsymbol{E}_\omega \in \varkappa^\dagger$ is a vector space isomorphism. In particular, for $\omega = d^V \mathscr{L}$, with $\mathscr{L} \in \overline{\Lambda}^n$ being a Lagrangian density locally given by $\mathscr{L} = L d^n x$, and L a local function on $C^\infty(J^\infty\pi)$, $\boldsymbol{E}(\mathscr{L}) := \boldsymbol{E}_\omega$ is locally given by $\boldsymbol{E}(\mathscr{L}) = \delta L / \delta u^\alpha d^V u^\alpha \otimes d^n x$ where*

$$\frac{\delta L}{\delta u^\alpha} := (-)^{|I|} D_I \frac{\partial L}{\partial u^\alpha_I}$$

are the Euler–Lagrange derivatives of L.

In view of the above theorem, $\boldsymbol{E}(\mathscr{L})$ does not depend on the choice of $\mathscr{L}$ in a cohomology class $[\mathscr{L}] \in H^n(\overline{\Lambda}, \overline{d})$ and it is naturally interpreted as the left hand side of the Euler–Lagrange (EL) equations determined by $\mathscr{L}$. Any $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ such that

$$\boldsymbol{E}(\mathscr{L}) - d^V\mathscr{L} = \overline{\mathrm{d}}\vartheta \tag{13}$$

will be called a *Legendre form* [14]. Eq. (13) may be interpreted as the *first variation formula* for the Lagrangian density $\mathscr{L}$. A local Legendre form is given by

$$\vartheta_{\mathrm{loc}} = (-)^{|J|} \binom{IJ}{J} D_J \frac{\partial L}{\partial u^\alpha_{IJi}} d^V u^\alpha_I \otimes d^{n-1}x_i,$$

where $\binom{IJ}{J}$ is the multinomial coefficient, and $d^{n-1}x_i := i_{D_i} d^n x$.

Remark 3. Notice that, if $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ is a Legendre form for a Lagrangian density $\mathscr{L} \in \overline{\Lambda}^n$, then $\vartheta + d^V\varrho$ is a Legendre form for the cohomologous Lagrangian density $\mathscr{L} + \overline{d}\varrho$, $\varrho \in \overline{\Lambda}^{n-1}$. Moreover, any two Legendre forms ϑ, ϑ' for the same Lagrangian density differ by a $\overline{d}$-closed, and, therefore, $\overline{d}$-exact form, i.e., $\vartheta - \vartheta' = \overline{d}\lambda$, for some $\lambda \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-2}$. Finally, for $\mathscr{L} \in \Lambda(J^{k+1}\pi)$ one can always find a Legendre form ϑ containing vertical derivatives of functions in $C^\infty(J^k\pi)$ only.

Remark 4. Complex (12) restricts to any diffiety $\mathscr{E}$ in the sense that there is a (unique) complex

$$0 \longrightarrow \mathscr{C}\Lambda^1|_{\mathscr{E}} \xrightarrow{\overline{d}|_{\mathscr{E}}} \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^1|_{\mathscr{E}} \xrightarrow{\overline{d}|_{\mathscr{E}}} \cdots \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q|_{\mathscr{E}} \xrightarrow{\overline{d}|_{\mathscr{E}}} \cdots, \tag{14}$$

such that the restriction map $\mathscr{C}\Lambda^1 \otimes \overline{\Lambda} \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}|_{\mathscr{E}}$ is a morphism of complexes. Moreover, complex (14) is acyclic in the qth term and the correspondence defined by $H^n(\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n|_{\mathscr{E}}, \overline{d}|_{\mathscr{E}}) \ni [\omega|_{\mathscr{E}}] \longmapsto \boldsymbol{E}_\omega|_{\mathscr{E}} \in \varkappa^\dagger|_{\mathscr{E}}$, $\omega \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$, is a vector space isomorphism.

3. Hamilton–Jacobi diffieties

In the following we simply write J^k for $J^k\pi$, $k \leq \infty$.

Definition 1. Let $\mathscr{E}$ be a diffiety. A finite dimensional diffiety $\mathscr{Y}$ will be called an *Hamilton–Jacobi (HJ) diffiety*. If $\mathscr{Y} \subset \mathscr{E}$, then $\mathscr{Y}$ will be called an *HJ subdiffiety* of $\mathscr{E}$.

Motivations for this definition can be found in the introduction and in Section 6 (see also Example 2). From an intrinsic point of view an HJ diffiety is nothing but a (finite dimensional) manifold with an involutive distribution, or, which is the same, a foliation. As recalled in the introduction, the equation for the leaves of the foliation is a compatible equation of the form (1).

A diffiety $\mathscr{E}$ is often presented together with an embedding $\mathscr{E} \subset J^\infty$. In this case, it is useful to restrict the attention to a special class of HJ subdiffieties of $\mathscr{E}$ that we define below.

A connection $\nabla : J^k \longrightarrow J^1\pi_k$ in $\pi_k : J^k \longrightarrow M$ will be said to be *holonomic* if it takes its values in $J^{k+1} \subset J^1\pi_k$. ∇ is holonomic iff locally

$$\begin{aligned} (\nabla^\alpha_I)_i &= u^\alpha_{Ii} && \text{if } |I| < k \\ (\nabla^\alpha_I)_i &= (\nabla^\alpha_J)_j && \text{if } |I| = |J| = k \text{ and } Ii = Jj. \end{aligned}$$

Let ∇ be a holonomic connection in π_k. For $|I| = k$, put $\nabla^\alpha_{Ii} := (\nabla^\alpha_I)_i$. Since ∇ is holonomic, the ∇^α_{Ii}'s are well-defined. Moreover, local ∇-constant sections are of the form $j_ks : M \longrightarrow J^k$, for some local section s of π. Notice that ∇ is flat iff, locally,

$$[\nabla_i, \nabla_j] = 0, \tag{15}$$

where

$$\nabla_i = \frac{\partial}{\partial x^i} + \sum_{|I|<k} u^\alpha_{Ii} \frac{\partial}{\partial u^\alpha_I} + \sum_{|I|=k} \nabla^\alpha_{Ii} \frac{\partial}{\partial u^\alpha_I}.$$

Eq. (15) can be rewritten as

$$\nabla_i \nabla^\alpha_{jI} - \nabla_j \nabla^\alpha_{iI} = \nabla^*(D_i \nabla^\alpha_{jI} - D_j \nabla^\alpha_{iI}) = 0, \quad |I| = k,$$

For $f \in C^\infty(J^k)$ and $I = i_1 \cdots i_s$, put

$$\nabla_I f := \nabla_{i_1} \cdots \nabla_{i_s} f \in C^\infty(J^k). \tag{16}$$

Definition (16) is a good one since ∇ is flat. In particular $\nabla_I u^\alpha = u^\alpha_I$, for $|I| < k$.

Now, let ∇ be a flat holonomic connection in π_k. Then for any $\theta \in J^k$ there is a locally unique (local section) s_θ of π such that (1) $\theta = [s_\theta]^k_x$, $x = \pi_k(\theta)$ and (2) $j_k s$ is a ∇-constant section of π_k. Define a map $\nabla_{[r]} : J^k \longrightarrow J^{k+r}$, $0 \leq r \leq \infty$, by putting

$$\nabla_{[r]}(\theta) := [s_\theta]^{k+r}_x, \quad x = \pi_k(\theta).$$

Proposition 2. *$\nabla_{[r]}$ is a smooth (closed) embedding locally given by*

$$\nabla^*_{[r]}(u^\alpha_I) = \nabla_I u^\alpha, \quad |I| \leq k + r.$$

Proof. $\nabla_{[r]}$ is clearly a section of the projection $\pi_{k+r,k}$. Now, suppose that $\nabla^*_{[r]}(u^\alpha_J) = \nabla_J u^\alpha$ for some $|J| > k$, $|J| < k + r$, and suppose that $\theta \in J^k$. Put $x = \pi_k(\theta)$. Then

$$\begin{aligned}
\nabla^*_{[r]}(u^\alpha_{Ji})(\theta) &= u^\alpha_{Ji}([s_\theta]^{k+r}_x) \\
&= \frac{\partial^{|J|+1} s_\theta}{\partial x^J \partial x^i}(x) \\
&= \left.\frac{\partial}{\partial x^i}\right|_x \frac{\partial^{|J|} s_\theta}{\partial x^J} \\
&= \left.\frac{\partial}{\partial x^i}\right|_x \nabla^*_{[r]}(u^\alpha_J) \circ j_{|J|} s_\theta \\
&= (D_i \nabla_J u^\alpha \circ j_{|J|+1} s_\theta)(x) \\
&= (D_i \nabla_J u^\alpha \circ j_{k+1} s_\theta)(x) \\
&= (D_i \nabla_J u^\alpha \circ \nabla)(\theta) \\
&= \nabla^*(D_i \nabla_J u^\alpha)(\theta) \\
&= (\nabla_{Ji} u^\alpha)(\theta).
\end{aligned}$$

By induction on $|J|$ the proposition follows. □

Corollary 3. *The vector fields ∇_i and $D_i \in D(J^\infty \pi)$ are $\nabla^*_{[\infty]}$-related, i.e., $\nabla^*_{[\infty]} \circ D_i = \nabla_i \circ \nabla^*_{[\infty]}$.*

Proof. Compute

$$(\nabla^*_{[\infty]} \circ D_i)(x^j) = \delta^j_i = (\nabla_i \circ \nabla^*_{[\infty]})(x^j)$$

and

$$\begin{aligned}
(\nabla^*_{[\infty]} \circ D_i)(u^\alpha_I) &= \nabla^*_{[\infty]}(u^\alpha_{Ii}) \\
&= \nabla_{Ii} u^\alpha \\
&= \nabla_i(\nabla_I u^\alpha) \\
&= (\nabla_i \circ \nabla^*_{[\infty]})(u^\alpha_I). \quad \square
\end{aligned}$$

One can interpret $\mathscr{Y}^\nabla := \mathrm{im}\nabla$ as a PDE on sections of π. Then ∇-constant sections are the lth jets of solutions of $\mathscr{Y}^\nabla$.

Corollary 4. *$\mathscr{Y}^\nabla_r = \mathrm{im}\nabla_{[r]}$ for all $r \leq \infty$.*

Proof. $\mathscr{Y}^\nabla$ is locally given by

$$F^\alpha_I = 0, \quad |I| \leq k,$$

where $F^\alpha_I := u^\alpha_I - \nabla_I u^\alpha$. Therefore, $\mathscr{Y}^\nabla_r$ is locally given by

$$D_J F^\alpha_I = 0, \quad |J| \leq r$$

Now,

$$\begin{aligned}
D_J F^\alpha_I &= D_J u^\alpha_I - D_J \nabla_I u^\alpha \\
&= u^\alpha_{IJ} - (D_J \circ \nabla^*_{[r]})(u^\alpha_I) \\
&= u^\alpha_{IJ} - (\nabla^*_{[r]} \circ \nabla_J)(u^\alpha_I).
\end{aligned}$$

But $u^\alpha_I = \nabla_I u^\alpha$ on $\mathscr{Y}^\nabla_r$. We conclude that $\mathscr{Y}^\nabla_r$ is locally given by

$$F^\alpha_{IJ} = 0, \quad |I| \leq k, \ |J| \leq r. \quad \square$$

Notice that $\mathrm{im}\nabla_{[\infty]} = \mathscr{Y}^\nabla_\infty$ is a HJ subdiffiety of J^∞. In particular, $\mathscr{Y}^\nabla_r$ is foliated by the graphs of the $(k + r)$th jets of ∇-constant sections.

Remark 5. $\nabla_{[\infty]} : (J^k, \nabla) \longrightarrow (\mathscr{Y}^\nabla_\infty, \mathscr{C})$ is an isomorphism of bundles (over M) with (flat) connections.

Now, let $\mathscr{E} \subset J^\infty$ be the infinite prolongation of a kth-order differential equation $\mathscr{E}_0 \subset J^k$ determined by a differential operator $F : J^k \longrightarrow V$, $V \longrightarrow M$ being a vector bundle, i.e., F is a morphism of the bundles π_k and $V \longrightarrow M$ and

$$\mathscr{E}_0 = \{\theta \in J^k : F(\theta) = 0\}.$$

Proposition 5. *Let ∇ be a flat holonomic connection in π_s, $s < k$. Then $\mathscr{Y}^\nabla_\infty \subset \mathscr{E}$ iff*

$$F \circ \nabla_{[k-s-1]} = 0. \tag{17}$$

Proof. The if implication is obvious. To prove the only if implication, notice that if $\mathscr{Y}^\nabla_\infty \subset \mathscr{E}$, then all ∇-constant sections are sth jets of solutions of $\mathscr{E}$. Since $\mathscr{Y}^\nabla_{k-s}$ is foliated by the graphs of k-jets of ∇-constant sections, the assertion follows. □

If $\mathscr{E}_0$ is locally given by

$$F_a(\ldots, x^i, \ldots, u^\alpha_I, \ldots) = 0, \quad |I| \leq k,$$

then (17) is locally given by

$$F_a(\ldots, x^i, \ldots, \nabla_I u^\alpha, \ldots) = 0, \quad |I| \leq k, \tag{18}$$

which is a system of $(k - s - 1)$th-order differential equation for the ∇_{Ji}'s $|J| = s$.

Definition 2. Let ∇ be a flat, holonomic connection in a jet bundle and let $\mathscr{E} \subset J^\infty$ be a diffiety. The HJ diffiety $\mathscr{Y}^\nabla_\infty$ will be called an *elementary Hamilton–Jacobi (HJ) diffiety*. If $\mathscr{Y}^\nabla_\infty \subset \mathscr{E}$, then $\mathscr{Y}^\nabla_\infty$ will be called an *elementary HJ subdiffiety* of $\mathscr{E}$. Eq. (17) for ∇ will be called the (sth, *generalized*) *HJ equation* of $\mathscr{E}$, $s = 0, 1, \ldots, k - 1$.

Example 2. Let $\pi : \mathbb{R} \times \mathbb{R}^n \ni (t, \mathbf{x}) \longmapsto t \in \mathbb{R}$ be the trivial bundle. Consider the Euler–Lagrange Eq. (2) in $J^2\pi$. Let

$$\nabla = (\mathrm{d}x^i - X^i \mathrm{d}t) \otimes \frac{\partial}{\partial x^i}$$

be a connection in π, $X^i = X^i(t, x)$. Since π is a bundle over a one-dimensional manifold, ∇ is automatically flat. Eq. (4) is then the 0th generalized HJ equation of Eq. (2). This motivates the choice of name for Eq. (17).

Remark 6. Notice that the definition of elementary HJ subdiffiety is not intrinsic (see Remark 2) to a given diffiety $\mathscr{E}$, and it actually depends on the embedding $\mathscr{E} \subset J^\infty$. Namely, it is easily seen by dimensional arguments that changing the embedding $\mathscr{E} \subset J^\infty$ could result in the transformation of an elementary HJ subdiffiety into a new HJ subdiffiety which does not correspond to any holonomic connection. This is why, urged by an anonymous referee, we gave Definition 1 as the more fundamental one. However, we will mainly consider the case when $\mathscr{E}$ emerges from a Lagrangian field theory as the diffiety corresponding to the (Euler–Lagrange) field equations. In this case $\mathscr{E}$ comes with a canonical embedding $\mathscr{E} \subset J^\infty = J^\infty\pi$, where sections of π are field configurations, and the use of elementary HJ diffieties is very natural (see Sections 5 and 6 for details).

In the remaining part of this section we briefly discuss the relation between general HJ diffieties and elementary ones.

Proposition 6. *Let $\mathscr{O} \subset J^\infty$ be an HJ diffiety. Then, locally, there exists k such that $\mathscr{O} \subset \mathscr{Y}^\nabla_\infty$ for some flat holonomic connection ∇ in π_k.*

Proof. Let $\mathscr{O} \subset J^\infty$ be an HJ diffiety. $\mathscr{O} = O_\infty$ for some differential equation $O \subset J^{k+1}$. Since $\mathscr{O}$ is finite dimensional, k can be chosen such that $\dim O = \dim O_1 = \cdots = \dim \mathscr{O}$. In particular, for any $\theta \in O$, $\mathscr{C}_k(\theta) \cap T_\theta O$ contains just one R-plane Θ. Therefore, O is $\pi_{k+1,k}$-horizontal (otherwise its tangent space would contain some $\pi_{k+1,k}$-vertical tangent vector and one could find more R-planes in $\mathscr{C}_k(\theta) \cap T_\theta O$). Put $N := \pi_{k+1,k}(O)$. Locally N is a submanifold in J^k such that $\dim N = \dim O$. If $\dim N = \dim J^k$, then O is already the image of a local, flat, holonomic connection in π_k. Thus assume that $\dim N < \dim O$. Suppose that $\theta \in O$ and $\theta' = \pi_{k+1,k}(\theta) \in N$. In view of Remark 1, there exists at least one Lie field $X \in D(J^k)$ such that $X_{\theta'}$ is transverse to N. Thus X is transverse to N locally around θ'. Points in O may be understood as R-planes at points of N [11]. They form an n-dimensional, involutive distribution on N. Transporting both N and the distribution on it along the flow of X, we may produce a new submanifold $N' \subset J^k$ with an involutive distribution on it made of R-planes. It corresponds to a $\pi_{k+1,k}$-horizontal submanifold O' in J^{k+1} locally containing O (in fact O' is obtained by transporting O along the flow of $X_1 \in D(J^{k+1})$). If $\dim N' = \dim J^k$, then O' is already the image of a local, flat, holonomic connection in π_k. Otherwise we may iterate the procedure. In the end we will produce the connection that we are searching for. □

Remark 7. Let $\mathscr{E} \subset J^\infty$ be the infinite prolongation of a kth-order differential equation $\mathscr{E}_0 \subset J^k$ and let $\mathscr{O} \subset \mathscr{E}$ be a finite dimensional subdiffiety. Let $\mathscr{O}$ be the infinite prolongation of a submanifold $O \subset J^s$, $s < k$, with $\dim O = \dim O_1 = \cdots = \dim \mathscr{O}$. Clearly $O_{k-s} \subset \mathscr{E}_0$. Suppose that $r \leq k - s$, ℓ is the codimension of O_r in J^{s+r}, and $\theta \in O_r$. If $\mathscr{E}_0$ possesses ℓ Lie symmetries transverse to O_r at θ, then $\mathscr{O}$ can be locally extended to $\mathscr{Y}^\nabla_\infty$ for some flat holonomic connection ∇ in π_{r-1}, such that $\mathscr{Y}^\nabla_\infty \subset \mathscr{E}$. This can be easily proved along the same lines as in the proof of the previous proposition.

4. Examples of HJ diffieties

Example 3. Consider the Burgers equation

$$u_t = u_{xx} + uu_x.$$

It may be understood as a submanifold in $J^2\pi$ with π the trivial bundle $\pi : \mathbb{R}^3 \ni (t, x, u) \longmapsto (t, x) \in \mathbb{R}$. Let

$$\nabla = (\mathrm{d}u - A\mathrm{d}t - B\mathrm{d}x) \otimes \frac{\partial}{\partial u}$$

be a connection in π, $A = A(t, x, u)$, $B = B(t, x, u)$. ∇ is flat iff

$$B_t - A_x + AB_u - BA_u = 0. \tag{19}$$

The 0th generalized HJ equation reads

$$A = B_x - BB_u - uB. \tag{20}$$

Substituting (20) into (19) we find

$$B_t - B_{xx} - 2BB_{ux} - B^2B_{uu} - uB_x - B^2 = 0. \tag{21}$$

Search for solutions of (21) in the form $B = \alpha(x, t)u$. We must have

$$\alpha_x = -\alpha^2$$
$$\alpha_t = 0$$

and, therefore $\alpha = 1/(x - x_0)$, x_0 being an integration constant. Thus $B = u/(x - x_0)$ and $A = u^2/(x - x_0)$. ∇-constant sections are the solutions of the (compatible) system

$$\begin{cases} u_t = u^2/(x - x_0) \\ u_x = u/(x - x_0) \end{cases},$$

i.e.,

$$u = -\frac{x - x_0}{t - t_0}$$

(t_0 being a new integration constant), which are indeed solutions of the Burgers equation.

Example 4. In the same bundle as in the previous example, consider the heat equation

$$u_t = u_{xx}.$$

Let ∇ be as above. The 0th generalized HJ equation reads

$$A = B_x + BB_u. \tag{22}$$

Substituting (22) into (19) we find

$$B_t - B_{xx} - 2BB_{ux} - B^2B_{uu} = 0. \tag{23}$$

Search for solutions of (21) in the form $B = \alpha(x, t)u$. We must have

$$\alpha_t - \alpha_{xx} - 2\alpha\alpha_x = 0.$$

One solution is $\alpha = \frac{1}{2}\phi$, ϕ being a solution of the Burgers equation. Choose, for instance, $\phi = -x/t$. Then

$$B = -\frac{x}{2t}u,$$
$$A = \frac{x^2 - 2t}{4t^2}u.$$

∇-constant sections are the solutions of the (compatible) system

$$\begin{cases} u_t = \dfrac{x^2 - 2t}{4t^2}u \\ u_x = -\dfrac{x}{2t}u \end{cases},$$

i.e.,

$$u = u_0 \exp\left[-\left(\frac{1}{\sqrt{t}} + \frac{x^2}{4t}\right)\right]$$

(u_0 being a new integration constant), which are indeed solutions of the heat equation.

Example 5. In the same bundle as in the previous examples, consider the KdV equation

$$u_t = 6uu_x - u_{xxx}, \tag{24}$$

and the corresponding diffiety $\mathscr{E}_{\mathrm{KdV}}$. The second-order system of PDEs

$$O : \begin{cases} u_t = 0 \\ u_{xx} = 3u^2 \\ u_{xt} = 0 \\ u_{tt} = 0 \end{cases}$$

is a four-dimensional one and determines a four-dimensional HJ subdiffiety $\mathscr{O} \subset \mathscr{E}_{\mathrm{KdV}}$. We now search for an elementary HJ subdiffiety of $\mathscr{E}_{\mathrm{KdV}}$ containing $\mathscr{O}$.

The Galilean boost

$$Y = -6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{25}$$

is a Lie symmetry of (24) and its second prolongation

$$Y_2 = Y + 6u_x\frac{\partial}{\partial u_t} + 6u_{xx}\frac{\partial}{\partial u_{xt}} + 12u_{xt}\frac{\partial}{\partial u_{tt}}$$

is transverse to O. Denote its flow by $\{A_\tau\}$. Then $\overline{O} = \bigcup_\tau A_\tau(O)$ is a five-dimensional second-order system of PDEs determining a five-dimensional subdiffiety $\overline{\mathscr{O}} \subset \mathscr{E}_{\mathrm{KdV}}$. Moreover, in view of Remark 7, $\overline{\mathscr{O}} = \mathscr{Y}_\infty^\nabla$ for some flat holonomic connection ∇ in π_1. In particular, ∇ is a solution of the first generalized HJ equation. Let us determine $\overline{O}$. A_τ is given by

$$\begin{aligned}
&A_\tau^*(x) = x - 6\tau t, \qquad A_\tau^*(t) = t, \qquad A_\tau^*(u) = u + \tau, \\
&A_\tau^*(u_x) = u_x, \qquad A_\tau^*(u_t) = u_t + 6\tau u_x, \\
&A_\tau^*(u_{xx}) = u_{xx}, \qquad A_\tau^*(u_{xt}) = u_{xt} + 6\tau u_{xx}, \qquad A_\tau^*(u_{tt}) = u_{tt} + 12\tau u_{xt} + 36\tau^2 u_{xx}.
\end{aligned}$$

Therefore, $\overline{O}$ is parametrically given by

$$\overline{O} : \begin{cases} x = y - 6\tau s \\ t = s \\ u = v + \tau \\ u_x = p \\ u_t = 6\tau p \\ u_{xx} = 3v^2 \\ u_{xt} = 18\tau v^2 \\ u_{tt} = 108\tau^2 v^2. \end{cases}$$

Eliminating the five parameters (y, s, v, p, τ) we get

$$\overline{O} : \begin{cases} u_{xx} = \dfrac{(u_t - 6uu_x)^2}{12u_x^2} \\ u_{xt} = \dfrac{u_t(u_t - 6uu_x)^2}{12u_x^3} \\ u_{tt} = \dfrac{u_t^2(u_t - 6uu_x)^2}{12u_x^4}, \end{cases}$$

and ∇ is given by

$$\nabla = (\mathrm{d}u - u_x\mathrm{d}x - u_t\mathrm{d}t) \otimes \frac{\partial}{\partial u} + (\mathrm{d}u_x - A\mathrm{d}x - C\mathrm{d}t) \otimes \frac{\partial}{\partial u_x} + (\mathrm{d}u_t - C\mathrm{d}x - B\mathrm{d}t) \otimes \frac{\partial}{\partial u_t}$$

with

$$A = \frac{(u_t - 6uu_x)^2}{12u_x^2}, \qquad B = -\frac{u_t^2(u_t - 6uu_x)^2}{12u_x^4}, \qquad C = \frac{u_t(u_t - 6uu_x)^2}{12u_x^3}.$$

A direct computation shows that ∇ is indeed flat and it is a solution of the first generalized HJ equation

$$A_x + u_x A_u + AA_{u_x} + CA_{u_t} + u_t - 6uu_x = 0.$$

Clearly, solutions of $\overline{O}$, or, which is the same, ∇-constant sections, are boosted solutions of O. Namely, solutions of O are

$$u = 2^{1/3}\wp(2^{-1/3}(x - c_0); 0, c_1), \tag{26}$$

$\wp(z; \omega_1, \omega_2)$ being the Weierstrass elliptic function of z, with periods ω_1, ω_2, and c_0, c_1 integration constants. Solutions of $\overline{O}$ are then found by transporting (26) along the flow of Y. They are

$$u = 2^{1/3}\wp(2^{-1/3}(x - ct - c_0); 0, c_1) + c/6,$$

where c is a new integration constant, and they are (local) solutions of the KdV equation.

5. The Lagrangian–Hamiltonian formalism in field theory

In this section we review the Lagrangian–Hamiltonian formalism for higher derivative field theories. Details can be found in [8] (see also [7]).

Definition 3. A *Lagrangian field theory* of order $\leq k+1$, $0 \leq k < \infty$, is a pair $(\pi, \mathscr{L})$, where $\pi : E \longrightarrow M$ is a fiber bundle and $\mathscr{L} \in \Lambda^n_n(J^{k+1}, \pi_{k+1}) \subset \overline{\Lambda}^n$, $n = \dim M$, is a Lagrangian density.

As already recalled, the horizontal cohomology class $[\mathscr{L}] \in H^n(\overline{\Lambda}, \overline{d})$ identifies with the action functional $\int_M \mathscr{L}$ which is extremized by solutions of the Euler–Lagrange (EL) field equations

$$\boldsymbol{E}(\mathscr{L}) = 0.$$

The EL equations are $2(k+1)$th-order PDEs. Denote by $\mathscr{E}_{EL} \subset J^\infty$ the corresponding diffiety.

The Lagrangian field theory $(\pi, \mathscr{L})$ determines a canonical PD Hamiltonian system $\omega_{\mathscr{L}} \in \Lambda^{n+1}(J^\dagger\pi_\infty)$ in $J^\dagger\pi_\infty \longrightarrow M$ (the reduced multimomentum bundle of π_∞). If $\mathscr{L}$ is locally given by $\mathscr{L} = Ld^n x$, $L \in C^\infty(J^{k+1})$, then $\omega_{\mathscr{L}}$ is locally given by

$$\omega_{\mathscr{L}} = \mathrm{d}p^{I.i}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge d^{n-1}x_i - \mathrm{d}E_{\mathscr{L}} \wedge d^n x, \qquad E_{\mathscr{L}} = u^\alpha_{Ii} p^{I.i}_\alpha - L,$$

the $p^{I.i}_\alpha$'s being momentum coordinates associated with the u^α_I. The corresponding PD Hamilton equations

$$i^{1,n}(j_1\sigma)\omega_{\mathscr{L}}|_\sigma = 0, \tag{27}$$

for sections σ of $J^\dagger\pi_\infty \longrightarrow M$, locally read

$$\begin{cases} p^{I.i}_\alpha{}_{,i} = \dfrac{\partial L}{\partial u^\alpha_I} - \delta^I_{Ji}\, p^{J.i}_\alpha \\ u^\alpha_I{}_{,i} = u^\alpha_{Ii} \end{cases}$$

where $(\bullet)_{,i}$ denotes differentiation of $(\bullet)$ with respect to x^i, $i = 1, \ldots, n$, and $\delta^I_K = 1$ when the multi-indices I, K coincide, and $\delta^I_K = 0$ otherwise. We will refer to Eq. (27) as the *Euler–Lagrange–Hamilton (ELH) equations on* $J^\dagger\pi_\infty$. They can be interpreted as Euler–Lagrange equations corresponding to a Hamilton–Pontryagin-like variational principle [15] and their solutions are characterized by the following:

Theorem 7. *A section σ of $J^\dagger\pi_\infty \longrightarrow M$ is a solution of Eq. (27) iff, locally, $\sigma = \vartheta \circ j_\infty s$, where s is a solution of the EL equations and ϑ is a Legendre form.*

In particular, Eq. (27) covers EL equations; i.e., if σ is a solution of (28), then the composition

$$M \xrightarrow{\sigma} J^\dagger\pi_\infty \longrightarrow J^\infty$$

is of the form $j_\infty s$ for a solution s of the EL equations, and all solutions of the EL equations can be obtained like this.

Lemma 8. *Let T be a section of $J^\dagger\pi_\infty \longrightarrow J^\infty$, i.e., $T \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$; then*

$$T^*\omega_{\mathscr{L}} = d(\mathscr{L} + T).$$

Proof. The lemma can be proved in a coordinate free manner, using the intrinsic definition of $\omega_{\mathscr{L}}$. We here propose a local proof. Let T be locally given by $T^*(p^{I.i}_\alpha) = T^{I.i}_\alpha \in C^\infty(J^\infty)$, i.e.,

$$T = T^{I.i}_\alpha d^V u^\alpha_I \otimes d^{n-1}x_i,$$

Notice that $\omega_{\mathscr{L}} = d\varrho_{\mathscr{L}}$, with $\varrho_{\mathscr{L}} \in \Lambda^n(J^\dagger\pi_\infty)$ locally given by

$$\varrho_{\mathscr{L}} = p^{I.i}_\alpha du^\alpha_I \wedge d^{n-1}x_i - Ed^n x.$$

Then

$$\begin{aligned} T^*\omega_{\mathscr{L}} &= dT^*\varrho_{\mathscr{L}} \\ &= d[T^{I.i}_\alpha du^\alpha_I \wedge d^{n-1}x_i - (u^\alpha_{Ii}T^{I.i}_\alpha - L)d^n x] \\ &= d(T^{I.i}_\alpha d^V u^\alpha_I \wedge d^{n-1}x_i + Ld^n x) \\ &= d(T + \mathscr{L}). \quad \square \end{aligned}$$

The Lagrangian field theory $(\pi, \mathscr{L})$ determines two more canonical PD Hamiltonian systems. First of all, the PD Hamiltonian system $\omega_{\mathscr{L}}$ is the pull-back of a unique PD Hamiltonian system ω in $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow M$. ω is locally given by

$$\omega = \sum_{|I|\leq k} dp^{I.i}_\alpha \wedge du^\alpha_I d^{n-1}x_i - dE \wedge d^n x, \qquad E = \sum_{|I|\leq k} u^\alpha_{Ii} p^{I.i}_\alpha - L.$$

The associated PD Hamilton equations

$$i^{1,n}(j_1\sigma)\omega|_\sigma = 0, \tag{28}$$

for sections σ of $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow M$ locally read

$$\begin{cases} \dfrac{\partial L}{\partial u^\alpha_K} - \delta^K_{Ji}\, p^{J.i}_\alpha = 0 & |K| = k+1 \qquad \mathrm{I} \\ p^{I.i}_\alpha{}_{,i} = \dfrac{\partial L}{\partial u^\alpha_I} - \delta^I_{Ji}\, p^{J.i}_\alpha & |I| \leq k \qquad \mathrm{II} \\ u^\alpha_J{}_{,i} = u^\alpha_{Ji} & |J| \leq k \qquad \mathrm{III.} \end{cases} \tag{29}$$

We will refer to them as the *ELH equations on* $\pi^\circ_{k+1,k}(J^\dagger\pi_k)$. They cover the EL equations as well, i.e., if σ is a solution of (28), then the composition

$$M \xrightarrow{\sigma} \pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^k$$

is of the form $j_k s$ for a solution s of the EL equations, and all solutions of the EL equations can be obtained like this. (29) shows that solutions of (28) take values in the *first constraint subbundle* $\mathscr{P} \longrightarrow M$, $\mathscr{P} \subset \pi^\circ_{k+1,k}(J^\dagger\pi_k)$, which is locally defined by

$$\frac{\partial L}{\partial u^\alpha_K} - \delta^K_{Ji}\, p^{J.i}_\alpha = 0, \quad |K| = k+1.$$

Let $\mathscr{P}_0 \subset J^\dagger\pi_k$ be the image of $\mathscr{P}$ under the projection $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^\dagger\pi_k$. Under the (not too restrictive) hypothesis that the projection $\mathscr{P} \longrightarrow \mathscr{P}_0$ is a smooth submersion with connected fibers, $i^*_{\mathscr{P}}\omega$ is the pull-back of a unique PD Hamiltonian system ω_0 in the bundle $\mathscr{P}_0 \longrightarrow M$. We will refer to the associated PD Hamilton equations

$$i^{1,n}(j_1\sigma_0)\omega_0|_{\sigma_0} = 0 \tag{30}$$

for sections σ_0 of $\mathscr{P}_0 \longrightarrow M$ as the *Hamilton–de Donder–Weyl (HDW) equations*. When $(\pi, \mathscr{L})$ is a (hyper)regular theory, i.e., $\mathscr{P}_0 = J^\dagger\pi_k$, they locally coincide with the de Donder (higher derivative, Hamilton-like) field equations [16]

$$\begin{cases} p^{I.i}_\alpha{}_{,i} = -\dfrac{\partial H}{\partial u^\alpha_I} \\ u^\alpha_I{}_{,i} = \dfrac{\partial H}{\partial p^{I.i}_\alpha}, \end{cases}$$

where the local function $H \in C^\infty(J^\dagger\pi_k)$ is uniquely defined by the condition that its pull-back via the projection $\mathscr{P} \longrightarrow J^\dagger\pi_k$ is $i^*_{\mathscr{P}}(E) \in C^\infty(\mathscr{P})$.

If σ is a solution of Eq. (28), then the composition

$$M \xrightarrow{\sigma} \pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^\dagger\pi_k$$

is a solution of (30). However, Eq. (30) generically possesses more solutions than are obtained like this, unless $(\pi, \mathscr{L})$ is a (hyper)regular theory. In this case, the ELH equations cover the HDW equations.

In [7], we presented a higher derivative, field theoretic version of the geometric HJ formalism [3,4] in the case of a hyperregular Lagrangian field theory. In the next section we generalize the constructions and results of [7] to possibly singular Lagrangian field theories. The concept of HJ diffiety will play a special role.

6. Hamilton–Jacobi subdiffieties of Euler–Lagrange equations

In [3,4,6], Cariñena, Gràcia, Marmo, Martínez, Muñoz-Lecanda, and Román-Roy presented a geometric formulation of the classical HJ theory of mechanical systems in both Lagrangian and Hamiltonian settings. They also presented a generalized HJ problem depending on the sole equations of motion, and not on the Lagrangian, nor the Hamiltonian. Their formulation "is based on the idea of obtaining solutions of a second order differential equations by lifting solutions of an adequate first order differential equation [4]". This idea can be generalized to the higher derivative, regular, Lagrangian field theoretic setting [7]. We here propose a further generalization to the (possibly) singular case.

Let $(\pi, \mathscr{L})$ be a kth-order Lagrangian field theory, and $\omega \in \Lambda^{n+1}(\pi^\circ_{k+1,k}(J^\dagger\pi_k))$ the associated PD Hamiltonian system in $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow M$. A section of the pull-back bundle $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^k$ can be understood as a pair (∇, T), where ∇ is a holonomic connection in π_k, and T is a section of $J^\dagger\pi_k \longrightarrow J^k$, i.e., $T \in V\Lambda^1(J^k, \pi_k) \otimes \Lambda^{n-1}_{n-1}(J^k, \pi_k)$. We will always adopt this point of view. Obviously, the diagram

$$\begin{tikzcd}
\pi^\circ_{k+1,k}(J^\dagger\pi_k) \arrow[r] \arrow[d] & J^\dagger\pi_k \\
J^{k+1} & J^k \arrow[l, "\nabla"] \arrow[u, "T"'] \arrow[ul, "(\nabla,T)"']
\end{tikzcd}$$

commutes.

Lemma 9. *Let (∇, T) be a section of $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^k$, with ∇ a flat (holonomic) connection. Then, one has*

$$(\nabla, T)^*\omega = d(\nabla^*\mathscr{L} + \mathrm{e}^{1,n-1}(\nabla)T).$$

Proof. We can take the pull-back $\pi^*_{\infty,k}(T) \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$. In the following, abusing the notation, we will indicate this pull-back by T again. Denote by $\mathfrak{p} : J^\dagger\pi_\infty \longrightarrow \pi^\circ_{k+1,k}(J^\dagger\pi_k)$ the canonical projection. Then (1) $(\nabla, T) = \mathfrak{p} \circ T \circ \nabla_{[\infty]}$ (where, in the lhs, we interpreted T as a section of $J^\dagger\pi_\infty$), and (2) $\mathfrak{p}^*\omega = \omega_{\mathscr{L}}$. Therefore,

$$\begin{aligned}
(\nabla, T)^*\omega &= \nabla^*_{[\infty]}T^*\mathfrak{p}^*\omega \\
&= \nabla^*_{[\infty]}T^*\omega_{\mathscr{L}} \\
&= \nabla^*_{[\infty]}d(\mathscr{L} + T) \\
&= d\nabla^*_{[\infty]}(\mathscr{L} + T) \\
&= d(\nabla^*\mathscr{L} + \mathrm{e}^{1,n-1}(\nabla)T),
\end{aligned}$$

where we used Lemma 8 and, in the last line, the (obvious) fact that $\nabla^*_{[\infty]}$ is a morphism of the variational bicomplex and the bicomplex defined by ∇. □

Definition 4. The *generalized HJ problem* for the Lagrangian theory $(\pi, \mathscr{L})$ consists in finding a holonomic flat connection ∇ in π_k and a section T of $J^\dagger\pi_k \longrightarrow J^k$ such that for every ∇-constant section σ, $(\nabla, T) \circ \sigma$ is a solution of the ELH equations.

The relevance of the generalized HJ problem resides in the following:

Theorem 10. *Let (∇, T) be a section of $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^k$, with ∇ a flat (holonomic) connection. The following conditions are equivalent:*

1. *(∇, T) is a solution of the generalized HJ problem;*
2. *$\mathrm{im}(\nabla, T) \subset \mathscr{P}$ and, for every ∇-constant section j, $T \circ j$ is a solution of the HDW equations;*
3. *$\mathrm{im}(\nabla, T) \subset \mathscr{P}$ (hence $\mathrm{im}T \subset \mathscr{P}_0$) and $i^{1,n}(\nabla)T^*\omega_0 = 0$;*
4. *$\mathrm{im}(\nabla, T) \subset \mathscr{P}$ and $i^{1,n}(\nabla)(\nabla, T)^*\omega = 0$.*

Moreover, each of the above conditions implies:

5. *$\mathscr{Y}^\nabla_\infty$ is an (elementary) HJ subdiffiety of $\mathscr{E}_{EL}$.*

Proof. 1. $\Longrightarrow$ 2. Let j be a ∇-constant section. Then $(\nabla, T) \circ j$ is a solution of the ELH equations and, therefore, takes values in $\mathscr{P}$. Since ∇-constant sections "foliate" J^k we conclude that (∇, T) itself takes values in $\mathscr{P}$. Finally, the projection $\mathscr{P} \longrightarrow \mathscr{P}_0$ maps solutions of the ELH equations to solutions of the HDW equations.

2. $\Longrightarrow$ 3. Let j be a ∇-constant section (and hence $T \circ j$ is a solution of the HDW equations) and X a π_k-vertical vector field on J^k along j. Compute

$$i_X i^{1,n}(j_1 j)T^*\omega_0|_j = i_{(\mathrm{d}T)(X)} i^{1,n}(j_1(T \circ j))\omega_0|_{T\circ j} = 0.$$

Since X is arbitrary, $i^{1,n}(j_1 j)T^*\omega_0|_j = 0$. Moreover, ∇-constant sections "foliate" J^k and, therefore, $i^{1,n}(\nabla)T^*\omega_0 = 0$.

3. $\Longrightarrow$ 4. Suppose that $\mathrm{im}(\nabla, T) \subset \mathscr{P}$. Then

$$(\nabla, T)^*\omega = (\nabla, T)^* i^*_{\mathscr{P}}\omega = T^*\omega_0.$$

4. $\Longrightarrow$ 1. Suppose that $\mathrm{im}(\nabla, T) \subset \mathscr{P}$, with $i^{1,n}(\nabla)(\nabla, T)^*\omega = 0$, and let j be a ∇-constant section. We prove that $(\nabla, T) \circ j$ is a solution of the ELH equations. Indeed, in view of Lemma 9,

$$\begin{aligned} 0 &= i^{1,n}(\nabla)(\nabla, T)^*\omega \\ &= i^{1,n}(\nabla)d(\nabla^*\mathscr{L} + \mathrm{e}^{1,n-1}(\nabla)T) \\ &= d^V\nabla^*\mathscr{L} + \overline{d}_\nabla T. \end{aligned}$$

Now, locally,

$$d^V\nabla^*\mathscr{L} + \overline{d}_\nabla T = \sum_{|J|,|I|\le k} \nabla^*\left(\frac{\partial L}{\partial u^\alpha_I} - D_i T^{I.i}_\alpha - \delta^I_{Ji} T^{J.i}_\alpha\right) d^V u^\alpha_I \otimes d^n x.$$

Thus,

$$\nabla^*\left(\frac{\partial L}{\partial u^\alpha_I} - D_i T^{I.i}_\alpha - \delta^I_{Ji} T^{J.i}_\alpha\right) = 0. \tag{31}$$

$\sigma := (\nabla, T) \circ j$ satisfies Eq. (29) *I* (because it takes values in $\mathscr{P}$) and (29) *III* (because j is ∇-constant). We show that it satisfies Eq. (29) *II* also. For $|I| \le k$,

$$\begin{aligned} \sigma^*(p^{I.i}_\alpha)_{,i} &= j^*(T^{I.i}_\alpha)_{,i} \\ &= (j_1 j)^* D_i T^{I.i}_\alpha \\ &= (\nabla \circ j)^* D_i T^{I.i}_\alpha \\ &= j^*\nabla^*(D_i T^{I.i}_\alpha) \\ &= j^*\nabla^*\left(\frac{\partial L}{\partial u^\alpha_I} - \delta^I_{Ji} T^{J.i}_\alpha\right) \\ &= j^*(\nabla, T)^*\left(\frac{\partial L}{\partial u^\alpha_I} - \delta^I_{Ji} p^{J.i}_\alpha\right) \\ &= \sigma^*\left(\frac{\partial L}{\partial u^\alpha_I} - \delta^I_{Ji} p^{J.i}_\alpha\right). \end{aligned}$$

1. $\Longrightarrow$ 5. Obvious, since the projection $\mathscr{P} \longrightarrow E$ maps solutions of the ELH equations to solutions of the EL equations. □

In view of the above theorem, given a solution (∇, T) of the generalized HJ problem we can obtain solutions of the ($2k$th-order) EL equations, finding solutions of the much simpler (kth-order) equation $\mathscr{Y}^\nabla$.

We now prove that the last implication in the above proof can be inverted as well in the following sense. If $\mathscr{Y}^\nabla_\infty$ is an HJ subdiffiety of the EL equations then there exists T such that (∇, T) is a solution of the generalized HJ problem. This result is obtained by observing the relation between the generalized HJ problem and Legendre forms.

Theorem 11. *Let (∇, T) be a section of $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^k$, with ∇ a flat (holonomic) connection. The following conditions are equivalent:*

1. *(∇, T) is a solution of the generalized HJ problem;*
2. *$(d^V\mathscr{L} + \overline{d}T)|_{\nabla_{[\infty]}} = 0$;*
3. *$\mathscr{Y}^\nabla_\infty$ is an (elementary) HJ subdiffiety of $\mathscr{E}_{EL}$, and there exists a Legendre form ϑ such that $(T - \vartheta)|_{\nabla_{[\infty]}} = 0$.*

Proof. 1. $\Longrightarrow$ 2. Recall, preliminarily, that, in view of Lemma 8, $T^*\omega_{\mathscr{L}} = d(\mathscr{L} + T) = d^V\mathscr{L} + \overline{d}T + d^V T$. Now let j be a ∇-constant section. Then $(\nabla, T) \circ j$ is a solution of the ELH equations on $\pi^\circ_{k+1,k}(J^\dagger\pi_k)$. Coordinate formulas then show that $T \circ \nabla_{[\infty]} \circ j$ is a solution of the ELH equations on $J^\dagger\pi_\infty$, i.e.,

$$i^{1,n}(j_1(T \circ \nabla_{[\infty]} \circ j))\omega_{\mathscr{L}}|_{T\circ\nabla_{[\infty]}\circ j} = 0.$$

Let X be a π_∞-vertical vector field over J^∞ along $\nabla_{[\infty]} \circ j$. Then

$$i_X i^{1,n}(j_1(\nabla_{[\infty]} \circ j))T^*\omega_{\mathscr{L}}|_{\nabla_{[\infty]}\circ j} = i_{(dT)(X)} i^{1,n}(j_1(T \circ \nabla_{[\infty]} \circ j))\omega_{\mathscr{L}}|_{T\circ\nabla_{[\infty]}\circ j} = 0.$$

Since X is arbitrary,

$$\begin{aligned} 0 &= i^{1,n}(j_1(\nabla_{[\infty]} \circ j))T^*\omega_{\mathscr{L}}|_{\nabla_{[\infty]}\circ j} \\ &= i^{1,n}(j_1(\nabla_{[\infty]} \circ j))(d^V\mathscr{L} + \overline{d}T + d^V T)|_{\nabla_{[\infty]}\circ j} \\ &= [i^{1,n}(\mathscr{C})(d^V\mathscr{L} + \overline{d}T + d^V T)]|_{\nabla_{[\infty]}\circ j} \\ &= (d^V\mathscr{L} + \overline{d}T)|_{\nabla_{[\infty]}\circ j}. \end{aligned}$$

Since ∇-constant sections foliate J^k, we get $(d^V\mathscr{L} + \overline{d}T)|_{\nabla_{[\infty]}} = 0$.

2. $\Longrightarrow$ 3. $0 = (d^V\mathscr{L} + \overline{d}T)|_{\nabla_{[\infty]}} = (\boldsymbol{E}(\mathscr{L}) + \overline{d}(T - \vartheta_0))|_{\nabla_{[\infty]}}$, where ϑ_0 is a Legendre form. Therefore $\boldsymbol{E}(\mathscr{L})|_{\nabla_{[\infty]}} = \overline{d}(\vartheta_0 - T)|_{\nabla_{[\infty]}} = \overline{d}|_{\nabla_\infty}(\vartheta_0 - T)|_{\nabla_\infty}$, where we used that $\mathscr{Y}^\nabla_\infty \subset J^\infty$ is a subdiffiety. Recall that, in view of Remark 4, $\boldsymbol{E}(\mathscr{L})|_{\nabla_{[\infty]}}$ cannot be $\overline{d}|_{\nabla_{[\infty]}}$-exact unless it is 0. We conclude that $\mathscr{Y}^\nabla_\infty \subset \mathscr{E}_{EL}$. Moreover, $(\vartheta_0 - T)|_{\nabla_{[\infty]}}$ is $\overline{d}|_{\nabla_{[\infty]}}$-closed, and hence $\overline{d}|_{\nabla_{[\infty]}}$-exact, i.e., there exists $\nu \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-2}$ such that $(\vartheta_0 - T)|_{\nabla_{[\infty]}} = \overline{d}|_{\nabla_{[\infty]}}\nu|_{\nabla_{[\infty]}}$ or, which is the same, $(T - \vartheta)|_{\nabla_{[\infty]}} = 0$ where we put $\vartheta = \vartheta_0 - \overline{d}\nu$. Finally, notice that ϑ itself is a Legendre form.

3. $\Longrightarrow$ 1. Let j be a ∇-constant section. Then $j = j_k s$ for some solution of the EL equations and $T \circ j_\infty s = \vartheta \circ j_\infty s$. In view of Theorem 7, $T \circ j_\infty s$ is a solution of the ELH equations on $J^\dagger\pi_\infty$. We conclude that

$$\pi^*_{k+1,k}(T) \circ j_{k+1}s = \pi^*_{k+1,k}(T) \circ \nabla \circ j_k s = (\nabla, T) \circ j_k s = (\nabla, T) \circ j$$

is a solution of the ELH equations on $\pi^\circ_{k+1,k}(J^\dagger\pi_k)$. □

Corollary 12. *Let ∇ be a holonomic flat connection in π_k. There exists T such that (∇, T) is a solution of the generalized HJ problem iff $\mathscr{Y}^\nabla_\infty$ is an (elementary) HJ subdiffiety of $\mathscr{E}_{EL}$.*

Proof. The if implication is already stated in Theorem 10, point 5. Conversely, if $\mathscr{Y}^\nabla_\infty \subset \mathscr{E}_{EL}$, then $\boldsymbol{E}(\mathscr{L})|_{\nabla_{[\infty]}} = 0$. Let ϑ be a Legendre form depending only on (vertical differentials of) derivatives up to the order k. Put $T := \vartheta|_{\nabla_{[\infty]}}$. T is a section of $J^\dagger\pi_k \longrightarrow J^k$. Moreover,

$$(d^V\mathscr{L} + \overline{d}T)|_{\nabla_{[\infty]}} = (\boldsymbol{E}(\mathscr{L}) - \overline{d}\vartheta + \overline{d}T)|_{\nabla_{[\infty]}} = 0.$$

Now, use Theorem 11. □

The above corollary shows that equation $i^{1,n}(\nabla)(\nabla, T)^*(\omega) = 0$ covers the generalized (kth-order) HJ equation of the EL equations. Since all the Legendre forms of a given Lagrangian density are known, it follows that solving the generalized HJ problem is basically equivalent to finding all (kth-order) HJ subdiffieties of the EL equations.

Corollary 13. *Let $\mathscr{L}$ and $\mathscr{L}'$ be Lagrangian densities (of the same order) determining the same action functional (i.e., $\mathscr{L}' = \mathscr{L} + \overline{d}\eta$ for some $\eta \in \overline{\Lambda}^{n-1}$), and $\mathscr{P}$ and $\mathscr{P}'$ the corresponding generalized HJ problems. Then (∇, T) is a solution of $\mathscr{P}$ iff $(\nabla, T - d^V\eta)$ is a solution of $\mathscr{P}'$.*

Proof. Let ϑ be a Legendre form for $\mathscr{L}$. Then $\vartheta' := \vartheta + d^V\eta$ is a Legendre form for $\mathscr{L}'$. Since, for $T \in \mathscr{C}\Lambda^1 \otimes \Lambda^{n-1}$, $T - \vartheta = T - d^V\eta + \vartheta'$, trivially, the assertion immediately follows from Theorem 11, point 3. □

The above corollary basically states that *the generalized HJ problem is independent of the Lagrangian density in the class of those determining the same action functional.*

Remark 8. Recall the example in the introduction. Among HJ subdiffieties of a system of regular, ordinary, second-order EL equations there are distinguished ones: namely, those determined (as in the introduction) by solutions of the standard HJ Eq. (5). It is well-known that, in its turn, Eq. (5) can be geometrically interpreted as follows. Consider the map

$$T : \mathbb{R} \times \mathbb{R}^n \in (t, x) \longmapsto (t, T_i(t, x)dx^i) \in \mathbb{R} \times T^*\mathbb{R}^n.$$

Then

$$T_i = \frac{\partial S}{\partial x^i},$$

with $S = S(x, t)$ a solution of (5), iff im$T$ is an isotropic submanifold with respect to the presymplectic structure

$$\Omega_0 := dp_i \wedge dx^i - dH \wedge dt$$

on $\mathbb{R} \times T^*\mathbb{R}^n$, i.e.,

$$T^*\Omega_0 = 0. \tag{32}$$

It is natural to wonder whether these considerations can be generalized to the field theoretic setting. A first guess would be to consider the equation

$$(\nabla, T)^* \omega = T^* \omega_0 = 0, \quad \mathrm{im}(\nabla, T) \subset \mathscr{P} \tag{33}$$

as the natural field theoretic generalization of (32). However, we think that this point of view (which is taken in [7]; see also [19]) is not completely satisfactory for the following reasons.

Let ϑ be a Legendre form. Consider

$$\Omega := i^*_{\mathscr{E}_{EL}}(d^V \vartheta) \in \mathscr{C}\Lambda^2(\mathscr{E}_{EL}) \otimes \overline{\Lambda}^{n-1}(\mathscr{E}_{EL}).$$

In view of the first variation formula (13), $\overline{d}\Omega = 0$. The horizontal cohomology class

$$\boldsymbol{\omega} := [\Omega] \in H^{n-1}(\mathscr{C}\Lambda^2(\mathscr{E}_{EL}) \otimes \overline{\Lambda}(\mathscr{E}_{EL}), \overline{d})$$

does only depend on the action functional and it is naturally interpreted as a (pre)symplectic form in the so called covariant phase space, i.e., the space of solutions of the EL equations (see, for instance, [17] for details). This functional symplectic structure should be understood as the fundamental symplectic structure in field theory. For instance, it is at the basis of the BV formalism [18].

Now, let (∇, T) be a solution of Eq. (33) and ϑ a Legendre form such that $(T - \vartheta)|_{\nabla_{[\infty]}} = 0$. It is easy to see that $d^V T = 0$. Therefore,

$$\nabla^*_{[\infty]}\boldsymbol{\omega} = \nabla^*_{[\infty]}[\Omega] = [d^V \nabla^*_{[\infty]}\vartheta] = [d^V T] = 0,$$

where the last two are cohomology classes in $H^{n-1}(V\Lambda^2(J^k, \pi_k) \otimes \Lambda^{\bullet}_{\bullet}(J^k, \pi_k), d^V)$. We conclude that $\mathscr{Y}^{\nabla}_{\infty}$ is an isotropic subdiffiety of the "(pre)symplectic diffiety" $(\mathscr{E}_{EL}, \boldsymbol{\omega})$. Unfortunately, this is not a special feature of solutions of (33). Namely, for $n > 1$, *every* finite dimensional subdiffiety of $(\mathscr{E}_{EL}, \boldsymbol{\omega})$ is isotropic, modulo topological obstructions. Indeed, the horizontal de Rham complex of a finite dimensional diffiety is locally acyclic in positive degree (see, for instance, the final comment of Section 1). On the other hand, for $n = 1$, solutions of (33) may be effectively characterized as those solutions of the generalized HJ problem defining isotropic subdiffieties of $(\mathscr{E}_{EL}, \boldsymbol{\omega})$. Because of this difference between the $n > 1$ and the $n = 1$ cases, we think that Eq. (33) is not as fundamental in field theory as Eq. (32) in mechanics and there possibly exists a different, more fundamental, field theoretic version of Eq. (32). This possibility will be explored elsewhere.

7. A final example

The "good" Boussinesq equation

$$u_{tt} = (u + u^2 + u_{xx})_{xx}$$

is obviously covered by the system of evolutionary equations

$$\begin{cases} v_t = u + u^2 + u_{xx} \\ u_t = v_{xx}, \end{cases} \tag{34}$$

which can be understood as a submanifold in $J^2\pi$ with π the trivial bundle $\pi : \mathbb{R}^4 \ni (t, x, u, v) \longmapsto (t, x) \in \mathbb{R}^2$. Further, Eqs. (34) are the EL equations determined by the action $\int L dt dx$ with [20]

$$L = \frac{1}{2}(u_x^2 + v_x^2 + v u_t - u v_t) + \frac{1}{3}u^3 + \frac{1}{2}u^2 \in C^\infty(J^1\pi).$$

The Lagrangian density L is singular in the sense that

$$\det \begin{pmatrix} \dfrac{\partial L}{\partial u_i \partial u_j} & \dfrac{\partial L}{\partial u_i \partial v_j} \\ \dfrac{\partial L}{\partial v_i \partial u_j} & \dfrac{\partial L}{\partial v_i \partial v_j} \end{pmatrix}_{i,j=t,x} = \det \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 0.$$

Denote by p^x, p^t, q^x, q^t momentum coordinates in $J^\dagger\pi$ associated with u, v respectively. Then

$$\omega = -dp^x \wedge du \wedge dt + dp^t \wedge du \wedge dx - dq^x \wedge dv \wedge dt + dq^t \wedge dv \wedge dx - dH \wedge dt \wedge dx,$$

with

$$H = \frac{1}{2}(u_x^2 + v_x^2) - \frac{1}{3}u^3 - \frac{1}{2}u^2.$$

It follows that

$$\mathscr{P}:\begin{cases} p^x - u_x = 0 \\ p^t + \frac{1}{2}v = 0 \\ q^x - v_x = 0 \\ q^t - \frac{1}{2}u = 0 \end{cases},$$

and

$$\mathscr{P}_0:\begin{cases} p^t + \frac{1}{2}v = 0 \\ q^t - \frac{1}{2}u = 0. \end{cases}$$

Consequently, fibers of $\mathscr{P}_0 \longrightarrow J^0\pi = \mathbb{R}^4$ are coordinatized by p^x, p^t only and

$$\omega_0 = du \wedge dv \wedge dx - dp^x \wedge du \wedge dt - dq^x \wedge dv \wedge dt - dH_0 \wedge dt \wedge dx,$$

with

$$H_0 = \frac{1}{2}[(p^x)^2 + (q^x)^2] - \frac{1}{6}u^3 - \frac{1}{2}u^2$$

Let

$$T = -P^x d^V u \otimes dt + P^t d^V u \otimes dx - Q^x d^V v \otimes dt + Q^t d^V v \otimes dx$$

be a section of $J^\dagger\pi \longrightarrow J^0\pi$ and

$$\nabla = (du - Adx - Bdt) \otimes \frac{\partial}{\partial u} + (dv - Cdx - Ddt) \otimes \frac{\partial}{\partial v}$$

be a connection in π. ∇ is flat iff

$$\nabla_t A = \nabla_x B, \quad \text{and} \quad \nabla_t C = \nabla_x D,$$

where $\nabla_t = \partial/\partial t + B\partial/\partial u + D\partial/\partial v$ and $\nabla_x = \partial/\partial x + A\partial/\partial u + C\partial/\partial v$. Moreover, $\mathrm{im}(\nabla, T) \subset \mathscr{P}$ iff

$$P^x = A, \qquad P^t = -\frac{1}{2}v, \qquad Q^x = C, \quad \text{and} \quad Q^t = \frac{1}{2}u.$$

In this case $\mathrm{im}T \subset \mathscr{P}_0$ and

$$\begin{aligned}(\nabla, T)^*(\omega) &= T^*(\omega_0) \\ &= du \wedge dv \wedge dx + (A_v - C_u)du \wedge dv \wedge Dt - (A_x + AA_u + CC_u - u^2 - u)du \wedge dt \wedge dx \\ &\quad - (C_x + AA_v + CC_v)dv \wedge dt \wedge dx.\end{aligned}$$

Notice that there is no T such that $T^*(\omega_0) = 0$. Finally

$$i^{1,n}(\nabla)T^*(\omega_0) = [(D - \nabla_x A + u^2 + u)d^V u - (B + \nabla_x C)d^V v] \otimes dt \wedge dx,$$

so $i^{1,n}(\nabla)T^*(\omega_0) = 0$ iff

$$\begin{cases} D - \nabla_x A + u^2 + u = 0 \\ B + \nabla_x C = 0, \end{cases} \tag{35}$$

which are precisely the 0th generalized HJ equations for (34). Let us search for solutions of the form

$$A = A(u), \qquad C = C(u), \qquad B = -cA, \qquad D = -cC, \qquad c = \text{const}.$$

Notice that, in these hypotheses, ∇ is identically flat. Eqs. (35) reduce to

$$\begin{aligned} AA_u &= cC + u^2 + u \\ AC_u &= -cA \end{aligned}$$

and for $A \neq 0$ we find

$$C = -cu + a, \qquad A^2 = \frac{1}{3}u^3 + \frac{1}{2}(1 - c^2)u^2 + au + b,$$

a, b being integration constants. We conclude that a (local) solution of the generalized HJ problem is (∇, T) with

$$A = \sqrt{\frac{1}{3}u^3 + \frac{1}{2}(1-c^2)u^2 + au + b}, \qquad B = -cA, \qquad C = -cu + a, \qquad D = -cC$$

$$P^x = A, \qquad P^t = -\frac{1}{2}v, \qquad Q^x = C, \qquad Q^t = \frac{1}{2}u,$$

and the system

$$\begin{cases} u_t = -c\sqrt{\frac{1}{3}u^3 + \frac{1}{2}(1-c^2)u^2 + au + b} \\ u_x = \sqrt{\frac{1}{3}u^3 + \frac{1}{2}(1-c^2)u^2 + au + b} \\ v_t = c^2 u - ca \\ v_x = -cu + a, \end{cases} \tag{36}$$

(locally) correspond to an HJ subdiffiety of (34). In the case $a = b = 0$, $0 < c^2 < 1$, solutions of (36) are

$$u = -\frac{3}{2}(1-c^2)\mathrm{sech}^2\left[\frac{1}{4}\sqrt{2(1-c^2)}(x - x_0 - ct)\right], \tag{37}$$

$$v = v_0 - 3\sqrt{2(1-c^2)}\tanh\left[\frac{1}{4}\sqrt{2(1-c^2)}(x - x_0 - ct)\right],$$

with x_0, v_0 integration constants. They are solutions of the EL equations (34). In particular (37) are well-known "*travelling wave*" solutions of the "good" Boussinesq equation.

References

[1] A.M. Vinogradov, Cohomological Analysis of Partial Differential Equations and Secondary Calculus, in: Transl. Math. Mon., vol. 204, Amer. Math. Soc., Providence, 2001.
[2] A.M. Vinogradov, Introduction to secondary calculus, in: M. Henneaux, I.S. Krasil'shchik, A.M. Vinogradov (Eds.), Secondary Calculus and Cohomological Physics, in: Contemp. Math., vol. 219, Amer. Math. Soc., Providence, 1998, pp. 241–272.
[3] J.F. Cariñena, et al., Geometric Hamilton–Jacobi theory, Int. J. Geom. Methods Mod. Phys. 3 (2006) 1417–1458. e-print: arXiv:math-ph/0604063.
[4] J.F. Cariñena, et al., Hamilton–Jacobi theory and the evolution operator, in: M. Asorey, J.V. García Esteve, M.F. Rañada, J. Sesma (Eds.), Mathematical Physics and Field Theory. Julio Abad in Memoriam, Prensas Universitarias de Zaragoza, Zaragoza, 2009, pp. 177–186. e-print: arXiv:0907.1039.
[5] H. Rund, The Hamilton–Jacobi Theory in the Calculus of Variations, Robert E. Krieger Publ. Co., Huntington, NY, 1973.
[6] J.F. Cariñena, et al., Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems, Int. J. Geom. Methods Mod. Phys. 7 (2010) 431–454. e-print: arXiv:0908.2453.
[7] L. Vitagliano, The Hamilton–Jacobi formalism for higher order field theories, Int. J. Geom. Methods Mod. Phys. 7 (2010) 1413–1436. e-print: arXiv:1003.5236.
[8] L. Vitagliano, The Lagrangian–Hamiltonian formalism for higher order field theories, J. Geom. Phys. 60 (2010) 857–887. e-print: arXiv:0905.4580.
[9] N. Román-Roy, Multisymplectic Lagrangian and Hamiltonian formalism of first-order classical field theories, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009) 100–125. e-print: arXiv:math-ph/0506022.
[10] L. Vitagliano, Partial differential Hamiltonian systems (2009) (submitted for publication). e-print: arXiv:0903.4528.
[11] A.V. Bocharov, et al., Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, in: Transl. Math. Mon., vol. 182, Amer. Math. Soc., Providence, 1999.
[12] A.M. Vinogradov, The $\mathscr{C}$-spectral sequence, Lagrangian formalism and conservation laws I, II, J. Math. Anal. Appl. 100 (1984) 1–129.
[13] I.M. Anderson, Introduction to the variational bicomplex, in: M. Gotay, J.E. Marsden, V.E. Moncrief (Eds.), Math. Aspects of Classical Field Theory, in: Contemp. Math., vol. 132, Amer. Math. Soc., Providence, 1992, pp. 51–73.
[14] R.J. Alonso-Blanco, A.M. Vinogradov, Green formula and Legendre transformation, Acta Appl. Math. 83 (1–2) (2004) 149–166.
[15] G.H. Livens, On Hamilton's principle and the modified function in analytical dynamics, Proc. Roy. Soc. Edinburgh 39 (1919) 113.
[16] Th. de Donder, Théorie Invariantive du Calcul des Variations, Gauthier Villars, Paris, 1935, pp. 95–108.
[17] L. Vitagliano, Secondary calculus and the covariant phase space, J. Geom. Phys. 59 (2009) 426–447.
[18] M. Henneaux, C. Teitelboim, Quantization of Gauge Systems, Princeton Univ. Press, Princeton, 1992.
[19] M. de Leon, J.C. Marrero, D. Martín de Diego, A geometric Hamilton–Jacobi theory for classical field theories. e-print: arXiv:0801.1181.
[20] J. Chen, H. Liu, Derivation of Lagrangian density for the "Good" Boussinesq equation and multisymplectic discretizations, Appl. Math. Comput. 204 (2008) 58–62.