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Geodesically equivalent metrics in general relativity

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ABSTRACT

We discuss whether it is possible to reconstruct a metric from its nonparameterized geodesics, and how to do it effectively. We explain why this problem is interesting for general relativity. We show how to understand whether all curves from a sufficiently big family are nonparameterized geodesics of a certain affine connection, and how to reconstruct algorithmically a generic 4-dimensional metric from its nonparameterized geodesics. The algorithm works most effectively if the metric is Ricci-flat. We also prove that almost every metric does not allow nontrivial geodesic equivalence, and construct all pairs of 4-dimensional geodesically equivalent metrics of Lorentz signature.

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1. Introduction

Let (M^n, g) be a connected Riemannian (=g is positive definite) or a pseudo-Riemannian manifold of dimension $n \geq 2$. We say that a metric $\bar{g}$ on M^n is *geodesically equivalent* to g, if every geodesic of g is a (reparameterized) geodesic of $\bar{g}$. We say that they are *affine equivalent*, if their Levi-Civita connections coincide.

The first examples of geodesically equivalent metrics are due to Lagrange [1]. He observed that the radial projection $f(x, y, z) = \left(-\frac{x}{z}, -\frac{y}{z}, -1\right)$ takes geodesics of the half-sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z < 0\}$ to the geodesics of the plane $E^2 := \{(x, y, z) \in \mathbb{R}^3 : z = -1\}$, since the geodesics of both metrics are intersection of the 2-plane containing the point $(0, 0, 0)$ with the surface. Later, Beltrami [2,3] generalized the example for the metrics of constant negative curvature, and for the pseudo-Riemannian metrics of constant curvature. In the example of Lagrange, he replaced the half-sphere by the half of one of the hyperboloids $H^2_{\pm} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = \pm 1\}$, with the restriction of the Lorentz metrics $dx^2 + dy^2 - dz^2$ to it. Then, the geodesics of the metric are also intersections of the 2-planes, containing the point $(0, 0, 0)$, with the surface, and, therefore, the stereographic projection sends it to the straight lines of the appropriate plane.

Though the examples of the Lagrange and Beltrami are 2-dimensional, one can easily generalize them for every dimension (for Riemannian metrics, it was done already in [2]) and for every signature.

Since the time of Hermann Weyl, geodesically equivalent metrics were actively discussed in the realm of general relativity theory. The context of general relativity poses the following restrictions: the dimension is 4, the metrics are pseudo-Riemannian of Lorentz signature $(-, +, +, +)$ or $(+, -, -, -)$, and sometimes the metrics satisfy additional assumptions such that one or both metrics are Ricci-flat ($R_{ij} = 0$), or Einstein $\left(R_{ij} = \frac{R}{4} g_{ij}\right)$, or, more generally, satisfy the Einstein equation $R_{ij} - \frac{R}{2} g_{ij} = T_{ij}$ with 'physically interesting' stress-energy tensor T_{ij}.

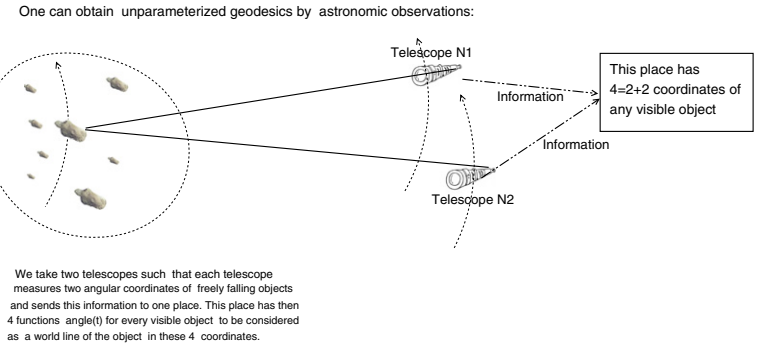
Let us explain (using a slightly naive language) one of the possible motivations for this interest. Suppose we would like to understand the structure of the space–time in a certain part of the universe. We assume that this part is far enough so the we can use only telescopes (in particular we cannot send a spaceship there). We still assume that the telescopes can see sufficiently many objects in this part of universe. Then, if the relativistic effects are not negligible (that happens for example if the objects in this part of space–time are sufficiently fast or if this region of the universe is big enough), we obtain as a rule

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the world lines of the objects as nonparameterized curves. Indeed, local coordinates on a 4-manifold are 4 smooth functions on the manifold such that their differentials are linearly independent. Now, for every freely falling object in this part of the universe such that it can be registered by telescopes, each telescope at every moment of time gives us two such functions, namely the spherical coordinates ϕ and θ (latitude and longitude) of the direction the light reflected from the object comes to the telescope from (in a naive language, the telescope 'sees' the direction where the object lies), see the picture below. Suppose we have two telescopes; then, altogether, we have 4 functions of t, $(\phi_1(t), \theta_1(t), \phi_2(t), \theta_2(t))$, that we consider to be the word line (i.e., geodesic) of the object in the coordinate system $(\phi_1, \theta_1, \phi_2, \theta_2)$. If we see sufficiently many objects, we have sufficiently many geodesics.

Of course, we cannot get lightlike or spacelike geodesics by this procedure. In the best case, we can reconstruct (numerically) sufficiently many geodesics, in the sense their velocity vectors are dense in a certain open subset of TM. See also the discussion in [4].



Now, as a rule, we cannot get the natural parameter (= proper time) of an object. Indeed, if the relativistic effects are not negligible, the proper time of the object is not our proper time t, i.e., the curve $(\phi_1(t), \theta_1(t), \phi_2(t), \theta_2(t))$ is a reparameterized geodesic only. If we cannot observe a periodic process on an object (note that the astronomical objects such that we can register a periodic process on, for example pulsars, are very rare) or any other way to measure the proper time of the object, we cannot obtain the proper time of the objects by astronomic observations (see also the discussion in [5]).

In view of this discussion, the following two problems (Problems 1 and 2) in the theory of geodesically equivalent metrics are interesting for general relativity:

Problem 1. How to reconstruct a metric from its nonparameterized geodesics?

The general setting is as follows: we have a family of smooth curves $\gamma(t;\alpha)$ in $U \subseteq \mathbb{R}^4$ depending on 6-dimensional[1] parameter $\alpha = (\alpha_1, \ldots, \alpha_6)$; we assume that the family is sufficiently big (we formalize 'sufficiently big' in the beginning of Section 2.1). We need to find a metric g such that for every fixed α the curve $\gamma(t;\alpha)$ is a reparameterized geodesic of g.

Mathematically, the problem has sense in every dimension and for every signature of the metric. In dimension 2, versions of this question were considered by Lie [6] and Liouville [7], and were also discussed by Veblen and Thomas [8–10] and Eisenhart [11] in the beginning of 20th century. In the realm of general relativity, the problem was explicitly stated by Ehlers et al. [12], where it was said that "*We reject clocks as basic tools for setting up the space–time geometry and propose …freely falling particles instead. We wish to show how the full space–time geometry can be synthesized …. Not only the measurement of length but also that of time then appears as a derived operation*".

This problem can be naturally divided in two subproblems.

Subproblem 1.1. *Given a family of curves $\gamma(t;\alpha)$, how to understand whether these curves are reparameterized geodesics of a certain affine connection? How to reconstruct this connection effectively?*

We will say that a metric *lies in a projective class* of a certain symmetric affine connection $\Gamma = \Gamma^i_{jk}$, if every geodesic of g is a reparameterized geodesic of Γ.

Subproblem 1.2. *Given an affine connection $\Gamma = \Gamma^i_{jk}$, how to understand whether there exists a metric g in the projective class of Γ? How to reconstruct this metric effectively?*

Both subproblems were actively discussed in the literature. In dimension 2, the answer on Subproblem 1.1 is classical and was known already to Sophus Lie; given a family of curves one constructs an ODE of the second order $y''(x) = f(x, y(x), y'(x))$; the curves $\gamma(t;\alpha)$ are reparameterized geodesics of a certain connection if and only if the right hand side

[1] Locally, the set of nonparameterized geodesics of an n-dimensional manifold has the structure of a manifold of dimension $2(n-1)$.

of the ODE is a 3rd degree polynomial in $y'(x)$,

$$f(x, y(x), y'(x)) = A(x, y(x)) + B(x, y(x))y'(x) + C(x, y(x))\left(y'(x)\right)^2 + D(x, y(x))\left(y'(x)\right)^3.$$

The answer in the multidimensional case can be obtained using the same idea as in dimension 2, we give it in Section 2.1.

The second subproblem is more complicated and is almost open. In dimension 2, the subproblem was considered in the recent paper [13] of Bryant et al.: given an affine connection, they construct a system of differential invariants that vanish if and only if there exists a metric (in a neighborhood of almost every point) in the projective class of this connection. The invariants are very complicated and are of very high orders.

In theory, one can also obtain a similar answer in every dimension. Indeed, by [14], in every dimension the existence of a metric in a projective class is equivalent to the existence of a nontrivial solution of a certain overdetermined system of linear PDE in the Cauchy–Frobenius form (i.e., the system is of first order and all derivatives of unknown functions are explicit (linear) expressions in the unknown functions). Given an overdetermined system of PDE in the Cauchy–Frobenius form, one can always, in theory, construct a system of differential invariants that vanish if and only if the system admits a nontrivial solution (in a neighborhood of almost every point). An effective construction of these differential invariants could be very complicated. The results of [13] show that it is indeed the case in dimension 2. It is hard to predict whether the system of differential invariants is easier in the multidimensional case (normally multidimensional cases are harder than low-dimensional; but sometimes overdetermined systems are easier to analyze in higher dimensions, because they can have higher degree of overdetermination).

In the present paper, in Section 2.2.2 we give an algorithmic answer to Subproblem 1.2 under the additional assumption that the metric g we are looking for is Ricci-flat and the projective class satisfies certain nondegeneracy assumption, i.e., in a situation most interesting from the viewpoint of general relativity. In Section 3.1, we also discuss the case of arbitrary metric: we show that also in this case one can algorithmically reconstruct the metric fro its projective class assuming certain nondegeneracy assumption on the projective class; though in this case the nondegeneracy assumption is harder to check.

Remark 1. Of course it is important in what form the geodesics $\gamma(t;\alpha)$ are given. Below, it will be clear what information we need from $\gamma(t;\alpha)$ in order our algorithm works. If the geodesics are given numerically (which is the case if they came from astronomic observations), this information could be extracted without difficulties.

Problem 2. In what situations is the reconstruction of a metric from the nonparameterized geodesics unique (up to the multiplication of the metric by a constant)?

The example of Lagrange/Beltrami above shows that in certain situations the reconstruction is not unique: the geodesics of every metric of constant curvature are straight lines, i.e., the geodesics of the standard flat metric, in a certain coordinate system. Constant curvature metrics are not the only metrics that allow nontrivial geodesical equivalence. For example, as shown by Dini, the following two metrics on $U^2 \subseteq \mathbb{R}^2$ are geodesically equivalent

$$g = (X(x) - Y(y))(\mathrm{d}x^2 + \mathrm{d}y^2) \quad \text{and} \quad \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{\mathrm{d}x^2}{X(x)} + \frac{\mathrm{d}y^2}{Y(y)}\right), \tag{1}$$

where X and Y are arbitrary (smooth) functions of the indicated variables such that the formulas (1) correspond to metrics (i.e., $0 \neq X \neq Y \neq 0$ for all $(x, y) \in U^2$). This example was generalized for all dimensions by Levi-Civita: from his results it is follows that the following two 4-dimensional metrics are geodesically equivalent:

$$\begin{aligned} g \;=\; & (X_0(x_0) - X_1(x_1))(X_0(x_0) - X_2(x_2))(X_0(x_0) - X_3(x_3))\mathrm{d}x_0^2 + (X_0(x_0) - X_1(x_1)) \\ & \times (X_1(x_1) - X_2(x_2))(X_1(x_1) - X_3(x_3))\mathrm{d}x_1^2 + (X_0(x_0) - X_2(x_2))(X_1(x_1) - X_2(x_2))(X_2(x_2) - X_3(x_3))\mathrm{d}x_2^2 \\ & + (X_0(x_0) - X_3(x_3))(X_1(x_1) - X_3(x_3))(X_2(x_2) - X_3(x_3))\mathrm{d}x_3^3 \end{aligned} \tag{2}$$

$$\begin{aligned} \bar{g} \;=\; & \frac{1}{X_0(x_0)}\frac{1}{X_0(x_0)X_1(x_1)X_2(x_2)X_3(x_3)}(X_0(x_0) - X_1(x_1))(X_0(x_0) - X_2(x_2))(X_0(x_0) - X_3(x_3))\mathrm{d}x_0^2 \\ & + \frac{1}{X_1(x_1)}\frac{1}{X_0(x_0)X_1(x_1)X_2(x_2)X_3(x_3)}(X_0(x_0) - X_1(x_1))(X_1(x_1) - X_2(x_2))(X_1(x_1) - X_3(x_3))\mathrm{d}x_1^2 \\ & + \frac{1}{X_2(x_2)}\frac{1}{X_0(x_0)X_1(x_1)X_2(x_2)X_3(x_3)}(X_0(x_0) - X_2(x_2))(X_1(x_1) - X_2(x_2))(X_2(x_2) - X_3(x_3))\mathrm{d}x_2^2 \\ & + \frac{1}{X_3(x_3)}\frac{1}{X_0(x_0)X_1(x_1)X_2(x_2)X_3(x_3)}(X_0(x_0) - X_3(x_3))(X_1(x_1) - X_3(x_3))(X_2(x_2) - X_3(x_3))\mathrm{d}x_3^3. \end{aligned} \tag{3}$$

Here $(x_0, \ldots, x_3)$ are local coordinates and the functions X_i depend on the indicated variables and are such that the metrics have sense.

In view of this, in the realm of general relativity, Problem 2 can be naturally divided in two subproblems.

We call a metric g *geodesically rigid*, if every metric $\bar{g}$, geodesic equivalent to g, is proportional to g.

Subproblem 2.1. *What metrics 'interesting' for general relativity are geodesically rigid?*

Subproblem 2.2. *Construct all pairs of nonproportional geodesically equivalent metrics.*

Let us comment on these subproblems. The part of the Subproblem 2.1 that is hard or even impossible to formalize is the word "interesting". Instead of formalizing this notion, let us give few results in this direction.

Probably the metrics that are most interesting in the context of general relativity are Ricci-flat nonflat metrics. As shown by Petrov in [15] (see also [4,16]),

4-dimensional Ricci-flat nonflat metrics of Lorentz signature cannot be geodesically equivalent, unless they are affinely equivalent (two metrics are *affinely equivalent*, if their Levi-Civita connections coincide; affine equivalent Ricci-flat 4-dimensional metrics are completely understood). It was one of the results for which Petrov obtained in 1972 the Lenin prize, the most important scientific award of the Soviet Union.

Recently, the answer of Petrov was generalized in [16] (see also [17]): it was shown that

if g and $\bar{g}$ are geodesically equivalent metrics on a 4-dimensional manifold, and g is Einstein and of nonconstant curvature, then the metrics are affinely equivalent.

Let us also give an example of a metric that is important for general relativity and that is not geodesically rigid. This is the so-called Friedman–Lemaitre–Robertson–Walker metric

$$g = -\mathrm{d}t^2 + R(t)^2 \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)}; \quad \kappa = +1; 0; -1, \tag{4}$$

where $R = R(t)$ is a real function (the scale factor) of the 'cosmic time' t. The metric is not geodesically rigid. Indeed, for every constant c such that the formula below has sense, the metric

$$\bar{g} = \frac{-1}{(R(t)^2 + c)^2}\mathrm{d}t^2 + \frac{R(t)^2}{c(R(t)^2 + c)} \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)} \tag{5}$$

is geodesically equivalent to g (one can see it directly as it was done for example [18] or [19], see also discussion in [5]. Actually, the pair of geodesically equivalent metrics (4) and (5) is a special case of geodesically equivalent metrics from Levi-Civita [20]).

For certain functions R, the metric (4) is the main ingredient of the so-called Standard Model of modern cosmology, and is of course very interesting for general relativity.

The metrics listed above, i.e., Einstein metrics and FLRW metrics, are without any doubt interesting for general relativity. Of course, there are other metrics that could be interesting for general relativity, and we consider it very important to understand what 'interesting' metrics are geodesically rigid. In the present paper, in Section 3.1, we prove that *almost every 4-dimensional metric is geodesically rigid*.

Let us explain what we understand under almost every. Our result is local, so we will work in a small neighborhood $U \subset \mathbb{R}^4$ with fixed coordinates $(x_1, \dots, x_4)$. We consider a metric g as the mapping $g : U \to \mathbb{R}^{\frac{n(n+1)}{2}} = \mathbb{R}^{10}$; the space $\mathbb{R}^{\frac{n(n+1)}{2}}$ should be viewed as the space of symmetric $n \times n$-matrices. On the space of metrics (viewed as mappings) we consider the standard uniform C^2-topology: the metric g is ε-close to the metric $\bar{g}$ in this topology, if the components of g and their first and second derivatives are ε-close to that of $\bar{g}$.

In the present paper, we prove that

for any metric g and every $\varepsilon > 0$ there exists a metric $\hat{g}$ such that $\hat{g}$ is ε-close to g in the C^2-sense, and such that $\hat{g}$ is geodesically rigid. Moreover, there exists $\varepsilon' > 0$ such that every metric that is ε'-close to $\hat{g}$ in the C^2-sense is also geodesically rigid.

The result is also true in dimensions ≥ 4; the proof is essentially the same. Now, concerning the lower dimensions, the result is true in dimension 3, if we replace the uniform C^2-topology by the uniform C^3-topology. The proof (will not be given here) is based on the same idea. In dimension 2, the result is again true, if we replace the uniform C^2-topology by the uniform C^8-topology.

This result was expected, at least if we replace C^2-topology by C^∞-topology. Indeed, by Sinjukov [21] and Eastwood et al. [14], the existence of a metric geodesically equivalent to a given one is equivalent to the existence of a nontrivial solution of a certain linear system of partial differential equations in the Cauchy–Frobenius form (18), whose coefficients are certain invariant expressions in the components of the given metrics and their derivatives. It is known that the existence of the solution of such system is equivalent to certain differential conditions on coefficients, that is, on the entries of the metrics. If there exists at least one real-analytic metric that is geodesically rigid, then the differential conditions are not identically fulfilled, and almost every (in the C^∞-sense) metric is geodesically rigid. Now, the existence of geodesically rigid metrics in dimensions $n \geq 3$ is well known (at least since Sinjukov [22]). The existence of geodesically equivalent metrics in dimension $n = 2$ is more tricky; it follows from [23] where all above mentioned differential conditions were constructed. So in a certain sense our result is the improving C^∞-closeness (which should be clear to experts, though we did not find a place where it is written) to C^2-closeness.

Let us now comment on Subproblem 2.2. First of all, the problem is very classical, and was explicitly asked by Beltrami[2] in [24]. In the Riemannian case, it was solved by Dini in dimension 2 and Levi-Civita in all dimensions. More precisely, Dini has shown that locally, in a neighborhood of almost every point of a 2-dimensional manifold, every two geodesically equivalent metrics are given by the form (1) in a certain coordinate system. Levi-Civita has generalized this result to every dimension, we recall his result in Section 3.2.1.

Unfortunately, the proofs of Dini and Levi-Civita require that the (1, 1)-tensor $g^{i\ell}\bar{g}_{\ell j}$ is semisimple (i.e., has no Jordan blocks), and that all its eigenvalues are real. If one of the metrics is Riemannian, this condition is fulfilled automatically. Examples show the existence of geodesically equivalent pseudo-Riemannian metrics such that the (1, 1)-tensor $g^{i\ell}\bar{g}_{\ell j}$ is not semisimple or/and its eigenvalues are not real. The examples exist already in dimension 2: as shown[3] in [25], the metrics from every column of the table are geodesically equivalent (we assume that the functions X and Y depend on the indicated variables only, and that the function h is a holomorphic function of the complex variable $z = x + i \cdot y$). Moreover, every pair of 2-dimensional geodesically equivalent pseudo-Riemannian metrics has this form in a neighborhood of almost every point in a certain coordinate system.

	Liouville case	Complex-Liouville case	Jordan-block case
g	$(X(x)-Y(y))(\mathrm{d}x^2-\mathrm{d}y^2)$	$\Im(h)\mathrm{d}x\mathrm{d}y$	$\big(1+xY'(y)\big)\,\mathrm{d}x\mathrm{d}y$
$\bar{g}$	$\left(\frac{1}{Y(y)}-\frac{1}{X(x)}\right)\left(\frac{\mathrm{d}x^2}{X(x)}-\frac{\mathrm{d}y^2}{Y(y)}\right)$	$-\left(\frac{\Im(h)}{\Im(h)^2+\Re(h)^2}\right)^2\mathrm{d}x^2$ $+2\frac{\Re(h)\Im(h)}{(\Im(h)^2+\Re(h)^2)^2}\mathrm{d}x\mathrm{d}y+\left(\frac{\Im(h)}{\Im(h)^2+\Re(h)^2}\right)^2\mathrm{d}y^2$	$\frac{1+xY'(y)}{Y(y)^4}\big(-2Y(y)\mathrm{d}x\mathrm{d}y$ $+(1+xY'(y))\mathrm{d}y^2\big)$

By direct calculations we see that the (1, 1)-tensor $g^{i\ell}\bar{g}_{\ell j}$ for these metrics is semisimple with two real eigenvalues in the Liouville case (we also see that the form of the metrics is very similar to (1), the only difference is the signature), has two complex-conjugated eigenvalues in the Complex-Liouville case, and is not semisimple in the Jordan-block case.

Actually, certain authors consider that Subproblem 2.2 is also solved; the solution is attributed to Aminova [27]. Unfortunately, the author of the present paper does not understand her result, and has certain doubts that it is correct. More precisely, in view of [27, Theorem 1.1] and the formulas [27, (1.17), (1.18)] for $k = 1$, $n = 4$ and all εs equal to $+1$, the following two metrics g and $\bar{g}$ given by the matrices (where ω is an arbitrary function of the variable x_4).

$$\begin{bmatrix} 0 & 0 & 0 & 3x_3+3\omega(x_4) \\ 0 & 0 & 1 & 2x_2 \\ 0 & 1 & 0 & x_1 \\ 3x_3+3\omega(x_4) & 2x_2 & x_1 & 4x_1x_2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 3\frac{x_3+\omega(x_4)}{{x_4}^5} \\ 0 & 0 & 2{x_4}^{-5} & \frac{-3x_3-3\omega(x_4)+2x_2x_4}{{x_4}^6} \\ 0 & 2{x_4}^{-5} & -{x_4}^{-6} & \frac{3x_3+3\omega(x_4)-2x_2x_4+x_1{x_4}^2}{{x_4}^7} \\ 3\frac{x_3+\omega(x_4)}{{x_4}^5} & \frac{-3x_3-3\omega(x_4)+2x_2x_4}{{x_4}^6} & \frac{3x_3+3\omega(x_4)-2x_2x_4+x_1{x_4}^2}{{x_4}^7} & \frac{(-3x_3-3\omega(x_4)+2x_2x_4)\left(2x_1{x_4}^2+3x_3+3\omega(x_4)-2x_2x_4\right)}{{x_4}^8} \end{bmatrix}$$

should be geodesically equivalent, though they are not (which can be checked by direct calculations). Note that the metrics above have signature (2, 2), so they are not that interesting for general relativity. In the case of Lorentz signature, the theorem of Aminova seems to be correct, but still it is very complicated to extract the precise formulas from her works.

Note also that according to [27], in the case of Lorentz signature, geodesically equivalent metrics were described by Petrov [28] in dimension 3, by Golikov [29] in dimension 4, and by Kruchkovich [30] in all dimensions. From these papers, we were able to find (and to check) the paper of Petrov [28] only.

In the present paper, we combine recent results of [31] *and above mentioned results of* [25,28] *to give an easy algorithm how to obtain a list of pairs of all possible geodesically equivalent* 4*-dimensional metrics* g, $\bar{g}$ *of Lorentz signature.*

More precisely, we explain (following [31]) that every such pair can be obtained by applying the explicit gluing construction from Theorem 3 to building blocks, and provide explicit formulas for all possible building blocks. One can easily obtain a complete list of metrics by this algorithm. There exist three possible 3-dimensional building blocks, three possible 2-dimensional, and one possible 1-dimensional, so all together there exist 10 normal forms for geodesically equivalent (nonproportional) metrics of Lorentz signature. The normal forms are given by explicit formulas and allow certain freedom as (almost) arbitrary choice of functions of one variable or constants or metrics on 2- or 3-dimensional disks. We also explain the (only) difficulty in applying this algorithm in higher dimensions.

[2] Italian original from [24]: La seconda …generalizzazione …del nostro problema, vale a dire: riportare i punti di una superficie sopra un'altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.

[3] As explained in [25], essential part of the result could be attributed to Darboux [26].

2. Problem 1: how to reconstruct a metric from its nonparameterized geodesics

2.1. Subproblem 1.1: how to reconstruct a connection by nonparameterized geodesics, and when it is possible

We will work in arbitrary dimension $n \geq 2$, in a small neighborhood $U \subset \mathbb{R}^n$. We assume that we are given a family of smooth curves $\gamma(t;\alpha)$. We assume that the family is sufficiently big in the sense that at any point $x_0 \in U$ the set of vectors

$$\Omega_{x_0} := \left\{ \xi \in T_{x_0}U \mid \text{ there exists } \alpha \text{ and } t_0 \text{ such that } \frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t;\alpha)\big)_{|t=t_0} \text{ is proportional to } \xi \right\}$$

contains an open subset of $T_{x_0}U$. We put $\Omega = \bigcup_{x\in U} \Omega_x$. We will call a pair $(t_0;\alpha)x_0$-*admissible*, if $\frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t;\alpha)\big)_{|t=t_0} \in \Omega_{x_0}$. We need to understand whether there exists a symmetric affine connection Γ such that every curve $\gamma(t;\alpha)$ is a reparameterized geodesic of Γ, and construct this connection if it exists.

It is well known (at least since the time of Levi-Civita [20]) that, in local coordinates, every geodesic $\gamma : I \to U$, $\gamma : t \mapsto \gamma^i(t) \in U \subset \mathbb{R}^n$ of a symmetric affine connection Γ is given in terms of arbitrary parameter t as solution of

$$\frac{\mathrm{d}^2\gamma^a}{\mathrm{d}t^2} + \Gamma^a_{bc}\frac{\mathrm{d}\gamma^b}{\mathrm{d}t}\frac{\mathrm{d}\gamma^c}{\mathrm{d}t} = f\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)\frac{\mathrm{d}\gamma^a}{\mathrm{d}t}. \tag{6}$$

Better known version of this formula assumes that the parameter is affine (we denote it by "s") and reads

$$\frac{\mathrm{d}^2\gamma^a}{\mathrm{d}s^2} + \Gamma^a_{bc}\frac{\mathrm{d}\gamma^b}{\mathrm{d}s}\frac{\mathrm{d}\gamma^c}{\mathrm{d}s} = 0, \tag{7}$$

it is easy to check that the change of the parameter $s \longrightarrow t$ transforms (7) in (6).

For further use, let us note that if we linearly change the parameter t of a curve $\gamma(t;\alpha)$ (by putting $t = \mathrm{const}\cdot t_{\mathrm{new}}$), the left hand side of (6) is multiplied by const^2 implying that the function f should be homogeneous of degree 1: $f(\mathrm{const}\cdot\xi) = \mathrm{const}\cdot f(\xi)$ for every ξ (such that $\xi \in \Omega$). This allows us to assume without loss of generality that for every x the subset $\Omega_x \subseteq T_xU$ contains a cone over a nonempty open subset.

Let us now take a point $x_0 \in U$. For every x_0-admissible $(t_0;\alpha)$, we view the Eq. (6) as a system of equations on the entries of $\Gamma(x_0)$ and on the function $f_{|\Omega_{x_0}}$; the coefficients in this system come from known data $\left(\frac{\mathrm{d}\gamma(t;\alpha)}{\mathrm{d}t}\right)_{|t=t_0}, \left(\frac{d^2\gamma(t;\alpha)}{\mathrm{d}t^2}\right)_{|t=t_0}$. Since we have infinitely many x_0-admissible $(t;\alpha)$'s, we have an infinite system of equations. Let us show that if this system of equations is solvable, then the solution is unique up to a certain 'gauge' freedom.

Let us first describe the gauge freedom: we consider two connections Γ and $\bar\Gamma$ related by Levi-Civita's formula

$$\Gamma^a_{bc} = \bar\Gamma^a_{bc} - \delta^a_b\phi_c - \delta^a_c\phi_b, \tag{8}$$

where $\phi = \phi_i$ is a 1-form. Suppose the curve γ satisfies Eq. (6) with a certain function f. Substituting Γ given by (8) in the left hand side of (6) and using

$$(\delta^a_b\phi_c + \delta^a_c\phi_b)\frac{\mathrm{d}\gamma^b}{\mathrm{d}t}\frac{\mathrm{d}\gamma^c}{\mathrm{d}t} = 2\left(\frac{\mathrm{d}\gamma^b}{\mathrm{d}t}\phi_b\right)\frac{\mathrm{d}\gamma^a}{\mathrm{d}t},$$

we obtain that the same curve γ satisfies the Eq. (6) with respect to the connection $\bar\Gamma$ and the function

$$\bar f(v) := f(v) + 2\left(v^b\phi_b\right). \tag{9}$$

Thus, if (Γ, f) is a solution of (6), then for every 1-form ϕ the pair $\left(\bar\Gamma, \bar f\right)$ given by (8), (9) is also a solution. Let us show that up to this gauge freedom the connection Γ and the function f are unique.

We again work at one point $x_0 \in U$ and again view (6) as equations on (Γ, f). Suppose we have two solutions (Γ, f) and $(\bar\Gamma, \bar f)$. We subtract one equation from the other to obtain

$$\tilde\Gamma^a_{bc}v^bv^c = \tilde f(v)v^a, \tag{10}$$

where $\tilde\Gamma = \bar\Gamma - \Gamma, \tilde f = \bar f - f$. This equation is fulfilled for all vectors $v = v^a$ lying in an open nonempty $\Omega_{x_0} \subset T_{x_0}U$. Since the mapping $\sigma(u, v) \mapsto \tilde\Gamma^a_{bc}u^bv^c$ is linear in u and v, it satisfies the parallelogram equality

$$0 = \sigma(u+v, u+v) + \sigma(u-v, u-v) - 2\sigma(u,u) - 2\sigma(v,v). \tag{11}$$

Combining (11) with (10), we obtain

$$\begin{aligned} 0 &= \tilde f(u+v)(v+u) + \tilde f(u-v)(u-v) - 2\tilde f(u)u - 2\tilde f(v)v \\ &= (\tilde f(u+v) + \tilde f(u-v) - 2\tilde f(u))u + (\tilde f(u+v) - \tilde f(u-v) - 2\tilde f(v))v. \end{aligned} \tag{12}$$

Taking u and v to be linearly independent, we obtain

$$\begin{cases}\tilde f(u+v)+\tilde f(u-v)-2\tilde f(u)=0\\ \tilde f(u+v)-\tilde f(u-v)-2\tilde f(v)=0\end{cases} \tag{13}$$

implying $\tilde f(u+v)=\tilde f(u)+\tilde f(v)$. As we explained above, the functions $f,\bar f$, and, therefore, $\tilde f$, also satisfy $\mathrm{const}\cdot\tilde f(v)=\tilde f(\mathrm{const}\cdot v)$. Then, the restriction of $\tilde f$ to a certain nonempty open subset $\Omega'_{x_0}\subset\Omega_{x_0}\subset T_{x_0}U$ is linear, i.e., is given by $\tilde f(v)=2\phi_a v^a$ for a certain 1-form $\phi=\phi_a$ and for all v from Ω'_{x_0}. Then, the connection

$$\hat\Gamma^a_{bc}:=\bar\Gamma^a_{bc}-\phi_b\delta^a_c-\phi_c\delta^a_b$$

has the property that for every $(t_0;\alpha)$ such that $\left(\frac{\mathrm d\gamma^a}{\mathrm dt}\right)_{|t=t_0}\in\Omega'_{x_0}$ the corresponding $\gamma(t;\alpha)$ satisfies (at $t=t_0$) the equation

$$\frac{\mathrm d^2\gamma^a}{\mathrm dt^2}+\Gamma^a_{bc}\frac{\mathrm d\gamma^b}{\mathrm dt}\frac{\mathrm d\gamma^c}{\mathrm dt}=\frac{\mathrm d^2\gamma^a}{\mathrm dt^2}+\hat\Gamma^a_{bc}\frac{\mathrm d\gamma^b}{\mathrm dt}\frac{\mathrm d\gamma^c}{\mathrm dt}\quad\left(\Longleftrightarrow\Gamma^a_{bc}\frac{\mathrm d\gamma^b}{\mathrm dt}\frac{\mathrm d\gamma^c}{\mathrm dt}=\hat\Gamma^a_{bc}\frac{\mathrm d\gamma^b}{\mathrm dt}\frac{\mathrm d\gamma^c}{\mathrm dt}\right)$$

implying $\Gamma=\hat\Gamma$ implying that Γ and $\bar\Gamma$ are as in (8) implying f and $\bar f$ are as in (9).

Finally, the connection Γ and the function f, if they exist, are uniquely determined by the nonparameterized curves $\gamma(t;\alpha)$ up to the gauge freedom

$$\Gamma^a_{bc}\mapsto\Gamma^a_{bc}+\delta^a_b\phi_c+\delta^a_c\phi_b,\qquad f\mapsto f+2\phi. \tag{14}$$

Remark 2. If the function f is linear, i.e., if $f(\xi)=2\phi_b\xi^b$ for a certain 1-form ϕ, then, up to the gauge freedom, we can take $f\equiv0$. Moreover, putting $f\equiv0$ we exhaust the gauge freedom.

Let us now explain how to reconstruct the pair (Γ,f) up to the gauge freedom. We give an algorithm how to do it. The algorithm gives also a possibility to understand whether there exists such (Γ,f): we will see it that in order to uniquely reconstruct the (possible) entries $\Gamma(x_0)^i_{jk}$ of the connection at a point x_0, we will need only finitely many $\gamma(t;\alpha)$ passing through this point. There exists such (Γ,f), if for all x_0 the entries of $\Gamma(x_0)^i_{jk}$ do not depend on the x_0-admissible $(t_0;\alpha)$ we used to construct $\Gamma(x_0)^i_{jk}$.

We will work at a point x_0; our goal is to reconstruct the components $\Gamma(x_0)^i_{jk}$. We take x_0-admissible $(t_0;\alpha)$ such that the first component $\left(\frac{\mathrm d\gamma^1}{\mathrm dt}\right)_{|t=t_0}\neq0$. For this geodesic $\gamma(t_0;\alpha)$, we rewrite Eq. (6) at $t=t_0$ in the following form:

$$\begin{aligned}&f\left(\frac{\mathrm d\gamma}{\mathrm dt}\right)=\left(\frac{\mathrm d^2\gamma^1}{\mathrm d^2t}+\Gamma^1_{ab}\frac{\mathrm d\gamma^a}{\mathrm dt}\frac{\mathrm d\gamma^b}{\mathrm dt}\right)\Big/\frac{\mathrm d\gamma^1}{\mathrm dt}\\ &\frac{\mathrm d\gamma^2}{\mathrm dt}\Gamma^1_{ab}\frac{\mathrm d\gamma^a}{\mathrm dt}\frac{\mathrm d\gamma^b}{\mathrm dt}-\frac{\mathrm d\gamma^1}{\mathrm dt}\Gamma^2_{ab}\frac{\mathrm d\gamma^a}{\mathrm dt}\frac{\mathrm d\gamma^b}{\mathrm dt}=\frac{\mathrm d^2\gamma^2}{\mathrm d^2t}\frac{\mathrm d\gamma^1}{\mathrm dt}-\frac{\mathrm d\gamma^2}{\mathrm dt}\frac{\mathrm d^2\gamma^1}{\mathrm d^2t}\\ &\vdots\\ &\frac{\mathrm d\gamma^n}{\mathrm dt}\Gamma^1_{ab}\frac{\mathrm d\gamma^a}{\mathrm dt}\frac{\mathrm d\gamma^b}{\mathrm dt}-\frac{\mathrm d\gamma^1}{\mathrm dt}\Gamma^n_{ab}\frac{\mathrm d\gamma^a}{\mathrm dt}\frac{\mathrm d\gamma^b}{\mathrm dt}=\frac{\mathrm d^2\gamma^n}{\mathrm d^2t}\frac{\mathrm d\gamma^1}{\mathrm dt}-\frac{\mathrm d\gamma^n}{\mathrm dt}\frac{\mathrm d^2\gamma^1}{\mathrm d^2t}.\end{aligned} \tag{15}$$

The first equation of (15) is equivalent to the equation of (6) for $a=1$ solved with respect to $f\left(\frac{\mathrm d\gamma}{\mathrm dt}\right)$. We obtain the second, third, etc. equations of (15) by substituting the first equation of (15) in the equations of (6) corresponding to $a=2,3$, etc.

We consider now a subsystem of (15) containing the second, third, etc. equations of (15). We see that the system does not contain the function f. Then, for every x_0-admissible (t_0,α), it is a linear (inhomogeneous) system on the components $\Gamma(x_0)^i_{jk}$. We take a sufficiently big number N and substitute N x_0-admissible generic $(t_0;\alpha)$'s in this subsystem.

Remark 3. If $n=4$, it is sufficient to take $N=12$. We understand the world '*generic*' in the following sense: for every n pairs $(t_0;\alpha)$, the velocity vectors $\left(\frac{\mathrm d\gamma}{\mathrm dt}\right)_{|t=t_0}$ are linearly independent.

At every point x_0, we obtained an inhomogeneous linear system of equations on $\frac{n^2(n+1)}{2}$ unknowns $\Gamma(x_0)^i_{jk}$.

In the case the solution of this system does not exist (at least at one point x_0), there exists no connection whose (reparameterized) geodesics are $\gamma(t;\alpha)$.

If the solution exists at all points, the solution is unique up to the gauge freedom (8). Indeed, a solution of the last $n-1$ equations of (15) gives us also the values f by the first equation of (15), so the gauge freedom in the Eqs. (15) is the same as

of Eq. (6). Thus, a solution, if it exists, gives us the only up to the gauge freedom candidate for the entries $\Gamma(x_0)^i_{jk}$ at every point x_0 such that its geodesics are (reparameterized) curves $\gamma(t;\alpha)$.

Assume now that at every point x_0, a solution $\Gamma(x_0)^i_{jk}$ exists. In order to construct the entries $\Gamma(x_0)^i_{jk}$ (up to the gauge freedom), we used N x_0-admissible curves. In order to understand whether all geodesics $\gamma(t;\alpha)$ are reparameterized geodesics of Γ, we need to substitute all geodesics $\gamma(t;\alpha)$ in Eq. (6), and check whether it is fulfilled; in this case, it is natural to rewrite the Eq. (6) in the f-free form

$$\left(\frac{\mathrm{d}^2\gamma^a}{\mathrm{d}t^2}+\Gamma^a_{bc}\frac{\mathrm{d}\gamma^b}{\mathrm{d}t}\frac{\mathrm{d}\gamma^c}{\mathrm{d}t}\right)\wedge\frac{\mathrm{d}\gamma^a}{\mathrm{d}t}=0.$$

2.2. Subproblem 1.2: given an affine connection $\Gamma=\Gamma^i_{jk}$, how to understand whether there exists a metric g in the projective class of Γ? How to reconstruct this metric effectively?

2.2.1. General theory

We are given a symmetric affine connection Γ^i_{jk} on M^n, we need to understand whether there exists a metric in the projective class of Γ. In this section we recall (following [32,14]) the general approach how to do it: the existence of a metric in the projective class is equivalent to the existence of a nondegenerate solution of a certain system of linear PDE in the Cauchy–Frobenius form, and, in theory, there exists an algorithmic way to understand the existence of such solutions.

Theorem 1 (*[14], see also references inside*)**.** *g lies in the projective class of a connection Γ^i_{jk} if and only if $\sigma^{ab}:=g^{ab}\cdot\det(g)^{1/(n+1)}$ is a solution of*

$$\left(\nabla_a\sigma^{bc}\right)-\frac{1}{n+1}\left(\nabla_i\sigma^{ib}\delta^c_a+\nabla_i\sigma^{ic}\delta^b_a\right)=0. \tag{16}$$

Here $\sigma^{ab}:=g^{ab}\cdot\det(g)^{1/(n+1)}$ should be understood as an element of $S^2M\otimes(\Lambda_n)^{2/(n+1)}M$. In particular,

$$\nabla_a\sigma^{bc}=\underbrace{\frac{\partial}{\partial x^a}\sigma^{bc}+\Gamma^b_{ad}\sigma^{dc}+\Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}}-\underbrace{\frac{2}{n+1}\Gamma^{\mathrm{d}}_{da}\sigma^{bc}}_{\text{addition coming from volume form}}.$$

The Eq. (16) is a system of $\left(\frac{n^2(n+1)}{2}-n\right)$ linear PDEs of the first order on $\frac{n(n+1)}{2}$ unknown components of σ.

2-dimensional version of these equations was essentially known to Liouville [7]: instead of working with $\sigma^{ab}:=g^{ab}\cdot\det(g)^{1/(n+1)}$, he worked with $a_{ij}=\frac{1}{\det(\bar g)^{2/3}}\bar g_{ij}$; in dimension 2 the entries of σ^{ij} and a_{ij} are linearly related. The 2-dimensional analogue of the Eq. (16) is then the Liouville system of 4 PDE's of the first order

$$\begin{aligned}
&\frac{\partial a_{11}}{\partial x}+2K_0a_{12}-2/3K_1a_{11}=0\\
&2\frac{\partial a_{12}}{\partial x}+\frac{\partial a_{11}}{\partial y}+2K_0a_{22}+2/3K_1a_{12}-4/3K_2a_{11}=0\\
&\frac{\partial a_{22}}{\partial x}+2\frac{\partial a_{12}}{\partial y}+4/3K_1a_{22}-2/3K_2a_{12}-2K_3a_{11}=0\\
&\frac{\partial a_{22}}{\partial y}+2/3K_2a_{22}-2K_3a_{12}=0,
\end{aligned} \tag{17}$$

where $K_0:=-\Gamma^2_{11}$, $K_1:=\Gamma^1_{11}-2\Gamma^2_{12}$, $K_2:=-\Gamma^2_{22}+2\Gamma^1_{12}$, $K_3:=\Gamma^1_{22}$.

Remark 4. One sees that the gauge freedom (14) does not affect the coefficients $K_0,\dots,K_3$ of Eq. (17). One can check by calculations that this is also true in all dimensions: the gauge freedom (14) does not change the Eq. (16).

The PDE-system (16) can be prolonged (see [14]) to the system

$$\begin{cases}
\nabla_a\sigma^{bc}=\delta_a{}^b\mu^c+\delta_a{}^c\mu^b\\
\nabla_a\mu^b=\delta_a{}^b\rho-\frac{1}{n}P_{ac}\sigma^{bc}+\frac{1}{n}W_{ac}{}^b{}_d\sigma^{cd}\\
\nabla_a\rho=-\frac{2}{n}P_{ab}\mu^b+\frac{4}{n}Y_{abc}\sigma^{bc}
\end{cases} \tag{18}$$

where P is the symmetrized Ricci tensor, Y the Cotton-York tensor and $W_{ab}{}^c{}_d$ the projective Weyl tensor for the connection Γ.

Remark 5. Here we use another index convention for the projective Weyl tensor than in Section 2.2.2 of our paper. This convention is the same as in [14], and is standard in the so-called tractor calculus, we refer to [14] for precise formulas. In Section 2.2.2 we will explain the convention used there by given the formula for Weyl tensor.

The system (18) is a linear system of PDE of the first order on the unknown functions σ^{bc}, μ^b, ρ. Moreover, all derivatives of unknowns are expressed as functions of unknowns, i.e., the system is in the Cauchy–Frobenius form. One can understood this system geometrically as a connection on the projective tractor bundle $\mathcal{E}^{(BC)} = \mathcal{E}^{(bc)}(-2) + \mathcal{E}^{b}(-2) + \mathcal{E}(-2)$, see [14] for details. The solutions of the system are then parallel sections of the connection; there exists an algorithmic way to understand whether a certain connection admits a nontrivial parallel section. In the 2-dimensional case, the algorithm was fulfilled for certain projectively homogeneous connections in [32]; for arbitrary 2-dimensional connection, the algorithm was fulfilled in [13], and the answer (i.e., the differential conditions on K_i such that its vanishing implies the existence of a nontrivial solution) appears to be very complicated. In theory, one can fulfill this algorithm for every dimension; it is clearly a nontrivial task. In the next section we will show that, under the additional assumption that the searched metric is Ricci-flat, there exists a trick that simplifies the algorithm.

2.2.2. The case $n = 4$, g is Ricci-flat

Let us now assume that we know the geodesics of a nonflat Ricci-flat metric. That is, we know a certain Γ such that for a certain ϕ_a which we do not know $\bar{\Gamma}^a_{bc} := \Gamma^a_{bc} + \delta^a_b\phi_c + \delta^a_c\phi_c$ is the Levi-Civita connection of a certain nonflat Ricci-flat metric which we again do not know. Our goal is to find this metric (which I call $\bar{g}$). By the above mentioned results of Petrov [28], Hall et al. [4,17], and Kiosak et al. [16], the metric is unique up to multiplication by a constant; the goal of this section is to explain how to find it algorithmically. The algorithm works under certain additional (generic) condition on the connection Γ.

We consider the projective Weyl tensor introduced in [33] (not to be confused with the conformal Weyl tensor)

$$W^i{}_{jk\ell} := R^i{}_{jk\ell} - \frac{1}{n-1}\left(\delta^i_\ell R_{jk} - \delta^i_k R_{j\ell}\right) \tag{19}$$

(in our convention $R_{jk} = R^a{}_{jka}$, so that $W^a{}_{jka} = 0$).

Weyl has shown that the projective Weyl tensor does not depend of the choice of connection within the projective class: if the connections Γ and $\bar{\Gamma}$ are related by the formula (8), then their projective Weyl tensors coincide. Now, from the formula (19), we know that, if the searched $\bar{g}$ is Ricci-flat, projective Weyl tensor coincides with the Riemann tensor $\bar{R}^i_{jk\ell}$ of $\bar{g}$. Thus, if we know the projective class of the Ricci-flat metric $\bar{g}$, we know its Riemann tensor.

Then, the metric $\bar{g}$ must satisfy the following system of equations due to the symmetries of the Riemann tensor:

$$\begin{cases} \bar{g}_{ia}W^a{}_{jkm} + \bar{g}_{ja}W^a{}_{ikm} = 0 \\ \bar{g}_{ia}W^a{}_{jkm} - \bar{g}_{ka}W^a{}_{mij} = 0. \end{cases} \tag{20}$$

The first portion of the Eq. (20) is due to the symmetry ($\bar{R}_{ijkm} = -\bar{R}_{jikm}$), and the second portion is due to the symmetry ($\bar{R}_{kmij} = \bar{R}_{ijkm}$) of the curvature tensor of $\bar{g}$.

We see that for every point $x_0 \in U$ (20) is a system of linear equations on $\bar{g}(x_0)_{ij}$. The number of equations (around 100) is much bigger than the number of unknowns (which is 10). It is expected therefore, that a generic projective Weyl tensor $W^i{}_{jkl}$ admits no more than 1-dimensional space of solutions (by assumptions, our W admits at least 1-dimensional space of solutions). The expectation is true, as the following classical result shows.

Theorem 2 *([15,34–37]). Let $W^i{}_{jk\ell}$ be a tensor in $\mathbb{R}^4$ such that it is skew-symmetric with respect to k, l and such that its traces $W^a{}_{ak\ell}$ and $W^a{}_{ja\ell}$ vanish. Assume that for all 1-forms $\xi_i \neq 0$ we have $W^a{}_{jk\ell}\xi_a \neq 0$. Then, the Eqs. (20) have no more than 1-dimensional space of solutions.*

Let us comment on the condition $W^a{}_{jk\ell}\xi_a \neq 0$. In this context, for every fixed indexes k, ℓ, $W^i{}_{j**}$ could be viewed as a $n \times n$-matrix; and the condition $W^a{}_{j**}\xi_a = 0$ means that the matrix has a nontrivial kernel (in particular, it is degenerate). Now, the condition $W^a{}_{jk\ell}\xi_a = 0$ means that for all indexes k, ℓ the kernels of the $n \times n$-matrices $W^a{}_{jk\ell}$ have nontrivial intersection. *Thus, it is a very restrictive condition on W, and, therefore, on Γ.*

This result shows that, under the assumptions that for all $\xi_i \neq 0$ we have $W^a{}_{jk\ell}\xi_a \neq 0$, we can reconstruct the conformal class of the metric $\bar{g}$ by solving the system of linear equations (20). This can be done algorithmically. Then, we also know the conformal class of σ in (16), i.e., we know that σ is of the form

$$\sigma^{ij} = \mathrm{e}^{\lambda}a^{ij}, \tag{21}$$

where a^{ij} is known and comes from the solution of the linear system (20), and the function λ is unknown. Substituting the ansatz (21) in the system (16), we obtain an inhomogeneous system of linear equations on the components $\frac{\partial\lambda}{\partial x^i}$. Direct calculations show that this system has at most one solution; since we assumed the existence of the metric in the projective class, one can always solve this system and obtain all $\frac{\partial\lambda}{\partial x^i}$. Finally, we can obtain the function λ, and, therefore, the metric $\bar{g}$, by integration.

Let us note that in all steps we assumed that a Ricci-flat metric $\bar g$ exists in the given projective class. But the algorithm also gives us an algorithmic check whether such metric exists: one should go along the steps of the algorithm and look whether something goes wrong.

For example, the system (20) could have no nontrivial solution (i.e., every solution $\bar g_{ij}$ of (20) has zero determinant). Then, no Ricci-flat metric $\bar g$ exists in our projective class.

If the system (20) has nontrivial solution, then, after plugging the ansatz (21) in (16), we obtain a system of nonhomogeneous linear equations on $\frac{\partial\lambda}{\partial x^i}$. If the system has no solution (the number of equations is much bigger than the number of unknowns; besides, the system is inhomogeneous), or the 1-form $\frac{\partial\lambda}{\partial x^i}\mathrm{d}x_i$ is not exact,[4] then no Ricci-flat metric $\bar g$ exists in our projective class.

Finally, if the system (20) has nontrivial solution, and if we can solve the system of linear equations we obtain after plugging the ansatz (21) in (16), and if the solution $\frac{\partial\lambda}{\partial x^i}$ satisfies the condition $\frac{\partial}{\partial x^k}\frac{\partial\lambda}{\partial x^i}=\frac{\partial}{\partial x^i}\frac{\partial\lambda}{\partial x^k}$, then, locally, we do obtain a metric g_{ij} in the projective class. The metric must not be Ricci-flat though.

Remark 6. In Section 3.1, we show that one can reconstruct an almost every (4-dimensional) metric by its projective class, see Remark 9 there. In the case of arbitrary metric, the nondegeneracy assumption on the projective class is more complicated, and it is harder to check it.

3. Problem 2: In what situations is the reconstruction of the metric by the nonparameterized geodesics unique (up to the multiplication of the metric by a constant)?

3.1. For generic 4-dimensional metric, the reconstruction of the metric by the nonparameterized geodesics is unique

Let us first construct one geodesically rigid metric in dimension $n=4$.

Using the formula (19), by short tensor calculations we see that the metric g_{ij} must satisfy the equation

$$ng^{a(i}W^{j)}{}_{akl}=g^{ab}W^{(i}{}_{ab[l}\delta^{j)}_{k]}, \tag{22}$$

where $n=4$, the brackets "[ ]" denote the skew-symmetrization without division, and the brackets "( )" denote the symmetrization without division.

Remark 7. Actually, Eq. (22) is a part of the curvature of the tractor connection (18); in this context, it was obtained in [14].

We take a 4-dimensional metric $\bar g$ such that at the point x_0 it is given by the identity matrix

$$\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix},$$

and such that its curvature tensor (with lowered indexes) R_{ijkl} at the point x_0 is given by

$$R_{ijkl}=h_{ik}h_{jl}-h_{il}h_{jk}+H_{ik}H_{jl}-H_{il}H_{jk}, \tag{23}$$

where the entries at x_0 of the $(0,2)$-tensors h and H are given by the diagonal matrices

$$h=\begin{bmatrix}1&&&\\&2&&\\&&-1&\\&&&0\end{bmatrix}\qquad H=\begin{bmatrix}0&&&\\&0&&\\&&1&\\&&&1\end{bmatrix}.$$

Such metric $\bar g$ exists by [38, Theorem 1.12.2] (see also [39, Theorem 1.1]), since the tensor (23) satisfies all symmetries of the curvature tensor.

Every metric g geodesically equivalent to $\bar g$ has the same projective Weyl tensor as $\bar g$. We view Eq. (22) as the system of homogeneous linear equations on the components of g; every metric g geodesically equivalent to $\bar g$ satisfies this system of equations (with the same coefficients W!). At the point x_0, this is a system on 10 unknowns $g(x_0)^{ij}$. Since the system is symmetric in i,j and skew-symmetric in k,l, the system contains 60 equations (actually, less because of certain hidden symmetries inside). By direct calculations, we see that the rank of this system is 9. Indeed, it has at least one nontrivial solution, namely $\bar g(x_0)^{ij}$, so its rank is at most 9. One can easily find 9 linear independent equations of this system (so the rank is at least 9), namely the equations corresponding to the followings indexes (i,j,k,l):

[4] The notation $\frac{\partial\lambda}{\partial x^i}$ is slightly misleading here: at this stage, the unknowns in the linear system we solve, i.e., $\frac{\partial\lambda}{\partial x^i}$, are not a priori partial derivatives of a certain function λ. They are partial derivatives (locally) of a certain function which we denote λ, if and only if the form $\frac{\partial\lambda}{\partial x^i}\mathrm{d}x^i$ is closed $\left(\Longleftrightarrow \frac{\partial}{\partial x^k}\frac{\partial\lambda}{\partial x^i}=\frac{\partial}{\partial x^i}\frac{\partial\lambda}{\partial x^k}\right)$.

(i, j, k, l)	Equation
$(1, 1, 2, 1)$	$-12g^{12} = 0$
$(1, 1, 3, 1)$	$2g^{13} = 0$
$(1, 1, 4, 1)$	$2g^{14} = 0$
$(2, 1, 2, 1)$	$5g^{11} - 6g^{22} + g^{33} = 0$
$(2, 1, 4, 1)$	$g^{24} = 0$
$(2, 2, 3, 2)$	$8g^{23} = 0$
$(3, 1, 3, 1)$	$-4g^{11} + g^{33} + 4g^{22} - g^{44} = 0$
$(3, 1, 4, 1)$	$2g^{34} = 0$
$(3, 2, 3, 2)$	$-6g^{22} + 4g^{33} + 3g^{11} - g^{44} = 0.$

We see that the equations in the table are linearly independent. Thus, at the point x_0, the set of solutions of this system is 1-dimensional, implying that every metric g, geodesically equivalent to $\bar{g}$, is proportional to $\bar{g}$.

Let us show that at every point in a small neighborhood of x_0, the system (22) also has rank 9. Indeed, the rank of a matrix is the biggest dimension of a nondegenerate quadratic submatrix and therefore is a lower semi-continuous (integer valued) function, i.e., rank of this system is at least 9 at every point of a small neighborhood of x_0. Now, at every point the components $\bar{g}^{ij}$ give us a nontrivial solution, so the rank cannot be bigger than 9. Thus, in a small neighborhood of x_0, every metric g geodesically equivalent to $\bar{g}$ is conformally equivalent to $\bar{g}$. Now, by Weyl [33], two conformally equivalent 4-dimensional metrics are proportional. Then, the metric $\bar{g}$ is geodesically rigid.

Now let $\tilde{g}$ be an arbitrary metric in a small neighborhood of x_0. We consider the metric

$$g_t := (1 - t)\tilde{g} + t\bar{g}.$$

The system (22) constructed for this metric has rank 9 for t lying in a small interval around 1. Since the coefficients of the system are algebraic expressions in t whose coefficients are algebraic expressions in the components of $\bar{g}$, $\tilde{g}$ and their first and second derivatives, for almost all t the system (22) constructed for the metric g_t has rank 9. We take t close to 0 such that the metric g_t is ε-close to $\tilde{g}$ and such that the system (22) constructed for the metric g_t has rank 9. As we explained above, this metric is geodesically rigid. Every metric $\hat{g}$ that is C^2-close to g_t is also geodesically rigid, since the entries of W for $\hat{g}$ are algebraic expressions in the components of $\hat{g}$ and its first and second derivatives. Hence, the coefficients in the system (22) constructed for $\hat{g}$ are close to that of the system (22) constructed for g_t implying the system also has rank 9 implying the metric $\hat{g}$ is geodesically rigid as well.

Thus, for every 4-dimensional metric $\tilde{g}$ and for any $\varepsilon > 0$ there exists a metric g_t that is ε-close in the C^2-sense to $\tilde{g}$ and $\varepsilon' > 0$ such that all metrics ε'-close in the C^2-sense to g_t are geodesically rigid.

Remark 8. As we mentioned in the introduction, a similar proof can be done for all dimensions $n \geq 4$. For dimensions 2 and 3, the proof does not work anymore, since the system (22) has corank at least 2 for all metrics g (one can prove it using the methods of [40, Section 2.3.2]). One can still modify the proof replacing the system (22) by another projectively invariant system of equations. This other projectively invariant system of equations requires higher derivatives of the components of g though. In dimension 3, one can construct (using the curvature of the tractor connection (18), see also [41]) such projectively invariant system such that its coefficients depend on the components of the metrics and its first, second and third derivatives. Therefore, for every 3-dimensional local metric $\tilde{g}$ and for any $\varepsilon > 0$ there exists a metric g_t that is ε-close in the C^3-sense to $\tilde{g}$ and $\varepsilon' > 0$ such that all metrics ε'-close in the C^3-sense to g_t are geodesically rigid. Now, in dimension 2, the construction of the projectively invariant system is much more involving (see [13]) and requires 8 derivatives of the components of the metric.

Remark 9. We also see that the projective class of almost every (in the C^2-sense) 4-dimensional metric determines its conformal class uniquely: one can find the conformal class by solving the system (22). Then, one can proceed along the algorithm from Section 2.2.2 and understand whether there exists a metric in the projective class, and find it.

3.2. Normal forms for pairs of geodesically equivalent 4-dimensional metrics such that one of them has Lorentz signature

3.2.1. Splitting and gluing constructions from [31]

Given two metrics g and $\bar{g}$ on the same manifold, we consider the $(1, 1)$-tensor $L = L(g, \bar{g})$ defined by

$$L^i_j := \left(\frac{\det(\bar{g})}{\det(g)}\right)^{\frac{1}{n+1}} \bar{g}^{ik} g_{kj}, \tag{24}$$

where $\bar{g}^{ik}$ is the contravariant inverse of $\bar{g}_{ik}$.

Remark 10. If n is even, the tensor L is always well defined. If n is odd, the ratio $\det(\bar g)/\det(g)$ may be negative, and the formula (24) may have no sense. In this case, we replace $\bar g$ by $-\bar g$ and make the ratio $\det(\bar g)/\det(g)$ positive and L well defined. In the cases interesting in our context, g and $\bar g$ have the same signature, and the problem with the sign does not appear at all.

Remark 11. The tensor L^i_j defined in (24) is essentially the same as the tensor introduced by Sinjukov (see Eqs. (32), (34) on the page 134 of the book [21], and also Theorem 4 on page 135) and which is often denoted by tensor a_{ij} in the related literature. More precisely, $L^i_j = a_{\ell j}g^{\ell i}$. It is also closely related to σ from Section 2.2.1: $\bar g$ is geodesically equivalent to g, if and only if $\bar\sigma^{ab} := L^a{}_\ell g^{\ell b}\cdot \det(g)^{1/(n+1)}$ is a solution of (16).

The simplified version of the *gluing construction* does the following. Consider two manifolds M_1 and M_2 with pairs of geodesically equivalent metrics $h_1 \sim \bar h_1$ on M_1 and $h_2 \sim \bar h_2$ on M_2. Assume that the corresponding (1, 1)-tensor fields $L_1 = L(h_1, \bar h_1)$ and $L_2 = L(h_2, \bar h_2)$ have no common eigenvalues in the sense that for any two points $x \in M_1$, $y \in M_2$ we have

$$\text{Spectrum } L_1(x) \cap \text{Spectrum } L_2(y) = \varnothing.$$

Then one can naturally construct a pair of geodesically equivalent metrics $g \sim \bar g$ on the direct product $M = M_1 \times M_2$. These new metrics g and $\bar g$ differ from the direct product metrics $h_1 + h_2$ and $\bar h_1 + \bar h_2$ on $M_1 \times M_2$ and are given by the following formulas involving L_1 and L_2: we denote by χ_i, $i = 1, 2$, the characteristic polynomial of L_i: $\chi_i = \det(t\cdot \mathbf{1} - L_i)$. We treat the (1, 1)-tensors L_i as linear operators acting on TM_i. A polynomial $f(L)$ in L is then the (1, 1)-tensor of the form $f(L) = a_0(x)\cdot \mathrm{Id} + a_1(x)L + a_2(x)L^2 + \cdots + a_m(x)L^m$. For two tangent vectors

$$u = (\underbrace{u_1}_{\in TM_1}, \underbrace{u_2}_{\in TM_2}), \qquad v = (\underbrace{v_1}_{\in TM_1}, \underbrace{v_2}_{\in TM_2}) \in TM$$

we put

$$g(u, v) = h_1\left(\chi_2(L_1)(u_1), v_1\right) + h_2\left(\chi_1(L_2)(u_2), v_2\right), \tag{25}$$

$$\bar g(u, v) = \frac{1}{\chi_2(0)}\bar h_1\left(\chi_2(L_1)(u_1), v_1\right) + \frac{1}{\chi_1(0)}\bar h_2\left(\chi_1(L_2)(u_2), v_2\right). \tag{26}$$

The corresponding (1, 1)-tensor $L = L(g, \bar g)$ is the direct sum of L_1 and L_2 in the natural sense: for every

$$v = (\underbrace{v_1}_{\in T_xM_1}, \underbrace{v_2}_{\in T_yM_2}) \in T_{(x,y)}(M_1 \times M_2) \quad \text{we have } L(v) = (L_1(v_1), L_2(v_2)).$$

It might be convenient to understand the formulas (25), (26) in matrix notation: we consider the coordinate system $(x^1, \dots, x^r, y^{r+1}, \dots, y^n)$ on M such that x-coordinates are coordinates on M_1 and y-coordinates are coordinates on M_2. Then, in this coordinate system, the matrices of g and $\bar g$ have the block diagonal form

$$g = \begin{pmatrix} h_1\chi_2(L_1) & 0 \\ 0 & h_2\chi_1(L_2) \end{pmatrix}, \qquad \bar g = \begin{pmatrix} \frac{1}{\chi_2(0)}\bar h_1\chi_2(L_1) & 0 \\ 0 & \frac{1}{\chi_1(0)}\bar h_2\chi_1(L_2) \end{pmatrix}. \tag{27}$$

Theorem 3 (*Gluing Lemma from [31]*)**.** *If h_1 is geodesically equivalent to $\bar h_1$, and h_2 is geodesically equivalent to $\bar h_2$, then the metrics $g, \bar g$ given by (25) and (26) are geodesically equivalent too.*

The *splitting construction* is the inverse operation. We will not describe it completely (and refer to [31]); we will use its following corollary explained in [31, Section 2.1]:

Every pair of geodesically equivalent metrics h and $\bar h$ in a neighborhood of almost every point can be obtained (up to a coordinate change) by applying splitting construction to building blocks.

By a *building block* we understand an open neighborhood $U \subset \mathbb{R}^m$ with a pair of geodesically equivalent metrics $h \sim \bar h$ such that at every point the tensor L given by (24) has only one real eigenvalue, or two complex-conjugate eigenvalues, and such that the geometric multiplicity of the eigenvalue is constant on U.

Remark 12. Riemannian version of the splitting/gluing constructions was known before, see for example [42, Lemma 2] and [43, Sections 2.2, 2.3].

Example 1. In the definition of the building block, we allow the dimension $m = 1$. Then, the following two metrics on the interval $I \subset \mathbb{R}^1$ with the following two geodesically equivalent metrics $h = \mathrm{d}x^2$ and $\bar h = X(x)\mathrm{d}x^2$ (where the function X never vanishes) form a building block. Actually, up to a coordinate change, $(U_1, h, \bar h)$ is the only 1-dimensional building block.

Example 2. All possible examples of 2-dimensional building blocks can be extracted from the table of 2-dimensional geodesically equivalent metrics from the introduction. The metrics from the first column of the table do not correspond to a building block, since the tensor L for these metrics has two different eigenvalues, $X(x)$ and $Y(y)$. But the metrics from the second and the third columns do correspond to the building block, since the tensors L for these metrics are given by the matrices

$$\begin{bmatrix} \Re(h) & \Im(h) \\ -\Im(h) & \Re(h) \end{bmatrix}, \qquad \begin{bmatrix} Y(x_2) & 0 \\ 1 + x_1 \dfrac{\mathrm{d}}{\mathrm{d}x_2} Y(x_2) & Y(x_2) \end{bmatrix}.$$

Of course, in every dimension, in particular in dimension two, there exists a trivial building block $(U, h, \bar{h} = \text{const} \cdot h)$; the tensor L for this metric is a multiple of δ^i_j. From the results of [25] it follows that every 2-dimensional building block has one of these three forms.

The formulas for the 3-dimensional building block can be obtained using Petrov [28] and Eisenhart [44]; we will give them later. From linear algebra it follows that if the metrics $g, \bar{g}$ have Lorentz signature, then 4-dimensional building blocks are not possible (except for the trivial block corresponding to proportional metrics $g \sim \bar{g} := \text{const} \cdot g$), since in the Lorentz signature a g-selfadjoint $(1, 1)$ tensor L cannot have a Jordan block of dimension ≥ 4 with real eigenvalue, and a Jordan block of dimension ≥ 2 with complex eigenvalue.

Example 3 (*Dini Formulas (1) Follow from splitting–gluing constructions*)**.** We consider the two 1-dimensional building blocks

$$\left(I_1, h_1 = \mathrm{d}x^2, \bar{h}_1 = \frac{1}{X(x)^2}\mathrm{d}x^2\right) \quad \text{and} \quad \left(I_2, h_2 = -\mathrm{d}y^2, \bar{h}_2 = -\frac{1}{Y(x)^2}\mathrm{d}y^2\right).$$

We assume that $X(x) > Y(y)$ for all (x, y). The corresponding tensors L_1 and L_2 (we view them as 1×1-matrices) and their characteristic polynomials are

$$L_1 = (X(x)); \qquad L_2 = (Y(y)); \qquad \chi_1(t) = t - X(x); \qquad \chi_2(t) = t - Y(y).$$

We see that the metrics h_1, h_2 satisfy the assumptions in Theorem 3. Plugging these data in the formulas (27), we obtain geodesically equivalent metrics g and $\bar{g}$ given by the matrices

$$g = \begin{pmatrix} X(x) - Y(y) & \\ & X(x) - Y(y) \end{pmatrix}, \qquad \bar{g} = \begin{pmatrix} \dfrac{X(x) - Y(y)}{X(x)^2 Y(y)} & \\ & \dfrac{X(x) - Y(y)}{X(x) Y(y)^2} \end{pmatrix}.$$

We see that these metrics are precisely the Dini metrics (1). For further use let us note that the tensor (24) for these metrics is given by $L = \begin{pmatrix} X(x) & \\ & Y(y) \end{pmatrix}$.

Example 4 (*Levi-Civita Metrics (2), (3) Follow from splitting–gluing constructions*)**.** We take 4 pairs of geodesically equivalent metrics on the interval I.

$$\begin{aligned} &g_1 = \mathrm{d}x_1^2 \sim \bar{g}_1 = \frac{1}{X_1(x_1)^2}\mathrm{d}x_1^2; \qquad g_2 = -\mathrm{d}x_2^2 \sim \bar{g}_2 = -\frac{1}{X_2(x_2)^2}\mathrm{d}x_2^2; \\ &g_3 = \mathrm{d}x_3^2 \sim \bar{g}_3 = \frac{1}{X_3(x_3)^2}\mathrm{d}x_3^2; \qquad g_4 = -\mathrm{d}x_4^2 \sim \bar{g}_4 = -\frac{1}{X_4(x_4)^2}\mathrm{d}x_4^4. \end{aligned} \tag{28}$$

We assume that for $i \neq j$ $X_i(x_i) \neq X_j(x_j)$ for all $x_i, x_j \in I$.

Gluing $(I, g_1, \bar{g}_1)$ and $(I, g_2, \bar{g}_2)$, ($(I, g_3, \bar{g}_3)$ and $(I, g_4, \bar{g}_4)$, respectively) we obtain two pairs of geodesically equivalent metrics (we denote them by $h_1 \sim \bar{h}_1$ ($h_2 \sim \bar{h}_2$, respectively)) on the 2-dimensional disk $U^2 = I \times I$. These metrics and the corresponding tensors (24) were essentially constructed in Example 3 and are given by matrices

$$h_1 = \begin{pmatrix} X_1(x_1) - X_2(x_2) & \\ & X_1(x_1) - X_2(x_2) \end{pmatrix} \sim \bar{h}_1 = \begin{pmatrix} \dfrac{X_1(x_1) - X_2(x_2)}{X_1(x_1)^2 X_2(x_2)} & \\ & \dfrac{X_1(x_1) - X_2(x_2)}{X_1(x_1) X_2(x_2)^2} \end{pmatrix},$$

$$h_2 = \begin{pmatrix} X_3(x_3) - X_4(x_4) & \\ & X_3(x_3) - X_4(x_4) \end{pmatrix} \sim \bar{h}_2 = \begin{pmatrix} \dfrac{X_3(x_3) - X_4(x_4)}{X_3(x_3)^2 X_4(x_4)} & \\ & \dfrac{X_3(x_3) - X_4(x_4)}{X_3(x_4) X_4(x_4)^2} \end{pmatrix},$$

$$L_1 = L(h_1, \bar{h}_1) = \begin{pmatrix} X_1(x_1) & \\ & X_2(x_2) \end{pmatrix},$$

$$L_2 = L(h_2, \bar{h}_2) = \begin{pmatrix} X_3(x_3) & \\ & X_4(x_4) \end{pmatrix}.$$

We see that the metrics h_1, h_2 satisfy the assumptions in Theorem 3. Gluing these metrics, we obtain the metrics (2), (3).

Remark 13. By changing the sign of the metrics (28) we can make geodesically equivalent metrics $g \sim \bar{g}$ of arbitrary signature.

Example 5 (*General Levi-Civita Metrics*)**.** We take m building blocks: the first r building blocks are 1-dimensional, and the last $m-r$ building blocks $h_{r+1} \sim \bar{h}_{r+1}, \dots, h_m \sim \bar{h}_m$ have dimensions $k_i \geq 2$, $i = r+1, \dots, m-r$. For cosmetic reasons we think that the first r building blocks are

$$\left(U_i^1, h_i = \pm \mathrm{d}x_i^2, \bar{h}_i = \pm \frac{1}{X_i(x_i)^2} \mathrm{d}x_i^2\right), \quad i = 1, \dots, r, \tag{29}$$

the sign $\pm$ in h_i and $\bar{h}_i$ is the same for each i, but may be different for different i's. The last $m-r$ building blocks are

$$\left(U_i^{k_i}, h_i = \sum_{\alpha_i, \beta_i = 1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} \mathrm{d}x_i^{\alpha_i} \mathrm{d}x_i^{\beta_i}, \bar{h}_i = \frac{1}{X_i^{k+1}} \sum_{\alpha_i, \beta_i = 1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} \mathrm{d}x_i^{\alpha_i} \mathrm{d}x_i^{\beta_i}\right), \quad i = r+1, \dots, m.$$

Here the functions X_i are constant for $i > r$ and depend only on the corresponding variable x_i for $i \leq r$. As above, we assume that $\mathrm{Image}(X_i) \cap \mathrm{Image}(X_j) = \varnothing$ for $i \neq j$. The metrics h_i, $i = r+1, \dots, m$ can be arbitrary, but their entries $(h_i)_{\alpha_i \beta_i}$ must depend on the coordinates $x_i = (x_i^1, \dots, x_i^{k_i})$ only.

Inductively applying the gluing procedure, we obtain for g and $\bar{g}$ the following form:

$$\begin{aligned} g &= \sum_{i=1}^{r} P_i \mathrm{d}x_i^2 + \sum_{i=r+1}^{m} \left[P_i \sum_{\alpha_i, \beta_i = 1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} \mathrm{d}x_i^{\alpha_i} \mathrm{d}x_i^{\beta_i} \right], \\ \bar{g} &= \sum_{i=1}^{r} P_i \rho_i \mathrm{d}x_i^2 + \sum_{i=r+1}^{m} \left[P_i \rho_i \sum_{\alpha_i, \beta_i = 1}^{k_i} (h_i(x_i))_{\alpha_i \beta_i} \mathrm{d}x_i^{\alpha_i} \mathrm{d}x_i^{\beta_i} \right], \end{aligned} \tag{30}$$

where

$$P_i := \pm \prod_{j \neq i} (X_i - X_j), \qquad \rho_i := \frac{1}{X_i \prod\limits_{\alpha} X_\alpha} \tag{31}$$

(the signs $\pm$ in (31) depend on the choice of the signs $\pm$ in (29) and can be arbitrary). This is precisely Levi-Civita's normal form for geodesically equivalent (Riemannian) metrics from [20].

Now, since every pair of geodesically equivalent metrics (in a neighborhood of almost every point) can be obtained by a gluing construction, and since in the Riemannian signature only the blocks used above can be used, every Riemannian geodesically equivalent metrics have the form (30) in a certain coordinate system. This is the famous Levi-Civita's Theorem from [20].

Note, than the Lorentz signature of g and $\bar{g}$ does not allow the tensor L to have complex eigenvalues of algebraic multiplicity greater than one. Similarly, it does not allow the tensor L to have a Jordan block of dimension 4, or two Jordan blocks. Thus, in order to obtain the description of nonproportional 4-dimensional geodesically equivalent metrics of Lorentz signature, one needs the building blocks of dimensions 1, 2, 3 only. In dimension 1, only one building block, namely the one from Example 1, is possible.

Geodesically equivalent metrics such that the tensor L has the 2-dimensional Jordan-block structure

$$\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

were described in Example 2. For the Jordan-block structure

$$\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}, \tag{32}$$

the description of the metrics follows from Petrov [28]: the metrics are given by

$$
\begin{aligned}
g &= \left(4x_2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)+2\right)\mathrm{d}x_1\mathrm{d}x_3+\mathrm{d}x_2{}^2+2x_1\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)\mathrm{d}x_2\mathrm{d}x_3+x_1{}^2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)^2\mathrm{d}x_3{}^2,\\
\bar g &= \frac{1}{\lambda\,(x_3)^6}\Bigg[\left(4x_2\lambda\,(x_3)^2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)+2\lambda\,(x_3)^2\right)\mathrm{d}x_1\mathrm{d}x_3+\lambda\,(x_3)^2\,\mathrm{d}x_2{}^2\\
&\quad-\left(4x_2\lambda\,(x_3)\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)+2\lambda\,(x_3)-2x_1\lambda\,(x_3)^2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)\right)\mathrm{d}x_2\mathrm{d}x_3\\
&\quad+\left(4x_2{}^2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)^2+4x_2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)-4x_1x_2\lambda\,(x_3)\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)^2\right)\mathrm{d}x_3{}^2\\
&\quad+\left(1+x_1{}^2\lambda\,(x_3)^2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)^2-2x_1\lambda\,(x_3)\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)\right)\mathrm{d}x_3{}^2\Bigg].
\end{aligned}
\tag{33}
$$

The corresponding L is given by the matrix

$$
\begin{bmatrix}
\lambda\,(x_3) & 1 & \left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)x_1\\
0 & \lambda\,(x_3) & 2\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\lambda\,(x_3)\right)x_2+1\\
0 & 0 & \lambda\,(x_3)
\end{bmatrix}.
$$

Remark 14. Actually, the formulas (33) are slightly more general than that of [28]. They are equivalent to the formulas from [28] (modulo a coordinate transformation) at the points such that $\mathrm{d}\lambda\neq 0$. The formulas [28] were obtained together with A. Bolsinov; they can be generalized for every dimension. We will publish this result elsewhere.

As it follows from [16, Lemma 6], if L has the Jordan-form $\begin{pmatrix}\lambda & 1 & \\ & \lambda & \\ & & \lambda\end{pmatrix}$, the eigenvalue λ is constant, and the metrics are affinely equivalent (i.e., Levi-Civita connections of g and $\bar g$ coincide). Affinely equivalent metrics whose tensor L has this form were essentially described by Eisenhart in [44], see also [45, Theorem 1]. From their description it follows, that, in a certain coordinate system, geodesically equivalent metrics $g\sim\bar g$ are given by

$$
\begin{aligned}
g &= 2\mathrm{d}x_3\mathrm{d}x_1+h\,(x_2,x_3)_{11}\,\mathrm{d}x_2{}^2+2h(x_2,x_3)_{12}\mathrm{d}x_2\mathrm{d}x_3+h(x_2,x_3)_{22}\mathrm{d}x_3{}^2,\\
\bar g &= 2\alpha\mathrm{d}x_3\mathrm{d}x_1+\alpha h\,(x_2,x_3)_{11}\,\mathrm{d}x_2{}^2+2\alpha h\,(x_2,x_3)_{12}\,\mathrm{d}x_2\mathrm{d}x_3+\beta\mathrm{d}x_3{}^2+\alpha h\,(x_2,x_3)_{22}\,\mathrm{d}x_3{}^2,
\end{aligned}
\tag{34}
$$

where α and β are constants.

Now, the metrics $g,\bar g$ such that $L=\begin{pmatrix}\lambda & & \\ & \lambda & \\ & & \lambda\end{pmatrix}$ are conformally equivalent. By the classical result of Weyl [33], they are proportional (i.e., $\bar g=\mathrm{const}\cdot g$).

Thus, we have described all building blocks that can be used in constructing metrics of Lorentz signature; Theorem 3 gives us the construction. Let us count the number of cases in dimension 4: we can represent 4 as the sum of natural numbers by 4 different ways:

Dim of blocks	Description of blocks	# of cases
$1+1+1+1$	All building blocks are as in Example 1, and $g\sim\bar g$ are essentially (2), (3) with the changed sign of $\mathrm{d}x_1^2$	1
$1+1+2$	The first two building blocks are as in Example 1, the third is as in Example 2	3
$2+2$	Both building blocks are as in Example 2; at least one of them is trivial	3
$1+3$	The first building block as is Example 1, the second is as in (33), as in (34), or trivial	3

Remark 15. The general schema also works in higher dimensions, but in this case there is the following essential difficulty (and this is the only difficulty): up to our knowledge, for dimensions $n-1\geq 5$, there is no description of all pairs of (g,L) such that g has Lorentz signature and L is an $(1,1)$-selfadjoint tensor such that it is covariantly constant, and such that the

Jordan normal form of L is

$$\begin{pmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & & \\ & & \lambda & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & \lambda & 0 & \\ & & & & & \lambda & \end{pmatrix}. \tag{35}$$

In dimension $n = 4$, since $n - 1 = 3$, the Jordan normal form (35) coincides with (32), and the local description follows from [28]. In dimension $n = 5$, we have $n - 1 = 4$ and one can obtain the local description (we will not do it in the present paper) combining the results of [44,45] with the algebraic description of possible holonomy groups of 4-dimensional metrics of Lorentz signature (see e.g. [46,47]).

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