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Weyl invariant polynomial and deformation quantization on Kähler manifolds

Hao Xu *

Center of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China
Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA

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ABSTRACT

Given a polynomial P of partial derivatives of the Kähler metric, we prove a simple criterion in terms of the coefficients for P to be an invariant polynomial, i.e. invariant under the transformation of coordinates. As applications, we prove an explicit composition formula for covariant differential operators under a canonical basis, also known as invariant differential operators in the case of bounded symmetric domains. We also prove a general explicit formula of star products on Kähler manifolds.

1. Introduction

Let $ds^2 = g_{ij}dx^i dx^j$ be the metric tensor in a local frame on a Riemannian manifold. Consider the algebra generated by partial derivatives of the metric $\{g_{ij\alpha}\}_{|\alpha|\geq 1}$, it can be shown that all polynomials in the variables $\{g_{ij\alpha}\}$ invariant under coordinate transformations, arise from complete contractions of covariant differentiations of curvature tensors. The proof requires H. Weyl's classical invariant theory for orthogonal groups by restriction to the normal coordinate systems. Weyl's invariant theory also played an important role in Liu's remarkable proof [1,2] of Witten's formula about intersection numbers of moduli spaces of principal bundles on a compact Riemann surface. Weyl's invariant theory also has important applications in Atiyah–Singer index theory (cf. [3]) and in Fefferman's program [4].

On the other hand, a natural question is: given a polynomial P in the variables $\{g_{ij\alpha}\}$, find a criterion solely in terms of the coefficients of P to determine whether P is invariant. This problem is still vague in general. In this paper, we give a somewhat satisfactory solution for Kähler manifolds.

Let (M, g) be a Kähler manifold of dimension n with Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=1}^{n} g_{i\bar{j}}dz_i \wedge dz_{\bar{j}}.$$

Thanks to the Kähler condition, we can canonically associate a polynomial in the variables $\{g_{i\bar{j}\alpha}\}_{|\alpha|\geq 1}$ to a semistable digraph H, such that each vertex represents a partial derivative of $g_{i\bar{j}}$ and each edge represents the contraction of a pair of indices.

* Corresponding author at: Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA. Tel.: +1 6179555709.
E-mail address: mathxuhao@gmail.com.

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Let $\sum_H c(H)H$ be a linear combination of semistable graphs. In Corollary 3.6, we shall give a simple criterion to characterize its invariantness. Similar criterion will be proved for covariant differential operators in Corollary 4.7.

Recall that a covariant differential operator $T^{\beta_1\cdots\beta_p}f_{/\beta_1\cdots\beta_p}$ is constructed through contractions of curvature tensors,

$$T^{\beta_1\cdots\beta_p} = g^{**}\cdots g^{**}R_{****/*\cdots*}\cdots R_{****/*\cdots*}.$$

Their linear combinations obviously form an algebra $\mathscr{R}$ under the Leibnitz rule of covariant derivatives. However, we do not have a canonical basis in terms of covariant derivatives of curvature tensors, due to additional relations like Bianchi identities and Ricci formulae. On the other hand, the polynomials in $\{g_{i\bar{j}\alpha}\}$ associated to stable graphs form a canonical basis of $\mathscr{R}$. In Theorem 4.10, we will prove an explicit composition formula under this basis. On a bounded symmetric domain Ω of rank r, Engliš [5] proved that $\mathscr{R}$ is equal to the algebra $\mathscr{D}(\Omega)$ of invariant differential operators. It is well known that $\mathscr{D}(\Omega)$ is a commutative algebra freely generated by r algebraically independent elements. It is an interesting and important problem to construct a set of generators explicitly (see [6–10]). In particular, Engliš [11] gave a set of generators using coefficients in the asymptotic expansion of the Berezin transform. From [12], Engliš' generators can be expressed in terms of a summation over graphs (cf. Eq. (22)).

The motivation of this paper comes from deformation quantization. Deformation quantization on a symplectic manifold M was introduced in the pioneering work of Bayen et al. [13] as a deformation of the usual pointwise product of $C^\infty(M)$ into a noncommutative associative $\star$-product of the formal series $C^\infty(M)[[\nu]]$. The celebrated work of Kontsevich [14,15] completely solved existence and classification of star-products up to gauge equivalence on general Poisson manifolds. Kontsevich's quantization formula was written as a summation over labeled directed graphs with two distinguished vertices and the coefficients are certain integrals over configuration spaces. Comprehensive surveys of deformation quantization can be found in [16] for Poisson manifolds, and [17] for Kähler manifolds.

Let us restrict to Kähler manifolds (M, g). A (differential) star product is an associative $\mathbb{C}[[\nu]]$-bilinear product $\star$ such that $\forall f_1, f_2 \in C^\infty(M)$,

$$f_1 \star f_2 = \sum_{j=0}^{\infty} \nu^j C_j(f_1, f_2), \tag{1}$$

where the $\mathbb{C}$-bilinear bidifferential operators C_j satisfy

$$C_0(f_1, f_2) = f_1 f_2, \qquad C_1(f_1, f_2) - C_1(f_2, f_1) = i\{f_1, f_2\}, \tag{2}$$

with the Poisson bracket $\{f_1, f_2\}$ given by

$$\{f_1, f_2\} = ig^{k\bar{l}}\left(\frac{\partial f_1}{\partial z^k}\frac{\partial f_2}{\partial \bar{z}^l} - \frac{\partial f_2}{\partial z^k}\frac{\partial f_1}{\partial \bar{z}^l}\right). \tag{3}$$

According to [18,19], a star product has the property of *separation of variables* (*Wick type*), if it satisfies $f \star h = f \cdot h$ and $h \star g = h \cdot g$ for any locally defined antiholomorphic function f, holomorphic function g and an arbitrary function h. If the role of holomorphic and antiholomorphic variables are swapped, we call it a star product of *anti-Wick type*.

There are earlier constructions of $\star$-products on restricted types of Kähler manifolds in [20–22]. Karabegov [18] solved the classification of deformation quantizations with separation of variables for Kähler manifolds. Schlichenmaier [23] showed that the Berezin–Toeplitz quantization gives rise to a star product, which turns out to be a very important quantization with many applications. See e.g. [24–31].

Feynman diagrams or directed graphs are effective tools in the construction and calculation of star products on Kähler manifolds. See [32–34,12,35]. Inspired by work of Reshetikhin and Takhtajan [33], Gammelgaard [32] obtained a remarkable universal formula in terms of acyclic graphs for a star product with separation of variables once a classifying Karabegov form is given. Gammelgaard's formula crucially relies on one's ability of writing down explicit Karabegov forms, a prototypical example is Karabegov–Schlichenmaier's identification theorem [36]. In [37,12], we obtained an explicit formula of Berezin star product in terms of strongly connected graphs, which was used to give a proof of an explicit formula of Berezin–Toeplitz star product due to Gammelgaard, Karabegov and Schlichenmaier. Karabegov [34,38] recently gave a very insightful algebraic proof of Gammelgaard's formula and generalized it to deformation quantization of an endomorphism bundle.

We will prove in Theorem 5.1 explicit formulae of star products whose Karabegov forms are summations over strongly connected graphs.

2. Covariant tensors in semistable trees

Throughout this paper, a *digraph* or simply a graph $G = (V, E)$ is defined to be a finite directed multigraph which is permitted to have multi-edges and loops.

A vertex v of a digraph G is called *stable* if $\deg^-(v) \geq 2$, $\deg^+(v) \geq 2$, i.e. both the inward and outward degrees of v are no less than 2. A vertex v is called *semistable* if we have

$$\deg^-(v) \geq 1, \qquad \deg^+(v) \geq 1, \qquad \deg^-(v) + \deg^+(v) \geq 3.$$

Definition 2.1. A *decorated tree* T is a directed tree that each vertex is decorated by a finite number of outward and inward external legs, corresponding to unbarred and barred indices respectively. T is called *semistable* (resp. *stable*) if each vertex is semistable (resp. stable). The inward (resp. outward) degree of a vertex v is defined to be the number of inward (resp. outward) half-edges at v. Note that a half-edge may refer to the head or tail of an edge of T or an external leg.

Definition 2.2. A directed edge uv of a semistable decorated tree or a semistable digraph is called *contractible* if $u \neq v$ and at least one of the following two conditions holds: (i) $\deg^+(u) = 1$; (ii) $\deg^-(v) = 1$.

Lemma 2.3. *Let T be a semistable decorated tree. Denote by T' a tree obtained by contracting a finite number of contractible edges in T. Then T' is also semistable and an edge in T' is contractible if and only if it is contractible in T.*

Proof. Let uv be a contractible edge of T. Let T' be the tree obtained by contracting uv and p the new vertex merging u and v. Then obviously $\deg_{T'} p \geq 4$. We also have $\deg^-_{T'} p \geq 1$ and $\deg^+_{T'} p \geq 1$, since u has at least one inward half-edge and v has at least one outward half-edge. So we proved that T' is semistable.

Let e be an edge of T other than uv. If e is not incident to u or v, then it is obvious that e is contractible in T if and only if it is contractible in T'.

If $e = vw$ is contractible in T, then there are two cases: (i) $\deg^-_T w = \deg^-_{T'} w = 1$; (ii) $\deg^-_T w \neq 1$ and $\deg^+_T v = 1$. In Case (i), it is obvious that e is also contractible in T'. In Case (ii), we have $\deg^-_T v \geq 2$, so the contractibility of uv in T implies $\deg^+_T u = 1$, namely $\deg^+_{T'} p = 1$. Thus pw is contractible in T'.

If $e = vw$ is non-contractible in T, then $\deg^-_{T'} w = \deg^-_T w \geq 2$ and $\deg^+_{T'} p \geq \deg^+_T v \geq 2$. So pw is non-contractible in T'.

The same argument works when $e = wu$. We conclude the proof. $\square$

Definition 2.4. A semistable decorated tree T is called *contractible* if all of its edges are contractible. Denote by $\mathscr{T}_g(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ the set of all contractible semistable decorated trees with external legs in the set $\{a_1 \cdots a_k, \bar{b}_1 \cdots \bar{b}_m\}$. Denote by $t_{k,m}(n)$ the number of n-vertex trees in $\mathscr{T}_g(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$. The first values of $t_{k,m}(n)$ were listed in Table 1 of Appendix.

Denote by $D(g_{a_1\bar{b}_1a_2\cdots a_k\bar{b}_2\cdots\bar{b}_m})$ the canonical invariant Weyl polynomial that equals $g_{a_1\bar{b}_1a_2\cdots a_k\bar{b}_2\cdots\bar{b}_m}$ at the center of a normal coordinate system. A proof of the following theorem was outlined in [37, Section 2], where a contractible tree was equivalently defined as an indecomposable admissible tree. Here we give a more direct proof.

Theorem 2.5. *Let $k, m \geq 2$. Then*

$$D(g_{a_1\bar{b}_1a_2\cdots a_k\bar{b}_2\cdots\bar{b}_m}) = \sum_{T \in \mathscr{T}_g(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)} (-1)^{|V(T)|+1} g_T, \tag{4}$$

where g_T is the Weyl invariant associated to T.

Proof. Note that a contractible semistable tree with no less than two vertices must have a vertex which is not stable; therefore the right-hand side of (4) is equal to $g_{a_1\bar{b}_1a_2\cdots a_k\bar{b}_2\cdots\bar{b}_m}$ at the center of a normal coordinate system. Thus we need only prove that the right-hand side of (4) is a covariant tensor with indices $(a_1 \cdots a_k, \bar{b}_1 \cdots \bar{b}_m)$. Let ϕ be a local biholomorphic mapping. Under the change of coordinates $x \mapsto \phi(x)$, we have

$$g_{i\bar{j}}(x) = g_{p\bar{q}}(\phi(x))(\partial_i\phi_p)(\overline{\partial_j\phi_q}) = \xrightarrow{\partial_{\bar{j}}\bar{\phi}} \circ \xrightarrow{\partial_i\phi},$$
$$g_{i\bar{j}l}(x) = g_{p\bar{q}r}(\phi(x))(\partial_l\phi_r)(\partial_i\phi_p)(\overline{\partial_j\phi_q}) + g_{p\bar{q}}(\phi(x))(\partial_{il}\phi_p)(\overline{\partial_j\phi_q})$$

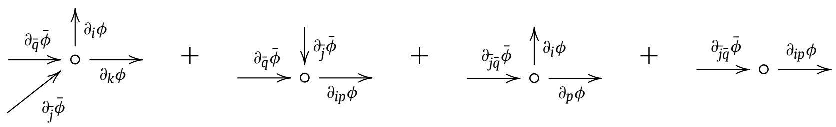
$$= \begin{array}{c} \uparrow{\scriptstyle \partial_i\phi} \\ \xrightarrow{\partial_{\bar{j}}\bar{\phi}} \circ \xrightarrow[\partial_l\phi]{} \end{array} + \xrightarrow{\partial_{\bar{j}}\bar{\phi}} \circ \xrightarrow{\partial_{il}\phi}$$

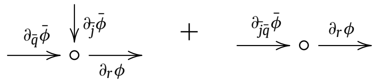
The graphical expressions will make the proof much easier. In general, it is not difficult to see that

$$g_{a_1\bar{b}_1a_2\cdots a_k\bar{b}_2\cdots\bar{b}_m}(x) = \sum_{P \in \text{partition}(a_1\cdots a_k|\bar{b}_1\cdots\bar{b}_m)} \circ_P, \tag{5}$$

where P runs over all partitions of the set $(a_1 \cdots a_k, \bar{b}_1 \cdots \bar{b}_m)$ such that no subset contains both barred and unbarred indices. $\circ_P$ denotes a single vertex decorated by external legs which are in one-to-one correspondence with elements of P. Trees in the right-hand side of (5) may be called ϕ-trees.

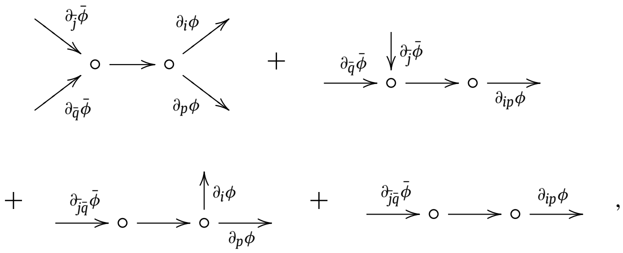
Let us first look at how Eq. (5) works by showing that $g_{i\bar{j}p\bar{q}} - g^{r\bar{s}} g_{r\bar{j}\bar{q}} g_{i\bar{s}p}$ is a covariant tensor. By (5), we have

$g_{i\bar{j}p\bar{q}}(x) =$ 

$g_{r\bar{j}\bar{q}}(x) =$ 

$g_{i\bar{s}p}(x) =$ 

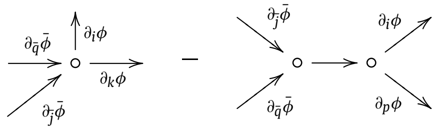
By $g^{r\bar{s}}(x) = g^{c\bar{d}}(\phi(x))(\partial_c \phi_r^{-1})(\overline{\partial_d \phi_s^{-1}})$, we have

$g^{r\bar{s}}(x) g_{r\bar{j}\bar{q}}(x) g_{i\bar{s}p}(x) =$ 

(6)

(6)

where we used $\sum_r (\partial_c \phi_r^{-1})(\partial_r \phi_t) = \delta_{ct}$. For the same reason, an internal edge $e = uv$ of a ϕ-tree could be contracted if both half-edges of e has no decoration and either $\deg^+ v = 1$ or $\deg^- u = 1$. Therefore the unique edge in the last three trees at the right-hand side of (6) could be contracted. The resulting ϕ-trees cancel with the corresponding ϕ-trees from $g_{i\bar{j}p\bar{q}}(x)$. Thus we get

$g_{i\bar{j}p\bar{q}}(x) - g^{r\bar{s}}(x) g_{r\bar{j}\bar{q}}(x) g_{i\bar{s}p}(x) =$ 

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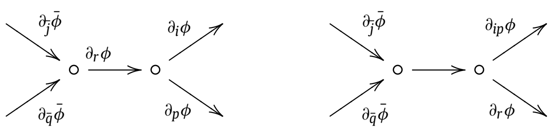
which implies that $g_{i\bar{j}p\bar{q}} - g^{r\bar{s}} g_{r\bar{j}\bar{q}} g_{i\bar{s}p}$ is a covariant tensor.

A half-edge of an internal edge is called an *internal half-edge*. From the above discussion, we see that under the change of coordinates $x \mapsto \phi(x)$, the right-hand side of (4)

$$\sum_{T \in \mathscr{T}_g(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)} (-1)^{|V(T)|+1} g_T(x) \tag{7}$$

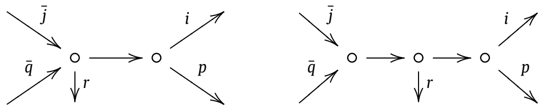
is equal to a summation of ϕ-trees whose internal half-edges or external legs are decorated by indices $(\partial_{a_1}\phi \cdots \partial_{a_k}\phi, \partial_{\bar{b}_1}\bar{\phi} \cdots \partial_{\bar{b}_m}\bar{\phi})$. In order to prove (4), it is enough to prove that for any ill-decorated ϕ-tree T_ϕ (i.e. some internal half-edge is decorated or some external leg has multiple derivatives), then its coefficient is zero in the above summation. We need to enumerate all trees in the summation (7) that may produce T_ϕ.

Again it is illuminating to look at an example first. Consider the following two ill-decorated ϕ-trees.

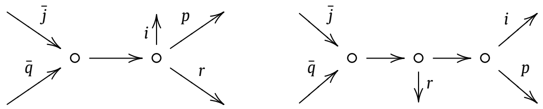


(8)

For the first ϕ-tree of (8), the left vertex is ill-decorated. It may come from two contractible semistable trees with opposite signs.

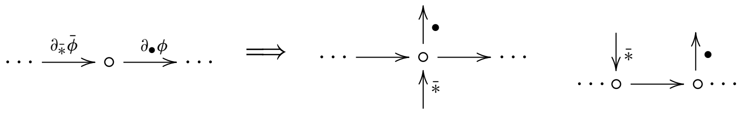


For the second ϕ-tree of (8), the right vertex is ill-decorated. It may come from two contractible semistable trees with opposite signs.



The above process may be called "freeing ill-decorated indices".

For a general ϕ-tree, we may treat each ill-decorated vertex separately. If the vertex has degree 2, then we have the following two ways to free ill-decorated indices, their numbers of vertices differ by 1.

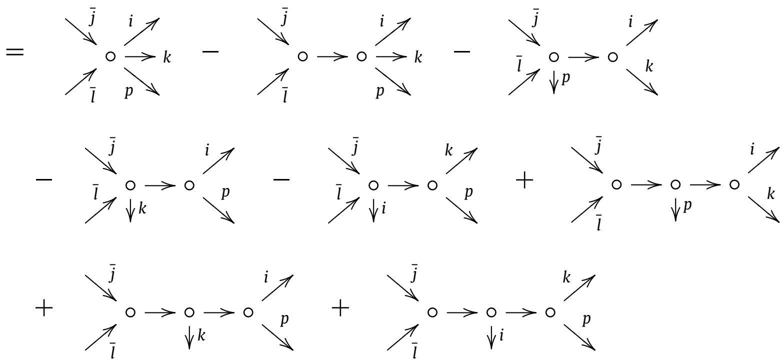


Namely the ill-decorated inward (resp. outward) indices may be separated and attached to either the original vertex or to a new vertex at the tail (resp. head) of the half-edge. It is obvious that the new edge is contractible.

If a vertex v has degree no less than 3, there are $2^{c(v)}$ ways of freeing ill-decorated indices, where $c(v)$ is the total number of decorated internal half-edges and external legs with multiple derivatives incident to v. It is easy to see that they add up to zero. So we conclude the proof. □

As an example, we can compute directly that

$$D(g_{i\bar{j}k\bar{l}p}) = -R_{i\bar{j}k\bar{l}/p} = -\partial_p R_{i\bar{j}k\bar{l}} + \Gamma^{\delta}_{pi} R_{\delta\bar{j}k\bar{l}} + \Gamma^{\delta}_{pk} R_{i\bar{j}\delta\bar{l}}$$



which agrees with (4).

3. Weyl invariants in semistable graphs

The weight of a digraph G is defined to be the integer $w(G) = |E| - |V|$. A digraph G is *stable* (resp. *semistable*) if each vertex of G is stable (resp. semistable). The set of semistable and stable graphs of weight k will be denoted by $\mathcal{G}^{ss}(k)$ and $\mathcal{G}(k)$ respectively.

A digraph G is called *strongly connected* or *strong* if there is a directed path from each vertex in G to every other vertex. The strongly connected components (SCC's) of a digraph G can each be contracted to a single vertex, the resulting graph is a directed acyclic graph (DAG), called the *condensation* of G. A *source* (resp. *sink*) of G is a SCC that has only outward (resp. inward) edges in the condensation of G.

Lemma 3.1. *Let $e = uv$ be a contractible edge of a semistable graph G. Denote by G' the graph obtained by contracting e in G. If $e' \neq e$ is an edge of G such that $e' \neq vu$, then e' is contractible in G if and only if it is contractible in G'.*

Proof. The proof is almost identical to the proof of Lemma 2.3. □

Definition 3.2. A semistable graph G is called *stabilizable* if after contractions of a finite number of contractible edges of G, the resulting graph becomes stable, which is called the *stabilization graph* of G and denoted by G^s.

If G is the stabilization graph of H, then $w(G) = w(H)$. By Lemma 3.1, the stabilizability of a semistable graph G is independent of the order of edge-contractions.

Lemma 3.3. *A strong semistable graph G is stabilizable.*

Proof. Let v be a nonstable vertex of G. Then v has no loop by the strongness of G. Moreover, v has either a unique inward or a unique outward edge, which is contractible. Thus we can always contract some edge until a stable graph is reached. □

A connected semistable graph may be not stabilizable, e.g. $1 \xrightarrow{1} 1$.

Lemma 3.4. *Let G be a stabilizable semistable graph. If the stabilization graph of G is strong, then G is also strong.*

Proof. Obviously G is connected. If G is not strong, first assume that G has two SCC's A, B. Then it is not difficult to see that any edge between A, B is not contractible, a contradiction. If G has more than two SCC's, consider its condensation G'. Choose any edge e in G' which is contractible in G, we can contract e to reduce the number SCC's of G by one. Since the stabilization graph of G is strong, we can always repeat this process until we get a graph with two SCC's, which is not contractible, a contradiction again. Therefore G must be strong. □

Theorem 3.5. *Let G be a stable graph of weight k. Then*

$$D(G) = \sum_{H\in\mathcal{G}^{ss}(k)}^{\text{stabilizable}} \frac{(-1)^{|V(H)|-|V(G)|}|\mathrm{Aut(G)}|}{|\mathrm{Aut(H)}|} H, \tag{9}$$

where H runs over stabilizable semistable graphs of weight k whose stabilization graph is G.

Proof. By definition, $D(G)$ is a sum of stabilizable semistable graphs obtained by expanding each vertex of G by (4) as a sum of contractible semistable trees, while keeping incidence relations of G. The group $\mathrm{Aut}(G)$ has a natural action on the above multiset of stabilizable semistable graphs H in the expansion of $D(G)$. Then it is not difficult to see that the set of orbits is in one-to-one correspondence with isomorphism classes of stabilizable semistable graphs of weight k and the isotropy group at H is $\mathrm{Aut}(H)$. Therefore the orbit of H has $|\mathrm{Aut(G)}|/|\mathrm{Aut(H)}|$ graphs. The factor $(-1)^{|V(H)|-|V(G)|}$ is clear from (4). So we conclude the proof of (9). □

Corollary 3.6. *A linear combination of stabilizable semistable graphs of weight k*

$$\sum_{H\in\mathcal{G}^{ss}(k)}^{\text{stabilizable}} c(H)\frac{(-1)^{|V(H)|}}{|\mathrm{Aut(H)}|}H \tag{10}$$

is a Weyl invariant (i.e. invariant under coordinate transformations) if and only if $c(H_1) = c(H_2)$ whenever H_1, H_2 have the same stabilization graph.

Proof. Note that (10) is a Weyl invariant if and only if it is equal to

$$\sum_{G\in\mathcal{G}_s(k)} c(G)\frac{(-1)^{|V(G)|}}{|\mathrm{Aut(G)}|}D(G).$$

So the corollary follows from Theorem 3.5. □

Definition 3.7. For convenience, a function $c(H)$ defined on the set of stabilizable semistable graphs is called a *Weyl function* if it satisfies $c(H_1) = c(H_2)$ whenever H_1, H_2 have the same stabilization graph.

Any constant function is a Weyl function. Below is a more nontrivial example.

Lemma 3.8. *Let $\mathscr{L}(H)$ be the set of linear subgraphs of H (note $\emptyset \in \mathscr{L}(H)$) and $p(L)$ the number of components of $L \in \mathscr{L}(H)$. Then*

$$\beta_C(H) = \sum_{L\in\mathscr{L}(H)} C^{p(L)}, \tag{11}$$

is a Weyl function for any constant C.

Proof. Let H' be a graph obtained by contracting a contractible edge $e = uv$ in H. For any given $L \in \mathscr{L}(H)$, define $L' \in \mathscr{L}(H')$ by

$$L' = \begin{cases} L, & e \notin L \\ L/\{e\}, & e \in L \end{cases}$$

where $L/\{e\}$ is the graph obtained by contracting e in L. Since we have either $\deg^+ u = 1$ or $\deg^- v = 1$, it is not difficult to see that $L \mapsto L'$ gives a one-to-one correspondence between $\mathscr{L}(H)$ and $\mathscr{L}(H')$. Moreover, $p(L) = p(L')$. So we have

$$\beta_C(H) = \sum_{L \in \mathscr{L}(H)} C^{p(L)} = \sum_{L' \in \mathscr{L}(H')} C^{p(L')} = \beta_C(H').$$

This implies $\beta_C(H_1) = \beta_C(H_2)$ whenever H_1, H_2 have the same stabilization graph. □

Corollary 3.9. (i) $\det(I - A(H))$ *is a Weyl function, where* I *is the identity matrix and* $A(H)$ *is the adjacency matrix of* H.
(ii) $|\mathscr{L}(H)|$, *the number of linear subgraphs of* H, *is a Weyl function.*

Proof. (i) follows by taking $C = -1$ in (11) and using the following Coefficient Theorem from spectral graph theory,

$$\det(I - A(H)) = \sum_{L \in \mathscr{L}(H)} (-1)^{p(L)}. \tag{12}$$

(ii) follows by taking $C = 1$ in (11). □

We remark that $\det(I - A(H))$ appears as the coefficients of asymptotic expansions of the Bergman kernel [37].

4. Covariant differential operators

Differential operators on Kähler manifolds can be encoded by digraphs with a distinguished vertex. The results in previous sections can be extended to this setting almost verbatim.

Definition 4.1. A *(one-)pointed tree* $T = (V \cup \{\bullet\})$ is defined to be a decorated tree with a distinguished vertex labeled by f. A *(one-)pointed graph* $\Gamma = (V \cup \{\bullet\}, E)$ is defined to be a digraph with a distinguished vertex labeled by f. T or Γ is called *semistable* (resp. *stable*) if each ordinary vertex $v \in V$ is semistable (resp. stable).

Definition 4.2. A directed edge uv of a semistable pointed tree or a semistable pointed graph is called *contractible* if $u \neq v$ and at least one of the following two conditions holds: (i) $u \in V$ and $\deg^+(u) = 1$; (ii) $v \in V$ and $\deg^-(v) = 1$.

A semistable pointed tree T is called *contractible* if all of its edges are contractible. Note that Lemma 2.3 still holds for pointed trees.

Theorem 4.3. *Let* $k, m \geq 0$. *Then*

$$D(f_{a_1 \cdots a_k \bar{b}_1 \cdots \bar{b}_m}) = \sum_{T = (V \cup \{\bullet\}) \in \mathscr{T}_f(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)} (-1)^{|V|} f_T, \tag{13}$$

where $\mathscr{T}_f(a_1 \cdots a_k | \bar{b}_1 \cdots \bar{b}_m)$ *the set of all contractible semistable pointed trees with external legs in the set* $\{a_1 \cdots a_k, \bar{b}_1 \cdots \bar{b}_m\}$ *and* f_T *is the Weyl invariant associated to the pointed tree* T.

Proof. The proof is similar to Theorem 2.5. □

Definition 4.4. The weight of a pointed graph $\Gamma = (V \cup \{\bullet\}, E)$ is defined to be $w(\Gamma) = |E| - |V|$. By abuse of notation, we denote $V(\Gamma) = V \cup \{\bullet\}$. The set of semistable and stable pointed graphs of weight k will be denoted by $\mathscr{G}_1^{ss}(k)$ and $\mathscr{G}_1(k)$ respectively. We denote by $\mathrm{Aut}(\Gamma)$ the set of all automorphisms of the pointed graph Γ fixing the distinguished vertex.

A semistable pointed graph Γ is called *stabilizable* if after contractions of a finite number of contractible edges of Γ, the resulting graph becomes stable, which is called the *stabilization graph* of Γ and denoted by Γ^s. Note that Lemma 3.1 still holds for pointed graphs.

Lemma 4.5. (i) *A strong semistable pointed graph* Γ *is stabilizable.*
(ii) *Let* Γ *be a stabilizable semistable graph. If the stabilization graph of* Γ *is strong, then* Γ *is also strong.*

Proof. The proof is similar to Lemmas 3.3 and 3.4. □

Theorem 4.6. *Let Γ be a stable pointed graph of weight k. Then*

$$D(\Gamma) = \sum_{Z} \frac{(-1)^{|V(Z)|-|V(\Gamma)|}|\mathrm{Aut}(\Gamma)|}{|\mathrm{Aut(Z)}|} Z, \tag{14}$$

where Z runs over stabilizable semistable pointed graphs of weight k whose stabilization graph is Γ.

Proof. The proof is similar to Theorem 3.5. □

Corollary 4.7. *A linear combination of stabilizable semistable pointed graphs*

$$\sum_{Z\in\mathcal{G}_1^{ss}}^{\text{stabilizable}} c(Z)\frac{(-1)^{|V(Z)|}}{|\mathrm{Aut(Z)}|} Z \tag{15}$$

is a covariant differential operator (i.e. invariant under coordinate transformations) if and only if $c(Z_1) = c(Z_2)$ whenever Z_1, Z_2 have the same stabilization graph.

Proof. It follows immediately from Theorem 4.6. □

Example 4.8. Engliš [5] proved the following asymptotic expansion for a Laplace integral on a domain $\Omega \in \mathbb{C}^n$ when $m \to \infty$,

$$\int_{\Omega} f(y)e^{-m(\Phi(x,x)+\Phi(y,y)-\Phi(x,y)-\Phi(y,x))}\frac{\omega_g^n(y)}{n!} = \frac{1}{m^n}\sum_{k\geq 0} m^{-k}R_k(f)(x), \tag{16}$$

where Φ is the Kähler potential and R_k are covariant differential operators.

In [12, Thm 3.2], we proved an explicit formula for R_k,

$$R_k(f) = \sum_{\Gamma\in\mathcal{G}_1^{ss}} \frac{\det(A(\Gamma_-)-I)}{|\mathrm{Aut}(\Gamma)|}\Gamma, \tag{17}$$

where Γ_- is obtained by removing the distinguished vertex of Γ.

We show that (17) is consistent with Corollary 4.7. Similar to Corollary 3.9 (i), we have $\det(I - A(\Gamma_-)) = \det(I - A(\Gamma'_-))$ where Γ' is obtained by contracting a contractible edge in Γ. Moreover, if Γ is a semistable pointed graph which is non-stabilizable, then $\det(I - A(\Gamma_-)) = 0$. In order to prove the last assertion, we may assume that each edge of Γ is non-contractible. If v is a strictly semistable ordinary vertex (i.e. $\deg(v) = 3$), then v must have a self-loop, namely Γ_- contains a SCC $\{1\}$. Therefore we must have $\det(I - A(\Gamma_-)) = 0$.

By Corollary 4.7, the graded algebra $\mathscr{R}$ of abstract covariant differential operators has a canonical basis $\mathcal{G}_1$ consisting of stable pointed graphs, graded by weights. Before we give a multiplication formula in this algebra, we need more definitions.

Definition 4.9. Let $\Gamma = (V \cup \{\bullet\}, E)$ be a pointed graph that can be obtained by inserting a finite number of vertices to edges of a semistable pointed graph Γ^{ss}, called the semistabilization graph of Γ. Such Γ is called *generalized stabilizable* (GS) if Γ^{ss} is stabilizable. The stabilization graph of Γ^{ss}, denoted by Γ^s, is also called the stabilization graph of Γ.

The reason we introduce GS pointed graphs is to account for the derivatives on edges $g^{i\bar{j}}$. See [12, Rem. 3.7] for detailed discussions.

We have the following explicit composition formula of covariant differential operators.

Theorem 4.10. *In terms of the basis of stable pointed graphs, we have*

$$\begin{aligned}&\left(\sum_{Z_1\in\mathcal{G}_1} c_1(Z_1)\frac{(-1)^{|V(Z_1)|}}{|\mathrm{Aut}(Z_1)|}Z_1\right)\circ\left(\sum_{Z_2\in\mathcal{G}_1} c_2(Z_2)\frac{(-1)^{|V(Z_2)|}}{|\mathrm{Aut}(Z_2)|}Z_2\right)\\ &\quad= \sum_{Z\in\mathcal{G}_1}\left(\sum_{\Gamma\subset Z}^{GS}(-1)^{|V((Z/\Gamma)^s)|+|V(\Gamma^s)|}c_1((Z/\Gamma)^s)c_2(\Gamma^s)\right)\frac{1}{|\mathrm{Aut(Z)}|}Z,\end{aligned} \tag{18}$$

where Γ runs over all GS pointed subgraphs of Z, and Z/Γ is the pointed graph obtained from Z by contracting Γ to a point.

Proof. Eq. (18) follows almost immediately from Corollary 4.7 and results proved in our previous paper [35, Lem. 3.10 & Rem. 3.7].

A further justification to (18) is the following lemma.

Lemma 4.11. *Let Z be a GS pointed graph and Γ a GS pointed subgraph of Z. Then Z/Γ is also a GS pointed graph.*

Proof. Let e be an edge in $(Z/\Gamma)^{ss}$. Then it is not difficult to see that e is contractible in $(Z/\Gamma)^{ss}$ if and only if e is contractible in Z^{ss}. Therefore $(Z/\Gamma)^{ss}$ is stabilizable, since Z^{ss} is stabilizable. □

By Lemma 4.5, each strong pointed graph is a GS pointed graph. All linear combinations of strong (stable) pointed graphs form a subalgebra $\mathscr{S}$, which contains certain interesting covariant differential operators arising from deformation quantization on Kähler manifolds (cf. Theorem 5.1). The composition formula in $\mathscr{S}$ is given by

$$\left(\sum_{Z_1\in\mathcal{G}_1}^{\text{strong}} c_1(Z_1)\frac{(-1)^{|V(Z_1)|}}{|\mathrm{Aut}(Z_1)|}Z_1\right)\circ\left(\sum_{Z_2\in\mathcal{G}_1}^{\text{strong}} c_2(Z_2)\frac{(-1)^{|V(Z_2)|}}{|\mathrm{Aut}(Z_2)|}Z_2\right)$$
$$=\sum_{Z\in\mathcal{G}_1}^{\text{strong}}\left(\sum_{\Gamma\subset Z}^{\text{strong}}(-1)^{|V((Z/\Gamma)^s)|+|V(\Gamma^s)|}c_1((Z/\Gamma)^s)c_2(\Gamma^s)\right)\frac{1}{|\mathrm{Aut}(Z)|}Z, \tag{19}$$

where Γ runs over all strong pointed subgraph of Z.

Recall that the Berezin transform has an asymptotic expansion (cf. [5,36]),

$$I_\alpha f(x)=\sum_{k=0}^{\infty}Q_kf(x)\alpha^{-k},\quad \alpha\to\infty. \tag{20}$$

The following explicit formula for the differential operators Q_k was proved in [12],

$$Q_k=\sum_{\Gamma\in\mathcal{G}_1(k)}^{\text{strong}}\frac{\det(A(\Gamma_-)-I)}{|Aut(\Gamma)|}\Gamma, \tag{21}$$

where Γ_- is obtained from Γ by removing the distinguished vertex from Γ.

We can also study $\mathscr{R}$ and $\mathscr{S}$ on a fixed Kähler manifold. For a bounded symmetric domain Ω of rank r equipped with the Bergman metric, it is obvious that $\mathscr{R}=\mathscr{S}$. Denote by $\mathscr{D}(\Omega)$ the algebra of invariant differential operators. Engliš proved that $\mathscr{S}$ coincides with $\mathscr{D}(\Omega)$ [5, Prop. 7] and $\mathscr{S}$ is freely generated by $Q_1, Q_3, Q_5, \dots, Q_{2r-1}$ [11, Thm. 1.1].

On a bounded symmetric domain, all $R_{i\bar{j}k\bar{l}/\alpha}=0$, $|\alpha|\geq 1$. So a pointed graph $\Gamma=0$ unless Γ is a balanced graph, i.e. $\deg^+(v)=\deg^-(v)$ for each vertex v. Combining Engliš' result and (21), we get a set of explicit generators for $\mathscr{D}(\Omega)$ in terms of balanced strong pointed graphs,

$$Q_k=\sum_{\Gamma\in\mathcal{G}_1(k)}^{\substack{\text{balanced}\\ \text{strong}}}\frac{\det(A(\Gamma_-)-I)}{|Aut(\Gamma)|}\Gamma,\quad k=1,3,\dots,2r-1, \tag{22}$$

whose composition formula is given by (19). Note that on a bounded symmetric domain, balanced strong pointed graphs in $\mathcal{G}_1(k)$ are not linearly independent. For $k=1,3,5$, G_k has 1, 5, 119 nonzero terms respectively.

5. Star products

On a Kähler manifold (M,ω_{-1}), a formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal $(1,1)$-form,

$$\hat{\omega}=\frac{1}{\nu}\omega_{-1}+\omega_0+\nu\omega_1+\nu^2\omega_2+\cdots, \tag{23}$$

where each ω_k is a closed, may be degenerate, $(1,1)$-form. Karabegov [18] showed that deformation quantizations with separation of variables on (M,ω_{-1}) are in one-to-one correspondence with such formal deformations. Given a star product $\star$ of anti-Wick type, its Karabegov form is computed as following: let $z^1,\dots,z^n$ be local holomorphic coordinates on an open subset U of M. Then there exists a set of formal functions on U, denoted by $u^1,\dots,u^n$,

$$u^k=\frac{1}{\nu}u^k_{-1}+u^k_0+\nu u^k_1+\nu^2u^k_2+\cdots,$$

satisfying $u^k\star z^l-z^l\star u^k=\delta^{kl}$. The Karabegov form of $\star$, which is independent of the coordinates chosen, is given by $\hat{\omega}|_U=-\sqrt{-1}\,\bar{\partial}(\sum_k u^kdz^k)$.

Let $G=(V,E)$ be a digraph that can be obtained by inserting a finite number of vertices to edges of a semistable graph G^{ss}. Similar to Definition 4.9, we may call such G a *generalized stabilizable graph* if G^{ss} is stabilizable. The stabilization graph of G^{ss}, denoted by G^s, is also called the stabilization graph of G.

By Lemmas 3.3 and 3.4, we know that any strong digraph must be one of the following: (i) a generalized stabilizable graph; (ii) a single vertex without loops; (iii) a connected linear digraph (i.e. a directed cycle with $n \geq 1$ vertices).

Let h be an arbitrary $\mathbb{C}$-valued function on the set of strong stable graphs and $\{①\}$. We define a function α_h on the set of all strong digraphs by

$$\alpha_h(G) = \begin{cases} (-1)^{|V(G)|-|V(G^s)|} h(G^s), & G \text{ is a generalized stabilizable graph}, \\ -1, & G \text{ is a single vertex without loops}, \\ (-1)^{n+1} h(①), & G \text{ is a directed cycle with } n \geq 1 \text{ vertices}. \end{cases}$$

Theorem 5.1. *Let h and α_h be the functions given above. For any functions f_1 and f_2 on a Kähler manifold, we have the following anti-Wick type star product*

$$f_1 \star f_2(x) = \sum_{\Gamma \in \mathcal{G}_1^{ss}}^{strong} \nu^{w(\Gamma)} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{G \in SCC(\Gamma_-)} \alpha_h(G)\, \Gamma^{op}(f_1, f_2), \tag{24}$$

where G runs over all strongly connected components of Γ_- and the partition function $\Gamma^{op}(f_1, f_2)$ is obtained by taking antiholomorphic and holomorphic derivatives of Γ separately on f_1 and f_2.

The Karabegov form of the star product (24) is given by

$$\hat{\omega} = \frac{1}{\nu}\omega_{-1} - h(①)Ric - \sqrt{-1}\partial\bar{\partial} \sum_{G \in \mathcal{G}^{ss}}^{strong} \nu^{w(G)} \frac{\alpha_h(G)}{|\mathrm{Aut}(G)|} G, \tag{25}$$

where $Ric = \sqrt{-1}\partial\bar{\partial} \log\det g$ is the Ricci curvature.

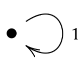
Proof. The first two terms of formal Berezin transform corresponding to (24) is

$I(f) = f + \Big[$  $\Big] + \cdots,$

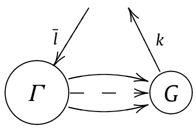
which implies that $\star$ satisfies (2). The associativity can be verified by the same argument as [35, Prop. 4.5]. By definition, in order to prove (25), we need only check that

$$u^k = \frac{1}{\nu}\frac{\partial \Phi}{\partial z^k} - h(①)\frac{\partial \log\det g}{\partial z^k} + \sum_{G \in \mathcal{G}_0^{ss}} \nu^{w(G)} \frac{-\alpha_h(G)}{|\mathrm{Aut}(G)|} \frac{\partial G}{\partial z^k} \tag{26}$$

satisfy $u^k \star z^l - z^l u^k = \delta^{kl}$ for $1 \leq k, l \leq n$. The coefficient of ν^0 in $u^k \star_B z^l - z^l u^k$ is equal to

$\Big[$  $\Big]^{op} \left(\frac{\partial \Phi}{\partial z^k}, z^l\right) = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l} = \delta^{kl}.$

In general, a graph H appearing in $u^k \star z^l - z^l u^k$ has the following form

$H =$  ,

where G is a strong graph. It may either come from $\dot{H}^{op}(\frac{\partial \Phi}{\partial z^k}, z^l)$ or $\Gamma^{op}(\frac{\partial G}{\partial z^k}, z^l)$, where $\dot{H}$ is obtained from H by gluing the head of k and the tail of $\bar{l}$. The coefficient of H in $u^k \star z^l - z^l u^k$ is equal to

$$\prod_{K \in SCC(\dot{H}_-)} \alpha_h(K) + (-\alpha_h(G)) \prod_{K \in SCC(\Gamma_-)} \alpha_h(K) = 0,$$

as claimed. □

By specializing the functions h and α_h, the above theorem recovers previous known star products: Berezin, Berezin–Toeplitz, Karabegov–Bordemann–Waldmann (standard) and its dual. For example, take $h(G) = \det(A(G) - I)$ for stable graphs, then $\alpha_h(G) = \det(A(G) - I)$ for all strong graphs (cf. [35, Lem. 3.9]). We get the Berezin star product.

From Lemma 4.5, Corollary 4.7 and the discussion in [35, Rem. 4.2], once (25) is given, the above theorem may be regarded as a special case of Gammelgaard's formula [32].

Proposition 5.2. *Let C be a constant and h_C be the function given by*

$$h_C(G) = \sum_{L\in\mathscr{L}(G)} \frac{(-1)^{|V(G)|+1+p(L)}C^{p(L)}}{|\mathrm{Aut}(G)|}, \tag{27}$$

where L runs over linear subgraphs (including empty subgraph) of G and $p(L)$ the number of components of $L \in \mathscr{L}(H)$. Then the corresponding star product (24) is

$$f_1 \star f_2(x) = \sum_{\Gamma\in\mathcal{G}_1^{ss}}^{strong} \frac{\nu^{w(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \sum_{L\in\mathscr{L}(\Gamma_-)} \frac{(-1)^{|V(\Gamma)|-1+p(L)}C^{p(L)}}{|\mathrm{Aut}(\Gamma)|} \Gamma^{op}(f_1, f_2). \tag{28}$$

The dual opposite of (28) is a star product of Wick type given by

$$f_1 \star' f_2(x) = \sum_{\Gamma} \nu^{w(\Gamma)} \frac{(-1)^{|E(\Gamma)|}C^{\ell(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Gamma(f_1, f_2), \tag{29}$$

where Γ runs over all semistable pointed graphs such that each SCC of Γ_- is either a single vertex or a linear digraph, $\ell(\Gamma)$ is the number of linear digraphs in the SCC's of Γ_- and $\Gamma(f_1, f_2) = \Gamma^{op}(f_2, f_1)$.

Proof. The proof of (28) is obvious. The proof of (29) is similar to the argument of [35, Thm. 4.3]. □

When $C = 1$, (24) and (29) are respectively the Berezin and Berezin–Toeplitz star products (cf. [35, Section 4]). When $C = 0$, (24) and (29) are respectively the Karabegov–Bordemann–Waldmann star product and its dual (cf. [35, Section 6]).

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Appendix. Enumeration of contractible semistable trees

Lemma A.1. *Let $k, m \geq 2$. Then $t_{k,m}(n) = 0$ when $n > k + m - 2$ and*

$$t_{k,m}(n) = t_{m,k}(n), \qquad t_{k,2}(k) = (2k-3)!!, \tag{30}$$

$$t_{k,m}(2) = 2^k + 2^m - k - m - 3, \tag{31}$$

$$\begin{aligned} t_{k,m}(3) \ =\ & \frac{1}{2}(3^{k+1} + 3^{m+1}) - (2^k + 2^m)(k+m) - 5(2^k + 2^m) \\ & + 2^{k+m} + \frac{1}{2}(k^2 + m^2) + \frac{7}{2}(k+m) + km + 7. \end{aligned} \tag{32}$$

Proof. The first two equations are obvious. Let us prove $t_{k,2}(k) = (2k-3)!!$. When $k = 2$, we have $t_{2,2}(2) = 1$. A k-vertex tree in $\mathscr{T}_g(a_1 \cdots a_k|\bar{b}_1\bar{b}_2)$ can be obtained by connecting the outward leg a_k to a new node in the middle of any of the edges and outward legs of a $(k-1)$-vertex tree in $\mathscr{T}_g(a_1 \cdots a_{k-1}|\bar{b}_1\bar{b}_2)$. There are $k-2$ edges and $k-1$ outward legs in a $(k-1)$-vertex tree in $\mathscr{T}_g(a_1 \cdots a_{k-1}|\bar{b}_1\bar{b}_2)$; therefore, $t_{k,2}(k)$ is larger than $t_{k-1,2}(k-1)$ by a factor of $2k-3$. So we get $t_{k,2}(k) = (2k-3)!!$.

There is a unique 2-vertex tree, so we have

$$t_{k,m}(2) = \sum_{i=0}^{k-2} \binom{k}{i} + \sum_{i=1}^{m-2} \binom{m}{i} = 2^k + 2^m - k - m - 3.$$

There are three 3-vertex directed trees,

$$\circ \rightarrow \circ \rightarrow \circ \qquad \circ \rightarrow \circ \leftarrow \circ \qquad \circ \leftarrow \circ \rightarrow \circ\,. \tag{33}$$

We compute their respective contributions to $t_{k,m}(3)$,

$$\begin{aligned} t_{k,m}(3) = & \sum_{i=1}^{k-3}\binom{k}{i}\sum_{j=1}^{k-2-i}\binom{k-i}{j} + \sum_{i=1}^{k-2}\binom{k}{i}\sum_{j=0}^{m-2}\binom{m}{j} + \sum_{i=0}^{m-3}\binom{m}{i}\sum_{j=1}^{m-2-i}\binom{m-i}{j} \\ & + \frac{1}{2}\sum_{i=2}^{m-2}\binom{m}{i}\sum_{j=2}^{m-i}\binom{m-i}{j} + \frac{1}{2}\sum_{i=2}^{k-2}\binom{k}{i}\sum_{j=2}^{k-i}\binom{k-i}{j}, \end{aligned}$$

Table 1
$t_{k,m}(n)$, numbers of contractible semistable trees.

(k, m)	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
(2, 2)	1	1				
(3, 2)	1	4	3			
(4, 2)	1	11	25	15		
(3, 3)	1	7	15	9		
(5, 2)	1	26	130	210	105	
(4, 3)	1	14	58	90	45	
(6, 2)	1	57	546	1750	2205	945
(5, 3)	1	29	208	628	765	325
(4, 4)	1	21	150	432	529	225

where the first three summations come from the first tree of (33), the last two summations come from the second and third trees of (33) respectively. We can simplify the binomials to get (32). □

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