Journal of Geometry and Physics 89 (2015) 74–81

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Neutral signature Walker-CSI metrics

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ARTICLE INFO

Article history:
Received 20 June 2014
Received in revised form 5 October 2014
Accepted 12 December 2014
Available online 23 December 2014

Keywords:
4D
Neutral signature
Walker
Scalar curvature invariant
CSI
Kundt

ABSTRACT

We will construct explicit examples of four-dimensional neutral signature Einstein Walker spaces for which all of the polynomial scalar curvature invariants are constant. We show that these Einstein Walker spaces are Kundt. We then investigate the mathematical properties of the spaces, including holonomy and universality.

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1. Introduction

Four-dimensional (4D) neutral signature Walker spaces are currently of interest. Pseudo-Riemannian spaces (of arbitrary signature) were algebraically classified in terms of the boost weight decomposition in [1,2]. Spaces which possess a geodesic, expansion-free, shear-free, and twist-free null-congruence and hence are pseudo-Riemannian Kundt metrics (of arbitrary signature), and Walker spaces which admit a k-dimensional invariant null plane [3], are of particular importance. The requirement of an invariant k-dimensional null plane implies that the (Walker) metric can be written in a canonical form [3].

A polynomial scalar curvature invariant is a scalar constructed from contractions of products of the Riemann tensor and its covariant derivatives. A vanishing scalar invariant (VSI) space is a space in which all polynomial scalar curvature invariants vanish. In [1] spaces with degenerate curvature structure of arbitrary dimension and signature were studied. In particular, 4D neutral signature spaces were emphasized and it was proven that VSI spacetimes are either of Kundt or Walker form. This signals that the Walker metrics play an important role for degenerate metrics of arbitrary signature.

In [4] we presented a number of 4D neutral signature exact Kundt solutions for which all of the polynomial scalar curvature invariants vanish or are all constant (CSI spaces), generalizing the 4D degenerate Kundt metrics [5] and the VSI metrics [6] and CSI spaces [7] in the Lorentzian case. In [8] we constructed explicit examples of 4D neutral signature Walker (but not necessarily degenerate Kundt) spaces for which all of the polynomial scalar curvature invariants vanish.

In this paper we will investigate examples of 4D neutral signature Einstein CSI metrics. These spaces are Walker, and we show that they are also Kundt. We study their holonomy structure. We also demonstrate that the spaces are universal.

2. 4D Neutral signature Walker CSI metrics

We shall present a class of Walker CSI metrics, using the following Walker form [2]:

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}v + A\mathrm{d}u + C\mathrm{d}U) + 2\mathrm{d}U(\mathrm{d}V + B\mathrm{d}U) : \tag{1}$$

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http://dx.doi.org/10.1016/j.geomphys.2014.12.008
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where

$$\begin{aligned} A &\equiv v^2 A_{02} + vA_{01}(u,U) + VA_{10} + A_0(u,U). \\ B &\equiv v^4 B_{04} + v^3 B_{03}(u,U) + v^2 VB_{12} + v^2 B_{02}(u,U) \\ &\quad + vVB_{11}(u,U) + V^2 B_{20} + vB_{01}(u,U) + VB_{10}(u,U) + B_0(u,U) \\ C &\equiv v^3 C_{03} + v^2 C_{02}(u,U) + vVC_{11} + vC_{01}(u,U) + VC_{10}(u,U) + C_0(u,U), \end{aligned}$$

and $\{A_{02}, A_{10}, B_{04}, B_{20}, B_{12}, C_{03}, C_{11}\}$ are all constants and the remaining arbitrary functions depend, in general, on $\{u, U\}$. Note that these solutions are not of the same form as the CSI Kundt solutions in [4].

In 4D neutral signature spaces, a null frame consists of null tetrad vectors $\{\ell_a, n_a, m_a, \tilde{m}_a\}$ that satisfy $\ell_a n^a = 1, m_a \tilde{m}^a = 1$, and all other inner products are zero. Here they are defined by:

$$\boldsymbol{\ell} = du, \qquad \mathbf{n} = dv + A(u,v,U,V)du + \frac{C(u,v,U,V)}{2}dU \tag{2}$$

$$\mathbf{m} = dU, \qquad \tilde{\mathbf{m}} = dV + \frac{C(u,v,U,V)}{2}du + B(u,v,U,V)dU \tag{3}$$

in which case the metric (1) takes the form

$$\mathrm{d}s^2 = 2\left(\boldsymbol{\ell} \otimes \mathbf{n} + \mathbf{m} \otimes \tilde{\mathbf{m}}\right). \tag{4}$$

We assume that the space is Einstein; i.e.,

$$R^a_b = \Lambda \delta^a_b, \qquad R = 4\Lambda \tag{5}$$

where Λ is a constant. Using the metric (1) in the Einstein equations, after a straightforward but lengthy calculation we find that the Einstein conditions imply that $C_{11} = A_{10} = 0$ and $B_{12} = B_{04} = C_{03} = 0$, and the only non-vanishing constants are

$$A_{02} = B_{20} = \frac{\Lambda}{2} \neq 0.$$

We also find that $C_{10}(u,U) = B_{03}(u,U) = 0$, and $C_{02} = -B_{11}, B_{02} = \frac{1}{\Lambda}B_{11}^2$, and

$$\Lambda C_{01} = 2\frac{\partial B_{11}}{\partial u} - 2A_{01}B_{11}.$$

The functions A, B, and C in the metric then become

$$\begin{aligned} A &= \frac{\Lambda}{2}v^2 + A_{01}v + A_0 \\ B &= \frac{1}{\Lambda}B_{11}^2 v^2 + B_{11}vV + \frac{\Lambda}{2}V^2 + B_{01}v + B_{10}V + B_0 \\ C &= -B_{11}v^2 + \frac{2}{\Lambda}\left(\frac{\partial B_{11}}{\partial u} - A_{01}B_{11}\right)v + C_0 \end{aligned}$$

where the remaining arbitrary functions of $\{u, U\}$ satisfy:

$$B_{01} = \frac{1}{\Lambda}\left\{B_{10}B_{11} + \frac{1}{\Lambda}A_{01}B_{11}^2 + \frac{\partial B_{11}}{\partial U} - \frac{1}{\Lambda}B_{11}\frac{\partial B_{11}}{\partial u}\right\},$$

$$C_0 = \frac{1}{\Lambda}\left\{\frac{\partial A_{01}}{\partial U} + \frac{1}{\Lambda}B_{11}\frac{\partial A_{01}}{\partial u} + \frac{\partial B_{10}}{\partial u} + \frac{1}{\Lambda}A_{01}\frac{\partial B_{11}}{\partial u} - \frac{1}{\Lambda}\frac{\partial^2 B_{11}}{\partial u^2} - 2A_0B_{11}\right\}.$$

As an illustration of the CSI property, we compute some scalar curvature invariants:

$$\begin{aligned} &C_{pqrs}C^{pqrs} = \frac{16\Lambda^2}{3} \\ &C_{pqrs}C^{pquv}C^{rs}{}_{uv} = \frac{32\Lambda^3}{9} \\ &C_{pqrs;t}C^{pqrs;t} = 0 \\ &C^{pqrs}\Box C_{pqrs} = 0. \end{aligned}$$

These invariants are indeed constant, as they should be in a CSI space. Note that it is necessary to compute invariants with fourth covariant derivatives of the Riemann tensor to fully verify they are CSI (see, for example, [2]). In Section 4.1 it is proved that the metric does indeed satisfy the CSI property.

2.1. Curvature

Since the space is Einstein, we have that $R_{ab} = 2\Lambda g_{ab} = 2\Lambda(n_{(a}\ell_{b)} + \tilde{m}_{(a}m_{b)})$. We obtain four independent bivectors from contractions of the Riemann tensor with the bivectors constructed from the null tetrad:

$$R_{abcd}\,\ell^c n^d = -2\Lambda\, n_{[a}\ell_{b]} + 2B_{11}\, n_{[a}m_{b]} + 2\Phi\, \ell_{[a}m_{b]} \tag{6}$$

$$R_{abcd}\,\ell^c \tilde{m}^d = 2B_{11}\, n_{[a}\ell_{b]} - \frac{4{B_{11}}^2}{\Lambda}\, n_{[a}m_{b]} - 2B_{11}\, \tilde{m}_{[a}m_{b]} \tag{7}$$

$$R_{abcd}\, n^c \tilde{m}^d = +2\Phi n_{[a}\,\ell_{b]} + 2\Psi\,\ell_{[a}\,m_{b]} + 2\Phi \tilde{m}_{[a}\,m_{b]} \tag{8}$$

$$R_{abcd}\, m^c \tilde{m}^d = -2B_{11}\, n_{[a}m_{b]} + 2\Phi\, \ell_{[a}m_{b]} - 2\Lambda\, \tilde{m}_{[a}m_{b]} \tag{9}$$

where Φ, Ψ are specific functions (and Φ is given explicitly in the Appendix).

3. Walker and Kundt properties

A metric is said to possess a 2-dimensional invariant plane if there exist vectors $\bar{\boldsymbol{\ell}}$ and $\bar{\mathbf{m}}$ such that the bivector $\bar{\boldsymbol{\ell}} \wedge \bar{\mathbf{m}}$ is recurrent; that is,

$$\nabla_a(\bar{\boldsymbol{\ell}} \wedge \bar{\mathbf{m}}) = k_a(\bar{\boldsymbol{\ell}} \wedge \bar{\mathbf{m}}) \tag{10}$$

for some covector k_a. If these vectors are also null and orthogonal, the invariant plane is called null. 4D neutral signature spaces possessing an invariant null plane are known as Walker spaces [3]. The definition of an invariant null plane in terms of spin coefficients was given in [9].

Here, the invariant null plane is defined by the null orthogonal vectors $\boldsymbol{\ell}$ and $\mathbf{m}$ (defined in (2)):

$$\nabla\,(\boldsymbol{\ell} \wedge \mathbf{m}) = (\boldsymbol{\ell} \wedge \mathbf{m}) \otimes \left((A_{01} + v\Lambda)\, du + \left(B_{10} + \frac{1}{2}C_{01} + \Lambda v \right) dU \right) \tag{11}$$

confirming that this is a Walker metric.

A Kundt metric is a metric which possesses a non-zero null vector $\mathbf{X}$ which is geodesic, expansion-free, twist-free, and shear-free, which implies a particular form for the covariant derivative of $\mathbf{X}$ (which can also be defined in terms of spin-coefficients [2]); namely, there exists a null frame in which the covariant derivative of X_a is of the form [10]: $X_{a;b} = L_{11}X_aX_b + \tau(X_am_b + m_aX_b) + \tilde{\tau}(X_a\tilde{m}_b + \tilde{m}_aX_b)$. We can project this tensor into the cotangent space orthogonal to $\mathbf{X}$ by using the projection tensor $h_a{}^b \equiv g_a{}^b - X_aY^b - Y_aX^b$, with Y_a chosen so that $h_a{}^bX_b = 0$. We then find that $L_{cd} \equiv X_{a;b}h_c{}^ah_d{}^b = 0$. Since any matrix may be broken into irreducible parts, $L_{ab} = Lg_{ab} + L_{(ab)} + L_{[ab]}$ (where L is the trace of L_{ab}, $L_{(ab)}$ is a trace-free symmetric matrix, and $L_{[ab]}$ is an anti-symmetric matrix), this implies that the projected covariant derivative of X_a is expansion-free, shear-free and vorticity-free [8].

For Walker metrics in 4D neutral spaces with an invariant 2-dimensional null plane it is always possible to find a null field that is geodesic, expansion-free, vorticity-free. But, in general, this null vector is not necessarily shear-free; i.e., the space does not necessarily have to be Kundt.

However, let us next show that the spaces under investigation here are Kundt. The geodesic null vector X_a for the CSI metric defined in (1) is expansion-free and vorticity-free when X_a is the gradient of some function $X(u, v, U, V)$, and $X_{,v} = X_{,V} = 0$. Thus $X_{,a} = X_{,u}\ell_a + X_{,U}m_a$ and its covariant derivative is of the form:

$$X_{a;b} = (\Lambda X_{,u}v + A_{01}X_{,u} + X_{,uu})\,\ell_a\ell_b \tag{12}$$

$$+ \left(-B_{11}X_{,u}v - \frac{B_{11}A_{01}X_{,u}}{\Lambda} + X_{,Uu} + \frac{B_{11,u}}{\Lambda} \right) (\ell_am_b + m_a\ell_b) \tag{13}$$

$$+ \left((\Lambda V + B_{11}v + B_{10})\,X_U + \frac{X_uB_{11,U}}{\Lambda} + B_{11}X_uV + \frac{{B_{11}}^2A_1X_u}{\Lambda^2} \right. \tag{14}$$

$$\left. + 2\,\frac{v{B_{11}}^2X_u}{\Lambda} + \frac{B_{10}B_{11}X_u}{\Lambda} + X_{UU} - \frac{B_{11}B_{11,u}X_u}{\Lambda^2} \right) m_am_b. \tag{15}$$

If we assume $X_{,U} \neq 0$ and $X_{,u} = 0$, so that $X_a = X_{,U}m_a$, and use the projection operator

$$h_a{}^b = g_a{}^b - \frac{1}{X_{,U}}\tilde{m}_aX^b - \frac{1}{X_{,U}}X_a\tilde{m}^b$$

(which has $h_a{}^bX_b = 0$ as required), we find that

$$X_{c;d}h_a{}^ch_b{}^d = 0 \tag{16}$$

so that $X = X_{,U}\, m_a$ is shear-free when projected onto the orthogonal cotangent space, and consequently $\mathbf{X}$ is a Kundt vector.

4. Algebraic classification

Consider the 2 independent boosts:

$$(\boldsymbol{\ell}, \boldsymbol{n}) \mapsto (e^{\lambda_1}\boldsymbol{\ell}, e^{-\lambda_1}\boldsymbol{n})$$
$$(\boldsymbol{m}, \tilde{\boldsymbol{m}}) \mapsto (e^{\lambda_2}\boldsymbol{m}, e^{-\lambda_2}\tilde{\boldsymbol{m}}).$$

For an order 2 tensor T, we can consider the boost weight of the components of this tensor as $\mathbf{b} \in \mathbb{Z}^2$ as follows. If the covariant component $T_{\mu_1\mu_2}$ transforms as:

$$T_{\mu_1\mu_2} \mapsto e^{-(b_1\lambda_1+b_2\lambda_2)}T_{\mu_1\mu_2},$$

then we will say the component $T_{\mu_1\mu_2}$ is of boost weight $\mathbf{b} \equiv (b_1, b_2)$ [11]. We can now decompose a tensor into boost weights, in particular,

$$T = \sum_{\mathbf{b}\in\mathbb{Z}^2} (T)_{\mathbf{b}}, \tag{17}$$

where $(T)_{\mathbf{b}}$ means the projection onto the components of boost weight $\mathbf{b}$ [11].

By successively using null-rotations at each level, which leave the neutral signature metric invariant, we can use the well-known boost-weight classification to give an algebraic classification of arbitrary-signature (Weyl) tensors. We can also define the $\mathbf{S}_i$-property and the $\mathbf{N}$-property using the algebraic boost weight decomposition [11].

4.1. Boost weight decomposition

By direct calculation, we find that the boost weights (b_1, b_2) of the non-trivial components of the Riemann tensor in Eqs. (6)–(9) are, respectively:

$$(0, 0),\ (1, -1),\ (-1, -1), \tag{18}$$
$$(1, -1),\ (2, -2),\ (0, -2), \tag{19}$$
$$(-1, -1),\ (-2, -2),\ (-1, -1), \tag{20}$$
$$(1, -1),\ (-1, -1),\ (0, 0). \tag{21}$$

These spaces are CSI [2]. Since the metric is Einstein, $R_{ab} = 2\Lambda g_{ab}$ and $R_{ab;c} = 0$, all scalar curvature invariants constructed from the Ricci tensor will be a constant multiple of powers of Λ.

The only boost weight (0, 0) terms in the Riemann tensor (see (6) and (9)) have coefficients which are a constant multiple of Λ, and hence all scalar curvature invariants constructed from the Riemann tensor will be a constant multiple of powers of Λ. See, for example, the curvature invariants displayed earlier.

All of the components of the covariant derivative of the Riemann tensor are of negative–negative boost weight $((b_1, b_2); b_1 < 0, b_2 < 0$ (see Appendix)). Therefore all scalar curvature invariants constructed from the covariant derivative of the Riemann tensor will be zero [1].

5. Holonomy

Since the metric has signature (2, 2), the holonomy groups will be Lie subgroups of $O(2, 2)$. Then the holonomy algebras will be isomorphic to subalgebras of $\mathfrak{o}(2, 2)$. Ghanam and Thompson [12] examined each of the possible Lie subalgebras of $\mathfrak{o}(2, 2)$ when represented by type (1, 1) tensors by exploiting an isomorphism between $\mathfrak{o}(2, 2)$ and $\mathfrak{su}(1, 1) \times \mathfrak{su}(1, 1)$. The geometric structure that must be possessed by each holonomy type (if it is possible) is determined and given in their Table II. Wang and Hall [13] have also examined the holonomy structure of 4D neutral metrics. They represented the members of $\mathfrak{o}(2, 2)$ by type 2-forms, and showed that $\Lambda_m M$, the vector space of 2-forms at $m \in M$, is six-dimensional. They gave an equivalent classification of the holonomy algebras.

We note that in the neutral 4D case, if the distribution is 2D null, it need not contain an invariant null line (i.e., a Walker space is not necessarily a Kundt space; e.g., see the example in [8]). Note that if a space admits an invariant null-line (which implies it admits a recurrent null vector and is a special Kundt spacetime) then it automatically admits an invariant null plane containing the null line. That is, a Walker space with an invariant 1D distribution is a Walker space with an invariant 2D distribution.

5.1. Computations

Since the space is Einstein, we have that $R_{ab} = 2\Lambda g_{ab} = 2\Lambda(n_{(a}\ell_{b)} + \tilde{m}_{(a}m_{b)})$.

From the earlier calculations of the contractions of the Riemann tensor with the bivectors from the null tetrad, we see that the generators of the infinitesimal holonomy algebra are:

$$n_{[a}\ell_{b]},\ n_{[a}m_{b]},\ \ell_{[a}m_{b]},\ \tilde{m}_{[a}m_{b]}$$

(the covariant derivatives of the Riemann tensor give no additional generators—see Appendix). Hence in some connected open neighbourhood the holonomy group is generally 4 dimensional and of type (4c) in [13]. Since $\mathbf{m}$ is an eigenvector of

each member of this holonomy algebra, it is recurrent (and it is not possible to find a scaling in order for it to be covariantly constant; i.e., it is 'properly' recurrent [14]). A number of 3 dimensional subgroups of type (3d) are possible when $\Phi = 0$ or $B_{11} = 0$ (in this latter case the metric is much simpler).

6. Universality

A metric is called *universal* if all conserved symmetric rank-2 tensors constructed from the metric, the Riemann tensor and its covariant derivatives of arbitrary order are multiples of the metric [15]. A universal metric thus has $S_{ab} = \lambda g_{ab}$ for any conserved symmetric rank-2 tensor S_{ab}. Universal spacetimes are vacuum solutions to all theories with Lagrangian consisting of the metric, the Riemann tensor and its derivatives of arbitrary order. Examples of universal metrics were given in [15].

In the Lorentzian case, it was shown that *a universal spacetime is necessarily a CSI spacetime* [16]; this was proven in an arbitrary number of dimensions and, since the proof essentially goes through in precisely the same way for any signature [17], it is also true in neutral signature.

Therefore, the CSI spacetimes play an important role in the question of universality. Note, however, that the CSI criterion is not sufficient. As far as universal spacetimes are concerned, the CSI constraint on universal spacetimes does allow for a systematic study of the necessary and sufficient conditions for universal metrics of Weyl types N and III in the classification of [18].

In particular, in [16] it was shown that a type N spacetime is universal if and only if it is an Einstein Kundt spacetime in the Lorentzian case (in arbitrary dimensions). However, for type III spacetimes the Einstein Kundt condition is not a sufficient condition for a spacetime to be universal and further constraints are necessary.

6.1. Neutral signature

For neutral signature, a universal space is necessarily a CSI space [16,17]. We also expect similar results (to those in the Lorentzian case discussed above); e.g., that algebraically special neutral signature Kundt spaces are universal.

The boost weight decomposition of the spaces were given above (i.e., their algebraic type was described). Let us first present several conserved tensors below to illustrate the universality of the spaces under consideration here.

Since the metric is Einstein, the only conserved symmetric rank-2 tensor constructed from the Ricci tensor is the Ricci tensor itself, $R_{ab} = 2\Lambda g_{ab}$ (or polynomials in the Ricci tensor), and consequently all scalar curvature invariants constructed from the Ricci tensor will be a constant multiple of powers of Λ.

Computing the left-hand sides of Eqs. (3, 7, 9–11) of [19] (for an Einstein space, we could replace the Weyl tensor with the Riemann tensor so that the constants of proportionality would change):

$$C_{almn}C_{b}{}^{lmn} + C_{blmn}C_{a}{}^{lmn} = \frac{8\Lambda^2}{3}g_{ab} \tag{22}$$

$$3\,C^{pqrs}{}_{;a}C_{pqrs;b} - 6\,C^{pqr}{}_{a;s}C_{pqrb}{}^{;s} + 12\,C^{prqs}C^{t}{}_{pqa}C_{trsb} + \frac{1}{2}g_{ab}[C_{pqrs}C^{pquv}C^{rs}{}_{uv}] = \frac{64\Lambda^3}{9}g_{ab} \tag{23}$$

$$2\,C^{pqrs}C_{pqrs;(ab)} + 2\,C^{pqrs}{}_{;a}C_{pqrs;b} + g_{ab}\Big[-2\,C_{pqrs;t}C^{pqrs;t} + 2\,C_{pqrs}C^{pquv}C^{rs}{}_{uv} + 8\,C_{prqs}C^{p}{}_{u}{}^{q}{}_{v}C^{rusv}\Big] = \frac{64\Lambda^3}{3}g_{ab} \tag{24}$$

$$\begin{aligned}&\frac{1}{2}C^{pqrs}C_{pqrs;(ab)} + \frac{1}{2}C^{pqrs}{}_{;a}C_{pqrs;b} - C^{pqr}{}_{a;s}C_{pqrb}{}^{;s} + C^{pqrs}C_{pqta}C_{rs}{}^{t}{}_{b} + 4\,C^{prqs}C^{t}{}_{pqa}C_{trsb} - C^{pqr}{}_{s}C_{pqrt}C^{s}{}_{a}{}^{t}{}_{b}\\ &\quad + \frac{1}{4}g_{ab}\Big[-C_{pqrs;t}C^{pqrs;t} + C_{pqrs}C^{pquv}C^{rs}{}_{uv} + 4C_{prqs}C^{p}{}_{u}{}^{q}{}_{v}C^{rusv}\Big] = \frac{16\Lambda^3}{3}g_{ab}\end{aligned} \tag{25}$$

$$\begin{aligned}&-\frac{3}{4}C^{pqrs}C_{pqrs;(ab)} + \frac{3}{4}C^{pqrs}{}_{;a}C_{pqrs;b} - \frac{3}{2}C^{pqrs}C_{pqta}C_{rs}{}^{t}{}_{b} - 9\,C^{prqs}C^{t}{}_{pqa}C_{trsb}\\ &\quad + \frac{3}{2}C^{pqr}{}_{s}C_{pqrt}C^{s}{}_{a}{}^{t}{}_{b} + \frac{1}{2}g_{ab}[C_{prqs}C^{p}{}_{u}{}^{q}{}_{v}C^{rusv}] = -\frac{40\Lambda^3}{9}g_{ab}.\end{aligned} \tag{26}$$

These are the essential conserved tensors in the lower covariant derivatives of the Riemann tensor. The expressions above illustrate universality. Note that the first equation is an identity for a CSI space in 4D [2]. We shall prove that these spaces are universal in the next subsection.

6.2. Analysis

Any tensor T can be decomposed according to (17). Let us define $B = b_1 + b_2$ and, with obvious notation, we can also decompose T according to

$$T^{(B)} \equiv \sum_{B=b_1+b_2} (T)_B.$$

We denote the maximum value of B for T as $B_{\max}(T)$.

Let us now show that the spaces under consideration are universal. For a rank-2 symmetric tensor S_{ab}, in general $B \in \{-2, 0, 2\}$ and $B_{\max}(S) = 2$. The metric only has boost weight (0, 0) terms and hence has $B \in \{0\}$ and all contractions of a tensor using the metric does not change the value of B.

Now, in our case, from Eqs. (18)–(21) we have that

$$R = R_{(0,0)} + R_{(1,-1)} + R_{(2,-2)} + R_{(-1,-1)} + R_{(-2,-2)}$$

and so

$$R = R^{(0)} + R^{(-2)} + R^{(-4)}$$

where, for example,

$$R^{(0)} \equiv R_{(0,0)} + R_{(1,-1)} + R_{(2,-2)}.$$

Note that $B_{\max}(R) = 0$. The only contributions to S_{ab} are boost weight (0, 0) terms with constant multiple powers of Λ.

Now, denoting the covariant derivative of the Riemann tensor by ∇R, from the Appendix we have that

$$\nabla R = (\nabla R)^{(-3)} + (\nabla R)^{(-5)},$$

with $B_{\max}(\nabla R) = -3$, and we have that

$$(\nabla^2 R) = (\nabla^2 R)^{(-4)} + (\nabla^2 R)^{(-6)},$$

so that $B_{\max}(\nabla^n R) \leq -3$ for all positive integers n.

Therefore, we note that any conserved tensor S_{ab} constructed from the Riemann tensor and any of its covariant derivatives has $B_{\max}(S) = -3$. Indeed, let S_{ab} be constructed by suitable contractions (using the metric) of n_0 multiples of the Riemann tensor, n_1 multiples of first covariant derivatives of the Riemann tensor, etc.; then

$$B_{\max}(S) \leq n_0 B_{\max}(R) + \sum_i n_i B_{\max}(\nabla^i R).$$

Thus if any $n_i \neq 0$ for any $i \geq 1$, then $B_{\max}(S) \leq -3$. Hence no $\nabla^i(R)$ terms contribute to S_{ab}. It is then a simple matter to investigate the contributions to S_{ab} from multiple copies of $R^{(0)}$ and $R^{(-2)}$ via multiplication and contraction (see Eqs. (18)–(21), (22)–(25), and the Appendix). Hence all such contractions can only contribute to S_{ab} by constant multiples of powers of Λ and the metric. Hence the spaces are universal.

Acknowledgements

The work was supported, in part, by NSERC of Canada (AC) (grant number NSERC 8532-2012) and by the David and Faye Sobey Foundation (NM).

Appendix

In this appendix let us display some important computations for the CSI metric (1). (See the text for the relevant notation and definitions.)

From [14] and $R_{ab} = \Lambda g_{ab}$, the Weyl and Riemann tensors differ by:

$$R_{abcd} - C_{abcd} = \frac{1}{n-2}\left(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{bc}g_{ad} - R_{ad}g_{bc}\right) \tag{27}$$

$$+ \frac{R}{(n-1)(n-2)}\left(g_{bc}g_{ad} - g_{bd}g_{ac}\right) \tag{28}$$

$$= \Lambda\left(g_{ac}g_{bd} - g_{bc}g_{ad}\right) + \frac{2\Lambda}{3}\left(g_{bc}g_{ad} - g_{bd}g_{ac}\right) \tag{29}$$

$$= \frac{\Lambda}{3}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right). \tag{30}$$

Next, we give the contractions of the Weyl tensor with bivectors from the null tetrad:

$$C_{abcd}\,\ell^c n^d = -\frac{4\Lambda}{3}\,n_{[a}\ell_{b]} + 2B_{11}\,n_{[a}m_{b]} + 2\Phi\,\ell_{[a}m_{b]} \tag{31}$$

$$C_{abcd}\,\ell^c m^d = -\frac{2\Lambda}{3}\,\ell_{[a}m_{b]} \tag{32}$$

$$C_{abcd}\,\ell^c \tilde{m}^d = 2B_{11}\,n_{[a}\ell_{b]} - \frac{4{B_{11}}^2}{\Lambda}\,n_{[a}m_{b]} - \frac{2\Lambda}{3}\,\ell_{[a}\tilde{m}_{b]} - 2B_{11}\,\tilde{m}_{[a}m_{b]} \tag{33}$$

$$C_{abcd}\, n^c m^d = -\frac{2\Lambda}{3}\, n_{[a}m_{b]} \tag{34}$$

$$C_{abcd}\, n^c \tilde{m}^d = +2\Phi\, n_{[a}\ell_{b]} - \frac{2\Lambda}{3}\, n_{[a}\tilde{m}_{b]} + 2\Psi\, \ell_{[a}m_{b]} + 2\Phi\, \tilde{m}_{[a}m_{b]} \tag{35}$$

$$C_{abcd}\, m^c \tilde{m}^d = -2B_{11}\, n_{[a}m_{b]} + 2\Phi\, \ell_{[a}m_{b]} - \frac{4\Lambda}{3}\, \tilde{m}_{[a}m_{b]} \tag{36}$$

where

$$\Phi \equiv \frac{1}{2\Lambda}\left(\Lambda\frac{\partial A_{01}}{\partial U} - \Lambda\left(\frac{\partial B_{10}}{\partial u}\right) + B_{11}\left(\frac{\partial A_{01}}{\partial u}\right) + A_{01}\frac{\partial B_{11}}{\partial u} - \frac{\partial^2 B_{11}}{\partial u^2}\right), \tag{37}$$

and Ψ is a complicated expression in terms of the metric coefficients and their partial derivatives. Therefore, we have that the contractions of the Riemann tensor with bivectors constructed from the null tetrad are given by:

$$R_{abcd}\, \ell^c n^d = -2\Lambda\, n_{[a}\ell_{b]} + 2B_{11}\, n_{[a}m_{b]} + 2\Phi\, \ell_{[a}m_{b]} \tag{38}$$

$$R_{abcd}\, \ell^c m^d = 0 \tag{39}$$

$$R_{abcd}\, \ell^c \tilde{m}^d = 2B_{11}\, n_{[a}\ell_{b]} - \frac{4{B_{11}}^2}{\Lambda}\, n_{[a}m_{b]} - 2B_{11}\, \tilde{m}_{[a}m_{b]} \tag{40}$$

$$R_{abcd}\, n^c m^d = 0 \tag{41}$$

$$R_{abcd}\, n^c \tilde{m}^d = +2\Phi n_{[a}\, \ell_{b]} + 2\Psi\, \ell_{[a}\, m_{b]} + 2\Phi \tilde{m}_{[a}\, m_{b]} \tag{42}$$

$$R_{abcd}\, m^c \tilde{m}^d = -2B_{11}\, n_{[a}m_{b]} + 2\Phi\, \ell_{[a}m_{b]} - 2\Lambda\, \tilde{m}_{[a}m_{b]}. \tag{43}$$

In addition, we compute contractions with the covariant derivative of the Riemann tensor to find:

$$R_{abcd;e}\, \ell^c n^d \ell^e = R_{abcd;e}\, \ell^c n^d m^e = 0 \tag{44}$$

$$R_{abcd;e}\, \ell^c n^d n^e = 2\chi_1 \ell_{[a}\, m_{b]} \tag{45}$$

$$R_{abcd;e}\, \ell^c n^d \tilde{m}^e = 2\chi_2 \ell_{[a}\, m_{b]} \tag{46}$$

$$R_{abcd;e}\, \ell^c m^d \ell^e = R_{abcd;e}\, \ell^c m^d n^e = R_{abcd;e}\, \ell^c m^d m^e = R_{abcd;e}\, \ell^c m^d \tilde{m}^e = 0 \tag{47}$$

$$R_{abcd;e}\, \ell^c \tilde{m}^d \ell^e = R_{abcd;e}\, \ell^c \tilde{m}^d n^e = R_{abcd;e}\, \ell^c \tilde{m}^d m^e = R_{abcd;e}\, \ell^c \tilde{m}^d \tilde{m}^e = 0 \tag{48}$$

$$R_{abcd;e}\, n^c m^d \ell^e = R_{abcd;e}\, n^c m^d n^e = R_{abcd;e}\, n^c m^d m^e = R_{abcd;e}\, n^c m^d \tilde{m}^e = 0 \tag{49}$$

$$R_{abcd;e}\, n^c \tilde{m}^d \ell^e = -2\chi_2 \ell_{[a}\, m_{b]} \tag{50}$$

$$R_{abcd;e}\, n^c \tilde{m}^d n^e = +2\, \chi_1 n_{[a}\ell_{b]} - 2\, \chi_3 \ell_{[a}m_{b]} + 2\, \chi_1 \tilde{m}_{[a}m_{b]} \tag{51}$$

$$R_{abcd;e}\, n^c \tilde{m}^d m^e = +2\chi_1 \ell_{[a}\, m_{b]} \tag{52}$$

$$R_{abcd;e}\, n^c \tilde{m}^d \tilde{m}^e = +2\, \chi_2 n_{[a}\ell_{b]} - 2\, \chi_4 \ell_{[a}m_{b]} + 2\, \chi_2 \tilde{m}_{[a}m_{b]} \tag{53}$$

$$R_{abcd;e}\, m^c \tilde{m}^d \ell^e = R_{abcd;e}\, m^c \tilde{m}^d m^e = 0 \tag{54}$$

$$R_{abcd;e}\, m^c \tilde{m}^d n^e = +2\chi_1 \ell_{[a}\, m_{b]} \tag{55}$$

$$R_{abcd;e}\, m^c \tilde{m}^d \tilde{m}^e = +2\chi_2 \ell_{[a}\, m_{b]} \tag{56}$$

where the χ_i are expressions in terms of the metric coefficients and their partial derivatives. These bivectors do not contribute to the infinitesimal holonomy algebra as they are not linearly independent of the bivectors (38)–(43).

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