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REDSHIFT AND CONTACT FORMS

VLADIMIR CHERNOV AND STEFAN NEMIROVSKI

ABSTRACT. It is shown that the redshift between two Cauchy surfaces in a globally hyperbolic spacetime equals the ratio of the associated contact forms on the space of light rays of that spacetime.

1. INTRODUCTION

Let X be a spacetime, that is, a connected time-oriented Lorentz manifold [2, §3.1]. The Lorentz scalar product on X will be denoted by $\langle \cdot, \cdot \rangle$ and assumed to have signature $(+, -, \dots, -)$ with $n \geq 2$ negative spatial dimensions.

Suppose that n_E (‘emitter’) and n_R (‘receiver’) are two infinitesimal observers, i.e. future-pointing unit Lorentz length vectors, at events $E, R \in X$ connected by a null geodesic γ . Then the *photon redshift* $z = z(n_E, n_R, \gamma)$ from n_E to n_R along γ is defined by the formula

$$1 + z = \frac{\langle n_E, \dot{\gamma}(E) \rangle}{\langle n_R, \dot{\gamma}(R) \rangle}$$

for any affine parametrisation of γ . In other words, $1 + z$ is the ratio of the frequencies of any lightlike particle travelling along γ measured by n_E and n_R , see e.g. [11, Appendix 9A] or [17, p. 354]. If $z > 0$, such particles appear ‘redder’ (having lower frequency) to n_R than to n_E , whence the terminology.

Assume now that X is globally hyperbolic [4, 5] and consider its space of light rays \mathfrak{N}_X . By definition, a point $\gamma \in \mathfrak{N}_X$ is an equivalence class of inextendible future-directed null geodesics up to an orientation preserving affine reparametrisation.

A seminal observation of Penrose and Low [18, 13, 14] is that the space \mathfrak{N}_X has a canonical structure of a contact manifold (see also [16, 12, 1]). A contact form α_M on \mathfrak{N}_X defining that contact structure can be associated to any smooth spacelike Cauchy surface $M \subset X$. Namely, consider the map

$$\iota_M : \mathfrak{N}_X \longrightarrow T^*M$$

taking $\gamma \in \mathfrak{N}_X$ represented by a null geodesic $\gamma \subset X$ to the 1-form on M at the point $x = \gamma \cap M$ collinear to $\langle \dot{\gamma}(x), \cdot \rangle|_M$ and having unit length with respect to the induced Riemann metric on M (see formula (2.2) below). This map identifies \mathfrak{N}_X with the unit cosphere

bundle \mathbb{S}^*M of the Riemannian manifold $(M, -\langle \cdot, \cdot \rangle|_M)$. Then

$$\alpha_M := \iota_M^* \lambda_{\text{can}},$$

where $\lambda_{\text{can}} = \sum p_k dq^k$ is the canonical Liouville 1-form on T^*M .

Contact forms defining the same contact structure are pointwise proportional. The purpose of the present note is to point out that the ratio of the contact forms on \mathfrak{N}_X associated to different Cauchy surfaces in X is given by the redshifts between infinitesimal observers having those Cauchy surfaces as their rest spaces.

Definition 1.1. Let M and M' be spacelike Cauchy surfaces in X . The redshift from M to M' along $\gamma \in \mathfrak{N}_X$ is defined by

$$z(M, M', \gamma) := z(n_M(x), n_{M'}(x'), \gamma),$$

where γ is any inextendible null geodesic representing γ , $x = \gamma \cap M$, $x' = \gamma \cap M'$, and n_M and $n_{M'}$ are the future pointing normal unit vector fields on M and M' .

Theorem 1.2. Let M and M' be spacelike Cauchy surfaces in X . For every $\gamma \in \mathfrak{N}_X$, we have

$$\frac{\alpha_{M'}}{\alpha_M}(\gamma) = 1 + z(M, M', \gamma).$$

Remark 1.3. The theorem remains true for *partial* Cauchy surfaces, i.e. locally closed acausal spacelike hypersurfaces $M, M' \subset X$, and for $\gamma \in \mathfrak{N}_X$ corresponding to null geodesics intersecting both M and M' .

Remark 1.4. If M and M' are Cauchy surfaces through a point $x \in X$ such that $n_M(x) = n_{M'}(x)$, then the theorem shows that the contact forms α_M and $\alpha_{M'}$ coincide on the tangent spaces to \mathfrak{N}_X at all points corresponding to null geodesics passing through x . In other words, an infinitesimal observer at an event x defines a contact form on $T\mathfrak{N}_X$ restricted to the sky $\mathfrak{S}_x \subset \mathfrak{N}_X$.

The contact geometry of \mathfrak{N}_X was previously used to recover the causal or, equivalently [15], conformal structure of X , see [14, 16, 10, 7, 8, 9]. Theorem 1.2 should make it possible to apply techniques from contact geometry to study the metric structure of a globally hyperbolic spacetime. A token application to the comparison of Liouville and Riemannian volumes on different Cauchy surfaces is given in §3 below.

2. PROOF OF THEOREM 1.2

The key fact is the following basic property of vector fields tangent to variations of pseudo-Riemannian geodesics by curves of the same speed. For Jacobi fields tangent to families of null geodesics in Lorentz manifolds, this computation appears in [18, p. 176], [16, pp. 252–253], and [1, pp. 10–11].

Lemma 2.1. *Let $\gamma_s : (a, b) \rightarrow X$, $0 \leq s < \varepsilon$, be a one-parameter family of curves in a pseudo-Riemannian manifold $(X, \langle \cdot, \cdot \rangle)$ such that γ_0 is a geodesic and $\langle \dot{\gamma}_s, \dot{\gamma}_s \rangle$ is independent of s . If*

$$J(t) := \left. \frac{d}{ds} \right|_{s=0} \gamma_s(t)$$

is the vector field along γ_0 tangent to this family, then

$$\langle \dot{\gamma}_0(t), J(t) \rangle = \text{const.}$$

Proof. Let us show that the t -derivative of this scalar product is zero. Indeed,

$$\frac{d}{dt} \langle \dot{\gamma}_0(t), J(t) \rangle = \langle \nabla_t \dot{\gamma}_0(t), J(t) \rangle + \langle \dot{\gamma}_0(t), \nabla_t J(t) \rangle, \quad (2.1)$$

where ∇ is the pull-back of the Levi-Civita connection of the pseudo-Riemannian metric on X to $(a, b) \times [0, \varepsilon)$ by the map $(t, s) \mapsto \gamma_s(t)$. The first term on the right hand side vanishes because the tangent vector of a geodesic is parallel along the geodesic. Note further that

$$\nabla_t \frac{\partial}{\partial s} = \nabla_s \frac{\partial}{\partial t}$$

since the Levi-Civita connection has no torsion and $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$ (see also [17, Proposition 4.44(1)]). Hence, the right hand side of (2.1) is equal to

$$\langle \dot{\gamma}_0(t), \nabla_s \dot{\gamma}_s(t) \Big|_{s=0} \rangle = \frac{1}{2} \left. \frac{d}{ds} \right|_{s=0} \langle \dot{\gamma}_s, \dot{\gamma}_s \rangle = 0$$

because the speed of γ_s is independent of s by assumption. \square

Remark 2.2. The relevance of torsion in this context is pointed out in the footnote on p. 184 of [18].

Let now M be a smooth spacelike Cauchy surface in a spacetime X and γ an inextendible future-directed null geodesic in X intersecting M at the (unique) point $x = \gamma \cap M$. Then

$$\iota_M(\gamma) = \frac{\langle \dot{\gamma}(x), \cdot \rangle|_M}{\langle \dot{\gamma}(x), n_M(x) \rangle}, \quad (2.2)$$

where n_M is the future-pointing unit normal vector field on M and $\gamma \in \mathfrak{N}_X$ is the equivalence class of γ . Indeed, since $\langle \dot{\gamma}(x), \cdot \rangle$ is a null covector, the Riemannian length of its restriction to $T_x M$ is equal to the Riemannian length of its restriction to the Lorentz normal direction, which is precisely $\langle \dot{\gamma}(x), n_M(x) \rangle (> 0)$.

Thus, if $\mathbf{v} \in T_\gamma \mathfrak{N}_X$ and $v = (\iota_M)_* \mathbf{v}$, then

$$\alpha_M(\mathbf{v}) = \lambda_{\text{can}}(v) = \frac{\langle \dot{\gamma}(x), (\pi_M)_* v \rangle}{\langle \dot{\gamma}(x), n_M(x) \rangle} \quad (2.3)$$

by the definition of the canonical 1-form λ_{can} and formula (2.2), where $\pi_M : T^*M \rightarrow M$ denotes the bundle projection.

Suppose that $\gamma_s : (a, b) \rightarrow X$, $s \in [0, \varepsilon]$, is a family of null geodesics intersecting M such that the maximal extension of γ_0 is γ and the corresponding curve in \mathfrak{N}_X has tangent vector \mathbf{v} at γ or, equivalently, $\frac{d}{ds}\big|_{s=0} \iota_M(\gamma_s) = v$. Let $x(s) = \gamma_s \cap M$ so that $x(0) = x$. Then

$$(\pi_M)_* v = \frac{d}{ds}\bigg|_{s=0} x(s)$$

because $x(s) = \pi_M \circ \iota_M(\gamma_s)$ by the definition of ι_M . Hence,

$$(\pi_M)_* v = J(x) + \tau'(0)\dot{\gamma}(x),$$

where $J = \frac{d}{ds}\big|_{s=0} \gamma_s$ is the Jacobi vector field along γ_0 tangent to the family γ_s and $\tau = \tau(s)$ is the function defined by $\gamma_s(\tau(s)) = x(s)$. Since $\dot{\gamma}(x)$ is null, it follows that

$$\langle \dot{\gamma}(x), (\pi_M)_* v \rangle = \langle \dot{\gamma}(x), J(x) \rangle. \quad (2.4)$$

If M' is another Cauchy surface and $x' = \gamma \cap M'$, we may choose $(a, b) \subseteq \mathbb{R}$ so that $\gamma(a, b) \ni x, x'$ and a family γ_s as above exists on (a, b) . By formulas (2.3) and (2.4), we obtain

$$\alpha_M(\mathbf{v}) = \frac{\langle \dot{\gamma}(x), J(x) \rangle}{\langle \dot{\gamma}(x), n_M(x) \rangle} \quad \text{and} \quad \alpha_{M'}(\mathbf{v}) = \frac{\langle \dot{\gamma}(x'), J(x') \rangle}{\langle \dot{\gamma}(x'), n_{M'}(x') \rangle}.$$

However,

$$\langle \dot{\gamma}(x), J(x) \rangle = \langle \dot{\gamma}(x'), J(x') \rangle$$

by Lemma 2.1 and therefore

$$\frac{\alpha_{M'}(\mathbf{v})}{\alpha_M(\mathbf{v})} = \frac{\langle \dot{\gamma}(x), n_M(x) \rangle}{\langle \dot{\gamma}(x'), n_{M'}(x') \rangle} = 1 + z(n_M(x), n_{M'}(x'), \gamma),$$

which proves Theorem 1.2.

Remark 2.3. The proof shows that the ratio $\frac{\alpha_{M'}(\mathbf{v})}{\alpha_M(\mathbf{v})}$, where \mathbf{v} is a tangent vector to \mathfrak{N}_X at a point $\gamma \in \mathfrak{N}_X$, is a positive function depending only on γ . Thus, the contact forms on \mathfrak{N}_X associated to different Cauchy surfaces in X define the same co-oriented contact structure indeed. This contact structure can also be described as the pull-back of the canonical contact structure on the spherical cotangent bundle ST^*M of a Cauchy surface M by the map $\rho_M = s_M \circ \iota_M$, where $s_M : T^*M - \{\text{zero section}\} \rightarrow ST^*M$ is the projection to the spherisation, see [16, pp. 252–253] and [7, §4].

3. LIOUVILLE MEASURE AND RIEMANNIAN VOLUME

Let M and M' be two spacelike Cauchy surfaces in a globally hyperbolic spacetime $(X, \langle \cdot, \cdot \rangle)$ and consider the contact forms $\alpha_M = \iota_M^* \lambda_{\text{can}}$ and $\alpha_{M'} = \iota_{M'}^* \lambda_{\text{can}}$ on \mathfrak{N}_X associated to M and M' . Then

$$\alpha_M = (1 + z(M, M', \gamma))^{-1} \alpha_{M'}$$

by Theorem 1.2 and therefore

$$\alpha_M \wedge (d\alpha_M)^{n-1} = (1 + z(M, M', \gamma))^{-n} \alpha_{M'} \wedge (d\alpha_{M'})^{n-1} \quad (3.1)$$

because $\alpha \wedge \alpha = 0$ and $d(f\alpha) = fd\alpha + df \wedge \alpha$ for any function f and 1-form α .

Recall that the *Liouville measure* on the unit cosphere bundle of a Riemannian manifold is defined by the non-vanishing $(2n - 1)$ -form

$$\Omega := \lambda_{\text{can}} \wedge (d\lambda_{\text{can}})^{n-1}.$$

Thus, formula (3.1) may be viewed as a general volume–redshift relation (cf. [11, §14.12 and §15.9]) for the Liouville measures on the unit cosphere bundles of M and M' with respect to the Riemann metrics $-\langle \cdot, \cdot \rangle_M$ and $-\langle \cdot, \cdot \rangle_{M'}$. Indeed, let

$$\iota_{M'M} = \iota_M \circ (\iota_{M'})^{-1} : \mathbb{S}^*M' \xrightarrow{\cong} \mathbb{S}^*M \quad (3.2)$$

be the map identifying the unit covectors corresponding to the same null geodesic at its intersection points with M and M' . Then (3.1) shows that

$$(\iota_{M'M})^* \Omega_M = (1 + z(M, M', \gamma))^{-n} \Omega_{M'} \quad (3.3)$$

at $\iota_{M'}(\gamma) \in \mathbb{S}^*M'$.

Let $\mathfrak{L} \subseteq \mathfrak{N}_X$ be a (Borel) subset of the space of light rays and denote by \mathfrak{L}_x the set of null geodesics from \mathfrak{L} passing through a point $x \in X$. Integrating (3.3) over $\iota_{M'}(\mathfrak{L})$, we obtain that

$$\int_{\iota_M(\mathfrak{L})} \Omega_M = \int_{\iota_{M'}(\mathfrak{L})} (1 + z(M, M', \gamma))^{-n} \Omega_{M'}. \quad (3.4)$$

The Liouville measure is locally the product of the Riemann measure on the base manifold and the surface area measure on the unit sphere in the standard Euclidean space \mathbb{R}^n , see [3, §5.2] or [6, Theorem VII.1.3]. Therefore both integrals in (3.4) can be converted to double integrals. Applying this to the left hand side first, we see that

$$\int_{\iota_M(\mathfrak{L})} \Omega_M = \int_M dV_M(x) \int_{\iota_M(\mathfrak{L}_x)} d\omega_x = \int_M \omega_M(x, \mathfrak{L}) dV_M(x),$$

where dV_M is the Riemann measure on M , $d\omega_x$ is the surface area measure on the fibre \mathbb{S}_x^*M , and

$$\omega_M(x, \mathfrak{L}) := \int_{\iota_M(\mathfrak{L}_x)} d\omega_x$$

is the area of the set $\iota_M(\mathfrak{L}_x)$ of unit covectors at $x \in M$ corresponding to null geodesics from \mathfrak{L} , i.e. the solid angle spanned by the light rays

from \mathfrak{L} at $x \in M$. Now (3.4) takes the form

$$\int_M \omega_M(x, \mathfrak{L}) dV_M(x) = \int_{M'} dV_{M'}(x') \int_{\iota_{M'}(\mathfrak{L}_{x'})} (1 + z(M, M', \gamma))^{-n} d\omega_{x'}. \quad (3.5)$$

Example 3.1. Assume that the redshift $z(M, M', \gamma) = z$ is the same for all $\gamma \in \mathfrak{L}$. Then (3.5) simplifies to

$$\int_M \omega_M(x, \mathfrak{L}) dV_M(x) = \frac{1}{(1+z)^n} \int_{M'} \omega_{M'}(x', \mathfrak{L}) dV_{M'}(x').$$

Example 3.2. Let $\mathfrak{L} = \mathfrak{N}_X$ be the set of all light rays. Then

$$\iota_M(\mathfrak{L}_x) = \mathbb{S}_x^* M$$

for every Cauchy surface M and every point $x \in M$. Hence,

$$\omega_M(x, \mathfrak{L}) = \mathfrak{c}_n,$$

where \mathfrak{c}_n is the area of the standard unit sphere in \mathbb{R}^n . Therefore (3.5) implies

$$\mathfrak{c}_n \text{Vol}(M) = \int_{M'} dV_{M'}(x') \int_{\mathbb{S}_{x'}^* M'} (1 + z(M, M', \gamma))^{-n} d\omega_{x'}.$$

If the redshift is constant as in Example 3.1, it follows that

$$\text{Vol}(M) = \frac{1}{(1+z)^n} \text{Vol}(M').$$

More generally, if $\underline{z} \leq z(M, M', \gamma) \leq \bar{z}$, then

$$\frac{1}{(1+\bar{z})^n} \text{Vol}(M') \leq \text{Vol}(M) \leq \frac{1}{(1+\underline{z})^n} \text{Vol}(M').$$

Example 3.3. Consider a subset $D \subseteq M$ and let

$$\mathfrak{L}^D = \{\gamma \in \mathfrak{N}_X \mid \gamma \cap D \neq \emptyset\}$$

be the set of all light rays passing through D . Then

$$\iota_M(\mathfrak{L}_x^D) = \begin{cases} \mathbb{S}_x^* M, & x \in D, \\ \emptyset, & x \in M \setminus D, \end{cases}$$

and therefore

$$\omega_M(x, \mathfrak{L}^D) = \begin{cases} \mathfrak{c}_n, & x \in D, \\ 0, & x \in M \setminus D. \end{cases}$$

Hence, (3.5) gives the following expressions for the volume of D in M :

$$\text{Vol}_M(D) = \frac{1}{\mathfrak{c}_n} \int_{M'} dV_{M'}(x') \int_{\iota_{M'}(\mathfrak{L}_{x'}^D)} (1 + z(M, M', \gamma))^{-n} d\omega_{x'} \quad (3.6)$$

$$= \frac{1}{\mathfrak{c}_n} \int_{\{\iota_{M'}(\gamma) \mid \gamma \cap D \neq \emptyset\}} (1 + z(M, M', \gamma))^{-n} \Omega_{M'}. \quad (3.7)$$

Thus, the volume of $D \subseteq M$ can be computed by integrating the redshift factor $(1 + z(M, M', \gamma))^{-n}$ with respect to the Liouville measure on \mathbb{S}^*M' over the subset of all unit covectors on M' corresponding to light rays γ passing through D . For constant redshift, (3.6) reduces to

$$\text{Vol}_M(D) = \frac{1}{\mathfrak{c}_n(1+z)^n} \int_{M'} \omega_{M'}(x', \mathfrak{L}^D) dV_{M'}(x').$$

Note that if M lies in the past of M' , then $\omega_{M'}(x', \mathfrak{L}^D)$ may be interpreted as the solid angle at $x' \in M'$ subtended by $D \subseteq M$.

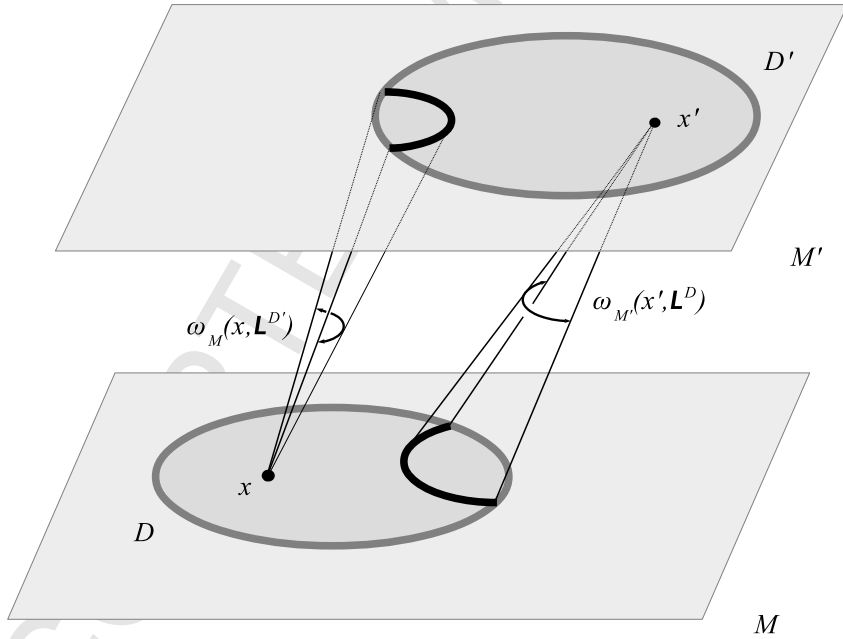


FIGURE 1. Cauchy surfaces and light rays ($n = 2$).

Example 3.4. Let now

$$\mathfrak{L}^{DD'} := \mathfrak{L}^D \cap \mathfrak{L}^{D'} = \{\gamma \in \mathfrak{N}_X \mid \gamma \cap D \neq \emptyset, \gamma \cap D' \neq \emptyset\}$$

be the set of all light rays intersecting $D \subseteq M$ and $D' \subseteq M'$. (Example 3.3 is a special case of this situation with $D' = M'$.) Then

$$\iota_M(\mathfrak{L}_x^{DD'}) = \begin{cases} \iota_M(\mathfrak{L}_x^{D'}), & x \in D, \\ \emptyset, & x \in M \setminus D, \end{cases}$$

and similarly

$$\iota_{M'}(\mathfrak{L}_{x'}^{DD'}) = \begin{cases} \iota_{M'}(\mathfrak{L}_{x'}^D), & x' \in D', \\ \emptyset, & x' \in M' \setminus D'. \end{cases}$$

Hence, it follows from (3.5) that

$$\int_D \omega_M(x, \mathfrak{L}^{D'}) dV_M(x) = \int_{D'} dV_{M'}(x') \int_{\iota_{M'}(\mathfrak{L}_{x'}^D)} (1 + z(M, M', \gamma))^{-n} d\omega_{x'}.$$

In the case of constant redshift z , we obtain

$$\int_D \omega_M(x, \mathfrak{L}^{D'}) dV_M(x) = \frac{1}{(1+z)^n} \int_{D'} \omega_{M'}(x', \mathfrak{L}^D) dV_{M'}(x').$$

If M is in the past of M' , then $\omega_{M'}(x', \mathfrak{L}^D)$ is the solid angle subtended by D at x' as in Example 3.3 and $\omega_M(x, \mathfrak{L}^{D'})$ is the solid angle at $x \in M$ spanned by rays emitted from x and received in D' , see Fig. 1.

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