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Rational solutions of CYBE for simple compact real Lie algebras

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Abstract

In [A.A. Stolin, On rational solutions of Yang–Baxter equation for $sl(n)$, Math. Scand. 69 (1991) 57–80; A.A. Stolin, On rational solutions of Yang–Baxter equation. Maximal orders in loop algebra, Comm. Math. Phys. 141 (1991) 533–548; A. Stolin, A geometrical approach to rational solutions of the classical Yang–Baxter equation. Part I, in: Walter de Gruyter & Co. (Ed.), Symposia Gaussiana, Conf. Alg., Berlin, New York, 1995, pp. 347–357] a theory of rational solutions of the classical Yang–Baxter equation for a simple complex Lie algebra $\mathbf{g}$ was presented. We discuss this theory for simple compact real Lie algebras $\mathbf{g}$. We prove that up to gauge equivalence all rational solutions have the form $X(u,v) = \frac{\Omega}{u-v} + t_1 \wedge t_2 + \cdots + t_{2n-1} \wedge t_{2n}$, where Ω denotes the quadratic Casimir element of $\mathbf{g}$ and $\{t_i\}$ are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$. The quantization of these solutions is also emphasized.

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Keywords: Classical Yang–Baxter equation; Rational solution; Bruhat–Tits building; Quasi-Frobenius Lie algebra; Yangian

1. Introduction

In their outstanding paper from 1982, A.A. Belavin and V.G. Drinfeld obtained an almost complete classification of solutions of the classical Yang–Baxter equation with spectral parameter for a simple complex Lie algebra $\mathbf{g}$. These solutions are functions $X(u,v)$ which depend only on the difference $u-v$ and satisfy the CYBE and some additional non-degeneracy condition. It was proved in [1] that non-degenerate solutions are of three types: rational, trigonometric and elliptic. The last two kinds were fully classified in [1]. However, the similar question for rational solutions remained open. This problem was solved in [8,9] by classifying instead solutions of the form

$$X(u,v) = \frac{\Omega}{u-v} + r(u,v), \tag{1.1}$$

where $r(u,v)$ is a polynomial with coefficients in $\mathbf{g} \otimes \mathbf{g}$ and Ω denotes the quadratic Casimir element of $\mathbf{g}$. This new type of solutions, which will also be called *rational*, look somewhat different from those in the Belavin–Drinfeld

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approach. However, as it turned out in [2], any solution of this type can be transformed into one which depends only on $u-v$, by means of a holomorphic transformation.

In [8–10] a correspondence was established between rational solutions of the form (1.1) and so-called *orders* in $\mathbf{g}((u^{-1}))$, i.e. subalgebras W of $\mathbf{g}((u^{-1}))$ which satisfy the condition

$$u^{-N_1}\mathbf{g}[[u^{-1}]] \subseteq W \subseteq u^{N_2}\mathbf{g}[[u^{-1}]] \tag{1.2}$$

for some non-negative integers N_1 and N_2. The study of rational solutions is essentially based on this correspondence and the description of the maximal orders.

In the present paper, we follow the method developed in [8–10] to study rational solutions of the CYBE for a simple compact Lie algebra $\mathbf{g}$ over $\mathbb{R}$. We establish a similar correspondence between solutions and orders and we are interested in the description of the maximal orders. We obtain that there is only one maximal order, the trivial one. Therefore all rational solutions will have the form

$$X(u,v) = \frac{\Omega}{u-v} + r, \tag{1.3}$$

where $r \in \mathbf{g}\wedge\mathbf{g}$ is a constant r-matrix. Here we would like to note that this theorem was communicated to the second author by V. Drinfeld without proof.

On the other hand, there exists a 1–1 correspondence between skew-symmetric constant r-matrices and pairs (L, B), where L is a subalgebra of $\mathbf{g}$ together with a non-degenerate 2-cocycle $B \in Z^2(L,\mathbb{R})$. A subalgebra L for which there exists a non-degenerate B is called *quasi-Frobenius*. We prove that any quasi-Frobenius subalgebra of a compact simple Lie algebra is commutative. Consequently, up to gauge equivalence, any rational solution has the form

$$X(u,v) = \frac{\Omega}{u-v} + t_1\wedge t_2 + \cdots + t_{2n-1}\wedge t_{2n}, \tag{1.4}$$

where $t_1,\ldots,t_{2n}$ are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$.

Finally we discuss the quantization of the Lie bialgebra structures corresponding to solutions of the form (1.4). The quantization is obtained by twisting the real Yangian $Y_\hbar(\mathbf{g})$.

2. Rational solutions and orders

Let $\mathbf{g}$ denote a simple compact Lie algebra over $\mathbb{R}$ and $U(\mathbf{g})$ its universal enveloping algebra. Let [,] be the usual Lie bracket on the associative algebra $U(\mathbf{g})^{\otimes 3}$.

We recall the following notation [1]: $\varphi_{12}, \varphi_{13}, \varphi_{23}, \varphi_{21}$: $\mathbf{g}\otimes\mathbf{g} \to U(\mathbf{g})^{\otimes 3}$ are the linear maps respectively defined by $\varphi_{12}(a\otimes b) = a\otimes b\otimes 1$, $\varphi_{13}(a\otimes b) = a\otimes 1\otimes b$, $\varphi_{23}(a\otimes b) = 1\otimes a\otimes b$ and $\varphi_{21}(a\otimes b) = b\otimes a\otimes 1$, for any $a, b\in\mathbf{g}$.

For a function $X:\mathbb{R}^2\to\mathbf{g}\otimes\mathbf{g}$, we consider X^{ij}: $\mathbb{R}^2\to U(\mathbf{g})^{\otimes 3}$ defined by $X^{ij}(u_i,u_j) = \varphi_{ij}(X(u_i,u_j))$.

Definition 2.1 (*[1]*). A solution of the classical Yang–Baxter equation (CYBE) is a function $X:\mathbb{R}^2\to\mathbf{g}\otimes\mathbf{g}$ such that the following conditions are satisfied:

$$[X^{12}(u_1,u_2), X^{13}(u_1,u_3)] + [X^{12}(u_1,u_2), X^{23}(u_2,u_3)] + [X^{13}(u_1,u_3), X^{23}(u_2,u_3)] = 0 \tag{2.1}$$

$$X^{12}(u,v) = -X^{21}(v,u). \tag{2.2}$$

Let us consider the Killing form K on $\mathbf{g}$. Then $(-K)$ is a positive definite invariant bilinear form on $\mathbf{g}$. Let $\{I_\lambda\}$ be an orthonormal basis in $\mathbf{g}$ with respect to $(-K)$. We denote by Ω the quadratic Casimir element of $\mathbf{g}$, i.e. $\Omega = -\sum I_\lambda\otimes I_\lambda$. Now we define rational solutions as in the complex case [8–10]:

Definition 2.2. A solution of the CYBE is called *rational* if it is of the form

$$X(u,v) = \frac{\Omega}{u-v} + r(u,v), \tag{2.3}$$

where $r(u,v)$ is a polynomial with coefficients in $\mathbf{g}\otimes\mathbf{g}$.

Remark 2.1. The simplest example of a rational solution is $X_0(u, v) = \frac{\Omega}{u-v}$. By adding to $X_0(u, v)$ any skew-symmetric constant r-matrix, we also obtain a rational solution.

We will consider rational solutions up to a certain equivalence relation:

Definition 2.3. Two rational solutions X_1 and X_2 are said to be *gauge equivalent* if there exists $\sigma(u) \in \mathrm{Aut}(\mathbf{g}[u])$ such that

$$X_1(u, v) = (\sigma(u) \otimes \sigma(v))X_2(u, v). \tag{2.4}$$

Here $\mathrm{Aut}(\mathbf{g}[u])$ denotes the group of automorphisms of $\mathbf{g}[u]$ considered as an algebra over $\mathbb{R}[u]$.

Remark 2.2. One can check that gauge transformations applied to rational solutions also give rational solutions.

Let $\mathbb{R}[[u^{-1}]]$ be the ring of formal power series in u^{-1} and $\mathbb{R}((u^{-1}))$ its field of quotients. Set $\mathbf{g}[u] := \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}[u]$, $\mathbf{g}[[u^{-1}]] := \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]$ and $\mathbf{g}((u^{-1})) := \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1}))$. There exists a non-degenerate $\mathbf{ad}$-invariant bilinear form on $\mathbf{g}((u^{-1}))$ given by

$$(x(u), y(u)) = \mathrm{Tr}(\mathbf{ad}\, x(u) \cdot \mathbf{ad}\, y(u))_{-1}, \tag{2.5}$$

meaning that we take the coefficient of u^{-1} in the series expansion of $\mathrm{Tr}(\mathbf{ad}\, x(u) \cdot \mathbf{ad}\, y(u))$.

In [8, Th. 1] a correspondence between rational solutions and a special class of subalgebras of $\mathbf{g}((u^{-1}))$ was presented. The same result holds when $\mathbf{g}$ is real compact:

Theorem 2.1. *Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. There is a natural one-to-one correspondence between rational solutions of the CYBE and subalgebras $W \subseteq \mathbf{g}((u^{-1}))$ such that*

(1) $W \supseteq u^{-N}\mathbf{g}[[u^{-1}]]$ *for some* $N > 0$;
(2) $W \oplus \mathbf{g}[u] = \mathbf{g}((u^{-1}))$;
(3) W *is a Lagrangian subspace with respect to the bilinear form on* $\mathbf{g}((u^{-1}))$ *given by* (2.5), *i.e.* $W = W^{\perp}$.

Proof. We briefly sketch the proof which is similar to that in the complex case. Let $V := \mathbf{g}[u]$. Then $V^* = u^{-1}\mathbf{g}[[u^{-1}]]$. If $f \in V^*$ and $x \in V$ then $f(x) := (f, x)$, where $(,)$ is the bilinear form given by (2.5).

Denote by $\mathrm{Hom}_{\mathrm{cont}}(V^*, V)$ the space of those linear maps $F : V^* \to V$ such that $\mathrm{Ker}(F) \supseteq u^{-N}V^*$ for some $N \geq 0$. To motivate the notation, we make the remark that this space consists of all linear maps F which are continuous with respect to the "u^{-1}-adic" topology. $\mathbb{R}[[u^{-1}]]$ is a topological valuation ring and V^* is a topological free $\mathbb{R}[[u^{-1}]]$-module. We also put the discrete topology on V.

There exists an isomorphism $\Phi : V \otimes V \to \mathrm{Hom}_{\mathrm{cont}}(V^*, V)$ defined by

$$\Phi(x \otimes y)(f) = f(y)x, \tag{2.6}$$

for any $x, y \in V$ and $f \in V^*$. The inverse map is given by

$$\Phi^{-1}(F) = -\sum_{i=1}^{n}\sum_{k=0}^{\infty} F(I_i u^{-k-1}) \otimes I_i u^k, \tag{2.7}$$

for any $F \in \mathrm{Hom}_{\mathrm{cont}}(V^*, V)$. We make the remark that $F(I_i u^{-k-1}) = 0$ for $k \geq N$ so that the sum which appears in (2.7) is finite.

There is a natural bijection between $\mathrm{Hom}_{\mathrm{cont}}(V^*, V)$ and the set of all subspaces W of $\mathbf{g}((u^{-1}))$ which are complementary to V and such that $W \supseteq u^{-N}V^* = u^{-N-1}\mathbf{g}[[u^{-1}]]$ for some $N \geq 0$. Indeed, for any $F \in \mathrm{Hom}_{\mathrm{cont}}(V^*, V)$, we consider the following subspace of $\mathbf{g}((u^{-1}))$:

$$W(F) := \{f + F(f) : f \in V^*\} \tag{2.8}$$

which satisfies the required properties.

The inverse mapping associates with any W the linear function F_W such that for any $f \in V^*$, $F_W(f) = -x$, uniquely defined by the decomposition $f = w + x$ with $w \in W$ and $x \in V$.

One can easily see that $W(\Phi(r))$ is Lagrangian with respect to the bilinear form (2.5) if and only if $r^{12}(u,v) = -r^{21}(v,u)$. Consequently, $\Omega/(u-v) + r(u,v)$ satisfies the unitarity condition (2.2) if and only if $W(\Phi(r))$ is a Lagrangian subspace.

Finally, if $\Omega/(u-v) + r(u,v)$ is a solution of (2.1) and (2.2), then

$$([f + \Phi(r)(f), g + \Phi(r)(g)], h + \Phi(r)(h)) = 0 \tag{2.9}$$

for any elements f, g, h in V^*. Because $W(\Phi(r))$ is Lagrangian, (2.9) implies that $W(\Phi(r))$ is a subalgebra of $\mathbf{g}((u^{-1}))$. □

Remark 2.3. One can easily see that if W is contained in $\mathbf{g}[[u^{-1}]]$ and satisfies the above properties, then the corresponding rational solution has the form $X(u,v) = \Omega/(u-v) + r$, where r is a constant polynomial.

Definition 2.4. An $\mathbb{R}$-subalgebra $W \subseteq \mathbf{g}((u^{-1}))$ is called an *order* in $\mathbf{g}((u^{-1}))$ if there exist two non-negative integers N_1, N_2 such that

$$u^{-N_1}\mathbf{g}[[u^{-1}]] \subseteq W \subseteq u^{N_2}\mathbf{g}[[u^{-1}]]. \tag{2.10}$$

Obviously $\mathbf{g}[[u^{-1}]]$ is an order.

Remark 2.4. Let W satisfy conditions (1) and (3) of Theorem 2.1. Then W is an order.

As regards gauge equivalence, the result of Theorem 2 in [8] remains true:

Theorem 2.2. *Let* $\mathbf{g}$ *be simple compact Lie algebra over* $\mathbb{R}$. *Let* X_1 *and* X_2 *be rational solutions of the CYBE and* W_1, W_2 *the corresponding orders in* $\mathbf{g}((u^{-1}))$. *Let* $\sigma(u) \in \mathrm{Aut}(\mathbf{g}[u])$. *Then the following conditions are equivalent:*

(1) $X_1(u,v) = (\sigma(u) \otimes \sigma(v))X_2(u,v)$;
(2) $W_1 = \sigma(u)W_2$.

Definition 2.5. Let V_1 and V_2 be subalgebras of $\mathbf{g}((u^{-1}))$. We say that V_1 and V_2 are *gauge equivalent* if there exists $\sigma(u) \in \mathrm{Aut}(\mathbf{g}[u])$ such that $V_1 = \sigma(u)V_2$.

3. Maximal orders for compact Lie algebras

We will prove the following result:

Theorem 3.1. *Let* $\mathbf{g}$ *be a simple compact Lie algebra over* $\mathbb{R}$. *Then any order* W *in* $\mathbf{g}((u^{-1}))$ *is gauge equivalent to an order contained in* $\mathbf{g}[[u^{-1}]]$.

Proof. Let G be a connected compact Lie group whose Lie algebra is $\mathbf{g}$. Then G is embedded into $SL(n,\mathbb{C})$ via any irreducible complex representation. Without any loss of generality, we may suppose that the image of a maximal torus T of G is included into the diagonal torus H of $SL(n,\mathbb{C})$.

Let W denote an order of $\mathbf{g}((u^{-1}))$. Since we have the following sequence of embeddings:

$$W \hookrightarrow W \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow (\mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}((u^{-1})) \hookrightarrow sl(n, \mathbb{C}((u^{-1}))), \tag{3.1}$$

we may view any $w \in W$ as a matrix in $sl(n, \mathbb{C}((u^{-1})))$.

Let us prove that for each $w \in W$, the exponential $\exp(w)$ defined formally by

$$\exp(w) := \sum_{k \geq 0} \frac{w^k}{k!} \tag{3.2}$$

makes sense as an element of $SL(n, \mathbb{C}((u^{-1})))$.

Without any loss of generality, we may suppose that W is an $\mathbb{R}[[u^{-1}]]$-module of finite rank. We set $\mathbb{O} := \mathbb{C}[[u^{-1}]]$ and consider the $\mathbb{O}$-module

$$M := \mathbb{O}^n + W\mathbb{O}^n + \cdots + WW\cdots W\mathbb{O}^n + \cdots. \tag{3.3}$$

Let us show that there exists some integer l such that

$$M \subseteq u^l \mathbb{O}^n. \tag{3.4}$$

If $x_1, \ldots, x_r$ is a basis of the $\mathbb{R}[[u^{-1}]]$-module W, then obviously

$$M \subseteq \sum_{k_i \geq 0} x_1^{k_1} \cdots x_r^{k_r} \mathbb{O}^n. \tag{3.5}$$

It is well known that the field $\mathbb{K} := \mathbb{C}((u^{-1}))$ may be endowed with the discrete valuation $v(\sum_{k \geq N} a_k u^{-k}) = N$. For any $f \in \mathbb{K}$, we consider its norm:

$$|f| = 2^{-v(f)}. \tag{3.6}$$

Note that $\mathbb{O}$ is the set of all f such that $|f| \leq 1$.

On the other hand, one can define a norm on $gl(n, \mathbb{K})$ which is compatible with the norm of $\mathbb{K}$. Given a matrix A of $gl(n, \mathbb{K})$, one sets

$$|A| = 2^s, \tag{3.7}$$

where $s := \inf k$ such that $A\mathbb{O}^n \subseteq u^k \mathbb{O}^n$.

This norm satisfies the properties: $|A_1 A_2| \leq |A_1||A_2|$, $|f(u) \cdot A| = |f(u)||A|$, $|A_1 + A_2| \leq \sup\{|A_1|, |A_2|\}$.

We make the remark that, since W is an order, there exists $N \geq 0$ such that $|w| \leq 2^N$ for all $w \in W$.

In order to prove (3.4), it is enough to show that

$$\sup_{(k_1, \ldots, k_r)} |x_1^{k_1} \cdots x_r^{k_r}| < \infty. \tag{3.8}$$

This means that for each $1 \leq i \leq r$ there exists a positive integer M_i such that

$$\sup_k |x_i^k| \leq M_i. \tag{3.9}$$

It suffices to prove that the norms of the eigenvalues of x_i for the action of x_i on $\mathbb{K}^n$ are less than or equal to 1. Indeed, let us suppose that this requirement is fulfilled. Then the coefficients of the characteristic polynomial of x_i have norm less or equal to 1, so they belong to $\mathbb{O}$. In follows that x_i is integral over $\mathbb{O}$, i.e. there exists $a_{ip_i}, \ldots, a_{i1}$ in $\mathbb{O}$ such that $x_i^{p_i} + a_{ip_i} x^{p_i - 1} + \cdots + a_{i1} = 0$. One can check by induction that x_i^k, for any $k \geq 0$, is a linear combination of $1, x_i, \ldots, x_i^{p_i - 1}$ with coefficients in $\mathbb{O}$. Since the elements of $\mathbb{O}$ have norm less or equal to 1, we get that

$$|x_i^k| \leq \sup\{1, |x_i|, \ldots, |x_i^{p_i - 1}|\} \tag{3.10}$$

for any $k \geq 0$ and thus (3.9) will be fulfilled.

Let w be an arbitrary element of W. Let $\varepsilon_1(w), \ldots, \varepsilon_n(w)$ be the eigenvalues of w for the action of w on $\mathbb{K}^n$. We will show that $|\varepsilon_i(w)| \leq 1$ for all i. Without any loss of generality, we may suppose that w is a diagonalizable element. Consider the eigenvalues $\alpha_1(w), \ldots, \alpha_m(w)$ for the action of w on $(\mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}((u^{-1}))$. Some of them are zero and some behave as roots. For any $\alpha_j(w)$ there exists a corresponding eigenvector which belongs to W. Since $W \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{O}$-module of finite type, it follows that $|\alpha_j(w)| \leq 1$ for all j. On the other hand, because the weights of a representation are linear combinations of simple roots, we have that $\varepsilon_1(w), \ldots, \varepsilon_n(w)$ are linear combinations of some $\alpha_j(w)$ with rational coefficients. This implies that $|\varepsilon_i(w)| \leq 1$ for all i.

Thus (3.9) holds and this implies (3.4). Since (3.4) holds for some integer l, $\exp(w)$ belongs to $SL(n, \mathbb{C}((u^{-1})))$, for any $w \in W$. We denote by S the connected subgroup generated by $\exp(w)$ for all $w \in W$. Its Lie algebra is W.

Recall that G is embedded into $SL(n, \mathbb{C})$ such that the image of a maximal torus T of G is contained in a maximal torus H of $SL(n, \mathbb{C})$. Let $\mathcal{T}$ be the affine Bruhat–Tits building associated with $G(\mathbb{R}((u^{-1})))$ and the valuation v. Let $\mathcal{T}'$ be the affine Bruhat–Tits building associated with $SL(n, \mathbb{C}((u^{-1})))$ and the valuation v. According to [4, p. 202–204] there exists an embedding

$$\mathcal{T} \hookrightarrow \mathcal{T}' \tag{3.11}$$

which is compatible with the preceding embedding $G \hookrightarrow SL(n, \mathbb{C})$.

Since W is contained in $\mathbf{g}((u^{-1}))$, one has that

$$S \subseteq G(\mathbb{R}((u^{-1}))) \hookrightarrow SL(n, \mathbb{C}((u^{-1}))). \tag{3.12}$$

The module M given by (3.3) satisfies the property $SM \subseteq M$. Since $\mathbb{O}^n \subseteq M \subseteq u^l \mathbb{O}^n$, it follows that $S\mathbb{O}^n \subseteq u^l \mathbb{O}^n$. Therefore S must be a bounded subgroup of $SL(n, \mathbb{C}((u^{-1})))$, i.e. there is an upper bound on the absolute values of the matrix entries of the elements of S.

According to [3, p. 161], S is bounded in the sense of Bruhat–Tits bornology for the building $\mathcal{T}'$ (see [3, p. 160]). Because the embedding $\mathcal{T} \hookrightarrow \mathcal{T}'$ is compatible with the building metric, it follows that S is a bounded subgroup of $G(\mathbb{R}((u^{-1})))$, in the sense of Bruhat–Tits bornology corresponding to the building $\mathcal{T}$.

Now the Bruhat–Tits fixed point theorem [3, p. 157, 161] implies that S fixes a point p of the building $\mathcal{T}$.

It was proved in [7] that the action of $G(\mathbb{R}[u])$ on the Bruhat–Tits building associated with $G(\mathbb{R}(u))$ and the valuation ω defined by $\omega(f/g) = \deg(g) - \deg(f)$ admits as simplicial fundamental domain a so-called "sector". This result remains true when we pass to our building $\mathcal{T}$ since, on taking the completion $\mathbb{R}((u^{-1}))$, the building does not change, only the apartment system gets completed. Moreover, the action of $G(\mathbb{R}[u])$ is continuous. Let $\mathcal{H}$ denote the Cartan subalgebra of $sl(n, \mathbb{C})$ corresponding to H and $\mathcal{H}_{\mathbb{R}}$ its real part. The simplicial fundamental domain for the action of $G(\mathbb{R}[u])$ on $\mathcal{T}$ is contained in the standard apartment of the building $\mathcal{T}'$ which is identified with $\mathcal{H}_{\mathbb{R}}$.

Let h be the point of $\mathcal{H}_{\mathbb{R}}$ which is equivalent to p via the action of $G(\mathbb{R}[u])$. There exists $X \in G(\mathbb{R}[u])$ such that $Xp = h$, which implies that XSX^{-1} is contained in the stabilizer P_h of h under the action of $G(\mathbb{R}((u^{-1})))$ on $\mathcal{T}$.

On the other hand, $P_h = P'_h \cap G(\mathbb{R}((u^{-1})))$, where P'_h is the stabilizer of h under the action of $SL(n, \mathbb{C}((u^{-1})))$ on $\mathcal{T}'$. It follows that

$$\mathbf{Ad}(X)W \subseteq \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathrm{Lie}(P'_h). \tag{3.13}$$

The stabilizer P'_h was computed in [4, p. 238] and its Lie algebra is

$$\mathcal{O}_h = \{(g_{ij}) \in sl(n, \mathbb{C}((u^{-1}))) : v(g_{ij}) \geq \alpha_{ij}(h)\}. \tag{3.14}$$

Let us prove that

$$\mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]. \tag{3.15}$$

We know that

$$\mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq su(n) \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h. \tag{3.16}$$

It is enough to show the following:

$$su(n) \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq su(n) \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]. \tag{3.17}$$

If a matrix (g_{ij}) belongs to $su(n) \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})) \cap \mathcal{O}_h$, then $v(g_{ij}) \geq \alpha_{ij}(h)$ for all i, j and $g_{ij} + \overline{g_{ji}} = 0$. We have $v(g_{ij}) = v(-\overline{g_{ij}}) = v(g_{ji})$. On the other hand, $v(g_{ji}) \geq -\alpha_{ij}(h)$. We conclude that $v(g_{ij}) \geq 0$ and therefore (g_{ij}) belongs to $su(n) \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]]$.

In conclusion, for some $X \in G(\mathbb{R}[u])$, one has that

$$\mathbf{Ad}(X)W \subseteq \mathbf{g} \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]] \tag{3.18}$$

which completes the proof. □

4. Description of rational solutions

Theorem 3.1 has an important consequence:

Corollary 4.1. *Let* $\mathbf{g}$ *be a simple compact Lie algebra over* $\mathbb{R}$. *Any rational solution of the CYBE for* $\mathbf{g}$ *is gauge equivalent to a solution of the form*

$$X(u, v) = \frac{\Omega}{u - v} + r, \tag{4.1}$$

where $r \in \mathbf{g} \wedge \mathbf{g}$ *is a constant* r*-matrix.*

Proof. We know that any order W of $\mathbf{g}((u^{-1}))$ is gauge equivalent to an order contained in $\mathbf{g}[[u^{-1}]]$. On the other hand, if a rational solution $X(u, v)$ corresponds to an order $W \subseteq \mathbf{g}[[u^{-1}]]$ then, by Remark 2.3, $X(u, v) = \Omega/(u - v) + r$, where r is a constant polynomial. Because $X(u, v)$ is a solution of the CYBE, it results that r itself is a solution of the CYBE. □

Let us recall a result which describes constant solutions in a different way. This theorem was formulated for the complex case in [1], but the proof obviously works for any simple compact Lie algebra $\mathbf{g}$ over $\mathbb{R}$.

Theorem 4.1. *Any rational solution of the CYBE of the form (4.1) induces a pair (L, B), where L is a subalgebra of $\mathbf{g}$ and B is a non-degenerate 2-cocycle on L. The Lie subalgebra L is the smallest vector subspace in $\mathbf{g}$ such that $r \in L \wedge L$ and B is the bilinear form on L which is the inverse of r. Conversely, any pair (L, B) provides a rational solution of the form (4.1), where $r \in L \wedge L$ is the inverse of B.*

Remark 4.1. In particular, if L is a commutative subalgebra of $\mathbf{g}$ and B is a non-degenerate skew-symmetric form on L, then there exists the corresponding solution of the form (4.1).

Recall that a subalgebra L of $\mathbf{g}$ is called *quasi-Frobenius* if there exists a non-degenerate 2-cocycle $B \in Z^2(L, \mathbb{R})$.

Theorem 4.2. *Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. Any quasi-Frobenius Lie subalgebra L of $\mathbf{g}$ is commutative.*

Proof. Any subalgebra of a compact Lie algebra is compact. Therefore L must be compact as well. Moreover (see for example [6, p. 97]), the derived algebra L' of L is semisimple and if $\zeta(L)$ denotes the center of L, then

$$L = L' \oplus \zeta(L). \tag{4.2}$$

Let us assume that $L' \neq 0$ and there exists a non-degenerate 2-cocycle B on L. We have the following identity:

$$B([x, y], z) + B([y, z], x) + B([z, x], y) = 0 \tag{4.3}$$

for any $x, y \in L'$ and $z \in \zeta(L)$. This implies $B([x, y], z) = 0$ for arbitrary $x, y \in L'$ and $z \in \zeta(L)$. Since L' is semisimple, its derived algebra coincides with L'. We obtain

$$B(w, z) = 0 \tag{4.4}$$

for any $w \in L'$ and $z \in \zeta(L)$.

On the other hand, since L' is semisimple, the restriction of B to L' is a coboundary, i.e. there exists a non-zero functional f on L' such that $B(w_1, w_2) = f([w_1, w_2])$, for all w_1, w_2 in L'. Let a_0 be the element of L' which corresponds to f via the isomorphism $L' \cong (L')^*$ defined by the Killing form. Then for all $w \in L'$ one has

$$B(a_0, w) = K(a_0, [a_0, w]) = K([a_0, a_0], w) = 0. \tag{4.5}$$

Together with (4.2) and (4.4) this implies that

$$B(a_0, l) = 0 \tag{4.6}$$

for all elements l of L. Thus B is degenerate on L, which is a contradiction. □

Corollary 4.2. *Up to gauge equivalence, any rational solution of the CYBE for a simple compact Lie algebra $\mathbf{g}$ over $\mathbb{R}$ has the form*

$$X(u, v) = \frac{\Omega}{u - v} + t_1 \wedge t_2 + \cdots + t_{2n-1} \wedge t_{2n}, \tag{4.7}$$

where $t_1, \ldots, t_{2n}$ are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$.

Proof. We have seen that rational solutions are determined by pairs (L, B), where L is a quasi-Frobenius Lie subalgebra and B a non-degenerate 2-cocycle on L. By the previous result, L is a commutative subalgebra and B is a non-degenerate skew-symmetric form on L. Then L is contained in a maximal commutative subalgebra $\mathbf{t}$ of $\mathbf{g}$ and the dimension of L is even, say $2n$.

Moreover, it is well known that there exists a basis $t_1, \ldots, t_{2n}$ in L such that $B(t_{2i-1}, t_{2i}) = -B(t_{2i}, t_{2i-1}) = -1$ for $1 \leq i \leq n$ and $B(t_j, t_k) = 0$ otherwise. The rational solution induced by the pair (L, B) is precisely (4.7). □

5. Quantization

Let $\mathbf{g}$ be a simple compact Lie algebra over $\mathbb{R}$. Let us recall that the rational solution $X_0(u, v) = \frac{\Omega}{u-v}$ induces a Lie bialgebra structure on $\mathbf{g}[u]$ via the 1-cocycle δ_0 given by

$$\delta_0(a(u)) = [a(u) \otimes 1 + 1 \otimes a(v), X_0(u, v)], \tag{5.1}$$

for any $a(u) \in \mathbf{g}[u]$.

We have seen that, up to gauge equivalence, rational solutions have the form (4.7). With any such solution one can associate a Lie bialgebra structure on $\mathbf{g}[u]$ by defining the 1-cocycle

$$\delta_r(a(u)) = [a(u) \otimes 1 + 1 \otimes a(v), X(u, v)]. \tag{5.2}$$

Here $r = t_1 \wedge t_2 + \cdots + t_{2n-1} \wedge t_{2n}$. In other words, the Lie bialgebra $(\mathbf{g}[u], \delta_r)$ is obtained from the Lie bialgebra $(\mathbf{g}[u], \delta_0)$ by so-called *twisting* via r.

Remark 5.1. This notion was introduced by V.G. Drinfeld in a more general setting for Lie quasi-bialgebras.

The purpose of this section is to give a quantization of the Lie bialgebra $(\mathbf{g}[u], \delta_r)$.

Let us begin by pointing out that the Lie bialgebra $(\mathbf{g}[u], \delta_0)$ admits a unique quantization which we will denote by $Y_\hbar(\mathbf{g})$ (here $\hbar$ is Planck's constant). The construction is analogous to that of the Yangian introduced in [5]. We recall that if K denotes the Killing form of a simple compact $\mathbf{g}$, then $(-K)$ is a positive definite invariant bilinear form. Let $\{I_\lambda\}$ be an orthonormal basis in $\mathbf{g}$ with respect to $(-K)$. Then $Y_\hbar(\mathbf{g})$ is the topological Hopf algebra over $\mathbb{R}[[\hbar]]$ generated by elements I_λ and J_λ with defining relations

$$[I_\lambda, I_\mu] = c^\nu_{\lambda\mu} I_\nu \tag{5.3}$$

$$[I_\lambda, J_\mu] = c^\nu_{\lambda\mu} J_\nu \tag{5.4}$$

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = \hbar^2\, a^{\alpha\beta\gamma}_{\lambda\mu\nu} \{I_\alpha, I_\beta, I_\gamma\} \tag{5.5}$$

$$[[J_\lambda, J_\mu], [I_r, J_s]] + [[J_r, J_s], [I_\lambda, J_\mu]] = \hbar^2 (a^{\alpha\beta\gamma}_{\lambda\mu\nu} c^\nu_{rs} + a^{\alpha\beta\gamma}_{rs\nu} c^\nu_{\lambda\mu}) \{I_\alpha, I_\beta, I_\gamma\}, \tag{5.6}$$

where $a^{\alpha\beta\gamma}_{\lambda\mu\nu} := \frac{1}{24} c^i_{\lambda\alpha} c^j_{\mu\beta} c^k_{\nu\gamma} c^k_{ij}$ and $\{x_1, x_2, x_3\} := \Sigma_{i \neq j \neq k} x_i x_j x_k$. The comultiplication, the co-unit and the antipode are given by the following:

$$\Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda \tag{5.7}$$

$$\Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda - \frac{\hbar}{2} c^\nu_{\lambda\mu} I_\nu \otimes I_\mu \tag{5.8}$$

$$\varepsilon(I_\lambda) = \varepsilon(J_\lambda) = 0, \qquad \varepsilon(1) = 1 \tag{5.9}$$

$$S(I_\lambda) = -I_\lambda \tag{5.10}$$

$$S(J_\lambda) = -J_\lambda + \frac{\hbar}{4} I_\lambda. \tag{5.11}$$

Clearly $Y_\hbar(\mathbf{g})$ contains $U(\mathbf{g})[[\hbar]]$ as a Hopf subalgebra.

Since the generators of $Y_\hbar(\mathbf{g})$ are simultaneously generators for the complex Yangian and all the structure constants are real, it follows immediately from [5, Th. 3] that $Y_\hbar(\mathbf{g})$ is a pseudotriangular Hopf algebra. More precisely, for any real number a, define an automorphism T_a of $Y_\hbar(\mathbf{g})$ by the formulae

$$T_a(I_\lambda) = I_\lambda \tag{5.12}$$

$$T_a(J_\lambda) = J_\lambda + a I_\lambda. \tag{5.13}$$

Then there exists an element $R(u) = 1 + \sum_{k=1}^{\infty} R_k u^{-k}$, where $R_1 = \Omega$ and $R_k \in Y_\hbar(\mathbf{g})^{\otimes 2}$, such that the following conditions are satisfied:

$$(T_a \otimes T_b) R(u) = R(u + a - b) \tag{5.14}$$

$$(T_u \otimes 1) \Delta^{\mathrm{op}}(x) = R(u)((T_u \otimes 1)\Delta(x)) R(u)^{-1} \tag{5.15}$$

$$(\Delta \otimes 1)R(u) = R^{13}(u)R^{23}(u) \tag{5.16}$$

$$R^{12}(u)R^{21}(-u) = 1 \otimes 1 \tag{5.17}$$

$$R^{12}(u_1 - u_2)R^{13}(u_1 - u_3)R^{23}(u_2 - u_3) = R^{23}(u_2 - u_3)R^{13}(u_1 - u_3)R^{12}(u_1 - u_2). \tag{5.18}$$

Here Δ^{op} denotes the opposite comultiplication.

In order to give a quantization of $(\mathbf{g}[u], \delta_r)$, we introduce a deformation of the Yangian $Y_\hbar(\mathbf{g})$ by a so-called *quantum twist*. The approach is based on [11, Th. 5] that we recall below:

Theorem 5.1. *Let $F \in (U(\mathbf{g})[[\hbar]])^{\otimes 2}$ such that*

$$F \equiv 1 (\mathrm{mod}\, \hbar) \tag{5.19}$$

$$(\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1 \otimes 1 \tag{5.20}$$

$$(\Delta \otimes 1)F \cdot F^{12} = (1 \otimes \Delta)F \cdot F^{23}. \tag{5.21}$$

Denote by $\widetilde{Y}_\hbar(\mathbf{g})$ the associative unital algebra which has the same multiplication m as $Y_\hbar(\mathbf{g})$ but the comultiplication is

$$\widetilde{\Delta} := F^{-1} \Delta F. \tag{5.22}$$

Then the following statements hold:

(1) *$\widetilde{Y}_\hbar(\mathbf{g})$ is a Hopf algebra with antipode*

$$\widetilde{S} := Q^{-1} S Q, \tag{5.23}$$

where $Q = m((S \otimes 1)(F))$.

(2) *Let $\widetilde{R}(u) := (F^{21})^{-1} R(u) F$. Then Eqs. (5.14)–(5.18) hold for $\widetilde{R}(u)$ and $\widetilde{\Delta}(u)$.*

Remark 5.2. In the literature, an element F satisfying (5.19)–(5.21) is called a *quantum twist* of $Y_\hbar(\mathbf{g})$. The Hopf algebra $\widetilde{Y}_\hbar(\mathbf{g})$ is the *twisted* (or *deformed*) *Yangian* by the tensor F.

We can easily construct a quantum twist in the following way:

Proposition 5.1. *Suppose that $t_1, \ldots, t_{2n}$ are linearly independent elements in a maximal torus $\mathbf{t}$ of $\mathbf{g}$. Then the 2-tensor*

$$F = \exp(\hbar(t_1 \otimes t_2 + \cdots + t_{2n-1} \otimes t_{2n})) \tag{5.24}$$

is a quantum twist of $Y_\hbar(\mathbf{g})$.

Proof. Conditions (5.19)–(5.21) can be checked by straightforward computations. □

Theorem 5.1 implies the following

Corollary 5.1. *The deformed Hopf algebra $\widetilde{Y}_\hbar(\mathbf{g})$, obtained by applying the quantum twist F given by (5.24), is a quantization of $(\mathbf{g}, \delta_r)$, where $r = t_1 \wedge t_2 + \cdots + t_{2n-1} \wedge t_{2n}$.*

Proof. For any $a \in \widetilde{Y}_\hbar(\mathbf{g})$, we have to check the following:

$$\hbar^{-1}(\widetilde{\Delta}(a) - \widetilde{\Delta}^{\mathrm{op}}(a)) \mathrm{mod}\, \hbar = \delta_r(a \,\mathrm{mod}\, \hbar). \tag{5.25}$$

Since $\widetilde{\Delta} = F^{-1}\Delta F$, we obtain

$$\widetilde{\Delta}(a) - \widetilde{\Delta}^{\mathrm{op}}(a) = F^{-1}\Delta(a)F - (F^{21})^{-1}\Delta^{\mathrm{op}}(a)F^{21}. \tag{5.26}$$

On the other hand, since $Y_\hbar(\mathbf{g})$ is a quantization of $(\mathbf{g}, \delta_0)$, we have that

$$\Delta(a) - \Delta^{\mathrm{op}}(a) = \hbar\delta_0(a \,\mathrm{mod}\, \hbar) + O(\hbar^2). \tag{5.27}$$

Using (5.26) and (5.27) and $(F^{21})^{-1}F = \exp(\hbar r)$, we obtain

$$\widetilde{\Delta}(a) - \widetilde{\Delta}^{\mathrm{op}}(a) = \hbar([\Delta(a), r] + \delta_0(a \,\mathrm{mod}\, \hbar)) + O(\hbar^2) = \hbar\delta_r(a \,\mathrm{mod}\, \hbar) + O(\hbar^2). \quad \square \tag{5.28}$$

Finally, we give the explicit formulae for the comultiplication and antipode of the twisted Yangian $\widetilde{Y}_\hbar(\mathbf{g})$. Let us recall the root system of $\mathbf{g}$ with respect to a torus, according to [6, p. 98–99]. We denote by $\mathbf{h}$ a Cartan subalgebra of $\mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}$ and let Λ be the root system with respect to $\mathbf{h}$, together with a lexicographic ordering of Λ. We choose the root vectors e_α, corresponding to each root α, such that $K(e_\alpha, e_{-\alpha}) = -1$. Let $\mathbf{h}_{\mathbb{R}} = \{h \in \mathbf{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha\}$. We put

$$C_\alpha := \frac{1}{\sqrt{2}}(e_\alpha + e_{-\alpha}) \tag{5.29}$$

$$S_\alpha := \frac{\mathrm{i}}{\sqrt{2}}(e_\alpha - e_{-\alpha}). \tag{5.30}$$

It is well known that

$$\mathbf{g} = \mathrm{i}\mathbf{h}_{\mathbb{R}} \oplus \sum_{\alpha>0}(\mathbb{R}C_\alpha \oplus \mathbb{R}S_\alpha). \tag{5.31}$$

An orthonormal basis in $\mathbf{g}$, with respect to the bilinear form $(-K)$, is formed by the elements C_α, S_α and $p_j := \mathrm{i}k_j$, where $\{k_j\}$ is an orthonormal basis in $\mathbf{h}_{\mathbb{R}}$. We choose this basis as our $\{I_\lambda\}$. The role of $\{J_\lambda\}$ is played correspondingly by some elements denoted by U_α, V_α and P_j. For any $h \in \mathbf{h}_{\mathbb{R}}$ we have the following:

$$[\mathrm{i}h, C_\alpha] = \alpha(h)S_\alpha \tag{5.32}$$

$$[\mathrm{i}h, S_\alpha] = -\alpha(h)C_\alpha \tag{5.33}$$

$$[\mathrm{i}h, U_\alpha] = \alpha(h)V_\alpha \tag{5.34}$$

$$[\mathrm{i}h, V_\alpha] = -\alpha(h)U_\alpha. \tag{5.35}$$

Let us consider now a quantum twist F as in (5.24). Since F is a product of exponents, it is enough to perform computations for

$$F = \exp(\hbar(t_1 \otimes t_2)), \tag{5.36}$$

where t_1 and t_2 are two linearly independent elements in the torus $\mathbf{t} = \mathrm{i}\mathbf{h}_{\mathbb{R}}$. Let $t_1 = \mathrm{i}h_1$ and $t_2 = \mathrm{i}h_2$, where h_1 and h_2 are elements of $\mathbf{h}_{\mathbb{R}}$.

Lemma 1. *Let $T_{1\alpha} := \mathrm{i}\hbar\alpha(h_1)h_2$ and $T_{2\alpha} := \mathrm{i}\hbar\alpha(h_2)h_1$. The following identities hold:*

$$F^{-1}(C_\alpha \otimes 1)F = C_\alpha \otimes \cos(T_{1\alpha}) - S_\alpha \otimes \sin(T_{1\alpha}) \tag{5.37}$$

$$F^{-1}(1 \otimes C_\alpha)F = \cos(T_{2\alpha}) \otimes C_\alpha - \sin(T_{2\alpha}) \otimes S_\alpha \tag{5.38}$$

$$F^{-1}(S_\alpha \otimes 1)F = S_\alpha \otimes \cos(T_{1\alpha}) + C_\alpha \otimes \sin(T_{1\alpha}) \tag{5.39}$$

$$F^{-1}(1 \otimes S_\alpha)F = \cos(T_{2\alpha}) \otimes S_\alpha + \sin(T_{2\alpha}) \otimes C_\alpha. \tag{5.40}$$

$$F^{-1}(U_\alpha \otimes 1)F = U_\alpha \otimes \cos(T_{1\alpha}) - V_\alpha \otimes \sin(T_{1\alpha}) \tag{5.41}$$

$$F^{-1}(1 \otimes U_\alpha)F = \cos(T_{2\alpha}) \otimes U_\alpha - \sin(T_{2\alpha}) \otimes V_\alpha \tag{5.42}$$

$$F^{-1}(V_\alpha \otimes 1)F = V_\alpha \otimes \cos(T_{1\alpha}) + U_\alpha \otimes \sin(T_{1\alpha}) \tag{5.43}$$

$$F^{-1}(1 \otimes V_\alpha)F = \cos(T_{2\alpha}) \otimes V_\alpha + \sin(T_{2\alpha}) \otimes U_\alpha. \tag{5.44}$$

Proof. To prove the first identity, we use relations (5.32), (5.33) and the formula

$$\exp(\lambda)\mu\exp(-\lambda) = \exp(\mathbf{ad}\,(\lambda))\mu = \mu + [\lambda, \mu] + \frac{1}{2!}[\lambda, [\lambda, \mu]] + \cdots \tag{5.45}$$

for $\lambda := -\hbar(\mathrm{i}h_1 \otimes \mathrm{i}h_2)$ and $\mu := C_\alpha \otimes 1$.

Identities (5.38)–(5.44) can be proved in a similar way. □

Consequently we obtain the following result:

Proposition 5.2. *The comultiplication $\widetilde{\Delta}$ of the twisted Yangian $\widetilde{Y}_\hbar(\mathbf{g})$ is given on its generators by the following:*

$$\begin{aligned}
\widetilde{\Delta}(C_\alpha) &= C_\alpha \otimes \cos(T_{1\alpha}) - S_\alpha \otimes \sin(T_{1\alpha}) + \cos(T_{2\alpha}) \otimes C_\alpha - \sin(T_{2\alpha}) \otimes S_\alpha \\
\widetilde{\Delta}(S_\alpha) &= S_\alpha \otimes \cos(T_{1\alpha}) + C_\alpha \otimes \sin(T_{1\alpha}) + \cos(T_{2\alpha}) \otimes S_\alpha + \sin(T_{2\alpha}) \otimes C_\alpha \\
\widetilde{\Delta}(U_\alpha) &= U_\alpha \otimes \cos(T_{1\alpha}) - V_\alpha \otimes \sin(T_{1\alpha}) + \cos(T_{2\alpha}) \otimes U_\alpha - \sin(T_{2\alpha}) \otimes V_\alpha \\
&\quad - \frac{\hbar}{2}[C_\alpha \otimes \cos(T_{1\alpha}) - S_\alpha \otimes \sin(T_{1\alpha}), \widetilde{\Omega}] \\
\widetilde{\Delta}(V_\alpha) &= V_\alpha \otimes \cos(T_{1\alpha}) + U_\alpha \otimes \sin(T_{1\alpha}) + \cos(T_{2\alpha}) \otimes V_\alpha + \sin(T_{2\alpha}) \otimes U_\alpha \\
&\quad - \frac{\hbar}{2}[S_\alpha \otimes \cos(T_{1\alpha}) + C_\alpha \otimes \sin(T_{1\alpha}), \widetilde{\Omega}] \\
\widetilde{\Delta}(p_j) &= p_j \otimes 1 + 1 \otimes p_j \\
\widetilde{\Delta}(P_j) &= P_j \otimes 1 + 1 \otimes P_j - \frac{\hbar}{2}[p_j \otimes 1, \widetilde{\Omega}],
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\Omega} = \sum_{\alpha>0} &(C_\alpha \cos(T_{2\alpha}) + S_\alpha \sin(T_{2\alpha})) \otimes (\cos(T_{1\alpha}) C_\alpha + \sin(T_{1\alpha}) S_\alpha) \\
&+ (C_\alpha \sin(T_{2\alpha}) - S_\alpha \cos(T_{2\alpha})) \otimes (\sin(T_{1\alpha}) C_\alpha - \cos(T_{1\alpha}) S_\alpha) + \sum_j p_j \otimes p_j.
\end{aligned}$$

We conclude by making explicit the antipode $\widetilde{S}$ of the twisted Yangian $\widetilde{Y}_\hbar(\mathbf{g})$. It is given by $\widetilde{S} = Q^{-1} S Q$, where $Q = \exp(\hbar h_1 h_2)$.

Like with Lemma 1, one can prove

Lemma 2. *Let $T_\alpha := \mathrm{i}\hbar(\alpha(h_2)h_1 + \alpha(h_1)h_2)$. The following identities hold:*

$$Q^{-1} C_\alpha Q = \exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha) C_\alpha + \sin(T_\alpha) S_\alpha) \tag{5.46}$$

$$Q^{-1} S_\alpha Q = \exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha) S_\alpha - \sin(T_\alpha) C_\alpha) \tag{5.47}$$

$$Q^{-1} U_\alpha Q = \exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha) U_\alpha + \sin(T_\alpha) V_\alpha) \tag{5.48}$$

$$Q^{-1} V_\alpha Q = \exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha) V_\alpha - \sin(T_\alpha) U_\alpha). \tag{5.49}$$

Proposition 5.3. *The antipode $\widetilde{S}$ of the deformed Yangian $\widetilde{Y}_\hbar(\mathbf{g})$ is given on its generators by*

$$\widetilde{S}(C_\alpha) = -\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha) C_\alpha + \sin(T_\alpha) S_\alpha) \tag{5.50}$$

$$\widetilde{S}(S_\alpha) = -\exp(\hbar\alpha(h_1)\alpha(h_2))(\cos(T_\alpha) S_\alpha - \sin(T_\alpha) C_\alpha) \tag{5.51}$$

$$\widetilde{S}(U_\alpha) = \exp(\hbar\alpha(h_1)\alpha(h_2)) \left(\cos(T_\alpha)\left(-U_\alpha + \frac{\hbar}{4} C_\alpha\right) + \sin(T_\alpha)\left(-V_\alpha + \frac{\hbar}{4} S_\alpha\right)\right) \tag{5.52}$$

$$\widetilde{S}(V_\alpha) = \exp(\hbar\alpha(h_1)\alpha(h_2)) \left(\cos(T_\alpha)\left(-V_\alpha + \frac{\hbar}{4} S_\alpha\right) + \sin(T_\alpha)\left(U_\alpha - \frac{\hbar}{4} C_\alpha\right)\right). \tag{5.53}$$

$$\widetilde{S}(p_j) = -p_j \tag{5.54}$$

$$\widetilde{S}(P_j) = -P_j + \frac{\hbar}{4} p_j. \tag{5.55}$$

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