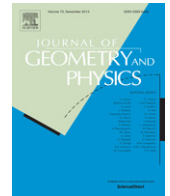




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# Cauchy problems related to integrable deformations of pseudo differential operators

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## ABSTRACT

In this paper we discuss the solvability of two Cauchy problems in the pseudo differential operators. The first is associated with a set of pseudo differential operators of negative order, the prominent example being the set of strict integral operator parts of the different powers of a solution of the KP hierarchy. We show that it can be solved, provided the setting possesses a compatibility completeness. In such a setting all solutions of the KP hierarchy are obtained by dressing with the solution of the related Cauchy problem. The second Cauchy problem is slightly more general and links up with a set of pseudo differential operators of order zero or less. The key example here is the collection of integral operator parts of the different powers of a solution of the strict KP hierarchy. This system is solvable as soon as exponential and compatibility completeness holds. Also under these circumstances, all solutions of the strict KP hierarchy are obtained by dressing with the solution of the corresponding Cauchy problem.

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## 1. Introduction

In [1], we discussed two types of deformations inside the algebra  $\text{Ps}d$  of pseudo differential operators, depending on different decompositions of this algebra. The evolution of these deformations was formulated in Lax equations depending of the projection on one of the components of the decomposition. This Lax form of the system was shown to be equivalent to a set of zero curvature equations. This indicates already that there might be linear systems of which they are the compatibility relations.

Here we present two Cauchy problems in  $\text{Ps}d$ . One that yields, for an appropriate choice of the operators involved, the zero curvature relations of the KP hierarchy and the other one, those of the strict KP hierarchy. The freedom one has in solving these systems consists of right multiplication with operators that are constant w.r.t. all the parameters involved. If  $\text{Ps}d$  has a suitable specialization, like putting the parameters to zero in the formal power series setting, this enables you to gauge the solutions of the Cauchy problems. For the systems linked with the two hierarchies this freedom does not affect the solutions of the hierarchies that one can get in this way. As for the solvability of the Cauchy problems, we need compatibility completeness in the first case and besides that, we also need in the second case exponential completeness. Both properties hold in the formal power series context. Under these conditions all the solutions of the KP hierarchy and its strict version can be obtained from the solutions of the corresponding Cauchy problems.

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The content of the various sections is as follows: Section 1 recalls the necessary results for Psd from [1] and it shows the equivalence of the zero curvature equations for the complementary set of pseudo differential operators with the zero curvature equations from [1]. The next section formulates the two associated Cauchy problems, we analyze the freedom they allow and show when they can be solved. We end with the implications of the results for both hierarchies.

## 2. Integrable deformations in Psd

In this section we shortly recall the results needed from [1] about integrable deformations in the pseudo differential operators. We start with an algebra  $R$  over a field  $k$  of characteristic zero and a privileged  $k$ -linear derivation  $\partial : R \mapsto R$ .

Given  $R$  and  $\partial$ , one forms the algebra  $R[\partial]$  of differential operators in  $\partial$  with coefficients from  $R$ . It consists of  $k$ -linear endomorphisms of  $R$  of the form  $\sum_{i=0}^n a_i \partial^i$ ,  $a_i \in R$ . If the powers of  $\partial$  are not  $R$ -linear independent, then we decouple these relations and pass to the algebra  $R[\xi]$  of formal expressions

$$\sum_{i=0}^n a_i \xi^i, \quad a_i \in R \text{ for all } i \geq 0.$$

Their addition and multiplication with scalars is done component wise; the product structure of  $R[\xi]$  is given by the following rule:

$$\left( \sum_{i=0}^n a_i \xi^i \right) \left( \sum_{j=0}^m b_j \xi^j \right) = \sum_{\substack{0 \leq i \leq n \\ 0 \leq k \leq m}} \sum_{0 \leq k \leq i} \binom{i}{k} a_i \partial^k (b_j) \xi^{i+j-k}. \quad (1)$$

Next one extends the algebra  $R[\xi]$  by adding the inverses of all the powers of  $\xi$  and by allowing infinite sums of these negative powers. Thus one arrives at the algebra  $\text{Psd} = R[\xi, \xi^{-1}]$  of all formal series

$$p = \sum_{j=-\infty}^N p_j \xi^j, \quad p_j \in R.$$

If one uses for each  $n \in \mathbb{Z}$ , the notation

$$\binom{n}{k} := \frac{n(n-1) \cdots (n-k+1)}{k!},$$

then the product of two series  $a = \sum_j a_j \xi^j$  and  $b = \sum_i b_i \xi^i$  is similar to (1) and is given by

$$ab := \sum_j \sum_i \sum_{s=0}^{\infty} \binom{j}{s} a_j \partial^s (b_i) \xi^{i+j-s}.$$

Inside Psd we will make use of a number of decompositions. For  $s \in \mathbb{Z}$ , any pseudo differential operator  $P = \sum_j p_j \xi^j \in \text{Psd}$  can be split as

$$P = P_{\geq s} + P_{< s}, \quad \text{where } P_{\geq s} = \sum_{j \geq s} p_j \xi^j \quad \text{and} \quad P_{< s} = \sum_{j < s} p_j \xi^j. \quad (2)$$

For  $s = 0$ , this yields in particular the splitting of  $P$  in the differential operator part  $P_{\geq 0}$  of  $P$  and its strict integral operator part  $P_{< 0}$ . Similarly, we have

$$P = P_{> s} + P_{\leq s}, \quad \text{where } P_{> s} = \sum_{j > s} p_j \xi^j \quad \text{and} \quad P_{\leq s} = \sum_{j \leq s} p_j \xi^j. \quad (3)$$

For  $s = 0$ , this corresponds to writing  $P$  as the sum of its pure differential operator part  $P_{> 0}$  and its integral operator part  $P_{\leq 0}$ .

As any associative  $k$ -algebra, also Psd is w.r.t. the commutator a Lie algebra over  $k$ . From the multiplication rules in Psd follows that for  $s = 0$  the two decompositions (2) and (3) yield two ways to split the Lie algebra Psd into the direct sum of two Lie subalgebras. The first being given by

$$\text{Psd} = \{P \in \text{Psd}, P = P_{\leq 0}\} \oplus \{P \in \text{Psd}, P = P_{> 0}\} := \text{Psd}_{\leq 0} \oplus \text{Psd}_{> 0},$$

and the second one by

$$\text{Psd} = \{P \in \text{Psd}, P = P_{< 0}\} \oplus \{P \in \text{Psd}, P = P_{\geq 0}\} := \text{Psd}_{< 0} \oplus \text{Psd}_{\geq 0}.$$

We write  $\pi_{\geq 0}$  for the projection from Psd on  $\text{Psd}_{\geq 0}$  consisting of taking the differential operator part of an element in Psd. Similarly, one defines the projections of Psd on  $\text{Psd}_{\leq 0}$ ,  $\text{Psd}_{> 0}$  and  $\text{Psd}_{< 0}$ , by  $\pi_{\leq 0}$ ,  $\pi_{> 0}$  and  $\pi_{< 0}$ , respectively.

To the Lie algebra  $\text{Psd}_{\leq 0}$  we associated the group

$$D(0) = \left\{ p_0 + \sum_{j < 0} p_j \xi^j \mid p_0 \in R^* \right\}$$

and its normal subgroup  $D(0)_1$  of all elements of the form

$$1 + \sum_{j < 0} p_j \xi^j$$

is seen as the group corresponding to the Lie algebra  $\text{Psd}_{\leq 0}$ .

Inside  $\text{Psd}$  we consider two integrable deformations. The first consists of a perturbation  $M$  of the basic direction  $\xi$  of the form

$$M = \xi + \sum_{j=0}^{\infty} m_{j+1} \xi^{-j}. \quad (4)$$

They are the prototype of deforming  $\xi$  by conjugating or *dressing* with an element from  $D(0)$ . The other deformation concerns elements  $L$  of the form

$$L = \xi + \sum_{j=1}^{\infty} \ell_{j+1} \xi^{-j}. \quad (5)$$

Examples of such elements are the  $D\xi D^{-1}$ , with  $D \in D(0)_1$ . In a suitable context, any element  $M$  resp.  $L$  of the form (4) resp. (5) can be obtained in this fashion. It requires the following notion:

**Definition 2.1.** The pair  $(R, \partial)$  is called exponentially complete, if for each  $r \in R$  the element

$$e^r = \sum_{i=0}^{\infty} \frac{r^i}{i!}$$

is a well-defined element of  $R^*$ , satisfying  $\partial(e^r) = \partial(r)e^r$ .

**Remark 2.2.** This property played a role in [2] at the construction of a suitable linear model for the KP hierarchy. It does not occur in the Cauchy problems related to the KP hierarchy presented here. It comes in naturally at the Cauchy problems linked to its strict version.

The following proposition gives sufficient conditions so that each element  $M$  resp.  $L$  of the form (4) resp. (5) can be obtained by conjugation with elements from  $D(0)$  resp.  $D(0)_1$ . Its proof can be found in [1].

**Proposition 2.3.** Consider monic elements  $P$  in  $\text{Psd}$  of order one.

1. If  $\partial$  is surjective, then any  $P = \xi + \sum_{i=1}^{\infty} p_{i+1} \xi^{-i}$  can be obtained by dressing the operator  $\xi$  by an element of  $D(0)_1$ .
2. Let  $R$  and  $\partial$  be such that  $\partial$  is surjective and that the pair  $(R, \partial)$  is exponentially complete. Then every pseudo differential operator  $P = \xi + \sum_{i=0}^{\infty} p_{i+1} \xi^{-i}$ , can be obtained by dressing the operator  $\xi$  by an element of  $D(0)$ .

**Example 2.4.** A concrete example of a  $k$ -algebra  $R$  and a  $k$ -linear derivation  $\partial$  of  $R$  that satisfies the conditions of Proposition 2.3 is the choice  $k = \mathbb{R}$  or  $\mathbb{C}$ ,  $R = k[[t_i, i \geq 1]]$  and  $\partial := \frac{\partial}{\partial t_1}$ .

Next one searches for deformations  $\{M^m, m \geq 1\}$  resp.  $\{L^m, m \geq 1\}$  of the elements  $\{\xi^m, m \geq 1\}$  such that their evolution is given by Lax equations whose form is determined by the projections  $\pi_{\leq 0}$  resp.  $\pi_{< 0}$ . More concretely, we assume for the deformations  $\{M^m, m \geq 1\}$  that the  $k$ -algebra  $R$  is, besides with a privileged  $k$ -linear derivation  $\partial$ , also equipped with a set  $\{\partial_r \mid r \geq 1\}$  of  $k$ -linear derivations commuting with  $\partial$ , that form the infinitesimal generators of the various flows of the evolution. The Lax equations each  $M^m$  should satisfy are

$$\partial_r(M^m) = [M^m, \pi_{\leq 0}(M^r)] = [\pi_{> 0}(M^r), M^m] = [C_r, M^m], \quad \text{for all } r \geq 1, \quad (6)$$

where  $C_r$  is a short hand notation for  $(M^r)_{> 0}$ . Since  $\partial_r$  and taking the commutator with  $C_r$  are both derivations of  $\text{Psd}$ , one sees that it suffices to find an  $M$  such that

$$\partial_r(M) = [C_r, M] = [M, \pi_{\leq 0}(M^r)]. \quad (7)$$

The Eqs. (7) for an operator  $M$  in  $\text{Psd}$  of the form (4), are called the *Lax equations of the strict KP hierarchy* and  $M$  is named a *solution* of the hierarchy. The data  $(R, \partial, \{\partial_r \mid r \geq 1\})$  are called a *setting* for this nonlinear system.

**Remark 2.5.** Note that any setting for the strict KP hierarchy admits the trivial solution  $M = \xi$ . Since  $C_1 = \xi$  for any  $M$ , the Lax equation for  $r = 1$  becomes

$$\partial_1(M) = \partial(M).$$

Moreover, it was shown in [1] that all the  $\{\partial_r\}$  commute on the  $k$ -subalgebra of  $R$  generated by the coefficients of the solution  $M$  and all its derivatives w.r.t.  $\partial$ . Hence, one often takes  $\partial = \partial_1$  and assumes that all the  $\{\partial_r\}$  commute among each other. We call such a setting  $(R, \partial, \{\partial_r\})$  *standard* and use the simplified notation  $(R, \{\partial_r\})$ . We always make the choice  $\partial = \partial_1$  for the basic setting of the strict KP hierarchy, consisting of the algebra  $R$  of formal power series  $k[[s_r, r \geq 1]]$  and the derivations  $\{\partial_r = \frac{\partial}{\partial s_r}, r \geq 1\}$ .

Likewise, we assume in the case of the deformations  $\{L^m, m \geq 1\}$  that  $R$  is equipped with a set  $\{\partial_i \mid i \geq 1\}$  of  $k$ -linear derivations commuting with  $\partial$ . Our search will then be for a set of deformations  $\{L^m, m \geq 1\}$ , with  $L$  of the form (5) that satisfies the equations

$$\partial_i(L^m) = [L^m, \pi_{<0}(L^i)] = [\pi_{\geq 0}(L^i), L^m] = [L^i_{\geq 0}, L^m] = [B_i, L^m], \quad i \geq 1, \quad (8)$$

where  $B_i$  is a short hand notation for  $(L^i)_{\geq 0}$ . The same argument as at the deformation  $M$  shows that it suffices to find an  $L$  that satisfies for all  $i, i \geq 1$ ,

$$\partial_i(L) = [B_i, L] = [L, \pi_{<0}(L^i)]. \quad (9)$$

The Eqs. (9) for an operator  $L$  in Psd of the form (5), are called the *Lax equations of the KP hierarchy* and  $L$  is named a *solution* of the hierarchy. Also here we call the data  $(R, \partial, \{\partial_i \mid i \geq 1\})$  a *setting* for this nonlinear system.

**Remark 2.6.** Also any setting for the KP-hierarchy admits the trivial solution  $L = \xi$ . Like in the strict KP-case, there holds again, see [1], that all the  $\{\partial_i\}$  commute on the  $k$ -subalgebra of  $R$  generated by the coefficients of a solution  $L$  and all the  $\partial^m(L), m \geq 1$ . Moreover, we have  $B_1 = \xi$  for any  $L$  and the Lax equation for  $i = 1$  becomes

$$\partial_1(L) = \partial(L).$$

Hence, one often takes  $\partial = \partial_1$  and assumes that all the  $\{\partial_i\}$  commute among each other. Also in this case one speaks then of a *standard setting*  $(R, \{\partial_i\})$ . In particular, we choose  $\partial = \partial_1$  at the central setting for the KP hierarchy, consisting of the algebra  $R$  of formal power series  $k[[t_i, i \geq 1]]$  and its set of derivations  $\{\partial_i = \frac{\partial}{\partial t_i}, i \geq 1\}$ .

With the help of a minimal setting it was shown in [1] that the strict differential operators  $\{C_r\}$  in Psd corresponding to a solution  $M$  of the strict KP-hierarchy, satisfy

$$\frac{\partial}{\partial t_1}(C_{r_2}) - \frac{\partial}{\partial t_2}(C_{r_1}) - [C_{r_1}, C_{r_2}] = 0. \quad (10)$$

We call the Eqs. (10) the *zero curvature relations* for the strict cut-off's  $\{C_r\}$  of the solution  $M$  of the strict KP-hierarchy. Similarly, one showed in [1] for a solution  $L$  of the KP-hierarchy that the differential operators  $\{B_i\}$  in Psd satisfy

$$\partial_{t_1}(B_{i_2}) - \partial_{t_2}(B_{i_1}) - [B_{i_1}, B_{i_2}] = 0. \quad (11)$$

The Eqs. (11) are called the *zero curvature relations* of the differential operators  $\{B_i\}$  corresponding to the solution  $L$  of the KP-hierarchy.

The zero curvature relations in both cases are also sufficient to get the Lax equations for  $M$  resp.  $L$ , for there holds.

**Theorem 2.7.** Let  $R$  be a  $k$ -algebra equipped with a privileged  $k$ -linear derivation  $\partial$  and let  $M$  resp.  $L$  be elements in Psd of the form (4) resp. (5).

1. Assume  $R$  possesses a set of  $k$ -linear derivations  $\{\partial_r \mid r \geq 1\}$  that all commute with  $\partial$ . Then  $M$  satisfies the Lax equations of the strict KP-hierarchy if and only if the zero curvature relations (10) for the  $\{C_r \mid r \geq 1\}$  hold.
2. Suppose  $R$  possesses a set of  $k$ -linear derivations  $\{\partial_i \mid i \geq 1\}$  that commute with  $\partial$ . Then  $L$  satisfies the Lax equations of the KP-hierarchy w.r.t.  $\{\partial_i\}$  if and only if the zero curvature relations (11) for the  $\{B_i, i \geq 1\}$  hold.

**Remark 2.8.** The zero curvature relations (10) and (11) point towards the existence of linear systems of which they form the compatibility conditions. Such systems are presented in the next section.

Note that to both a solution  $L$  of the KP-hierarchy as well as a solution  $M$  of the strict KP-hierarchy there is associated still another set of pseudo differential operators that satisfy zero curvature relations. For, if we write respectively for each  $i \geq 1$  and each  $r \geq 1$

$$A_i := -(L^i)_{<0} \quad \text{resp.} \quad D_r := -(M^r)_{\leq 0},$$

then we know that there hold respectively the Lax equations

$$\begin{aligned} \partial_i(L) &= [B_i, L] = [A_i, L], \\ \partial_r(M) &= [C_r, M] = [D_r, M] \end{aligned}$$

and in that light it is not surprising that the collection  $\{A_i\}$  satisfies

$$\partial_{t_1}(A_{i_2}) - \partial_{t_2}(A_{i_1}) - [A_{i_1}, A_{i_2}] = 0, \quad (12)$$

and likewise that the set  $\{D_r\}$  satisfies

$$\frac{\partial}{\partial t_1}(D_{r_2}) - \frac{\partial}{\partial t_2}(D_{r_1}) - [D_{r_1}, D_{r_2}] = 0. \quad (13)$$

To show this, one takes the zero curvature equations for the  $\{B_i\}$  resp. the  $\{C_r\}$ , one substitutes  $B_i = L^i + A_i$  resp.  $C_r = M^r + A_r$  and uses the Lax equations for the relevant powers of  $L$  resp.  $M$ . This yields, first of all,

$$\begin{aligned} \partial_{i_1}(A_{i_2} + L^{i_2}) - \partial_{i_2}(A_{i_1} + L^{i_1}) - [A_{i_1} + L^{i_1}, A_{i_2} + L^{i_2}] \\ = \partial_{i_1}(A_{i_2}) - \partial_{i_2}(A_{i_1}) + \partial_{i_1}(L^{i_2}) - [A_{i_1}, L^{i_2}] - \partial_{i_2}(L^{i_1}) - [L^{i_1}, A_{i_2}] - [A_{i_1}, A_{i_2}] \\ = \partial_{i_1}(A_{i_2}) - \partial_{i_2}(A_{i_1}) - [A_{i_1}, A_{i_2}] = 0 \end{aligned}$$

and, secondly,

$$\begin{aligned} \partial_{r_1}(D_{r_2} + M^{r_2}) - \partial_{r_2}(D_{r_1} + M^{r_1}) - [D_{r_1} + M^{r_1}, D_{r_2} + M^{r_2}] \\ = \partial_{r_1}(D_{r_2}) - \partial_{r_2}(D_{r_1}) + \partial_{r_1}(M^{r_2}) - [D_{r_1}, M^{r_2}] - \partial_{r_2}(M^{r_1}) - [M^{r_1}, D_{r_2}] - [D_{r_1}, D_{r_2}] \\ = \partial_{r_1}(D_{r_2}) - \partial_{r_2}(D_{r_1}) - [D_{r_1}, D_{r_2}] = 0. \end{aligned}$$

In a standard setting there holds also the reverse of both statements

**Theorem 2.9.** *Let  $L$  resp.  $M$  be candidate solutions to the KP-hierarchy resp. strict KP-hierarchy in a standard setting. Then there holds*

1.  $L$  is a solution of the KP-hierarchy if and only if all the  $\{A_i\}$  satisfy the zero curvature equations (12).
2.  $M$  is a solution of the strict KP-hierarchy if and only if all the  $\{D_r\}$  satisfy the zero curvature equations (13).

**Proof.** For each  $P \in \text{Psd}$ , we have  $\partial(P) = [\xi, P]$ . Hence, in a standard setting we have  $\partial_1(P) = [\xi, P]$  resp.  $\partial_1(P) = [\xi, P]$ . Now we only need to show still that the zero curvature equations are sufficient in both cases. Thereto we use these equations for the case  $i_1 = 1$  and  $r_1 = 1$  respectively and we substitute  $A_1 = \xi - L$  resp.  $D_1 = \xi - M$ . This yields respectively

$$\partial_1(C_i) - \partial_i(\xi - L) - [\xi - L, C_i] = \partial_1(C_i) - [\xi, C_i] + \partial_i(L) + [L, C_i] = \partial_i(L) - [C_i, L] = 0$$

and

$$\partial_1(D_r) - \partial_r(\xi - M) - [\xi - M, D_r] = \partial_1(D_r) - [\xi, D_r] + \partial_r(M) + [M, D_r] = \partial_r(M) - [D_r, M] = 0,$$

which are the Lax equations one is looking for. This proves the result.  $\square$

In the next section, we treat the Cauchy problems that give rise to the zero curvature equations from Theorem 2.9.

### 3. The associated Cauchy problems

Before treating the Cauchy problems in  $\text{Psd}$ , we shortly recall the finite dimensional problems of which they are a generalization and indicate, where the terminology zero curvature comes from.

#### 3.1. A classical Cauchy problem

Here we describe a classical complex Cauchy problem that serves as a basic example for the Cauchy problems in  $\text{Psd}$  that we consider.

We start with a point  $x_0 \in \mathbb{C}^r$  and a set of local coordinates  $z_1, \dots, z_r$  around this point. Assume we have a collection  $\{C_1, \dots, C_r\}$  of  $n \times n$ -matrices that are holomorphic on a neighborhood of  $x_0$ . The classical problem we think of, is the following: given a vector  $\alpha \in \mathbb{C}^n$ , is there locally around  $x_0$  a holomorphic  $\mathbb{C}^n$ -valued function  $g$

$$\frac{\partial}{\partial z_i}(g) = C_i g, \quad 1 \leq i \leq r, \quad \text{and} \quad g(x_0) = \alpha, \quad (14)$$

and is it unique? In particular, we are interested in the case that system (14) can be solved for all initial values  $\alpha$ . Then we can unite the problems (14) for a set of linear independent initial values into a matrix-valued question. Let  $g_0 \in \text{GL}_n(\mathbb{C})$  be given. The question becomes now: does there exist on a neighborhood of  $x_0$  a holomorphic map  $\Phi$  from that neighborhood into the complex  $n \times n$ -matrices such that

$$\frac{\partial}{\partial z_i}(\Phi) = C_i \Phi, \quad 1 \leq i \leq r, \quad \text{and} \quad \Phi(x_0) = g_0, \quad (15)$$

and to what extent this  $\Phi$  is unique? The local unique solvability of system (15) requires a condition on the functions  $\{C_i\}$  and goes back to Cauchy and Kovalevskaya, see e.g. [3]. There holds namely

**Theorem 3.2.** *Let  $x_0$  be a point in  $\mathbb{C}^r$  with local coordinates  $z_1, \dots, z_r$  and let the  $C_1, \dots, C_r$  be holomorphic matrix-valued functions on a neighborhood of  $x_0$ . Then the Eq. (15) possess locally a unique solution  $\Phi$ , if and only if the equations*

$$[C_i, C_j] - \left( \frac{\partial}{\partial z_i}(C_j) - \frac{\partial}{\partial z_j}(C_i) \right) = 0 \quad (16)$$

hold.

Let  $\Omega$  be the holomorphic matrix differential 1-form defined by

$$\Omega = \sum_{i=1}^r C_i dz_i.$$

**Theorem 3.2** gives a necessary and sufficient condition for the solvability of the linear Pfaffian system

$$dy = \Omega y, \quad y(z) \in \mathbb{C}^n. \quad (17)$$

The conditions (16) are equivalent to the Pfaffian system being integrable in the sense of Frobenius, see [4,5]. In more geometric terms, one can express the fact that the solution space of the Pfaffian system (17) is  $n$ -dimensional as the space of horizontal sections of the connection

$$\nabla = d - \Omega = d - \sum_{i=1}^r C_i dz_i$$

has dimension  $n$ . According to [6], this property is equivalent to the curvature of  $\nabla$  being zero. Therefore, the Eqs. (16) are also called *zero curvature* equations or relations.

### 3.3. The Cauchy problems in Psd

In the setting of the pseudo differential operators  $R[\xi, \xi^{-1}]$ , we have no proper definition of curvature as we had in Section 3.1, but we will introduce two linear systems for which the zero curvature relations correspond to the compatibility conditions of each system.

Our starting point for the Cauchy systems linked with the KP hierarchy and its strict version is **Theorem 2.9**. It states in the KP-case that in a standard setting  $(R, \{\partial_i\})$  the Lax equations for a potential solution  $L$  of the KP hierarchy are equivalent to the zero curvature equations (12) for the strict integral operator parts  $\{A_i = -(L^i)_{<0}\}$ . The first thing one notices is that the specific form of the elements  $\{A_i\}$  is not so important for the associated Cauchy system. It suffices that each  $A_i$  has strict negative order. So we assume that we have given a set of pseudo differential operators of strict negative order

$$\left\{ A_i \mid i \geq 1, A_i = \sum_{m>0} a_m(i) \xi^{-m} \right\}.$$

Now we consider for a  $K = 1 + \sum_{j<0} k_j \xi^j$ ,  $k_i \in R$ , the system

$$\partial_i(K) = A_i K, \quad i \geq 1, \quad (18)$$

which makes sense, since both sides are pseudo differential operators of strict negative order. The system (18) is an analogue in the present setting of the finite dimensional Cauchy problem (15). Note that for  $\{A_i = -(L^i)_{<0}\}$ , the equation for  $i = 1$  in system (18) corresponds in the standard setting to

$$\partial_1(K) = (\xi - L)K = \partial(K) + K\xi - LK \Leftrightarrow L = K\xi K^{-1}. \quad (19)$$

If  $K_1$  is another solution of this system for the same set of  $\{A_i\}$ , then  $K_1 = KK_0$ , where  $K_0$  is a constant pseudo differential operator for all the  $\{\partial_i\}$ , i.e.  $\partial_i(K_0) = 0$ , for all  $i \geq 1$ . There holds namely,

$$\partial_i(K_1) = A_i K_1 = \partial_i(K) K^{-1} K_1 + K \partial_i(K_0) = A_i K_1 + K \partial_i(K_0).$$

This implies  $K \partial_i(K_0) = 0$  and, as  $K$  is invertible, the desired identity holds. Reversely, for any pseudo differential operator  $K_0$  of the form  $K_0 = 1 + \sum_{j<0} k_j \xi^j$ , with  $\partial_i(k_j) = 0$  for all  $i \geq 1$ , and any solution  $K$  of system (18), the operator  $KK_0$  is another solution of (18). In the case of  $\{A_i = -(L^i)_{<0}\}$ , with  $L$  a solution of the KP hierarchy, this freedom does not affect the solution  $L$  thanks to relation (19)

$$L = KK_0 \xi (KK_0)^{-1} = K \xi K^{-1}.$$

Next we focus on the existence of solutions of system (18). A crucial property for that is the following:

**Definition 3.4.** The standard setting  $(R, \{\partial_i\})$  is said to be compatibility complete, if for each collection  $\{g(i) \in R, i = 1, 2, \dots\}$  that satisfies the compatibility conditions

$$\partial_{i_1}(g(i_2)) = \partial_{i_2}(g(i_1)), \quad \text{for all } i_1 \text{ and } i_2 \geq 1, \quad (20)$$

there exists a  $\kappa \in R$  satisfying

$$\partial_i(\kappa) = g(i) \quad \text{for all } i \geq 1.$$

The standard setting  $(k[[t_i \mid i \geq 1]], \{\partial_i := \frac{\partial}{\partial t_i}\})$  satisfies this property. Before showing this, we fix a number of notations for this algebra. We will use a multi-index notation for monomials in the variables  $\{t_i \mid i \geq 1\}$ : take any  $\alpha = (\alpha_i) \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}$  with only a finite number of  $\alpha_i$  nonzero. Then one writes

$$t^\alpha := \prod_{i \in \mathbb{N}} t_i^{\alpha_i}.$$

On these multi-indices one uses the order relation

$$\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \quad \text{for all } i \in \mathbb{N}$$

and the inequality  $\alpha < \beta$  means  $\alpha \leq \beta$  and for one index  $\alpha_i < \beta_i$ . For simplicity the zero index is denoted by 0. The degree  $\deg(\alpha)$  of the multi-index  $\alpha$  is given by

$$\deg(\alpha) := \sum_{i \in \mathbb{N}} \alpha_i.$$

Let, for each  $i \geq 1$ , the multi-index  $1(i)$  be defined by

$$1(i)_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } j = i. \end{cases}$$

With these notations the derivative w.r.t.  $t_i$  of an element  $f = \sum_{\alpha \geq 0} f_\alpha t^\alpha$  equals

$$\sum_{\alpha \geq 1(i)} \alpha_i f_\alpha t^{\alpha - 1(i)}. \quad (21)$$

Now we can state

**Lemma 3.5.** *The basic setting  $(k[[t_i \mid i \geq 1]], \{\frac{\partial}{\partial t_i}\})$  is compatibility complete. In particular, if  $\{g(j)\}$  is a set of formal power series that satisfies the compatibility conditions (20), then there is for each  $c \in k$  a unique  $\kappa = \kappa(t) \in k[[t_i]]$  satisfying*

$$\partial_j(\kappa) = g(j) \quad \text{for all } j \geq 1 \quad \text{and} \quad \kappa(0) = c.$$

**Proof.** Let  $g(i)$ , for each  $i \geq 1$ , be the series  $\sum_{\alpha \geq 0} g(i)_\alpha t^\alpha$ . Then there holds for a solution  $\kappa(t) = \sum_{\alpha \geq 0} \kappa_\alpha t^\alpha$  that

$$\sum_{\alpha \geq 0} \alpha_i \kappa_\alpha t^{\alpha - 1(i)} = \sum_{\alpha \geq 0} g(i)_\alpha t^\alpha.$$

Hence for each  $\alpha > 0$  choose an  $i$  such that  $\alpha_i > 0$ , then we have

$$\kappa_\alpha = \frac{g(i)_{\alpha - 1(i)}}{\alpha_i} \quad (22)$$

and that shows that all  $\kappa_\alpha$  with  $\alpha > 0$  are fixed. Now the boundary condition  $\kappa(0) = c$  determines  $\kappa_0$  and this gives uniqueness. Thus we merely have to show still that formula (22) is independent of the choice. If  $\alpha = m \cdot 1(i)$ , with  $m > 0$ , then the choice is unique. Assume now that  $\alpha \geq 1(i_1) + 1(i_2)$ . The compatibility conditions for the  $\{g(i)\}$  yield on the power series level

$$\sum_{\delta \geq 1(i_1)} \delta_{i_1} g(i_2)_\delta t^{\delta - 1(i_1)} = \sum_{\beta \geq 1(i_2)} \beta_{i_2} g(i_1)_\beta t^{\beta - 1(i_2)}.$$

Consider  $\delta$  and  $\beta$  such that  $\delta - 1(i_1) = \beta - 1(i_2)$ . For those indices the coefficients in the power series are equal and yield

$$\frac{g(i_2)_\delta}{\beta_{i_2}} = \frac{g(i_1)_\beta}{\delta_{i_1}}. \quad (23)$$

Now put

$$\alpha := \delta + 1(i_2) = \beta + 1(i_1).$$

Then we have  $\delta = \alpha - 1(i_2)$ ,  $\beta = \alpha - 1(i_1)$  and there holds  $\alpha_{i_1} = \delta_{i_1}$  and  $\alpha_{i_2} = \beta_{i_2}$ . Substituting these identities in Eq. (23) gives for all  $\alpha \geq 1(i_1) + 1(i_2)$  that

$$\frac{g(i_2)_{\alpha - 1(i_2)}}{\alpha_{i_2}} = \frac{g(i_1)_{\alpha - 1(i_1)}}{\alpha_{i_1}}.$$

This proves the claim.  $\square$

At considering the existence of solutions in Psd of the system (18), the zero curvature relations (12) make their appearance. There holds namely

**Theorem 3.6.** *If the standard setting  $(R, \{\partial_i\})$  is compatibility complete, then there is a solution  $K$  in Psd of the system (18) if and only if the  $\{A_i\}$  satisfy the zero curvature relations (12).*

**Proof.** We first show that the zero curvature relations are necessary to be able to solve the system. Let  $K$  be a solution of the Eqs. (18). Then we have on one hand

$$\partial_i \partial_j(K) = \partial_i(A_j)K + A_j \partial_i(K) = (\partial_i(A_j) + A_j A_i) K$$



and on the other

$$\partial_j \partial_i (K) = \partial_j (A_i) K + A_i \partial_j (K) = (\partial_j (A_i) + A_i \partial_j) K.$$

Hence we have

$$(\partial_i (A_j) + A_j \partial_i) K - (\partial_j (A_i) + A_i \partial_j) K = (\partial_i (A_j) - \partial_j (A_i) - [A_i, A_j]) K = 0$$

and, since  $K$  is invertible, this gives you the zero curvature relations (11).

Now assume that the zero curvature relations (11) hold for the  $\{A_i\}$ . We will show for all  $j \geq 1$  by induction w.r.t.  $j$  that there are  $k_j \in R$  such that  $K = 1 + \sum_{j \geq 1} k_j \xi^{-j}$  satisfies the equations in (18). We start with the construction of  $k_1$ . The Eqs. (18) for the coefficient of  $\xi^{-1}$  are

$$\partial_i (k_1) = a_1(i), \quad i \geq 1,$$

and the zero curvature relations at the level  $\xi^{-1}$  amount to

$$\partial_{i_1} (a_1(i_2)) = \partial_{i_2} (a_1(i_1)).$$

Hence, by the compatibility completeness we know that there is a  $k_1$  that satisfies these equations. The Eq. (18) for the coefficients of  $\xi^{-t}$ ,  $t \geq 2$ , are

$$\partial_i (k_t) = b_i[t] := a_t(i) + \sum_{1 \leq p \leq t-1} \sum_{0 \leq \ell \leq t-2} \binom{-t+p+\ell}{\ell} a_{t-p-\ell}(i) \partial^\ell (k_p). \quad (24)$$

From this formula we see that the right hand side exists of polynomial expressions in the  $a_s(i)$ ,  $s \leq t$ , and the  $\partial^\ell (k_p)$ ,  $p < t$ , that we may assume to have been found already. Therefore, to find a  $k_t$ , the fact that the setting is integrally complete yields that it is sufficient to show for all  $i_1$  and  $i_2 \geq 1$

$$\partial_{i_1} (b_{i_2}[t]) = \partial_{i_2} (b_{i_1}[t]). \quad (25)$$

We will prove the identities (25) by induction. Since we already constructed  $k_1$ , we may assume that we have solved the Eq. (18) up to level  $\xi^{-s}$ , i.e. we have found  $K_{\geq -s}$  such that for all  $i \geq 1$  there holds

$$\partial_i (K_{\geq -s}) = ((A_i)_{\geq -s} K_{\geq -s})_{\geq -s}. \quad (26)$$

The Eqs. (25) for  $t = s+1$  enable you to find a  $k_{s+1}$  such that the Eqs. (26) also hold for  $s+1$  instead of  $s$ . To get the Eqs. (25) for  $t = s+1$ , it suffices to prove for all  $i_1$  and  $i_2 \geq 1$  that

$$\partial_{i_1} (((A_{i_2})_{\geq -s-1} K_{\geq -s})_{\geq -s-1}) = \partial_{i_2} (((A_{i_1})_{\geq -s-1} K_{\geq -s})_{\geq -s-1}). \quad (27)$$

By using the fact that  $\partial_{i_1}$  and  $\partial_{i_2}$  are derivations and by substituting Eq. (26) the left hand side of the Eq. (27) reduces to

$$\{\partial_{i_1} ((A_{i_2})_{\geq -s-1}) K_{\geq -s} + (A_{i_2})_{\geq -s-1} (A_{i_1})_{\geq -s} K_{\geq -s}\}_{\geq -s-1} = \{[\partial_{i_1} ((A_{i_2})_{\geq -s-1}) + (A_{i_2})_{\geq -s-1} (A_{i_1})_{\geq -s}] K_{\geq -s}\}_{\geq -s-1}. \quad (28)$$

Similarly the right hand side becomes

$$\{\partial_{i_2} ((A_{i_1})_{\geq -s-1}) K_{\geq -s} + (A_{i_1})_{\geq -s-1} (A_{i_2})_{\geq -s} K_{\geq -s}\}_{\geq -s-1} = \{[\partial_{i_2} ((A_{i_1})_{\geq -s-1}) + (A_{i_1})_{\geq -s-1} (A_{i_2})_{\geq -s}] K_{\geq -s}\}_{\geq -s-1}. \quad (29)$$

One directly verifies that the operators  $\partial_{i_1} ((A_{i_2})_{\geq -s-1}) + (A_{i_2})_{\geq -s-1} (A_{i_1})_{\geq -s}$  resp.  $\partial_{i_2} ((A_{i_1})_{\geq -s-1}) + (A_{i_1})_{\geq -s-1} (A_{i_2})_{\geq -s}$  differ from  $\partial_{i_1} (A_{i_2}) + A_{i_2} A_{i_1}$  resp.  $\partial_{i_2} (A_{i_1}) + A_{i_1} A_{i_2}$  by a pseudo differential operator of order smaller or equal to  $-s-2$ . Hence the zero curvature relations for the  $\{A_i\}$  imply that the two cut-off's (28) and (29) at order  $-s-1$  have to be equal. This proves that there is a  $k_{s+1} \in R$  such that  $K_{\geq -s-1} = \xi^0 + \sum_{i=1}^{s+1} k_i \xi^{-i}$  satisfies for all  $i \geq 1$

$$\partial_i (K_{\geq -s-1}) = ((A_i)_{\geq -s-1} K_{\geq -s})_{\geq -s-1}. \quad (30)$$

This proves the theorem.  $\square$

Because of the relation (19), we get

**Corollary 3.7.** *If the standard setting  $(R, \{\partial_i\})$  is compatibility complete, then every solution  $L \in R[\xi, \xi^{-1}]$  of the KP hierarchy is obtained by dressing with a solution  $K$  of system (18) for the  $\{A_i = -(L^i)_{<0}\}$ .*

In the basic standard setting  $(k[[t_i \mid i \geq 1]], \{\frac{\partial}{\partial t_i}\})$  the constant pseudo differential operators are the elements of  $k[\xi, \xi^{-1}]$  and we have the algebra map

$$\pi : k[[t_i \mid i \geq 1]][\xi, \xi^{-1}] \rightarrow k[\xi, \xi^{-1}]$$

of substituting  $t_i = 0$  for all  $i \geq 1$ . This offers one the possibility to gauge the solution  $K$  of system (18).



**Corollary 3.8.** Let  $\{A_i\}$  be a set of pseudo differential operators in the basic standard setting that satisfies the zero curvature equations (12) and let  $K_0 = \xi^0 + \sum_{j<0} \kappa_j \xi^j$ ,  $\kappa_j \in k$ , be a constant pseudo differential operator. Then the Eqs. (18) possess a unique solution  $K$  such that  $\pi(K) = K_0$ .

**Remark 3.9.** In [2], it was stated without a proof that for the standard setting  $(k[[t_i \mid i \geq 1]], \{\frac{\partial}{\partial t_i}\})$  the zero curvature relations for the cut-off's  $\{A_i = (L^i)_{<0}\}$  imply the solvability of the system (18). Theorem 3.6 and its proof show that this statement holds in a more general setting.

Also in the case of the strict KP hierarchy we start at Theorem 2.9 that states that in a standard setting  $(R, \{\partial_r\})$  the Lax equations for a potential solution  $M$  of the strict KP hierarchy are equivalent to the zero curvature equations (13) for the integral operator parts  $\{D_r = (M^r)_{\leq 0}\}$  of different powers of  $M$ . As in the KP-case one notices that the specific form of the elements  $\{D_r\}$  is not so important for the associated Cauchy system. It suffices that each  $D_r$  has order smaller than or equal to zero. Therefore one starts with a set of pseudo differential operators with this property

$$\left\{ D_r \mid r \geq 1, D_r = \sum_{n \leq 0} d_n(r) \xi^n \right\}.$$

Since the degrees of all the  $\{D_r\}$  are zero or less, it makes sense to consider for integral operators of the form  $K = \sum_{j \leq 0} k_j \xi^j$ , with invertible leading term  $k_0$ , the following system:

$$\partial_r(K) = D_r K, \quad r \geq 1. \quad (31)$$

This system is a slight generalization of (18) and as such another analogue in Psd of Cauchy problem (15). Note that in the case that each  $D_r$  equals the cut-off  $-(M^r)_{\leq 0}$ , of the  $r$ -th power of a solution  $M$  of the strict KP hierarchy, then the equation for  $r = 1$  in system (31) corresponds in the standard setting to

$$\partial_1(K) = (\xi - M)K = \partial(K) + K\xi - MK \Leftrightarrow M = K\xi K^{-1}. \quad (32)$$

Similar to the Cauchy problem for the KP hierarchy, the freedom we have in solving the system (31) can be described by: multiplying a found solution  $K$  of (31) from the right with a constant pseudo differential operator

$$K_0 = \kappa_0 \xi^0 + \sum_{j < 0} \kappa_j \xi^j, \quad \kappa_0 \in R^*,$$

i.e. one satisfying  $\partial_r(\kappa_j) = 0$ , for all  $r \geq 1$  and all  $j \geq 0$ . In the case of  $\{D_r = -(M^r)_{\leq 0}\}$ , with  $M$  a solution of the strict KP hierarchy, this freedom does not affect the solution  $M$  thanks to relation (32)

$$M = KK_0 \xi (KK_0)^{-1} = K\xi K^{-1}.$$

To prove the existence of solutions of system (31) we need compatibility completeness of the setting and the property that allows you to describe the image of the dressing procedure with elements from  $D(0)$ , see Proposition 2.3. Both hold in the basic standard setting  $(k[[s_r \mid r \geq 1]], \{\frac{\partial}{\partial s_r}\})$ , with  $k = \mathbb{R}$  or  $\mathbb{C}$ .

**Theorem 3.10.** Let the standard setting  $(R, \{\partial_i\})$  be compatibility complete and assume that the pair  $(R, \partial)$  is exponentially complete. Then there is a solution  $K$  in Psd of the system (31) if and only if the set  $\{D_r\}$  satisfies the zero curvature relations (13).

**Proof.** Since the integral operators  $K$  in system (31) are invertible, the proof of the necessity of the zero curvature relations follows the same line as that for system (18). So what remains is to show the sufficiency.

Now assume that the zero curvature relations (10) hold for the  $\{D_r\}$ . To find an operator  $K = k_0 + \sum_{m \geq 1} k_j \xi^{-m}$  that satisfies the system (31), we first want to find  $k_0 \in R^*$ . The Eqs. (31) for the coefficient of  $\xi^0$  are

$$\partial_i(k_0) = d_0(i)k_0, \quad i \geq 1, \quad (33)$$

and the zero curvature relations at the level  $\xi^0$  amount to

$$\partial_{i_1}(d_0(i_2)) = \partial_{i_2}(d_0(i_1)).$$

Hence, the collection  $\{d_0(i)\}$  satisfies the compatibility conditions and there is a  $\tilde{k}_0 \in R$  that satisfies for all  $r \geq 1$

$$\partial_r(\tilde{k}_0) = d_0(r).$$

By exponential completeness, the element

$$k_0 = e^{\tilde{k}_0}$$

satisfies the Eqs. (33) and belongs to  $R^*$ . If we write  $K = k_0 \tilde{K}$ , with  $\tilde{K}$  a pseudo differential operator of degree zero with leading coefficient equal to one, then the Eqs. (31) are equivalent to

$$\partial_r(\tilde{K}) = (k_0^{-1} D_r k_0 - k_0^{-1} \partial_r(k_0)) \tilde{K} =: \hat{D}_r \tilde{K}.$$

Note that by the Eqs. (33) all the  $\{\hat{D}_r\}$  are pseudo differential operators of degree less than zero. Thanks to Theorem 3.6, we get a solution  $\tilde{K}$  of these equations as soon as the zero curvature relations hold for all the  $\{\hat{D}_r\}$ . Now we have

$$\begin{aligned}\partial_{r_1}(\hat{D}_{r_2}) &= \partial_{r_1}(k_0^{-1})D_{r_2}k_0 + k_0^{-1}\partial_{r_1}(D_{r_2})k_0 + k_0^{-1}D_{r_2}\partial_{r_1}(k_0) - \partial_{r_1}(k_0^{-1}\partial_{r_2}(k_0)) \\ &= -\partial_{r_1}(k_0)k_0^{-2}D_{r_2}k_0 + k_0^{-1}\partial_{r_1}(D_{r_2})k_0 + k_0^{-1}D_{r_2}\partial_{r_1}(k_0) + \partial_{r_1}(k_0)k_0^{-2}\partial_{r_2}(k_0) - k_0^{-1}\partial_{r_1}\partial_{r_2}(k_0)\end{aligned}$$

and likewise

$$\begin{aligned}\partial_{r_2}(\hat{D}_{r_1}) &= \partial_{r_2}(k_0^{-1})D_{r_1}k_0 + k_0^{-1}\partial_{r_2}(D_{r_1})k_0 + k_0^{-1}D_{r_1}\partial_{r_2}(k_0) - \partial_{r_2}(k_0^{-1}\partial_{r_1}(k_0)) \\ &= -\partial_{r_2}(k_0)k_0^{-2}D_{r_1}k_0 + k_0^{-1}\partial_{r_2}(D_{r_1})k_0 + k_0^{-1}D_{r_1}\partial_{r_2}(k_0) + \partial_{r_2}(k_0)k_0^{-2}\partial_{r_1}(k_0) - k_0^{-1}\partial_{r_2}\partial_{r_1}(k_0).\end{aligned}$$

Further the commutator  $[\hat{D}_{r_2}, \hat{D}_{r_1}]$  is equal to

$$k_0^{-1}[D_{r_2}, D_{r_1}]k_0 - [k_0^{-1}\partial_{r_2}(k_0), k_0^{-1}D_{r_1}k_0] - [k_0^{-1}D_{r_2}k_0, k_0^{-1}\partial_{r_1}(k_0)].$$

As there holds

$$-[k_0^{-1}\partial_{r_2}(k_0), k_0^{-1}D_{r_1}k_0] = -\partial_{r_2}(k_0)k_0^{-2}D_{r_1}k_0 + k_0^{-1}D_{r_1}\partial_{r_2}(k_0)$$

and

$$-[k_0^{-1}D_{r_2}k_0, k_0^{-1}\partial_{r_1}(k_0)] = -k_0^{-1}D_{r_2}\partial_{r_1}(k_0) + \partial_{r_1}(k_0)k_0^{-2}D_{r_2}k_0,$$

all these relations combine to

$$\partial_{r_1}(\hat{D}_{r_2}) - \partial_{r_2}(\hat{D}_{r_1}) - [\hat{D}_{r_1}, \hat{D}_{r_2}] = 0$$

and this proves the required relations for the  $\{\hat{D}_r\}$ . The operator  $K = k_0\tilde{K}$  is a solution of system (31).  $\square$

Using relation (32), Theorem 3.10 implies

**Corollary 3.11.** *If the standard setting  $(R, \{\partial_i\})$  is exponentially and compatibility complete, then every solution  $M \in R[\xi, \xi]$  of the strict KP hierarchy is obtained by dressing with a solution  $K$  of system (31) for the  $\{D_r = -(M')_{\leq 0}\}$ .*

In the standard setting  $(k[[s_r]], \{\frac{\partial}{\partial s_r}\})$  the constant pseudo differential operators are the elements of  $k[\xi, \xi^{-1}]$  and we have the specialization map

$$\pi : k[[s_r]][\xi, \xi^{-1}] \rightarrow k[\xi, \xi^{-1}]$$

of substituting  $s_r = 0$  for all  $r \geq 1$ . This enables you to gauge the solution  $K$  of system (31).

**Corollary 3.12.** *Let  $\{D_r\}$  be a set of pseudo differential operators in the basic standard setting that satisfies the zero curvature equations (13) and let  $K_0 = \xi^0 + \sum_{j \geq 0} \kappa_j \xi^j$ ,  $\kappa_j \in k$ ,  $\kappa_0 \neq 0$ , be a constant pseudo differential operator. Then the Eqs. (31) possess a unique solution  $K$  such that  $\pi(K) = K_0$ .*

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## References

- [1] G.F. Helminck, A.G. Helminck, E.A. Panasenkov, Integrable deformations in the algebra of pseudo differential operators from a Lie algebraic perspective, Theoret. Math. Phys. 174 (1) (2013) 134–153.
- [2] M. Mulase, Complete integrability of the Kadomtsev–Petviashvili equation, Adv. Math. 54 (1) (1984) 57–66.
- [3] S.V. Kovalevskaya, Nauchnye Raboty, Izdat. Akad. Nauk, Moscow, 1948, p. 368.
- [4] R.R. Gontsov, G.F. Helminck, V.A. Poberezhny, On deformations of linear differential systems, Russian Math. Surveys 66 (1) (2011) 63–105.
- [5] F. Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer-Verlag, New York, 1983.
- [6] S. Kobayashi, Differential Geometry of Complex Vector Bundles, in: Publications of the Mathematical Society of Japan, vol. 15, Iwanami Shoten Publishers and Princeton University Press, 1987.