



# The Chaplygin case in dynamics of a rigid body in fluid is orbitally equivalent to the Euler case in rigid body dynamics and to the Jacobi problem about geodesics on the ellipsoid



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## ABSTRACT

The main goal of this paper is to demonstrate how the theory of invariants for integrable Hamiltonian systems with two degrees of freedom created by A.T. Fomenko, H. Zieschang, and A.V. Bolsinov helps to establish Liouville and orbital equivalence of some classical integrable systems. Three such systems are treated in the article: the Euler case in rigid body dynamics, the Jacobi problem about geodesics on the ellipsoid and the Chaplygin case in dynamics of a rigid body in fluid. The first two systems were known to be Liouville and even topologically orbitally equivalent (Fomenko, Bolsinov). Now we show that the Chaplygin system is orbitally equivalent to the Euler and Jacobi systems.

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## 1. Introduction

Integrable Hamiltonian systems occur in many problems of geometry, mechanics, and physics and have been of growing interest within the past years. For their study many methods have been developed, from classical works of Poincaré, Liouville, Moser up to modern treatments (e.g., bi-Hamiltonian approach). At the end of the past century the theory of invariants based on the topological approach was suggested by A.T. Fomenko, H. Zieschang, A.V. Bolsinov, and others for the study of integrable Hamiltonian systems with two degrees of freedom. This theory allows to investigate different qualitative properties of such systems and to conclude whether two systems are equivalent (in some sense) or not. Primarily, we mean the following three types of equivalence: Liouville equivalence [1–3], topological [4,5] and smooth [6] orbital equivalence. For each type of equivalence, a non-degenerate integrable system restricted to a 3-dimensional isoenergy surface is assigned with an invariant (molecule) which is a graph with some numerical marks. The main result of the theory in question can now be formulated in the following way: two integrable Hamiltonian systems considered on non-degenerate isoenergy 3-surfaces are equivalent (in one of the senses mentioned) if and only if the corresponding invariants are the same. We should underline that this approach enables one to establish the equivalence of different systems without writing out analytic formulas which seems to be quite a complicated task. Moreover, it allows us to conclude that two systems are not equivalent if their invariants differ. Possibility of establishing the fact of non-equivalence without using the theory of invariants seems doubtful.

Since its creation, the theory of topological invariants was applied to the study of many classical integrable systems, in particular, integrable cases in rigid body dynamics [7–12] and integrable geodesic flows [7,13–19]. This way, A.V. Bolsinov

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and A.T. Fomenko proved the topological orbital equivalence between two famous integrable systems: the Euler case in rigid body dynamics and the Jacobi problem about geodesics on the ellipsoid [20]. At the same time, the smooth orbital equivalence does not hold [21]. Moreover, as shown in [22], these two systems are not topologically conjugate (for any values of their parameters). In this paper we calculate topological invariants for the integrable Chaplygin case in dynamics of a rigid body in fluid and conclude that this classical system is also orbitally equivalent to the two mentioned above.

## 2. Main definitions and outline of the general theory

In this section we recall some basic definitions and main results of the theory of invariants for integrable Hamiltonian systems. Many notions of this theory will be discussed rather briefly. For details we address the readers to the papers [1–6] and the book [7].

**Definition 1.** A Hamiltonian system with  $n$  degrees of freedom is a triple  $(M^{2n}, \omega, H)$  where  $M^{2n}$  is a  $2n$ -dimensional smooth manifold endowed with a symplectic form  $\omega$ , and  $H \in C^\infty(M^{2n})$  is a smooth function on  $M^{2n}$  called the *Hamiltonian function*. A Hamiltonian vector field is defined as  $v = \text{sgrad } H = \omega^{-1}dH$ . We shall also denote a Hamiltonian system by  $(M^{2n}, v)$ .

**Theorem 1** (Liouville). Suppose that the smooth functions  $f_1, \dots, f_n$  are the first integrals of the Hamiltonian system  $(M^{2n}, \omega, H)$  satisfying the following conditions:

- (1)  $f_1, \dots, f_n$  are functionally independent, i.e. their gradients are linearly independent almost everywhere on  $M$ ;
- (2)  $\{f_i, f_j\} = 0$  for  $i, j = 1, \dots, n$ , i.e. the integrals commute with respect to the Poisson bracket determined by the symplectic structure;
- (3) the vector fields  $\text{sgrad } f_1, \dots, \text{sgrad } f_n$  are complete, i.e. the natural parameter on their integral trajectories is defined on the whole real axis.

Let  $T_\xi = \{x \in M \mid f_i(x) = \xi_i, i = 1, \dots, n\}$  be a regular common level for the functions  $f_1, \dots, f_n$  (regularity means that the differentials  $df_1, \dots, df_n$  are linearly independent for all  $x \in T_\xi$ ). Then

- (a)  $T_\xi$  is a smooth Lagrangian submanifold which in case it is connected and compact is diffeomorphic to the  $n$ -dimensional torus  $T^n$  (the Liouville torus);
- (b) a neighbourhood  $U$  of the torus  $T^n$  is fiberwise diffeomorphic to the direct product  $T^n \times D^n$ , i.e. the pre-image of  $T^n \times \{\text{pt}\}$  under this diffeomorphism is a Liouville torus;
- (c) in the neighbourhood  $U$  there exists a coordinate system (the action–angle variables)  $s_1, \dots, s_n, \varphi_1, \dots, \varphi_n$  where  $s_1, \dots, s_n$  are coordinates on the disk  $D^n$  depending only on the integrals  $f_1, \dots, f_n$  and  $\varphi_1, \dots, \varphi_n$  are standard angle coordinates on the torus, such that
  - (i)  $\omega = \sum_{i=1}^n d\varphi_i \wedge ds_i$ ,
  - (ii) the Hamiltonian vector field takes the form  $\dot{s}_i = 0, \dot{\varphi}_i = q_i(s_1, \dots, s_n), i = 1, \dots, n$ , and therefore determines a rectilinear winding (rational or irrational) on each of the Liouville tori.

This classical result suggests the following definitions.

**Definition 2.** A Hamiltonian system  $(M^{2n}, \omega, H)$  is called *Liouville integrable* if there exists a set of smooth functions  $f_1, \dots, f_n$  satisfying the conditions 1–3 of Theorem 1.

**Definition 3.** The decomposition of the manifold  $M^{2n}$  into connected components of common level surfaces of the integrals  $f_1, \dots, f_n$  is called the *Liouville foliation* corresponding to the Liouville integrable Hamiltonian system  $(M^{2n}, \omega, H)$ .

The Liouville foliation consists of regular leaves (they are Liouville tori in the compact case) which fill  $M$  almost in the whole and singular ones filling a set of zero measure on  $M$ . Theorem 1 describes therefore the local structure of the Liouville foliation in the neighbourhood of a regular leaf. According to this theorem, the Liouville foliation is trivial in the neighbourhood of a Liouville torus. From now on we shall consider only systems with compact leaves.

**Definition 4.** The mapping  $\mathcal{F} = f_1 \times \dots \times f_n: M^{2n} \rightarrow \mathbb{R}^n$  is called the *momentum mapping*.

Let  $K \subset M^{2n}$  be the set of all critical points of the momentum mapping (i.e. the set of all points  $x \in M$  such that  $\text{rank } d\mathcal{F}(x) < n$ ).

**Definition 5.** The set  $\Sigma = \mathcal{F}(K) \subset \mathbb{R}^n$  of all critical values of  $\mathcal{F}$  is called the *bifurcation diagram* of the momentum mapping.

Obviously, the points from  $\Sigma$  are images of the singular leaves of the Liouville foliation under the momentum mapping, and the regular values of  $\mathcal{F}$  (belonging to  $\mathcal{F}(M) \setminus \Sigma$ ) are images of the regular leaves (Liouville tori).

Now we want to answer a natural question: what integrable Hamiltonian systems should be called equivalent? Several definitions are possible. Let  $X_1 = (M_1^{2n}, \omega_1, H_1)$  and  $X_2 = (M_2^{2n}, \omega_2, H_2)$  be two such systems.

**Definition 6.** The systems  $X_1$  and  $X_2$  are called *topologically* (resp. *smoothly*) *conjugate* if there exists a homeomorphism (resp. diffeomorphism)  $\tau: M_1 \rightarrow M_2$  mapping the flow  $\sigma_1^t$  corresponding to  $X_1$  into the flow  $\sigma_2^t$  corresponding to  $X_2$ :  $\tau \circ \sigma_1^t = \sigma_2^t \circ \tau$ . In other words, the systems  $X_1$  and  $X_2$  can be obtained from each other by a change of coordinates (continuous or smooth).

**Definition 7.** The systems  $X_1$  and  $X_2$  are called *topologically* (resp. *smoothly*) *orbitally equivalent* if there exists a homeomorphism (resp. diffeomorphism)  $\tau : M_1 \rightarrow M_2$  mapping the oriented trajectories of the first system into those of the second one. Note that the parameter (time) on the trajectories is not necessarily preserved.

**Definition 8.** The systems  $X_1$  and  $X_2$  are called *Liouville equivalent* if there exists a diffeomorphism  $\tau : M_1 \rightarrow M_2$  transforming the Liouville foliation of the first system to that of the second one.

**Definition 9.** The systems  $X_1$  and  $X_2$  are called *roughly Liouville equivalent* if there exists a homeomorphism between the bases of the corresponding Liouville foliations that can locally (i.e. in the neighbourhood of each point of the base) be lifted up to a fibre homeomorphism of the Liouville foliations.

In the non-resonant case each subsequent equivalence relation above is weaker than the previous one. The last definition seems rather complicated but it has a clear topological interpretation which will be explained further.

**Remark 1.** In the four definitions above we may consider systems restricted to their invariant isoenergy surfaces  $\{H = h = \text{const}\}$ . Moreover, Definitions 6 and 7 can obviously be given for arbitrary dynamical systems on smooth manifolds.

**Remark 2.** It is reasonable to consider also more strict notions of equivalence requiring in Definitions 6–8 that  $\tau$  is a symplectomorphism, i.e.  $\tau^* \omega_2 = \omega_1$ . We shall not deal with such notions in the framework of the theory of invariants discussed (see for instance [23,24]).

Suppose we are given two integrable Hamiltonian systems  $(M_1^4, v_1)$  and  $(M_2^4, v_2)$  with two degrees of freedom. Can we understand whether they are equivalent (in one of the senses) or not? The theory of invariants gives the answer to this question in some cases. Namely, for systems restricted to their regular isoenergy surfaces there exist invariants which permit to classify such systems up to the following types of equivalence: rough Liouville, (fine) Liouville and orbital (topological and smooth) equivalence. Now we shall briefly describe these invariants.

Let  $X = (Q^3, v = \text{sgrad } H)$  be an integrable Hamiltonian system with 2 degrees of freedom restricted to the compact and regular surface of constant energy  $Q = \{H = h = \text{const}\}$  (regularity means that  $dH|_Q \neq 0$ ). Suppose that the additional integral  $f$  of  $X$  is a Bott function on  $Q$  [7, Section 1.8] and that  $X$  is non-resonant on  $Q$  (i.e. the winding determined by  $X$  on the Liouville tori is irrational for almost all tori). We shall start with constructing the invariant for the rough Liouville equivalence. First, we consider a graph which is the base of the Liouville foliation on  $Q$ . The points of this graph correspond to the connected components of the level surfaces of the function  $f$  on  $Q$ , i.e. to the leaves of the Liouville foliation. Hence, by definition, it is the Reeb graph of the function  $f$  on  $Q$ . The edges of this graph represent regular one-parameter families of Liouville tori. The vertices correspond to the singular leaves.

Now we endow each vertex with an invariant picturing the Liouville foliation in the neighbourhood of the corresponding singular leaf. Such invariant is a *3-atom* defined as a small neighbourhood of a singular leaf of the Liouville foliation viewed up to fibre diffeomorphism. Thus, a 3-atom is a compact 3-dimensional surface with boundary foliated to Liouville tori and one singular leaf. Its boundary consists of several (possibly one) Liouville tori. Each boundary torus corresponds to a certain edge of the graph adjacent to the given vertex.

There exists a one-to-one correspondence between 3-atoms and 2-atoms. By definition, a *2-atom* is a small invariant neighbourhood of a critical level of a Morse function on a 2-dimensional compact manifold considered up to fibre diffeomorphism. A 2-atom is a compact 2-dimensional surface with boundary foliated to circles and one singular leaf. Its boundary consists of one or several circles.

The connection between 3-atoms and 2-atoms is that each 3-atom is a Seifert fibration over a certain 2-atom ([7, Theorem 3.4]). So we may assume that the vertices of the graph under consideration are 2-atoms, and there is a one-to-one correspondence between the boundary circles of each 2-atom and the edges of the graph adjacent to the given vertex.

In Fig. 1 we give an example of two 2-atoms which appear in the systems discussed in this paper. The corresponding 3-atoms can be obtained from them by multiplying by a circle.

**Definition 10.** The constructed graph  $W$  is called the *molecule* (or the *Fomenko invariant*) of the integrable system on the given isoenergy surface  $Q$ .

**Theorem 2** (Theorem 3.5 in [7]). *The integrable systems  $(Q_1^3, v_1)$  and  $(Q_2^3, v_2)$  are roughly Liouville equivalent if and only if their molecules  $W_1$  and  $W_2$  are the same.*

**Proof.** The definition of the rough Liouville equivalence means that, firstly, the bases of the Liouville foliations on  $Q_1$  and  $Q_2$  are the same graphs and, secondly, that the systems  $v_1$  and  $v_2$  are Liouville equivalent in neighbourhoods of singular leaves (the Liouville equivalence in neighbourhoods of regular leaves is true for any two systems of the same dimension by the Liouville theorem). But this exactly means that the molecules of the systems  $v_1$  and  $v_2$  coincide.  $\square$

To construct the invariant for the (fine) Liouville equivalence we need to provide the molecule  $W$  of the integrable system  $(Q^3, v)$  with some additional information. From the molecule  $W$  we know which atoms should be glued together to obtain the Liouville foliation on  $Q^3$ . On each edge of  $W$  we need to glue together the adjacent atoms along their boundary tori

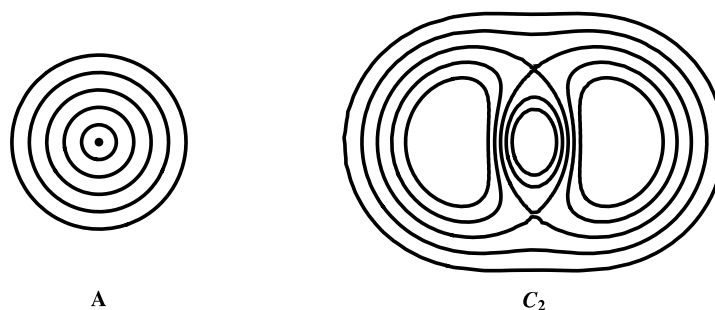


Fig. 1. Examples of 2-atoms.

corresponding to the given edge. We can do this in different ways by means of arbitrary diffeomorphisms of tori. As a result, we shall obtain different Liouville foliations and, generally speaking, different 3-manifolds  $Q$ . That is why we need to fix somehow the gluing diffeomorphisms. Consider on every edge a natural integer-valued gluing matrix defining an isomorphism of the fundamental groups of the boundary tori and hence a diffeomorphism of the tori up to isotopy. Of course, this matrix depends on the choice of coordinate systems on the tori. These coordinate systems cannot be chosen arbitrarily. For each atom there exists a class of admissible coordinate systems on its boundary tori (see [7, Section 4.1]). Now we endow the molecule  $W$  with numerical marks  $r_i$ ,  $\varepsilon_i$ , and  $n_k$  being functions of the gluing matrices. The marks  $r_i \in \mathbb{R}/\mathbb{Z} \cup \{\infty\}$  and  $\varepsilon_i = \pm 1$  are assigned to each edge of the molecule and the marks  $n_k \in \mathbb{Z}$  are assigned to groups of atoms called families. If we cut the molecule along all the edges with finite  $r$ -mark, it will split into several connected pieces. Those pieces which do not contain atoms of type  $A$  (minimax atoms) are said to be *families*. For explicit definitions of the marks  $r_i$ ,  $\varepsilon_i$ ,  $n_k$  see [7, Section 4.3].

**Definition 11.** The molecule  $W$  endowed with the marks  $r_i$ ,  $\varepsilon_i$ ,  $n_k$  is called the *marked molecule* (or *Fomenko–Zieschang invariant*) of the integrable system on the isoenergy surface  $Q$ .

**Theorem 3** (Theorem 1.6 in [3]). *The integrable systems  $(Q_1^3, v_1)$  and  $(Q_2^3, v_2)$  are Liouville equivalent if and only if their marked molecules  $W_1^* = (W_1, r_i, \varepsilon_i, n_k)$  and  $W_2^* = (W_2, r_i, \varepsilon_i, n_k)$  coincide.*

Analogously, the complete invariant for the orbital equivalence is obtained from the marked molecule  $W^*$  by providing it with some additional marks. We shall describe this invariant in the case of *simple* systems, i.e. systems containing only atoms with a single critical circle (these are atoms  $A$ ,  $B$ , and  $A^*$ ).

The most important characteristic of the orbital portrait of the system is the rotation function  $\rho(t)$  defined on every edge of the molecule (here  $t$  is a parameter on the edge). By definition,  $\rho(t) = q_1(t)/q_2(t)$  where  $q_1(t)$  and  $q_2(t)$  are frequencies on the corresponding Liouville torus  $T^2(t)$  (see Theorem 1). This ratio of frequencies is a complete orbital invariant of the system on the given torus and the rotation function gives the orbital characterization of the system on the one-parameter family of tori. We shall consider the class of integrable systems whose rotation functions are “good”, i.e. have a finite number of critical points and points of infinity.

Consider a rotation function  $\rho(t)$  on some edge of the molecule and suppose that the parameter  $t$  changes from 0 to 1. Construct a finite sequence  $R$  of real numbers and symbols  $\pm\infty$  by the following rule. The first element of this sequence is the limit of  $\rho$  at zero (finite or infinite). Then, as  $t$  varies from 0 to 1, we successively write out the values of  $\rho$  at its local minima, local maxima, and its infinite values. Every point of infinity of  $\rho$  is depicted with two symbols (each being  $+\infty$  or  $-\infty$ ) indicating the left and right limits. Finally, the last element of  $R$  is the limit of  $\rho$  at the point 1.

**Definition 12.** The sequence  $R$  is called the *rotation vector* (or *R-vector*) of the integrable system on the given one-parameter family of tori.

The  $R$ -vector is not well-defined since it depends on the basis on the tori of the given family. To obtain an invariant object from it we need to eliminate this dependence. Consider two pairs of basic cycles  $(\lambda^-, \mu^-)$  and  $(\lambda^+, \mu^+)$  on the tori corresponding to a certain edge of the molecule. These pairs of cycles are defined as admissible bases on the boundary tori of the two atoms at the ends of the given edge. Thus, we have two rotation functions  $\rho^-$  and  $\rho^+$ , and a gluing matrix connecting the two bases:

$$\begin{pmatrix} \lambda^+ \\ \mu^+ \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda^- \\ \mu^- \end{pmatrix}.$$

Now set  $R = \beta R^- - \alpha$  if  $\beta \neq 0$  and  $R = R^- \bmod 1$  if  $\beta = 0$ . In the first case the number  $\alpha$  is subtracted from each component of the vector  $\beta R^-$ .

**Definition 13.** The constructed vector  $R$  is called the *R-invariant* of the integrable system on the given one-parameter family of tori (or on the given edge of the molecule).

**Theorem 4** (see Proposition 6.3 in [4]). The  $R$ -invariant is a well-defined complete topological orbital invariant of the integrable system considered on the given one-parameter family of Liouville tori. This means that two such systems are topologically orbitally equivalent if and only if their  $R$ -invariants coincide. If, in addition, the rotation functions of these systems on the corresponding families of tori are Morse functions, then the above statement holds for the smooth orbital equivalence as well.

Now we shall describe one more orbital invariant of the integrable system on the isoenergy surface. Let  $W^*$  be its marked molecule. An edge of  $W^*$  is called *finite* if the  $r$ -mark on it is finite, *infinite* if the  $r$ -mark on it is infinite, and *superinfinite* if it is infinite and all the components of its rotation vector are also infinite. If we cut the molecule  $W^*$  along all its edges which are not superinfinite, it will split into a disjoint union of subgraphs. Those subgraphs which do not contain  $A$ -atoms are called *radicals*. Now to each edge  $e_j$  of  $W^*$  incident to the given radical  $U$  we assign an integer  $[\Theta]_j$  by the following rule:

$$[\Theta]_j = \begin{cases} [\alpha_j/\beta_j] & \text{if } e_j \text{ is finite and exits the radical } U, \\ [-\delta_j/\beta_j] & \text{if } e_j \text{ is finite and enters the radical } U, \\ [MR_j^+] & \text{if } e_j \text{ is infinite and enters the radical } U, \\ -[-MR_j^-] & \text{if } e_j \text{ is infinite and exits the radical } U, \\ -\gamma_j/\alpha_j & \text{if } e_j \text{ is a superinfinite interior edge of the radical } U. \end{cases}$$

Here  $\alpha, \beta, \gamma, \delta$  are the entries of the gluing matrix  $C_j$ ,  $MR^+$  and  $MR^-$  are the arithmetic means of the finite components of the rotation vectors  $R^+$  and  $R^-$ . It is easy to see that the five possibilities above cover all possible cases and hence the number  $[\Theta]_j$  is well defined.

**Definition 14.** The number  $b(U) = \sum [\Theta]_j$  is called the  $b$ -invariant of the radical  $U$ .

**Remark 3.** The meaning of the numbers  $MR^+$  and  $MR^-$  for the construction of the orbital invariants can shortly be explained in the following way. Recall that in the case of an infinite edge the  $R$ -invariant is defined only modulo 1 (see Definition 13). It is easy to notice that fixing the number  $MR^-$  eliminates this indeterminacy. Of similar meaning is the number  $MR^+$ .

**Definition 15.** The marked molecule  $W^*$  endowed with the  $R$ -invariants of all edges and the  $b$ -invariants of all radicals is called the  $t$ -molecule of a simple integrable system on the given isoenergy surface.

**Theorem 5** (Theorem 2.1 in [5]). Two simple integrable Hamiltonian systems  $(Q_1^3, v_1)$  and  $(Q_2^3, v_2)$  are topologically orbitally equivalent if and only if their  $t$ -molecules  $W_1^{*t} = (W_1^*, R_1, b_1)$  and  $W_2^{*t} = (W_2^*, R_2, b_2)$  coincide.

Recall that the molecule of a simple system by definition contains only atoms of complexity one (i.e. atoms  $A, B$ , and  $A^*$ ). For the systems of general type the  $t$ -molecule as defined above does not contain enough information for the complete orbital classification. In this case the  $b$ -invariant should be supplemented by some other invariants and becomes much more complicated. But for some systems which though are not simple but possess certain symmetries the above theorem remains true. This important remark will be used further for establishing the equivalence between systems containing atom  $C_2$ .

Theorem 5 has an analogue for the smooth orbital equivalence. The appropriate invariant in this case is the  $st$ -molecule  $W^{*st} = (W^*, R, \Lambda_m)$  where the invariants  $\Lambda_m$  are certain power series assigned to the vertices of the saddle atoms (see [6, Definition 1]).

Thus, we have given a review of the key results of the theory of topological invariants for integrable Hamiltonian systems with two degrees of freedom. Somewhere we omitted details, since our aim was just to present a general nature of the objects we deal with. In the concrete examples below this objects take much more simple form than in the general case and some facts in the proof of the main theorem could be explained independently, without references to the general theory. However, we decided that our main result is a good example demonstrating applications of the theory of invariants. This is the reason why the proof (and the whole article) may seem too formal.

### 3. Equivalence of the three classical systems. Main results

In this section we formulate main results concerning the equivalence (Liouville and orbital) of three famous integrable systems: the Jacobi problem about geodesics on the two-dimensional ellipsoid [25], the Euler case in rigid body dynamics [26] and the Chaplygin case in dynamics of a rigid body in fluid [27]. These systems were studied from the point of view of the topological classification and the theory of invariants in [20,5,21,22,28,29].

The geodesic flow on the two-dimensional ellipsoid is an integrable Hamiltonian system with two degrees of freedom on the cotangent bundle  $T^*E^2$  with the standard symplectic structure where  $E^2 = \left\{ \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \right\}$  is the ellipsoid with squares of semi-axes  $a, b, c$  ( $0 < a < b < c$ ) embedded in the Euclidean space  $\mathbb{R}^3(x, y, z)$ . The Hamiltonian function and the additional integral of this system (written in the coordinates of the embedding space  $\mathbb{R}^6 = T^*\mathbb{R}^3$ ) are

$$H_j = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad f_j = abc \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \left( \frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} + \frac{\dot{z}^2}{c} \right).$$

Here  $(\dot{x}, \dot{y}, \dot{z})$  is the tangent vector to a geodesic (we identify tangent and cotangent vectors in the natural way).



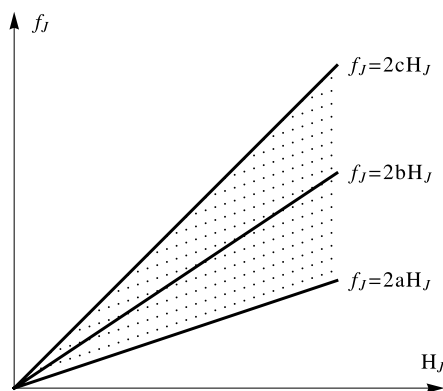


Fig. 2. Bifurcation diagram for the Jacobi problem.

The bifurcation diagram for this system is shown in Fig. 2. Each isoenergy surface (except for the zero one) is a unit covector bundle over  $E^2$  and is diffeomorphic to  $\mathbb{R}P^3$ . Note that, due to the homogeneity of the Hamiltonian function, the orbital structure of this system does not depend on the energy level. Moreover, proportional triples  $(a, b, c)$  and  $(a', b', c')$  define orbitally equivalent systems [20, Theorem A]. Thus, the Jacobi problem describes a two-parameter family of integrable systems (the triple  $(a, b, c)$  viewed up to proportionality gives two parameters).

Another famous integrable system is the Euler case which describes the motion of a rigid body fixed at its centre of mass. It is given by the Euler–Poisson equations:

$$\dot{K} = K \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega.$$

Here vector  $K = (s_1, s_2, s_3)$  is the kinetic momentum vector of the body,  $\Omega = (As_1, Bs_2, Cs_3)$  is its angular velocity vector,  $\gamma = (r_1, r_2, r_3)$  is the unit vertical vector. The coordinates of these vectors are written in the orthonormal basis which is fixed in the body and whose axes coincide with the principal axes of inertia. The parameters  $A, B, C$  ( $A < B < C$ ) of the problem are the inverses of the principal moments of inertia of the rigid body.

The Euler case is a Hamiltonian system in the space  $\mathbb{R}^6(s_1, s_2, s_3, r_1, r_2, r_3)$  considered as the dual space of the Lie algebra  $e(3) = so(3) + \mathbb{R}^3$ . The Poisson structure here is given by the Lie–Poisson bracket:

$$\{s_i, s_j\} = \varepsilon_{ijk} s_k, \quad \{s_i, r_j\} = \varepsilon_{ijk} r_k, \quad \{r_i, r_j\} = 0.$$

The Hamiltonian function of the system is  $H_E = (As_1^2 + Bs_2^2 + Cs_3^2)/2$ .

Recall that the Lie–Poisson bracket on  $e(3)^*$  has two Casimir functions  $f_1 = r_1^2 + r_2^2 + r_3^2, f_2 = s_1 r_1 + s_2 r_2 + s_3 r_3$  and defines a symplectic structure on the four-dimensional submanifold  $M_{a,g}^4 = \{f_1 = a^2, f_2 = g\}$  diffeomorphic to  $T^*S^2$ . The Euler–Poisson equations can be viewed as a Hamiltonian system with two degrees of freedom on  $M_{1,g}^4$ . This system is integrable by means of the additional integral  $f_E = s_1^2 + s_2^2 + s_3^2$ . The bifurcation diagram is shown in Fig. 3. There are three zones for the regular energy values:

$$(1)_E = \left( \frac{Ag^2}{2}, \frac{Bg^2}{2} \right), \quad (2)_E = \left( \frac{Bg^2}{2}, \frac{Cg^2}{2} \right), \quad (3)_E = \left( \frac{Cg^2}{2}, +\infty \right).$$

In the case  $g = 0$  only the third zone remains and the bifurcation diagram is the same as in the Jacobi problem.

As follows from [5], proportional triples  $(A, B, C)$  and  $(A', B', C')$  define orbitally equivalent Euler type systems. Moreover, for the third zone of energies (“large” energies) the class of orbital equivalence does not depend on the energy value. Assuming that the value  $g$  of the area integral  $f_2$  is fixed, we obtain two three-parameter families of the Euler type systems corresponding to zones  $(1)_E, (2)_E$  and a two-parameter family of systems corresponding to zone  $(3)_E$ . Here two of the parameters are given by the triple  $(A, B, C)$  viewed up to proportionality and the third parameter (for zones  $(1)_E, (2)_E$ ) is the energy value  $\chi$ .

It is well known that there exists the orbital equivalence between the Jacobi and Euler systems. More precisely, gathering all the information concerning the equivalence of these systems we may formulate the following result.

**Theorem 6.** Let  $v_J(a, b, c)$  ( $a < b < c$ ) and  $v_E(A, B, C)$  ( $A < B < C$ ) be the Jacobi and Euler systems on surfaces of constant (non-zero) energy. In the Euler case we consider either of the two possibilities:  $g = 0$  and arbitrary value of energy  $\chi > 0$  or  $g \neq 0$  and the value of energy from zone  $(3)_E$ . Then the following statements hold.

- (a) (Bolsinov, Fomenko, [20, Theorem B]) The systems  $v_J(a, b, c)$  and  $v_E(A, B, C)$  are Liouville equivalent. Moreover, they are topologically orbitally equivalent in the following sense. For any triple  $(a, b, c)$  there exists a unique up to proportionality triple  $(A, B, C)$  (and vice versa, for any triple  $(A, B, C)$  there exists a unique up to proportionality triple  $(a, b, c)$ ) such that the systems  $v_J(a, b, c)$  and  $v_E(A, B, C)$  are topologically orbitally equivalent. The relation between the triples  $(A, B, C)$  and  $(a, b, c)$  corresponding to each other under this equivalence is established effectively (i.e. is given by explicit formulae).

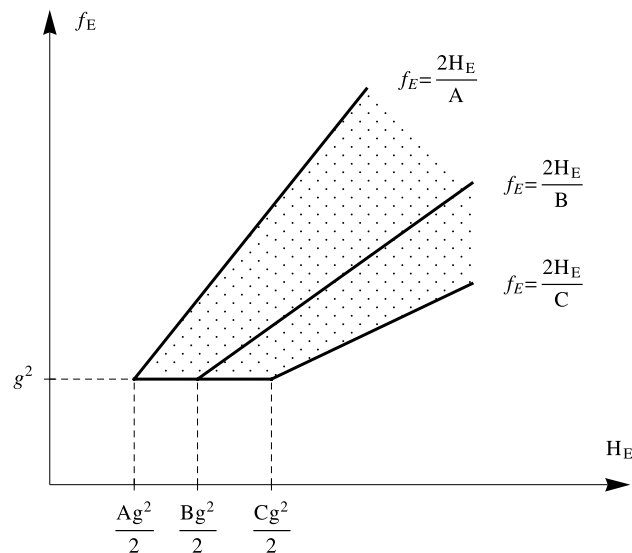


Fig. 3. Bifurcation diagram for the Euler case.

- (b) (Bolsinov, Dullin, [21, pp. 21–24]) Any two systems  $v_j(a, b, c)$  and  $v_E(A, B, C)$  are not smoothly orbitally equivalent (even in the sense of  $C^1$ -smoothness). Here we should remark that this statement rests on the results of the computer analysis.
- (c) (Orel, [22, Main theorem]) Any two systems  $v_j(a, b, c)$  and  $v_E(A, B, C)$  are not topologically conjugate (and hence also not smoothly conjugate).

The proof of this theorem is based on the comparison of the corresponding invariants of the systems. In particular, for the proof of the first statement  $t$ -molecules of the systems were constructed and conditions for their coincidence were then investigated. The second statement is based on the computer analysis of the possible values of some smooth invariants.

Consider now one more classical integrable system – the Chaplygin case in dynamics of a rigid body in fluid. It is a partial integrable case of the Kirchhoff equations which describe the free motion of a rigid body in the ideal incompressible fluid:

$$\dot{s} = s \times \frac{\partial H}{\partial s} + r \times \frac{\partial H}{\partial r}, \quad \dot{r} = r \times \frac{\partial H}{\partial s}. \quad (1)$$

Here the vectors  $s = (s_1, s_2, s_3)$  and  $r = (r_1, r_2, r_3)$  are the total moment and momentum (written in a fixed basis rigidly connected with the body),  $H$  is the full energy (Hamiltonian function) of the system.

As the Euler–Poisson equations, the Kirchhoff equations can be viewed as a Hamiltonian system with two degrees of freedom on the symplectic leaf  $M_{a,g}^4$  in the space  $e(3)^*$ . The Chaplygin case is given by  $H = (s_1^2 + s_2^2 + 2s_3^2)/2 + c(r_1^2 - r_2^2)/2$  and is integrable by means of the additional integral  $K = (s_1^2 - s_2^2 + cr_3^2)^2 + 4s_1^2s_2^2$ . The rough topology of this system (including the bifurcation diagram and the Fomenko invariants) was studied in [28]. The fine topology in the terms of the Fomenko–Zieschang invariants was investigated in [29].

**Remark 4.** Though the Chaplygin system includes two parameters  $a$  and  $c$  (here  $a^2$  is the value of the first Casimir function), it turns out that the class of orbital equivalence of this system does not depend on these parameters (they disappear after an appropriate linear transformation of the phase variables). So we shall assume that  $a = c = 1$  and that the Chaplygin case describes a one-parameter family of integrable systems. The only parameter is the value of energy  $h$ .

The bifurcation diagram for the Chaplygin case is shown in Fig. 4. We can distinguish three zones of energy:

$$(1)_{Ch} = \left(-\frac{1}{2}, 0\right), \quad (2)_{Ch} = \left(0, \frac{1}{2}\right), \quad (3)_{Ch} = \left(\frac{1}{2}, +\infty\right).$$

Now we formulate our main theorem concerning the equivalence of the Chaplygin case to the two systems discussed above.

**Theorem 7 (Main theorem).** Let  $v_{Ch}(h)$  and  $v_E(A, B, C, \chi)$  be the Chaplygin and Euler systems on regular surfaces of constant energy  $\{H_{Ch} = h\}$  and  $\{H_E = \chi\}$ . We suppose that the energy values  $h$  and  $\chi$  belong to zones with the same number. Let  $v_j(a, b, c)$  denote, as above, the Jacobi system on a non-zero level of energy. (The triples  $(A, B, C)$  and  $(a, b, c)$  are viewed up to proportionality.) Then the following statements hold.

- (1) If the energy value  $h$  of the Chaplygin system  $v_{Ch}(h)$  belongs to zone  $(1)_{Ch}$ , then it is Liouville equivalent to the Euler system  $v_E(A, B, C, \chi)$  for any  $A, B, C$  and for any  $\chi$  from zone  $(1)_E$ . There also exists a one-parameter family of the Euler

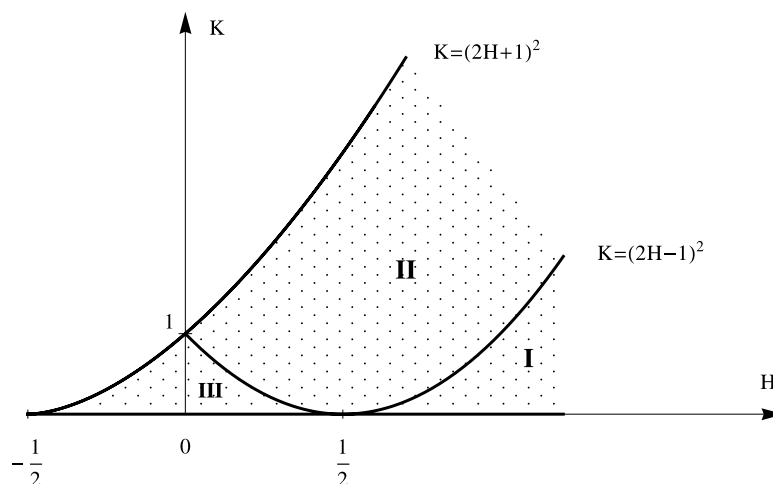


Fig. 4. Bifurcation diagram for the Chaplygin case.

systems  $v_E(A, B(A), C = 1, \chi(A))$  orbitally (topologically and smoothly) equivalent to  $v_{Ch}(h)$ . This orbital equivalence can be extended on the four-dimensional neighbourhoods of the isoenergy surfaces. But for any  $A, B, C, \chi$  the systems  $v_{Ch}(h)$  and  $v_E(A, B, C, \chi)$  are not topologically conjugate.

- (2) If the energy value  $h$  of the Chaplygin system  $v_{Ch}(h)$  belongs to zone  $(2)_{Ch}$ , then it is Liouville equivalent to the Euler system  $v_E(A, B, C, \chi)$  for any  $A, B, C$  and for any  $\chi$  from zone  $(2)_E$ . There also exists a one-parameter family of the Euler systems  $v_E(A, B(A), C = 1, \chi(A))$  topologically orbitally equivalent to  $v_{Ch}(h)$ . But for any  $A, B, C, \chi$  the systems  $v_{Ch}(h)$  and  $v_E(A, B, C, \chi)$  are not smoothly orbitally equivalent (even in the sense of  $C^1$ -smoothness) and are not topologically conjugate.
- (3) If the energy value  $h$  of the Chaplygin system  $v_{Ch}(h)$  belongs to zone  $(3)_{Ch}$ , then it is Liouville equivalent to the Euler system  $v_E(A, B, C)$  for any  $A, B, C$  (with the energy value from zone  $(3)_E$ ) and to the Jacobi system  $v_J(a, b, c)$  for any  $a, b, c$ . If  $h$  is large enough, there exist unique up to proportionality triples  $(A, B, C)$  and  $(a, b, c)$  such that the system  $v_{Ch}(h)$  is topologically orbitally equivalent to  $v_E(A, B, C)$  and  $v_J(a, b, c)$ . But this orbital equivalence cannot be made smooth (even in the sense of  $C^1$ -smoothness). Moreover, for any  $A, B, C$  and  $a, b, c$  the system  $v_{Ch}(\chi)$  is not topologically conjugate with  $v_E(A, B, C)$  or  $v_J(a, b, c)$ .

**Comment 1.** It is convenient to present the results of Theorem 7 in the following table. For each of the four types of equivalence and for each of the three zones of energy we put “+” if the corresponding Euler and Chaplygin systems are equivalent and “–” otherwise.

Each cell in the first row of this table corresponds to the Liouville equivalence between the one-parameter family of the Chaplygin systems with parameter  $h$  and the three-parameter family of the Euler systems with parameters  $A, B$ , and  $\chi$ . This means that any two systems from both families are Liouville equivalent. We assume  $C = 1$  since the triples  $(A, B, C)$  are viewed up to proportionality.

	zone (1)	zone (2)	zone (3)
Liouville equivalence	+	+	+
Topological orbital equivalence	+	+	+
Smooth orbital equivalence	+	–	–
Topological conjugacy	–	–	–

The first two cells in the second row and the first cell in the third row establish the orbital equivalence (topological or smooth) between the one-parameter family of the Chaplygin systems with parameter  $h$  and the two-parameter family of the Euler systems with parameters  $A$  and  $\chi$ . The parameter  $B$  is now uniquely defined by  $A: B = B(A)$ . Each point in the one-parameter family of the Chaplygin systems given by the value of energy  $h$  corresponds to a curve in the two-parameter family of the Euler systems given by  $B = B(A)$ ,  $\chi = \chi(h, A)$  (see Fig. 5).

Finally, the third cell in the second row of the table corresponds to the topological orbital equivalence between the one-parameter family of the Chaplygin systems with parameter  $h$  and the one-parameter family of the Euler systems. Each point in the first family given by  $h$  corresponds to a unique point in the second family given by  $A = A(h)$ ,  $B = B(h)$ .

**Comment 2.** The orbital isomorphisms stated by Theorem 7 admit an explicit description in the terms of the parameters  $A, B, C$  of the Euler problem. For the first and second zones of energy the one-parameter family of the Euler systems orbitally



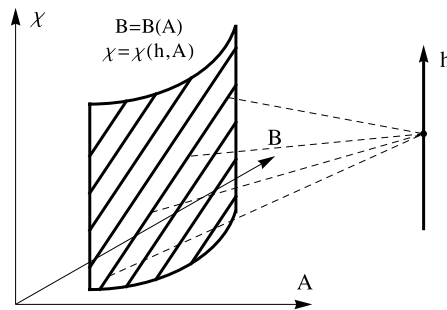


Fig. 5. Orbital equivalence of the Chaplygin and Euler cases: zones (1) and (2).

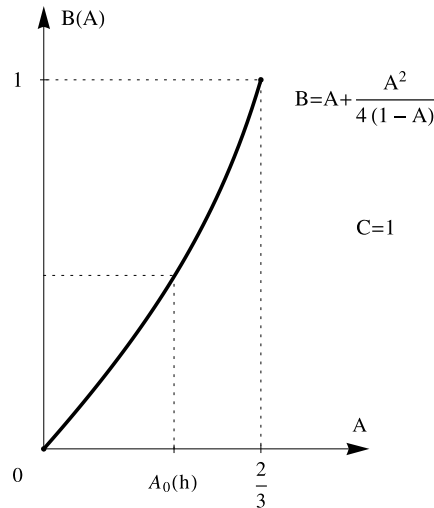


Fig. 6. One-parameter family of the triples  $(A, B, C)$ .

equivalent to the given Chaplygin system is described by a simple formula  $A[(C - A)(B - A)]^{-1/2} = 2$ . Since the triples  $(A, B, C)$  are viewed up to proportionality, we may set  $C = 1$  and express  $B$  via  $A$ :

$$B = A + \frac{A^2}{4(1 - A)}.$$

From the condition  $0 < A < B < 1$  we find the interval of possible values for the parameter  $A$ :  $0 < A < 2/3$ . (In the case of zones  $(2)_{Ch}$  and  $(2)_E$  this interval should be shorter:  $A \in (0, A_0(h))$  where  $A_0(h) < 2/3$ , see below.) The one-parameter family of the triples  $(A, B, C)$  is represented in Fig. 6.

For the case of “large” energies (zones  $(3)_{Ch}$  and  $(3)_E$ ) the one-parameter family of the Euler systems orbitally equivalent to the one-parameter family of the Chaplygin systems is given by the formula  $C[(C - A)(C - B)]^{-1/2} = 2$ . Again, setting  $C = 1$  we express  $B$  via  $A$ :

$$B = 1 - \frac{1}{4(1 - A)}.$$

Here the interval of possible values for the parameter  $A$  is  $[A_1, 1/2)$  where  $A_1 > 0$  is a fixed number defined below. The corresponding one-parameter family of the triples  $(A, B, C)$  is shown in Fig. 7.

Thus, Theorems 6 and 7 establish relations between the three famous integrable systems: the Jacobi problem, the Euler case and the Chaplygin case. All these systems happen to be orbitally equivalent in the above sense. However, the explicit formulae for these orbital isomorphisms are not known.

#### 4. Proof of the main theorem

The Liouville equivalence of the Chaplygin system to the Euler system (and hence the Jacobi one on the corresponding energy levels) was proved in [29, Corollary 1]. The marked molecules calculated there for the Chaplygin case (see Fig. 8) appeared to be the same as in the Euler case. For the computation of the marks the separation of variables and some standard

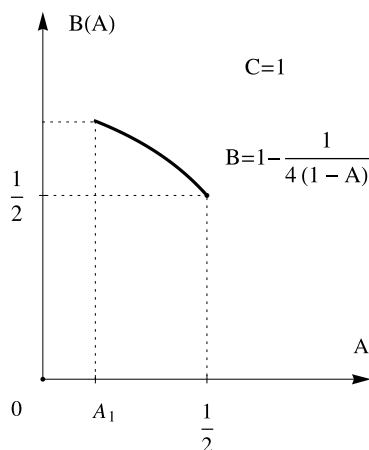
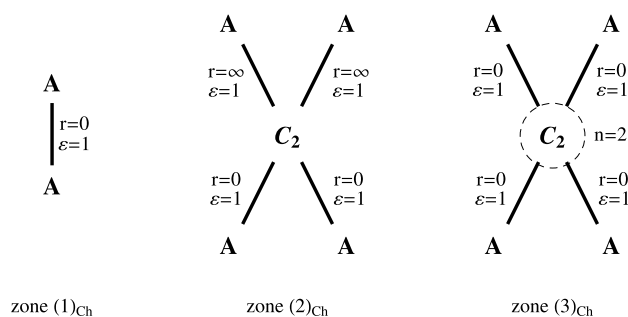
Fig. 7. One-parameter family of the triples  $(A, B, C)$ .

Fig. 8. Marked molecules for the Chaplygin case.

methods, such as the loop molecule method, the rule for summation of the marks, and the Topalov formula, were used. It was also shown how the admissible coordinate systems on the Liouville tori can be chosen directly.

To prove the topological orbital equivalence between the three systems we need to compare their  $t$ -molecules. We emphasize that all the three systems have a natural symmetry that permits to treat them as simple systems and to use the simple  $t$ -molecule defined above for their orbital classification. This remark was used in [5], [20] and now we use it in the Chaplygin case.

To obtain the  $t$ -molecules for the Chaplygin systems, we need to add the  $R$ -invariants and the  $b$ -invariants to the corresponding marked molecules.

**Step 1. Calculation of the rotation function.** To calculate the rotation functions for the Chaplygin systems we use the separation of variables given in [27]. The separated differential equations on the Liouville torus  $T_{h,k}^2 \subset \{H = h, K = k\}$  (we consider one of the two Liouville tori over the point  $(h, k)$ ) take the form:

$$\dot{u}_1 = \sqrt{Q_1(u_1)}, \quad \dot{u}_2 = \sqrt{Q_2(u_2)} \quad (2)$$

$$\text{where } Q_1(u) = 2(u+1)(u-1)(u-\alpha_1), \quad Q_2(u) = 2(1+u_2)(1-u_2)(\alpha_2-u_2).$$

Here  $\alpha_1 = 2h + \sqrt{k}$ ,  $\alpha_2 = 2h - \sqrt{k}$ , and  $u_1, u_2$  are smooth functions on the torus  $T_{h,k}^2$ . The image of the mapping  $u_1 \times u_2 : T_{h,k}^2 \rightarrow \mathbb{R}^2(u_1, u_2)$  is the infinite rectangle  $[\max\{\alpha_1, 1\}, +\infty] \times [-1, \min\{\alpha_2, 1\}]$ .

Let  $P_0$  be a fixed point on the torus  $T_{h,k}^2$  and let  $\gamma_1 \subset I_1 = \{u_2 = u_2(P_0)\}$ ,  $\gamma_2 \subset I_2 = \{u_1 = u_1(P_0)\}$  be the cycles on  $T_{h,k}^2$  containing the point  $P_0$ . It can be shown that  $\gamma_1$  and  $\gamma_2$  form a basis in the one-dimensional homology group of  $T_{h,k}^2$ . Consider the following functions on  $T_{h,k}^2$ :

$$\varphi_1(P) = 2\pi \frac{\int_{P_0}^P \frac{du_1}{\sqrt{Q_1(u_1)}}}{\int_{\gamma_1} \frac{du_1}{\sqrt{Q_1(u_1)}}} \pmod{2\pi}, \quad \varphi_2(P) = 2\pi \frac{\int_{P_0}^P \frac{du_2}{\sqrt{Q_2(u_2)}}}{\int_{\gamma_2} \frac{du_2}{\sqrt{Q_2(u_2)}}} \pmod{2\pi}. \quad (3)$$

The integrals in the nominators are taken along any path in  $T_{h,k}^2$  connecting the points  $P_0$  and  $P$ . It is easy to see that they are well defined modulo  $2\pi$ . We state that  $\varphi_1$  and  $\varphi_2$  are the angle coordinates on the torus  $T_{h,k}^2$  and

$$\dot{\varphi}_1 = \frac{2\pi}{\int_{\gamma_1} \frac{du_1}{\sqrt{Q_1(u_1)}}}, \quad \dot{\varphi}_2 = \frac{2\pi}{\int_{\gamma_2} \frac{du_2}{\sqrt{Q_2(u_2)}}}.$$

Indeed, the integrals in the denominators of (3) are constant and the differentiation of the nominators gives 1 in view of (2).

It can be shown that each of the cycles  $\gamma_1$  and  $\gamma_2$  is a 4-fold ramified covering over the corresponding segment  $[\max\{\alpha_1, 1\}, +\infty]$  or  $[-1, \min\{\alpha_2, 1\}]$ . Hence for the rotation function we have

$$\rho(h, k) = \frac{\int_{\gamma_2} \frac{du_2}{\sqrt{Q_2(u_2)}}}{\int_{\gamma_1} \frac{du_1}{\sqrt{Q_1(u_1)}}} = \frac{\int_{-1}^{\min\{\alpha_2, 1\}} \frac{du_2}{\sqrt{Q_2(u_2)}}}{\int_{\max\{\alpha_1, 1\}}^{+\infty} \frac{du_1}{\sqrt{Q_1(u_1)}}} = \frac{E_2}{E_1}.$$

By means of standard substitutions it is not difficult to bring the elliptic integrals  $E_1, E_2$  to the canonical form. As a result we have

$$E_1 = \begin{cases} \pm 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+\alpha_1-2t^2)}} & \text{if } \alpha_1 > 1, \\ \pm 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-(1+\alpha_1)t^2)}} & \text{if } \alpha_1 < 1; \end{cases} \quad (4)$$

$$E_2 = \begin{cases} \pm 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+\alpha_2-2t^2)}} & \text{if } \alpha_2 > 1, \\ \pm 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-(1+\alpha_2)t^2)}} & \text{if } \alpha_2 < 1. \end{cases} \quad (5)$$

The sign in these formulae depends on the choice of the connected component.

Note that the rotation function can be calculated in a similar way for the systems reducible to Abel equations (see [11]).

The formulae obtained for the rotation function are written in the basis  $(\gamma_1, \gamma_2)$ . For each of the regular domains (chambers)  $I = \{h > 1/2, 0 < k < (2h-1)^2\}$ ,  $II = \{h > 0, (2h-1)^2 < k < (2h+1)^2\}$ ,  $III = \{-1/2 < h < 1/2, 0 < k < \min\{(2h-1)^2, (2h+1)^2\}\}$  which lie in the image of the momentum mapping (see Fig. 4) we need to write the rotation function in some admissible coordinate system continuously depending on the Liouville tori. Based on the results of [29, Paragraphs 4, 5], we can write out the admissible bases and the gluing matrices on the edges of the marked molecules corresponding to zones (1)<sub>Ch</sub>, (2)<sub>Ch</sub>, (3)<sub>Ch</sub> (see Figs. 9 and 10). Each edge is assigned a certain orientation. The rotation functions should be written in the bases corresponding to the beginnings of the edges. For this we use a simple relation between the rotation functions for two different bases (see [7, Proposition 1.13]). Note that the left molecule in Fig. 9 and the left edges in Fig. 10 correspond to different signs in the formulae (4) and (5) whereas the right molecule in Fig. 9 and the right edges in Fig. 10 correspond to the same signs.

We have  $\alpha_1 > \alpha_2 > 1$  for chamber *I*,  $\alpha_1 > 1 > \alpha_2$  for chamber *II*, and  $1 > \alpha_1 > \alpha_2$  for chamber *III*. The corresponding rotation functions are the following (we recall that  $\alpha_1 = 2h + \sqrt{k}$ ,  $\alpha_2 = 2h - \sqrt{k}$ ):

$$\rho_I(h, k) = -1 - \frac{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+\alpha_2-2t^2)}}}{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+\alpha_1-2t^2)}}}, \quad (6)$$

$$\rho_{II,(3)}(h, k) = 1 + \frac{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-(1+\alpha_2)t^2)}}}{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+\alpha_1-2t^2)}}} \quad (\text{for zone (3)}_{Ch}), \quad (7)$$

$$\rho_{II,(2)}(h, k) = \frac{\rho_{II,(3)}(h, k)}{1 - \rho_{II,(3)}(h, k)} \quad (\text{for zone (2)}_{Ch}), \quad (8)$$

$$\rho_{III}(h, k) = 1 + \frac{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-(1+\alpha_1)t^2)}}}{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(2-(1+\alpha_2)t^2)}}}. \quad (9)$$

We are interested in the behaviour of these functions on the isoenergy surfaces, that is, for fixed  $h$ . It is easy to see that the functions  $\rho_I$  and  $\rho_{III}$  are monotonous with respect to  $k$ . For the functions  $\rho_{II,(2)}$  and  $\rho_{II,(3)}$  we have carried out a computer analysis. The qualitative behaviour of the level curves of the function  $\rho_{II,(3)}$  is shown in Fig. 11. From here we can conclude

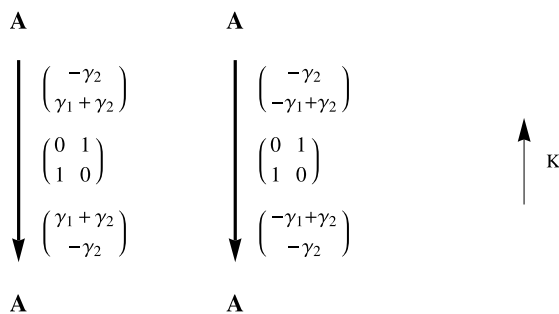


Fig. 9. Admissible bases and gluing matrices, zone  $(1)_{Ch}$  (two connected components).

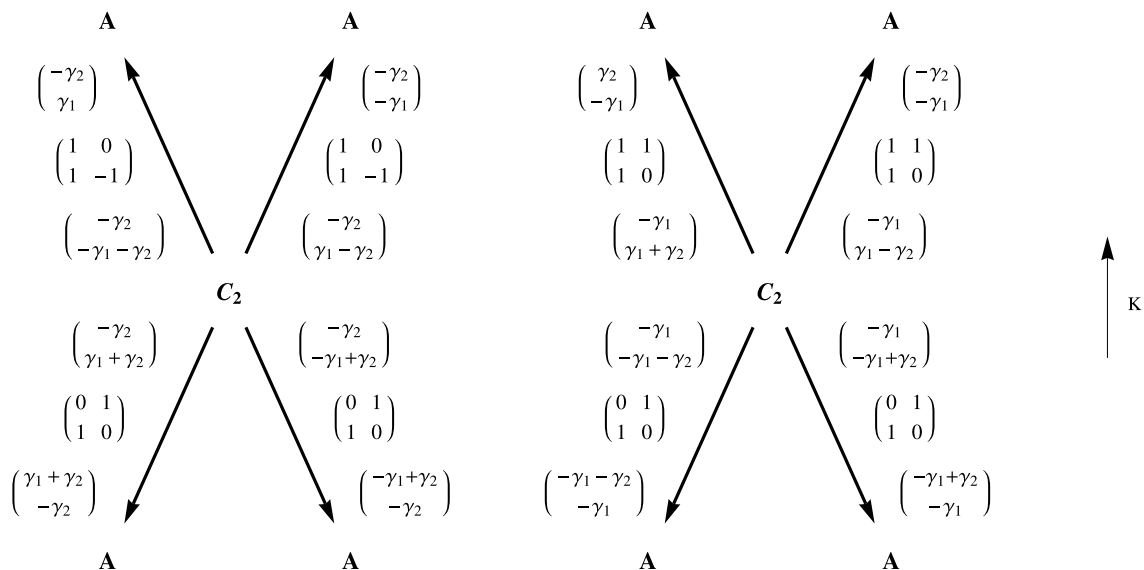


Fig. 10. Admissible bases and gluing matrices, zones  $(2)_{Ch}$  and  $(3)_{Ch}$ .

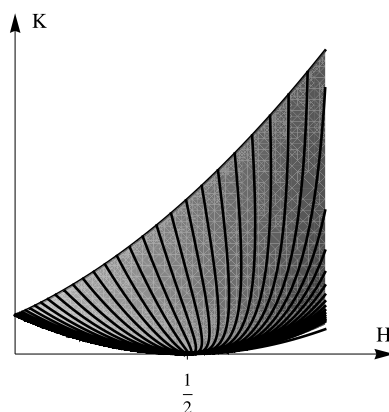


Fig. 11. Level curves of the rotation function  $\rho_{II,(3)}$ .

rather confidently that  $\rho_{II,(3)}$  is monotonous for large enough values of energy from zone  $(3)_{Ch}$  and for energy values from zone  $(2)_{Ch}$  (hence  $\rho_{II,(2)}$  is also monotonous). But there exists an energy value  $h_0 > 1/2$  such that for  $h \in (1/2, h_0)$  the function  $\rho_{II,(3)}$  has a critical point on the interval  $((2h - 1)^2, (2h + 1)^2)$ .

The reviewers of this paper suggested an analytical proof of the monotonicity of the function  $\rho_{II,(3)}$  for large values of energy which we have pleasure to present. Denote  $q = (1 + \alpha_2)/2$ . Then we need to prove the monotonicity of the

function

$$G(q) = \frac{E(q)\sqrt{1+2h-q}}{E\left(\frac{1}{1+2h-q}\right)}, \quad q \in (0, 1)$$

where

$$E(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$$

is the complete elliptic integral of the first kind. To show the monotonicity of  $G(q)$  consider the logarithmic derivative

$$(\ln G(q))' = \frac{G'(q)}{G(q)} = \frac{E'(q)}{E(q)} - \frac{1}{2(1+2h-q)} - \frac{1}{(1+2h-q)^2} \frac{E'\left(\frac{1}{1+2h-q}\right)}{E\left(\frac{1}{1+2h-q}\right)}.$$

The functions  $E(q)$  and  $E'(q)$  are continuous on  $[0, 1)$ , hence the ratio  $\frac{E'(q)}{E(q)}$  is bounded on the segment  $[0, \sigma]$  for any  $\sigma < 1$ . From the representation of  $E(q)$  as a power series

$$E(q) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left( \frac{(2n)!}{2^{2n}(n!)^2} \right)^2 q^n$$

it is easy to see that the ratio  $\frac{E'(q)}{E(q)}$  tends to infinity as  $q \rightarrow 1-0$ . Therefore, this ratio is bounded from below on  $[0, 1)$ .

Analogously, the ratio  $\frac{E'\left(\frac{1}{1+2h-q}\right)}{E\left(\frac{1}{1+2h-q}\right)}$  is bounded from above, since the argument of the functions  $E, E'$  ranges here from  $\frac{1}{2h+1}$  to  $\frac{1}{2h} < 1$ . It is now clear that for large enough values of  $h$  we have  $G'(q) > 0$  for  $q \in [0, 1)$ .

A similar idea can be used to show the monotonicity of the function  $\rho_{II,(3)}$  for energy values from zone  $(2)_{Ch}$ . However, we do not know a short proof of this fact for all  $h \in (0, 1/2)$ .

**Step 2. Proof of the topological orbital equivalence.** Now we can write out the  $t$ -molecules for the Chaplygin type systems. From the explicit formulae for the rotation functions we obtain the rotation vectors on the edges of the molecules. Then, depending on the gluing matrices, we “improve” these vectors and transform them into  $R$ -invariants. Finally, we write out the  $b$ -invariants. The molecules corresponding to zones  $(2)_{Ch}$  and  $(3)_{Ch}$  contain only one radical consisting of the unique atom  $C_2$ .

**Lemma 8.** *There exist four types of  $t$ -molecules for the Chaplygin systems shown in Figs. 12–15.*

Here

$$m(h) = \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-(2h+1)t^2)}},$$

$$n(h) = \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(2h+1-t^2)}},$$

and  $\xi(h)$  denotes the local minimum of the rotation function when  $h \in (1/2, h_0)$ .

Now we want to compare the  $t$ -molecules in the Chaplygin case with those for the Euler and Jacobi systems. The latter were computed in [5], [20] (see Figs. 16–19).

Here

$$K(A, B, C) = \frac{A}{\sqrt{(C-A)(B-A)}}, \quad L(A, B, C) = \frac{C}{\sqrt{(C-A)(C-B)}},$$

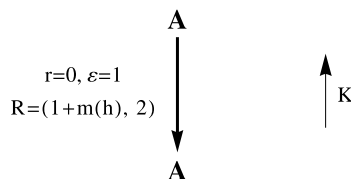
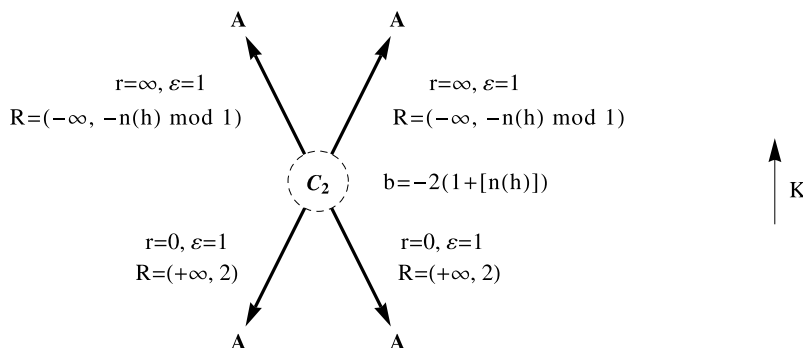
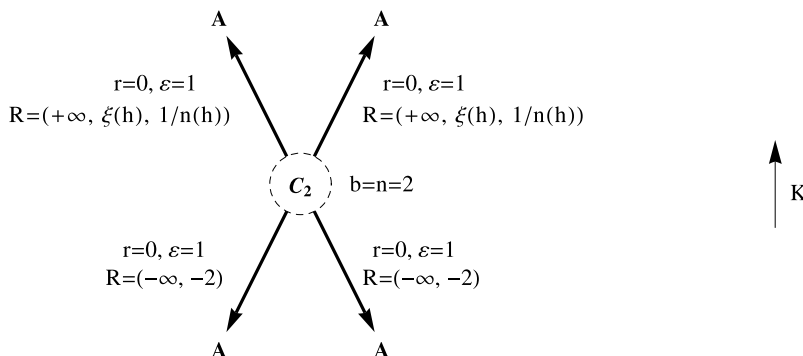
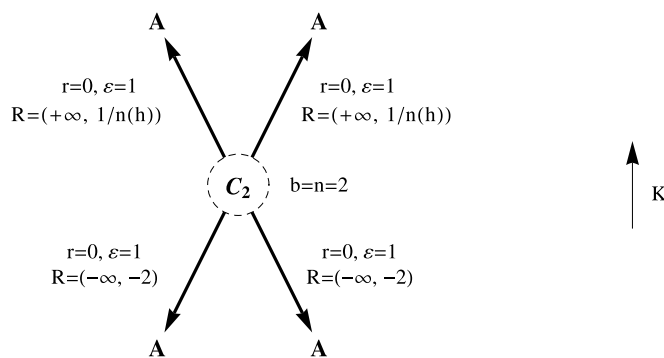
$$M(A, B, C, \chi) = \frac{1}{\pi} \int_A^{2\chi/g^2} \left[ \left( \frac{2\chi}{g^2} - u \right) (C-u)(B-u)(u-A) \right]^{-1/2} u du,$$

$$N(A, B, C, \chi) = \frac{1}{\pi} \int_A^B \left[ \left( \frac{2\chi}{g^2} - u \right) (C-u)(B-u)(u-A) \right]^{-1/2} u du;$$

$$k(a, b, c) = \frac{\pi \sqrt{\frac{a}{(b-a)(c-a)}}}{\int_0^{+\infty} \Phi(u, a) du}, \quad l(a, b, c) = \frac{\pi \sqrt{\frac{c}{(c-a)(c-b)}}}{\int_0^{+\infty} \Phi(u, c) du},$$

where  $\Phi(u, t) = u[(u+a)(u+b)(u+c)(u+t)]^{-1/2}$ .

The functions  $M(A, B, C, \chi)$  and  $N(A, B, C, \chi)$  are monotonous with respect to the energy value  $\chi$ .

Fig. 12.  $t$ -molecule for the Chaplygin case, zone  $(1)_{Ch}$ .Fig. 13.  $t$ -molecule for the Chaplygin case, zone  $(2)_{Ch}$ .Fig. 14.  $t$ -molecule for the Chaplygin case, zone  $(3)_{Ch} : h < h_0$ .Fig. 15.  $t$ -molecule for the Chaplygin case, zone  $(3)_{Ch} : h \ge h_0$ .

Consider the molecules corresponding to zones  $(1)_{Ch}$  and  $(1)_E$  in the systems of Chaplygin and Euler type. They have no saddle atoms. In this case the only orbital invariant (topological and smooth) is the  $R$ -invariant. Suppose that  $K(A, B, C) = 2$ , i.e. that the second components of the  $R$ -invariants coincide. Since the functions  $m(h)$  and  $M(A, B, C, \chi)$  (the first components) change monotonously from  $2 = K(A, B, C)$  to  $+\infty$  with the change of energy  $h$  (resp.  $\chi$ ), we conclude that for any  $h \in (-1/2, 0)$  there exists  $\chi \in (Ag^2/2, Bg^2/2)$  such that  $M(A, B, C, \chi) = m(h)$  and hence the two  $R$ -invariants coincide. This implies the smooth orbital equivalence of the corresponding systems on the isoenergy surfaces  $Q_h^3$  and  $Q_\chi^3$ . The



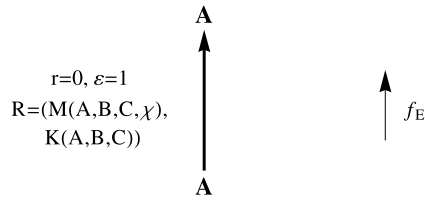


Fig. 16.  $t$ -molecule for the Euler case, zone  $(1)_E$ .

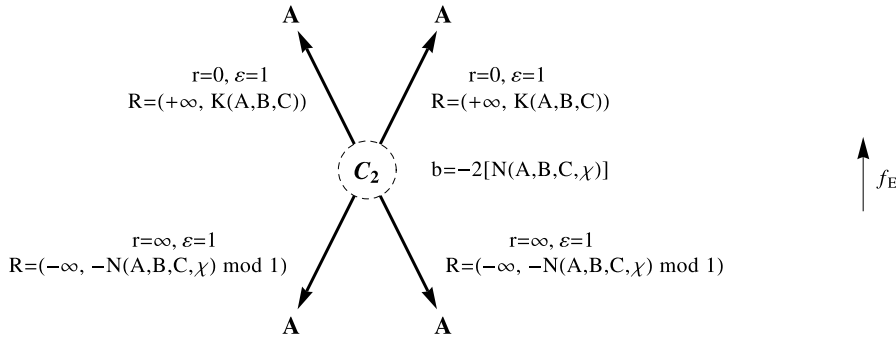


Fig. 17.  $t$ -molecule for the Euler case, zone  $(2)_E$ .

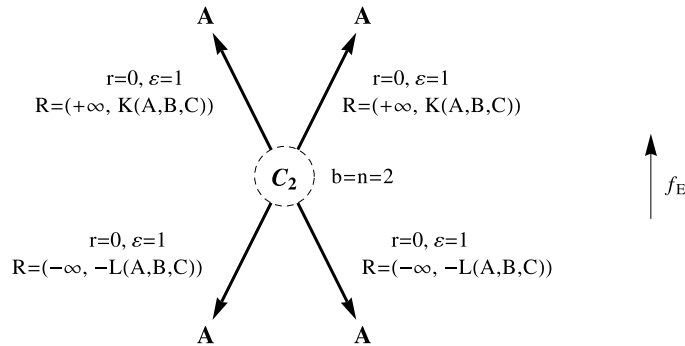


Fig. 18.  $t$ -molecule for the Euler case, zone  $(3)_E$ .

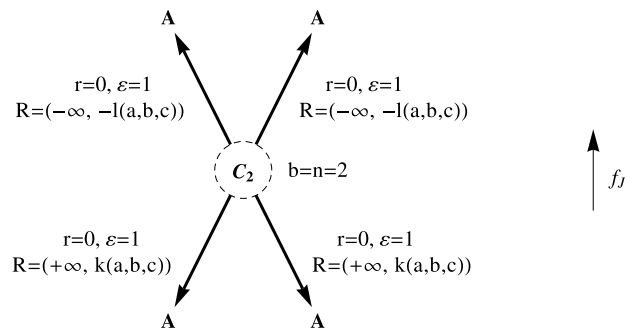


Fig. 19.  $t$ -molecule for the Jacobi problem.

one-parameter family of the Euler systems orbitally equivalent to the Chaplygin system  $v_{Ch}(h)$  is given by the two equations:  $K(A, B, C) = 2$  and  $M(A, B, C, \chi) = m(h)$ . The first equation defines a curve  $B = B(A)$  in the space of the triples  $(A, B, C)$  (see Fig. 6) and the second one establishes the correspondence between the energy levels  $\chi$  and  $h$  under fixed values of  $A, B, C$ .

Consider now the Chaplygin and Euler systems on the manifolds  $D_{Ch}^4 = \{h \in [-1/2, 0)\}$  and  $D_E^4 = \{\chi \in [Ag^2/2, Bg^2/2)\}$ . Each of them is a disconnected pair of four-dimensional disks foliated into concentric isoenergy three-dimensional spheres. The centres of the disks are fixed points of elliptic type (corresponding to the absolute minimum of energy). We have

established the orbital equivalence of the systems on separate leaves of this foliation (i.e. on the 3-spheres). These orbital isomorphisms between 3-spheres can be glued together into a single orbital isomorphism between 4-disks. As a result, we obtain the orbital equivalence of the systems under consideration on the disks  $D_{Ch}^4$  and  $D_E^4$ .

When the values of energy of the Chaplygin and Euler systems belong to zones  $(2)_{Ch}$  and  $(2)_E$ , the coincidence of the  $t$ -molecules means that  $K(A, B, C) = 2$  and  $N(A, B, C, \chi) = n(h) + 1$ . Suppose that the first condition is true. The function  $N(A, B, C, \chi)$  under fixed  $A, B, C$  changes monotonously from  $+\infty$  to  $L(A, B, C)$  as  $\chi$  ranges from  $Bg^2/2$  to  $Cg^2/2$  and the function  $n(h)$  changes monotonously from  $+\infty$  to  $n(1/2)$  as  $h$  ranges from 0 to  $1/2$ . Hence for  $h \in (0, 1/2)$  the equation  $1 + n(h) = N(A, B, C, \chi)$  is solvable with respect to  $\chi$  if and only if  $1 + n(h) > L(A, B, C)$ . It was shown in [20, Theorem 3.3] that the possible values of the invariant  $L(A, B, C)$  belong to the interval  $(1, +\infty)$ . Therefore the equation  $1 + n(h) = N(A, B, C, \chi)$  is solvable for any  $h \in (0, 1/2)$  and the corresponding values of the parameters  $A, B, C$  satisfy the condition  $1 < L(A, B, C) < 1 + n(h)$ . The one-parameter family of the Euler systems orbitally equivalent to the Chaplygin system  $v_{Ch}(h)$  is given by the two equations:  $K(A, B, C) = 2$  and  $n(h) = N(A, B, C, \chi)$ . As in the previous case, the first equation defines a curve  $B = B(A)$  in the space of the triples  $(A, B, C)$  and the second one establishes the correspondence between the energy levels under fixed  $A, B, C$ . But now the interval of possible values for the parameter  $A$  is shorter. Indeed, from the condition  $1 < L(A, B, C) < 1 + n(h)$  we find  $0 < A < A_0(h)$  where

$$A_0(h) = \frac{2}{3} \left( 2 - \sqrt{1 + \frac{3}{(1 + n(h))^2}} \right) < \frac{2}{3}.$$

Finally, consider the case of “large” energies. Suppose that  $h \geq h_0$  so that the rotation functions in the Chaplygin case have no critical points. Here we have the following conditions for the coincidence of the  $t$ -molecules for the Chaplygin and Euler systems:  $K(A, B, C) = 1/n(h)$ ,  $L(A, B, C) = 2$ . As shown in [20, Theorem 3.3], the region of possible values for the pair of invariants  $K, L$  is an infinite rectangle  $(0, +\infty) \times (1, +\infty)$  and for each point from this rectangle there exists a unique up to proportionality triple  $(A, B, C)$  with such invariants  $K, L$ . Since  $n(h) > 0$ , we conclude that for each value of energy  $h \geq h_0$  there exists a unique (up to proportionality) Euler type system  $v_E(A, B, C)$  topologically orbitally equivalent to the Chaplygin system  $v_{Ch}(h)$ . This Euler system is given by the parameters

$$A(h) = \frac{3}{2 \left( 2 + \sqrt{1 + 3(n(h))^2} \right)}, \quad B(h) = 1 - \frac{1}{4(1 - A(h))}, \quad C = 1.$$

The one-parameter family of the Euler systems orbitally equivalent to the Chaplygin systems is represented in Fig. 7 with  $A_1 = A(h_0) = 3 \left[ 2 \left( 2 + \sqrt{1 + 3(n(h_0))^2} \right) \right]^{-1}$ . Owing to Theorem 6, there also exists a unique up to similarity ellipsoid with squares of semi-axes  $a, b, c$  such that the Jacobi system  $v_J(a, b, c)$  is topologically orbitally equivalent to the Chaplygin system  $v_{Ch}(h)$ . Note that in this case orbital isomorphisms cannot be extended on the four-dimensional neighbourhoods of the isoenergy surfaces. This is explained by the fact that the function  $n(h)$  is strictly monotonous whereas the corresponding elements of the  $R$ -invariants in the Euler case and the Jacobi problem are constant with respect to the energy.

**Step 3. Proof of the smooth orbital non-equivalence for zones (2) and (3).** To prove that the orbital equivalence for energy zones 2 and 3 cannot be made smooth, we follow the idea used in [21, Section 5]. Namely, we add to the  $t$ -molecules of the systems under consideration an additional invariant of the smooth orbital equivalence and decide whether the obtained composed invariants are the same for some values of the parameters. This additional invariant ( $\Lambda$ -invariant) is defined in the following way.

Consider the rotation function  $\rho(t)$  on some edge of the molecule. Suppose that the Liouville tori  $T^2(t)$  tend to a singular leaf as  $t \rightarrow t_0$  and that this singular leaf contains a closed hyperbolic trajectory of the system. In particular, this means that the corresponding vertex of the molecule adjacent to the given edge is a saddle atom. Let  $\rho$  be related with the admissible basis corresponding to this saddle atom. Then, as  $t \rightarrow t_0$ , the following representation holds ([21, Proposition 6 in Section 5]):

$$\rho(t) = \Lambda \ln |t - t_0| + q(t),$$

where  $q(t)$  is a function continuous at  $t_0$ . The number  $\Lambda$  does not change under  $C^1$ -smooth orbital isomorphisms, i.e. it is an invariant of the smooth orbital equivalence (see [21]).

For the computation of the  $\Lambda$ -invariant, we use the following simple statement.

**Lemma 9** (Lemma in Section 5 of [21]). As  $t \rightarrow b$ , we have

$$\int_a^b \frac{g(u) du}{\sqrt{(u-b)(u-t)}} = -g(b) \ln |t - b| + c(t),$$

where the function  $c(t)$  is continuous at  $t = b$ .

Using formulae (6)–(9) and Lemma 9, it is easy to calculate the  $\Lambda$ -invariant for the Chaplygin case:

$$\Lambda_{(2),Ch} = \left( 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-2ht^2)}} \right)^{-1} \quad \text{for zone } (2)_{Ch} \text{ and}$$

$$\Lambda_{(3),Ch} = \left( 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(2h-t^2)}} \right)^{-1} \quad \text{for zone } (3)_{Ch}.$$

For the Euler case and the Jacobi problem we have

$$\Lambda_E = \frac{1}{\pi} \frac{B}{\sqrt{(C-B)(B-A)}}, \quad \Lambda_J = \frac{\sqrt{\frac{b}{(c-b)(b-a)}}}{\int_0^{+\infty} \Phi(u, b) du},$$

where the function  $\Phi(u, t)$  was defined above.

Consider energy zones  $(2)_{Ch}$  and  $(2)_E$ . The coincidence of the  $t$ -molecules and the  $\Lambda$ -invariants for the Chaplygin and Euler cases requires that

$$\frac{A}{\sqrt{(C-A)(B-A)}} = 2 \quad \text{and}$$

$$\frac{B}{\sqrt{(C-B)(B-A)}} = \pi \left( 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-2ht^2)}} \right)^{-1}.$$

But this is impossible since for zone  $(2)_{Ch}$  we have  $h \in (0, 1/2)$  and

$$\pi \left( 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-2ht^2)}} \right)^{-1}$$

$$< \pi \left( 2 \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-2ht^2)}} \right)^{-1} \Big|_{h=0} = 1 < 2$$

whereas  $\frac{B}{\sqrt{(C-B)(B-A)}} > \frac{A}{\sqrt{(C-A)(B-A)}}.$

Similar inequalities show that in the case of zone  $(3)_{Ch}$  the coincidence of the  $t$ -molecule and the  $\Lambda$ -invariant for the Chaplygin system with those for the Euler and Jacobi systems is also not possible.

**Step 4. Proof of the topological non-conjugacy.** When we want to prove that the systems under consideration are not conjugate (topologically), we need to add some “exact” invariants to their  $t$ -molecules. Following [22], we consider the periods of the three singular closed trajectories of the systems. Two of these trajectories are elliptic and correspond to atom A, the third one is hyperbolic and corresponds to atom  $C_2$ . Some observations similar to those in step 3 show that for any values of the parameters the coincidence of the  $t$ -molecules and the three new invariants introduced above for the systems in question is not the case.

We shall confine ourselves to demonstrating how the described idea works for the case of “large” energies. We denote by  $(t^1 : t^2 : t^3)$  the triple of the periods of the closed trajectories introduced above. The numbers  $t^1$  and  $t^3$  are the periods of the elliptic trajectories and  $t^2$  is the period of the hyperbolic one. The triple  $(t^1 : t^2 : t^3)$  is viewed up to proportionality (this means that we do not take into account the energy level).

For the Euler system  $v_E(A, B, C)$  (the case of zone  $(3)_E$ ) and for the Jacobi system  $v_J(a, b, c)$  the invariants  $(t^1 : t^2 : t^3)$  were calculated in [22] and take the following values:

$$(t_E^1 : t_E^2 : t_E^3) = \frac{1}{\sqrt{C}} : \frac{1}{\sqrt{B}} : \frac{1}{\sqrt{A}};$$

$$(t_J^1 : t_J^2 : t_J^3) = t(a, b) : t(a, c) : t(b, c) \quad \text{where } t(\alpha, \beta) = \int_0^{2\pi} \sqrt{\alpha \cos^2 \varphi + \beta \sin^2 \varphi} d\varphi.$$

Consider now zone  $(3)_{Ch}$  of energy values for the Chaplygin case. The number  $t_{Ch}^1$  corresponds to the critical trajectories whose image under the momentum mapping belongs to the ray  $\{K = 0, H > 1/2\}$  in the bifurcation diagram (see Fig. 4). According to [29, Lemma 4], such critical circles are given in the space  $\mathbb{R}(\mathbf{s}, \mathbf{r})$  by

$$s_1 = s_2 = r_3 = 0, \quad r_1^2 + r_2^2 = 1, \quad s_3 = \pm \sqrt{h + 1/2 - r_1^2}.$$

In this case Eq. (1) implies

$$\begin{cases} \dot{r}_1 = 2s_3 r_2, \\ \dot{r}_2 = -2s_3 r_1. \end{cases}$$

It can be easily found from here that

$$t_{Ch}^1 = \frac{1}{2} \int_0^{2\pi} \frac{d\varphi}{\sqrt{h + 1/2 - \cos^2 \varphi}}.$$

Analogously, we obtain that

$$t_{Ch}^2 = \int_0^{2\pi} \frac{d\varphi}{\sqrt{2h - \cos^2 \varphi}} \quad \text{and} \quad t_{Ch}^3 = \int_0^{2\pi} \frac{d\varphi}{\sqrt{2h + 1 - \cos^2 \varphi}}.$$

Thus, the triple of invariants  $(t^1 : t^2 : t^3)$  for the Chaplygin case is

$$(t_{Ch}^1 : t_{Ch}^2 : t_{Ch}^3) = \left( \frac{1}{2} \int_0^{2\pi} \frac{d\varphi}{\sqrt{h + 1/2 - \cos^2 \varphi}} : \int_0^{2\pi} \frac{d\varphi}{\sqrt{2h - \cos^2 \varphi}} : \int_0^{2\pi} \frac{d\varphi}{\sqrt{2h + 1 - \cos^2 \varphi}} \right).$$

It is easy to see that  $t_{Ch}^2 > t_{Ch}^3$  whereas  $t_E^2 < t_E^3$  and  $t_J^2 < t_J^3$ . Therefore the triple  $(t_{Ch}^1 : t_{Ch}^2 : t_{Ch}^3)$  cannot be equal to the analogous triples for the Euler case and the Jacobi problem (independently of the values of the parameters  $A, B, C, a, b, c$ ). This fact shows that for the case of “large” energies any Chaplygin type system is not topologically conjugate with some Euler type system or some Jacobi type system. For the case of energy zones (1) and (2) this result can be obtained in a similar way.

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