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The minimal dimensions of faithful representations for Heisenberg Lie superalgebras

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ABSTRACT

This paper aims to determine the minimal dimensions and super-dimensions of faithful representations for Heisenberg Lie superalgebras over an algebraically closed field of characteristic zero.

1. Introduction

Throughout $\mathbb{F}$ is an algebraically closed field of characteristic zero and all vector spaces and algebras are over $\mathbb{F}$ and of finite dimensions.

Ado's theorem says that every finite dimensional Lie (super)algebra has a finite-dimensional faithful representation [1]. Let $\mathfrak{g}$ be a Lie (super)algebra and write

$$\mu(\mathfrak{g}) = \min\{\dim V \mid V \text{ is a faithful } \mathfrak{g}\text{-module}\}.$$

It is in general difficult to determine $\mu(\mathfrak{g})$. The earliest result is that $\mu(\mathfrak{g}) = \lceil 2\sqrt{\dim \mathfrak{g} - 1}\rceil$ for an abelian Lie algebra $\mathfrak{g}$, which is due to Schur for $\mathbb{F} = \mathbb{C}$ and to Jacobson for arbitrary $\mathbb{F}$ (see also [2] for a simple proof due to Mirzakhani). In 1998 Burde concluded that $\mu(\mathfrak{h}_m) = m + 2$ for Heisenberg Lie algebra $\mathfrak{h}_m$ of dimension $2m + 1$ [3]. In 2008 Burde and Moens established an explicit formula of $\mu(\mathfrak{g})$ for semi-simple and reductive Lie algebras [4]. In 2009 Cagliero and Rojas obtained a formula $\mu(\mathfrak{h}_{m,p})$ for the current Heisenberg Lie algebra $\mathfrak{h}_{m,p}$ [5]. One can also find the formula $\mu(\mathfrak{J})$ for a Jordan algebra $\mathfrak{J}$ with the trivial multiplication [6].

However, very little is known about the function μ for Lie superalgebras. In 2012 Liu and Wang determined $\mu(\mathfrak{g}) = \lceil 2\sqrt{\dim \mathfrak{g}}\rceil$ for any purely odd Lie superalgebra $\mathfrak{g}$ [6] and it remains open to determine $\mu(\mathfrak{g})$ for an abelian Lie superalgebra $\mathfrak{g}$ with nontrivial even part. In this paper, we shall determine the minimal (super-)dimensions of the faithful representations for Heisenberg Lie superalgebras.

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A two-step nilpotent Lie superalgebra with 1-dimensional center is called a Heisenberg Lie superalgebra. Then Heisenberg Lie superalgebras split into the following two types according to the parities of their centers [7]. Write $\mathfrak{h}_{m,n}$ for the Heisenberg Lie superalgebra with 1-dimensional even center $\mathbb{F}z$, which has a $\mathbb{Z}_2$-homogeneous basis

$$(u_1, \ldots, u_m, v_1, \ldots, v_m; z \mid w_1, \ldots, w_n)$$

with multiplication given by

$$[u_i, v_i] = -[v_i, u_i] = z = [w_j, w_j], \quad i = 1, \ldots, m, j = 1, \ldots, n,$$

the remaining brackets being zero. Hereafter $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the group of order 2.

Write $\mathfrak{h}_n$ for the Heisenberg Lie superalgebra with 1-dimensional odd center $\mathbb{F}z$, which has a $\mathbb{Z}_2$-homogeneous basis

$$(v_1, \ldots, v_n \mid z; w_1, \ldots, w_n)$$

with multiplication given by

$$[v_i, w_i] = z = -[w_i, v_i], \quad i = 1, \ldots, n,$$

the remaining brackets being zero.

Both $\mathfrak{h}_{m,n}$ and $\mathfrak{h}_n$ are nilpotent. Note that $\mathfrak{h}_{m,0}$ is a Heisenberg Lie algebra and $\mathfrak{h}_{0,n}$ is isomorphic to the Heisenberg Lie superalgebra considered in [1, p. 18], whose even part coincides with 1-dimension center. However, the Heisenberg Lie superalgebras with odd centers, $\mathfrak{h}_n$, have no analogs in Lie algebras. We should also mention that Hegazi studied representations of the Heisenberg Lie superalgebras of even center, $\mathfrak{h}_{m,n}$, and tried to find a finite-dimensional faithful representation of $\mathfrak{h}_{m,n}$ [8, Section 3].

Throughout this paper, subalgebras and (sub)modules of Lie superalgebras are assumed to be $\mathbb{Z}_2$-graded. Hereafter we write $\mathfrak{g}$ for $\mathfrak{h}_{m,n}$ or $\mathfrak{h}_n$. A main result of this paper is that

$$\mu(\mathfrak{g}) = \begin{cases} m + \lceil n/2 \rceil + 2 & \mathfrak{g} = \mathfrak{h}_{m,n} \\ n + 2 & \mathfrak{g} = \mathfrak{h}_n. \end{cases}$$

To formulate the super-dimensions of the faithful representations, write for $i \in \{0, 1\}$,

$$\mu_i(\mathfrak{g}) = \min\{\dim V_{\bar{i}} \mid V \text{ is a faithful } \mathfrak{g}\text{-module}\};$$
$$\mu_i^*(\mathfrak{g}) = \min\{\dim V \mid V \text{ is a faithful } \mathfrak{g}\text{-module with } \dim V_{\bar{i}} = \mu_i(\mathfrak{g})\}.$$

In this paper we also determine the values $\mu_i(\mathfrak{g})$ and $\mu_i^*(\mathfrak{g})$.

2. Minimal dimensions

Since Engel's theorem holds for Lie superalgebras, as in Lie algebra case [3, Lemma 1], we have

Lemma 2.1. *Let L be a nilpotent Lie superalgebra with a 1-dimensional center $\mathbb{F}z$. Then a representation $\lambda : L \to \mathfrak{gl}(V)$ is faithful if and only if z acts nontrivially.*

Proof. The "only if" part is obvious. Suppose z acts nontrivially. If $\ker(\lambda) \neq 0$, then Engel's theorem ensures that $\ker(\lambda)$ contains a nonzero element killed by L and hence $\ker(\lambda)$ contains the center $\mathbb{F}z$, showing that $\rho(z) = 0$, a contradiction. □

Let

$$\zeta(\mathfrak{g}) = \max\{\dim \mathfrak{a} \mid \mathfrak{a} \text{ is an abelian subalgebra of } \mathfrak{g} \text{ not containing the center of } \mathfrak{g}\}.$$

Let $\sqrt{-1}$ denote a fixed root of the equation $x^2 = -1$ in $\mathbb{F}$. We have

Lemma 2.2. *Let $\mathfrak{a}$ be an abelian subalgebra not containing z of $\mathfrak{g}$ and having dimension $\zeta(\mathfrak{g})$. Then*

- *for $\mathfrak{g} = \mathfrak{h}_{m,n}$, the super-dimension $(\dim \mathfrak{a}_{\bar{0}}, \dim \mathfrak{a}_{\bar{1}})$ must be $(m, \lfloor n/2 \rfloor)$;*
- *for $\mathfrak{g} = \mathfrak{h}_n$, the super-dimension $(\dim \mathfrak{a}_{\bar{0}}, \dim \mathfrak{a}_{\bar{1}})$ has $n + 1$ possibilities:*

$$(i, n - i), \quad i = 0, \ldots, n.$$

In particular,

$$\zeta(\mathfrak{g}) = \begin{cases} m + \lfloor n/2 \rfloor & \mathfrak{g} = \mathfrak{h}_{m,n} \\ n & \mathfrak{g} = \mathfrak{h}_n. \end{cases}$$

Proof. Since $\mathfrak{a}$ does not contain the center $\mathbb{F}z$, there is a $\mathbb{Z}_2$-graded subspace $\mathfrak{k}$ containing $\mathfrak{a}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathbb{F}z$. Let $B : \mathfrak{k} \times \mathfrak{k} \to \mathbb{F}$ be the form determined by $[x, y] = B(x, y)z$ for all $x, y \in \mathfrak{k}$. It is clear that B is bilinear and non-degenerate. Since $\mathfrak{a}$ is abelian, $B(x, y) = 0$ for all $x, y \in \mathfrak{a}$. Therefore, $\mathfrak{a}$ is a B-isotropic subspace of $\mathfrak{k}$. It follows that $\dim \mathfrak{a} \le \frac{\dim \mathfrak{k}}{2} = \frac{\dim \mathfrak{g}-1}{2}$.

Suppose $\mathfrak{g} = \mathfrak{h}_{m,n}$. Then $\dim \mathfrak{a} \le m + \lfloor n/2 \rfloor$. Let $\mathfrak{b}$ be the subspace spanned by

$$u_1, u_2, \dots, u_m, \qquad w_1 + \sqrt{-1}w_2, \qquad w_3 + \sqrt{-1}w_4, \dots, w_{n-1} + \sqrt{-1}w_n$$

if n is even and by

$$u_1, u_2, \dots, u_m, \qquad w_1 + \sqrt{-1}w_2, \qquad w_3 + \sqrt{-1}w_4, \dots, w_{n-2} + \sqrt{-1}w_{n-1}$$

if n is odd. One can check that $\mathfrak{b}$ is an abelian subalgebra of dimension $m + \lfloor n/2 \rfloor$ and $\mathfrak{b}$ does not contain z. Hence, $\zeta(\mathfrak{g}) = \dim \mathfrak{a} = m + \lfloor n/2 \rfloor$.

Clearly, $\mathfrak{a}_{\bar 0}$ is a B-isotropic subspace of $\mathfrak{k}_{\bar 0}$ and $\mathfrak{a}_{\bar 1}$ is a B-isotropic subspace of $\mathfrak{k}_{\bar 1}$. Since $B|_{\mathfrak{k}_{\bar 0}\times\mathfrak{k}_{\bar 0}}$ and $B|_{\mathfrak{k}_{\bar 1}\times\mathfrak{k}_{\bar 1}}$ are non-degenerate, we have $\dim \mathfrak{a}_{\bar 0} \le m$, $\dim \mathfrak{a}_{\bar 1} \le \lfloor n/2 \rfloor$. Note that $\dim \mathfrak{a} = m + \lfloor n/2 \rfloor$. It follows that $\dim \mathfrak{a}_{\bar 0} = m$, $\dim \mathfrak{a}_{\bar 1} = \lfloor n/2 \rfloor$.

Suppose $\mathfrak{g} = \mathfrak{h}_n$. Then $\dim \mathfrak{a} \le n$. Let $\mathfrak{b}'$ be the subspace spanned by $v_1, v_2, \dots, v_n$. Clearly, $\mathfrak{b}'$ is an abelian subalgebra of dimension n of $\mathfrak{h}_n$ and $\mathfrak{b}'$ does not contain z. Hence, $\zeta(\mathfrak{g}) = \dim \mathfrak{a} = n$. From the definition of $\mathfrak{h}_n$, one may easily find abelian subalgebras not containing z and having the indicated super-dimension $(i, n-i)$ with $i = 0, \dots, n$. □

Lemma 2.3. *Let V be a faithful $\mathfrak{g}$-module. Then there exists a nonzero homogeneous element v_0 in V such that $zv_0 \neq 0$. Moreover, let ρ_{v_0} be the linear mapping defined by*

$$\rho_{v_0} : \mathfrak{g} \longrightarrow V, \qquad x \longmapsto xv_0$$

and let $\mathfrak{a} = \ker(\rho_{v_0})$ and $V_0 = \mathrm{im}(\rho_{v_0})$. Then $\mathfrak{a}$ is an abelian subalgebra not containing z and if $\dim \mathfrak{a} = \zeta(\mathfrak{g})$, then $v_0 \notin V_0$.

Proof. Lemma 2.1 ensures that there exists a nonzero homogeneous element v_0 in V such that $zv_0 \neq 0$. It follows that $\mathfrak{a}$ does not contain z. Since ρ_{v_0} is homogeneous, $\mathfrak{a}$ is a $\mathbb{Z}_2$-graded subspace of $\mathfrak{g}$. For $x, y \in \mathfrak{a}$, it is obvious that $[x, y] \in \mathfrak{a} \cap \mathbb{F}z = 0$ and it follows that $\mathfrak{a}$ is an abelian subalgebra.

Suppose $\dim \mathfrak{a} = \zeta(\mathfrak{g})$. Assume in contrary that $v_0 \in V_0$. Then there exists an $x \in \mathfrak{g}_{\bar 0}$ such that $xv_0 = v_0$, since v_0 is a nonzero homogeneous element of V. Clearly, $(\mathfrak{h}_{m,n})_{\bar 0}$ is a solvable Lie algebra. Since $[u_i, v_i] = z$, by Lie's theorem, z acts nilpotently on V. For $\mathfrak{h}_n$, z is odd. Therefore, $x \notin \mathbb{F}z$. Moreover, it is clear that $x \notin \mathfrak{a}$. Then by the maximality of $\mathfrak{a}$, we have $[x, \mathfrak{a}] \neq 0$. There must be some $y \in \mathfrak{a}$ such that $[x, y] = z$. Since $x \in \mathfrak{g}_{\bar 0}$, we have

$$zv_0 = [x, y]v_0 = x(yv_0) - y(xv_0) = 0,$$

using that $yv_0 = 0$ and $xv_0 = v_0$. This is a contradiction. Hence $v_0 \notin V_0$. □

Proposition 2.4. *Let $\mathfrak{g} = \mathfrak{h}_{m,n}$ or $\mathfrak{h}_n$. Then*

$$\mu(\mathfrak{g}) \ge \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1.$$

That is,

- $\mu(\mathfrak{h}_{m,n}) \ge m + \lceil n/2 \rceil + 2$;
- $\mu(\mathfrak{h}_n) \ge n + 2$.

Proof. Assume that $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation. Let $v_0, \mathfrak{a}, V_0$ be as in Lemma 2.3. By Lemmas 2.2 and 2.3, we have

$$\dim V \ge \dim V_0 = \dim \mathfrak{g} - \dim \mathfrak{a} \ge \dim \mathfrak{g} - \zeta(\mathfrak{g}).$$

If $\dim V_0 \ge \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$, we are done. Suppose $\dim V_0 = \dim \mathfrak{g} - \zeta(\mathfrak{g})$. Then $\dim \mathfrak{a} = \zeta(\mathfrak{g})$. By Lemma 2.3, we have $v_0 \notin V_0$. Therefore,

$$\dim V \ge \dim V_0 + 1 = \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1.$$

That is, $\mu(\mathfrak{h}_{m,n}) \ge m + \lceil n/2 \rceil + 2$; $\mu(\mathfrak{h}_n) \ge n + 2$. □

Theorem 2.5. *We have*

$$\mu(\mathfrak{g}) = \begin{cases} m + \lceil n/2 \rceil + 2 & \mathfrak{g} = \mathfrak{h}_{m,n} \\ n + 2 & \mathfrak{g} = \mathfrak{h}_n. \end{cases}$$

Proof. By Proposition 2.4, it is enough to establish a faithful representation of the desired dimension for $\mathfrak{g}$. Consider the even linear mapping

$$\pi : \mathfrak{h}_{m,n} \longrightarrow \mathfrak{gl}(m+2 \mid \lceil n/2 \rceil)$$

given by

$$\begin{aligned} &\pi(u_i) = e_{1,i+1}, \qquad \pi(v_i) = e_{i+1,m+2}, \qquad \pi(z) = e_{1,m+2}, \\ &\pi(w_{2k-1}) = \frac{1}{2} e_{m+2+k,m+2} + e_{1,m+2+k}, \\ &\pi(w_{2k}) = \frac{\sqrt{-1}}{2} e_{m+2+k,m+2} - \sqrt{-1} e_{1,m+2+k}, \end{aligned}$$

where $1 \le i \le m$, $1 \le 2k, 2k-1 \le n$. Under π, an element of $\mathfrak{h}_{m,n}$,

$$\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{m} b_i v_i + cz + \sum_{j=1}^{n} d_j w_j \quad (a_i, b_i, c, d_j \in \mathbb{F}) \tag{2.1}$$

is presented as

$$\left(\begin{array}{cccccc|cccc} 0 & a_1 & a_2 & \cdots & a_m & c & d_{1,2} & d_{3,4} & \cdots & d_{n-1,n} \\ & & & & & b_1 & & & & \\ & & & & & b_2 & & & & \\ & & & & & \vdots & & & & \\ & & & & & b_m & & & & \\ & & & & & 0 & & & & \\ \hline & & & & & \widetilde{d}_{1,2} & & & & \\ & & & & & \widetilde{d}_{3,4} & & & & \\ & & & & & \vdots & & & & \\ & & & & & \widetilde{d}_{n-1,n} & & & & \end{array}\right) \quad (n \text{ even}) \tag{2.2}$$

or

$$\left(\begin{array}{cccccc|ccccc} 0 & a_1 & a_2 & \cdots & a_m & c & d_{1,2} & d_{3,4} & \cdots & d_{n-2,n-1} & d_n \\ & & & & & b_1 & & & & & \\ & & & & & b_2 & & & & & \\ & & & & & \vdots & & & & & \\ & & & & & b_m & & & & & \\ & & & & & 0 & & & & & \\ \hline & & & & & \widetilde{d}_{1,2} & & & & & \\ & & & & & \widetilde{d}_{3,4} & & & & & \\ & & & & & \vdots & & & & & \\ & & & & & \widetilde{d}_{n-2,n-1} & & & & & \\ & & & & & \frac{1}{2} d_n & & & & & \end{array}\right) \quad (n \text{ odd}), \tag{2.3}$$

where $d_{i,i+1} = d_i - \sqrt{-1} d_{i+1}$, $\widetilde{d}_{i,i+1} = \frac{1}{2}(d_i + \sqrt{-1} d_{i+1})$. It is routine to verify that π is a faithful representation of dimension $m + \lceil n/2 \rceil + 2$.

Let us consider the even linear mapping

$$\pi' : \mathfrak{h}_n \longrightarrow \mathfrak{gl}(n+1 \mid 1)$$

given by

$$\begin{aligned} &\pi'(v_i) = e_{1,i+1}, \\ &\pi'(z) = e_{1,n+2}, \qquad \pi'(w_i) = e_{i+1,n+2}, \end{aligned}$$

where $1 \le i \le n$. Under π', an element of $\mathfrak{h}_n$,

$$\sum_{i=1}^{n} a_i v_i + cz + \sum_{i=1}^{n} b_i w_i \quad (a_i, c, b_i \in \mathbb{F}) \tag{2.4}$$

is presented as

$$\left(\begin{array}{ccccc|c} 0 & a_1 & a_2 & \cdots & a_n & c \\ & & & & & b_1 \\ & & & & & b_2 \\ & & & & & \vdots \\ & & & & & b_n \\ \hline & & & & & 0 \end{array}\right).$$

It is routine to verify that π' is a faithful representation of dimension $n+2$. $\square$

3. Super-dimensions

In this section we discuss the super-dimensions of the faithful representations for Heisenberg Lie superalgebras. We first establish a technical lemma, for which we shall use a result due to Burde [3]: the formula $\mu(L)$ for Heisenberg Lie algebras.

Lemma 3.1. *Let V be a faithful module of $\mathfrak{h}_{m,n}$. Let v_0 be as in Lemma 2.3. If v_0 is even, then $\dim V_{\bar{0}} \geq m+2$; if v_0 is odd, then $\dim V_{\bar{1}} \geq m+2$.*

Proof. Note that $(\mathfrak{h}_{m,n})_{\bar{0}}$ is a Heisenberg Lie algebra. Obviously, $V_{\bar{0}}$ is a module of the Lie algebra $(\mathfrak{h}_{m,n})_{\bar{0}}$. If v_0 is even, then $v_0 \in V_{\bar{0}}$. Since $zv_0 \neq 0$, $V_{\bar{0}}$ is a faithful module of $(\mathfrak{h}_{m,n})_{\bar{0}}$ by Lemma 2.1. According to the minimal dimensions of faithful representations for Heisenberg Lie algebras [3], we have $\dim V_{\bar{0}} \geq \mu((\mathfrak{h}_{m,n})_{\bar{0}}) = m+2$. Similarly, if v_0 is odd, then $V_{\bar{1}}$ is a faithful module of $(\mathfrak{h}_{m,n})_{\bar{0}}$ and hence $\dim V_{\bar{1}} \geq m+2$. $\square$

Theorem 3.2. *Suppose V is a faithful $\mathfrak{g}$-module of the minimal dimension $\mu(\mathfrak{g})$. Then*

- *For $\mathfrak{h}_{m,n}$, the super-dimension $(\dim V_{\bar{0}}, \dim V_{\bar{1}})$ has 2 possibilities:*

  $$(m+2, \lceil n/2 \rceil), \qquad (\lceil n/2 \rceil, m+2).$$

- *For $\mathfrak{h}_n$, the super-dimension $(\dim V_{\bar{0}}, \dim V_{\bar{1}})$ has $n+1$ possibilities:*

  $$(i+1, n-i+1), \quad i = 0, \ldots, n.$$

Proof. Let v_0, $\mathfrak{a}$, V_0 be as in Lemma 2.3. Since $\mathfrak{a}$ does not contain the center $\mathbb{F}z$, there exists a subalgebra $\mathfrak{a}'$ containing z such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}'$. Since $\dim V = \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$ and $\dim \mathfrak{a} \leq \zeta(\mathfrak{g})$, we have $\dim \mathfrak{g} - \zeta(\mathfrak{g}) \leq \dim V_0 \leq \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$. It is enough to consider the following two cases.

Case 1: $\dim V_0 = \dim \mathfrak{g} - \zeta(\mathfrak{g})$. Then $\dim \mathfrak{a} = \zeta(\mathfrak{g})$ and Lemma 2.3 yields $v_0 \notin V_0$. Then we have $\dim \mathfrak{a}' = \dim \mathfrak{g} - \zeta(\mathfrak{g})$. Since $\dim V = \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$, it easy to see that V has an $\mathbb{F}$-basis

$$\{v_0, xv_0 \mid x \text{ runs over a homogeneous basis of } \mathfrak{a}'\}. \tag{3.1}$$

For $\mathfrak{g} = \mathfrak{h}_{m,n}$, by Lemma 2.2 we have $\dim \mathfrak{a}_{\bar{0}} = m$ and $\dim \mathfrak{a}_{\bar{1}} = \lfloor n/2 \rfloor$. Hence, $\dim \mathfrak{a}'_{\bar{0}} = m+1$, $\dim \mathfrak{a}'_{\bar{1}} = \lceil n/2 \rceil$. By (3.1), if $v_0 \in V_{\bar{0}}$ then $\dim V_{\bar{0}} = m+2$ and $\dim V_{\bar{1}} = \lceil n/2 \rceil$; if $v_0 \in V_{\bar{1}}$, then $\dim V_{\bar{0}} = \lceil n/2 \rceil$ and $\dim V_{\bar{1}} = m+2$.

For $\mathfrak{g} = \mathfrak{h}_n$, by Lemma 2.2, $\dim \mathfrak{a}_{\bar{0}} = i$ and $\dim \mathfrak{a}_{\bar{1}} = n-i$, $i = 0, \ldots, n$. Hence, $\dim \mathfrak{a}'_{\bar{0}} = i$ and $\dim \mathfrak{a}'_{\bar{1}} = n+1-i$, $i = 0, \ldots, n$. Therefore we have $\dim V_{\bar{0}} = i+1$ and $\dim V_{\bar{1}} = n+1-i$, where $i = 0, \ldots, n$.

Case 2: $\dim V_0 = \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$. Then $\dim \mathfrak{a} = \zeta(\mathfrak{g}) - 1$ and $\dim \mathfrak{a}' = \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$. Since $\dim V = \dim \mathfrak{g} - \zeta(\mathfrak{g}) + 1$, one sees that V has an $\mathbb{F}$-basis

$$\{xv_0 \mid x \text{ runs over a homogeneous basis of } \mathfrak{a}'\}. \tag{3.2}$$

For $\mathfrak{g} = \mathfrak{h}_{m,n}$, clearly, $\dim \mathfrak{a}'_{\bar{0}} = m+i$ and $\dim \mathfrak{a}'_{\bar{1}} = \lceil n/2 \rceil + 2 - i$ for some $i \in \{1, 2\}$. By (3.2), if $v_0 \in V_{\bar{0}}$, then $\dim V_{\bar{0}} = m+i$ and $\dim V_{\bar{1}} = \lceil n/2 \rceil + 2 - i$; if $v_0 \in V_{\bar{1}}$, then $\dim V_{\bar{0}} = \lceil n/2 \rceil + 2 - i$ and $\dim V_{\bar{1}} = m+i$ for some $i \in \{1, 2\}$. By Lemma 3.1, it must be $i = 2$.

For $\mathfrak{g} = \mathfrak{h}_n$, then $\dim \mathfrak{a} = n-1$. Clearly, $\dim \mathfrak{a}' = n+2$, $\dim \mathfrak{a}'_{\bar{0}} = i+1$ and $\dim \mathfrak{a}'_{\bar{1}} = n+1-i$, $i = 0, \ldots, n-1$. Therefore, we have either $\dim V_{\bar{0}} = i+1$ and $\dim V_{\bar{1}} = n+1-i$, or $\dim V_{\bar{0}} = n+1-i$ and $\dim V_{\bar{1}} = i+1$, for some $i \in \{0, \ldots, n-1\}$.

Up to now, we have shown that:

- For $\mathfrak{h}_{m,n}$, the super-dimension $(\dim V_{\bar{0}}, \dim V_{\bar{1}})$ has at most 2 possibilities:

  $$(m+2, \lceil n/2 \rceil), \qquad (\lceil n/2 \rceil, m+2).$$

- For $\mathfrak{h}_n$, the super-dimension $(\dim V_{\bar{0}}, \dim V_{\bar{1}})$ has at most $n+1$ possibilities:

  $$(i+1, n-i+1), \quad i = 0, \ldots, n.$$

Next let us realize the faithful representations of the super-dimensions indicated above. For $\mathfrak{h}_{m,n}$, (2.2) and (2.3) give a minimal faithful representation of $\mathfrak{h}_{m,n}$ with super-dimension $(m+2, \lceil n/2 \rceil)$. Consider the even linear mapping

$$\pi : \mathfrak{h}_{m,n} \longrightarrow \mathfrak{gl}(\lceil n/2 \rceil \mid m+2)$$

given by

$$\pi(u_i) = e_{\lceil n/2\rceil+1,\lceil n/2\rceil+i+1}, \qquad \pi(v_i) = e_{\lceil n/2\rceil+i+1,\lceil n/2\rceil+m+2}, \qquad \pi(z) = e_{\lceil n/2\rceil+1,\lceil n/2\rceil+m+2},$$
$$\pi(w_{2k-1}) = \frac{1}{2} e_{k,\lceil n/2\rceil+m+2} + e_{\lceil n/2\rceil+1,k}, \qquad \pi(w_{2k}) = \frac{\sqrt{-1}}{2} e_{k,\lceil n/2\rceil+m+2} - \sqrt{-1} e_{\lceil n/2\rceil+1,k},$$

where $1 \le i \le m$, $1 \le 2k, 2k-1 \le n$. Under π, an element of form (2.1) is presented as

$$\left(\begin{array}{cccc|cccccc}
 & & & & & & & & & \widetilde{d}_{1,2}\\
 & & & & & & & & & \widetilde{d}_{3,4}\\
 & & & & & & & & & \vdots\\
 & & & & & & & & & \widetilde{d}_{n-1,n}\\
\hline
d_{1,2} & d_{3,4} & \cdots & d_{n-1,n} & 0 & a_1 & a_2 & \cdots & a_m & c\\
 & & & & & & & & & b_1\\
 & & & & & & & & & \vdots\\
 & & & & & & & & & b_m\\
 & & & & & & & & & 0
\end{array}\right) \quad (n \text{ even}) \tag{3.3}$$

or

$$\left(\begin{array}{ccccc|ccccc}
 & & & & & & & & & \widetilde{d}_{1,2}\\
 & & & & & & & & & \widetilde{d}_{3,4}\\
 & & & & & & & & & \vdots\\
 & & & & & & & & & \widetilde{d}_{n-1,n}\\
 & & & & & & & & & \frac{1}{2}d_n\\
\hline
d_{1,2} & d_{3,4} & \cdots & d_{n-1,n} & d_n & 0 & a_1 & \cdots & a_m & c\\
 & & & & & & & & & b_1\\
 & & & & & & & & & b_2\\
 & & & & & & & & & \vdots\\
 & & & & & & & & & b_m\\
 & & & & & & & & & 0
\end{array}\right) \quad (n \text{ odd}), \tag{3.4}$$

where $d_{i,j} = d_i - \sqrt{-1} d_j$, $\widetilde{d}_{i,j} = \frac{1}{2}(d_i + \sqrt{-1} d_j)$. It is routine to verify that π is a faithful representation with super-dimension $(\lceil n/2 \rceil, m+2)$.

For $0 \le r \le n$, let us consider the even linear mapping

$$\pi' : \mathfrak{h}_n \longrightarrow \mathfrak{gl}(r+1 \mid n-r+1)$$

given by

$$\pi'(v_i) = e_{1,i+1}, \qquad \pi'(v_j) = -e_{j+1,n+2},$$
$$\pi'(z) = e_{1,n+2}, \qquad \pi'(w_k) = e_{k+1,n+2}, \qquad \pi'(w_l) = e_{1,l+1},$$

where $1 \le i, k \le r$ and $r+1 \le j, l \le n$. Under π', an element (2.4) of $\mathfrak{h}_n$ is presented as

$$\left(\begin{array}{cccc|cccc}
0 & a_1 & \cdots & a_r & b_{r+1} & \cdots & b_n & c\\
 & & & & & & & b_1\\
 & & & & & & & \vdots\\
 & & & & & & & b_r\\
\hline
 & & & & & & & -a_{r+1}\\
 & & & & & & & \vdots\\
 & & & & & & & -a_n\\
 & & & & & & & 0
\end{array}\right). \tag{3.5}$$

It is routine to verify that π' is a faithful representation with super-dimension $(r+1, n-r+1)$ for all $r = 0, \ldots, n$. $\square$

Recall that for $i \in \{0, 1\}$,

$$\mu_i(\mathfrak{g}) = \min\{\dim V_{\bar{i}} \mid V \text{ is a faithful } \mathfrak{g}\text{-module}\},$$
$$\mu_i^*(\mathfrak{g}) = \min\{\dim V \mid V \text{ is a faithful } \mathfrak{g}\text{-module with } \dim V_{\bar{i}} = \mu_i(\mathfrak{g})\}.$$

Theorem 3.3. *We have*

$$\mu_0(\mathfrak{g}) = \mu_1(\mathfrak{g}) = \begin{cases} \min\{m+2, \lceil n/2\rceil\} & \mathfrak{g} = \mathfrak{h}_{m,n} \\ 1 & \mathfrak{g} = \mathfrak{h}_n \end{cases}$$

and

$$\mu_0^*(\mathfrak{g}) = \mu_1^*(\mathfrak{g}) = \begin{cases} m + \lceil n/2\rceil + 2 & \mathfrak{g} = \mathfrak{h}_{m,n} \\ n+2 & \mathfrak{g} = \mathfrak{h}_n. \end{cases}$$

Proof. Let (λ, V) be a faithful representation of $\mathfrak{g}$. Evidently,

$$\mu_0^*(\mathfrak{g}) \geq \mu(\mathfrak{g}); \qquad \mu_1^*(\mathfrak{g}) \geq \mu(\mathfrak{g}). \tag{3.6}$$

Keep the notations in Lemma 2.3. As in the proof of Theorem 3.2, there exists a subalgebra $\mathfrak{a}'$ containing z such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}'$. By Lemma 2.2, $\dim \mathfrak{a}' \geq \dim \mathfrak{g} - \zeta(\mathfrak{g})$. Hence, by Lemma 2.3(4), if v_0 is even, then $\dim V_{\bar{1}} \geq \dim \mathfrak{a}'_{\bar{1}}$; if v_0 is odd, then $\dim V_{\bar{0}} \geq \dim \mathfrak{a}'_{\bar{1}}$.

Let $\mathfrak{g} = \mathfrak{h}_{m,n}$. By Lemma 2.2, we have $\dim \mathfrak{a}'_{\bar{1}} \geq \lceil n/2\rceil$. So, if v_0 is even, then $\dim V_{\bar{1}} \geq \lceil n/2\rceil$; if v_0 is odd, then $\dim V_{\bar{0}} \geq \lceil n/2\rceil$. By Lemma 3.1, if v_0 is even, then $\dim V_{\bar{0}} \geq m+2$; if v_0 is odd, then $\dim V_{\bar{1}} \geq m+2$. Therefore, $\dim V_{\bar{0}} \geq \min\{m+2, \lceil n/2\rceil\}$ and $\dim V_{\bar{1}} \geq \min\{m+2, \lceil n/2\rceil\}$. Since (2.2) and (2.3) define a faithful representation of $\mathfrak{h}_{m,n}$ with super-dimension $(m+2, \lceil n/2\rceil)$, and (3.3) and (3.4) define a faithful representation of $\mathfrak{h}_{m,n}$ with super-dimension $(\lceil n/2\rceil, m+2)$, we have

$$\mu_0(\mathfrak{h}_{m,n}) = \mu_1(\mathfrak{h}_{m,n}) = \min\{m+2, \lceil n/2\rceil\}.$$

It follows from (3.6) that

$$\mu_0^*(\mathfrak{h}_{m,n}) = \mu_1^*(\mathfrak{h}_{m,n}) = m + \lceil n/2\rceil + 2.$$

Let $\mathfrak{g} = \mathfrak{h}_n$. By Lemma 2.2, we have $\dim \mathfrak{a}'_{\bar{1}} \geq 1$. So, if v_0 is even, then $\dim V_{\bar{1}} \geq 1$; if v_0 is odd, then $\dim V_{\bar{0}} \geq 1$. On the other hand, if v_0 is even, then $v_0 \in V_{\bar{0}}$ and $\dim V_{\bar{0}} \geq 1$; if v_0 is odd, then $v_0 \in V_{\bar{1}}$ and $\dim V_{\bar{1}} \geq 1$. Then by (3.5), we have

$$\mu_0(\mathfrak{h}_n) = \mu_1(\mathfrak{h}_n) = 1.$$

It follows from (3.6) that

$$\mu_0^*(\mathfrak{h}_n) = \mu_1^*(\mathfrak{h}_n) = n+2. \quad \square$$

Remark 3.4. Let L be a Lie superalgebra and Π the parity functor of the category of $\mathbb{Z}_2$-graded vector spaces. It is well known that if V is an L-module, then so is $\Pi(V)$ with respect to the original module action. Therefore, in general we have

$$\mu_0(L) = \mu_1(L), \qquad \mu_0^*(L) = \mu_1^*(L).$$

This fact may be used to shorten the proofs of Theorems 3.2 and 3.3.

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