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Spherical T-duality II: An infinity of spherical T-duals for non-principal SU(2)-bundles

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ABSTRACT

Recently we initiated the study of spherical T-duality for spacetimes that are principal SU(2)-bundles (Bouwknegt et al., to appear). In this paper, we extend spherical T-duality to spacetimes that are oriented *non-principal* SU(2)-bundles. There are several interesting new examples in this case and a new phenomenon appearing in the non-principal case is the existence of infinitely many spherical T-duals.

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1. Introduction

T-duality for pairs consisting of a circle bundle, together with a degree 3 H-flux, was originally studied in detail in [1–4] with contributions by several others later. In string theory, T-dual pairs are distinct compactification manifolds that cannot be distinguished by any experiment, which is the notion of isomorphism relevant in physics. This equivalence in physics implies the isomorphisms of a number of other mathematical structures, such as Courant algebroids [5], generalized complex structures [5] and twisted K-theory [1], see also [6,7]. It turns out that all of these structures are physically relevant.

Recently we initiated the study of spherical T-duality for principal SU(2)-bundles in [8]. Let P be a principal SU(2)-bundle over M and H a 7-cocycle on P,

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & P \\ & & \pi \downarrow \\ & & M \end{array} \tag{1.1}$$

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Principal SU(2)-bundles over a compact oriented four dimensional manifold M are classified by $H^4(M;\mathbb{Z}) \cong \mathbb{Z}$ via the 2nd Chern class $c_2(P)$. This can be seen using the well known isomorphism, $H^4(M;\mathbb{Z}) \cong [M, S^4] \cong \mathbb{Z}$ and noting that there is a canonical principal SU(2)-bundle $P \to S^4$, known as the Hopf bundle, whose 2nd Chern class is the generator of $H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$. The orientation of M and SU(2) imply that π_* is a canonical isomorphism $H^7(P;\mathbb{Z}) \cong H^4(M;\mathbb{Z}) \cong \mathbb{Z}$. The dual principal SU(2)-bundle,

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & \widehat{P} \\ & & \Big\downarrow{\scriptstyle \widehat{\pi}} \\ & & M \end{array} \tag{1.2}$$

is defined by $c_2(\widehat{P}) = \pi_* H$ while the dual 7-cocycle $\widehat{H} \in H^7(\widehat{P})$ satisfies $c_2(P) = \widehat{\pi}_* \widehat{H}$ by the isomorphism $\widehat{\pi}_* : H^7(\widehat{P};\mathbb{Z}) \cong H^4(M;\mathbb{Z}) \cong \mathbb{Z}$. We proved that this spherical T-duality map induces degree-shifting isomorphisms between the real and integral twisted cohomologies of P and $\widehat{P}$ and also between the 7-twisted K-theories.

Beyond dimension 4 the situation becomes more complicated as not all integral 4-cocycles of M are realized as the 2nd Chern class of a principal SU(2)-bundle $\pi : P \to M$, and multiple bundles can have the same $c_2(P)$. We refer the reader to [8] for precise statements of spherical T-duality in the higher dimensional case. In string theory, spherical T-duality does not imply an isomorphism of two compactifications, but only of some of the data of these compactifications corresponding to certain conserved charges [8]. The isomorphism of conserved charges is implied by the fact that spherical T-duality induces an isomorphism on twisted cohomology.

In this paper, we extend spherical T-duality to (oriented) non-principal SU(2)-bundles. While principal SU(2)-bundles correspond to unit sphere bundles of quaternionic line bundles, (oriented) non-principal SU(2)-bundles correspond to unit sphere bundles of rank 4 oriented real Riemannian vector bundles. A striking new phenomenon in the non-principal case is that when the base M is a compact oriented simply-connected 4 dimensional manifold, and given an oriented non-principal SU(2)-bundle E with H a 7-cocycle on E, then for each integer, we will show that there is an *infinite* lattice of spherical T-duals with 7-cocycle flux over M, in stark contrast to the case of principal SU(2)-bundles as described above. One reason is because in the non-principal bundle case, the Euler class does not determine (oriented) non-principal SU(2)-bundles when the base is as above, but in addition the 2nd Stiefel–Whitney class and the Pontryagin class are also needed for the classification. However, it is only the Euler class (and its transgression) that is needed in the Gysin sequence and also to prove the isomorphisms of 7-twisted integral cohomologies, and in addition the 2nd Stiefel–Whitney class is needed to prove the isomorphisms of 7-twisted K-theories. Since we no longer insist that the SU(2) bundles be principal, there are now a number of interesting new examples available, such as the spherical T-duality of Aloff–Wallach spaces with 7-cocycle flux described in Section 4. One relevant class of compactifications in string theory which has been of great interest over the past 30 years is that of Sasakian manifolds, described in Section 7.

2. Oriented non-principal SU(2)-bundles

We begin with a few definitions. Recall that

Definition 2.1. A *principal* SU(2)*-bundle* P over a space M,

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & P \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & M \end{array} \tag{2.1}$$

is a map $P \xrightarrow{\pi} M$, where P is a space equipped with a free action of SU(2) for which π is the projection onto the orbit space $M = P/\mathrm{SU}(2)$ and the map is locally trivial, in the sense that for any $x \in X$, there exists an open neighborhood U containing x for which one has a commuting diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times \mathrm{SU}(2) \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array} \tag{2.2}$$

An (oriented) *non-principal* SU(2)*-bundle* E over M, on the other hand, is a fiber bundle over M with fiber SU(2) and structure group $\mathrm{Diff}_+(\mathrm{SU}(2))$, the orientation preserving diffeomorphisms of SU(2). Since the inclusion $\mathrm{SO}(4) = \mathrm{Isom}_+(\mathrm{SU}(2)) \hookrightarrow \mathrm{Diff}_+(\mathrm{SU}(2))$ of the orientation preserving isometries of SU(2) into the orientation preserving diffeomorphisms of SU(2) is a homotopy equivalence by Hatcher's theorem [9], it follows that the quotient space $\mathrm{SO}(4) \setminus \mathrm{Diff}_+(\mathrm{SU}(2))$ is contractible, therefore *there is no loss of generality* in assuming that a general (oriented) non-principal SU(2) bundle E is a fiber bundle over M with fiber SU(2) and structure group SO(4).

Recall that $\mathrm{SU}(2) = \mathrm{Sp}(1)$ is canonically identified with the group of unit quaternions, and consider the surjective morphism

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(4) \tag{2.3}$$

sending a pair of unit quaternions (z, w) to the map $\mathbb{R}^4 \to \mathbb{R}^4, x \mapsto zxw$ using quaternion multiplication, where we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. The kernel of this map consists of $(1, 1)$ and $(-1, -1)$ generating $\mathbb{Z}_2$, since $\mathbb{R}$ is central in $\mathbb{H}$. Therefore

$$\mathrm{SO}(4) \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2. \tag{2.4}$$

It follows that $\pi_3(\mathrm{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$. We can choose an explicit identification as follows: given $(p, q) \in \mathbb{Z}$, we have a map $\phi_{(p,q)} : \mathrm{Sp}(1) \to \mathrm{SO}(4)$ which sends a unit quaternion u to the map

$$\phi_{(p,q)}(u) : \mathbb{R}^4 \to \mathbb{R}^4, \qquad \phi_{(p,q)}(u)(x) = u^p x u^q. \tag{2.5}$$

Recall that principal $\mathrm{SO}(4)$-bundles Q over S^4 are classified by $\pi_3(\mathrm{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$. Let $(p, q) \in \pi_3(\mathrm{SO}(4))$ define a principal $\mathrm{SO}(4)$-bundle $Q(p, q)$ over S^4, by using the clutching function $\phi_{(p,q)}$. By [10], the first Pontryagin class $p_1(Q(p, q)) = 2(p-q)$ and by [11], the Euler class $e(Q(p, q)) = p + q$. Therefore the first Pontryagin class together with the Euler class determine all principal $\mathrm{SO}(4)$-bundles over S^4. An example of a principal $\mathrm{SO}(4)$-bundle over S^4 is the oriented frame bundle

$$\begin{array}{ccc} \mathrm{SO}(4) & \longrightarrow & \mathrm{SO}(5) \\ & & \pi \big\downarrow \\ & & S^4 \end{array} \tag{2.6}$$

We calculate (p, q) for this example. The Euler class of this bundle is just the Euler class of S^4 and is equal to 2. So $p + q = 2$. The Pontryagin class of this bundle is just the Pontryagin class of S^4, which is zero since S^4 is the oriented boundary of the 5 dimensional ball. So $2(p - q) = 0$. Therefore $p = 1 = q$, and $Q(1, 1) = \mathrm{SO}(5)$.

We summarize the above, studied in [12].

Lemma 2.2. *Any principal* $\mathrm{SO}(4)$*-bundle* Q *over* S^4 *is classified by the invariants* $p_1(Q)$ *and* $e(Q)$ *in* $H^4(S^4; \mathbb{Z}) \cong \mathbb{Z}$.

Let M be a compact, oriented, 4 dimensional manifold. Then for each pair of integers (p, q), define a principal $\mathrm{SO}(4)$-bundle $Q_M(p, q)$ over M as follows. For any degree one map $f : M \to S^4$, set $Q_M(p, q) = f^*(Q(p, q))$. Then the Euler number of $Q_M(p, q)$ is $e(Q_M(p, q)) = p+q \in \mathbb{Z} \cong \mathrm{H}^4(M; \mathbb{Z})$ and the 1st Pontryagin number of $Q_M(p, q)$ is $p_1(Q_M(p, q)) = 2(p-q) \in \mathbb{Z} \cong \mathrm{H}^4(M; \mathbb{Z})$.

However these are not the only principal $\mathrm{SO}(4)$-bundles over M. We assume that M is simply-connected here. A simple obstruction theory argument (cf. [13]) shows that a principal $\mathrm{SO}(4)$-bundle Q over $M \setminus B(R, x_0)$, where $B(R, x_0)$ is a small $4D$-ball of radius R centered at $x_0 \in M$, is classified by the second Stiefel–Whitney class $w_2(Q) \in \mathrm{H}^2(M \setminus B(R, x_0); \mathbb{Z}_2) \cong \mathrm{H}^2(M; \mathbb{Z}_2)$. Now the restriction of Q to the boundary $\partial(M \setminus B(R, x_0)) = S^3$ is determined by a map $h : S^2 \to \mathrm{SO}(4)$. But any such map is null homotopic since $\pi_2(\mathrm{SO}(4)) \cong 0$, therefore the restriction of Q to the boundary S^3 is trivializable. Gluing the bundle over the $4D$ ball $B(R, x_0)$ corresponds to fixing trivializations over the boundary S^3 and also prescribing a map $S^3 \mapsto \mathrm{SO}(4)$. This fixes the Pontryagin class p_1 and the Euler class e as shown earlier. Then the main Theorem in [13] (with useful clarifications in [12,14]) shows that

Lemma 2.3. *Any principal* $\mathrm{SO}(4)$*-bundle* Q *over a simply-connected, compact, oriented* 4 *dimensional manifold* M *is classified by the invariants* $w_2(Q) \in H^2(M; \mathbb{Z}_2)$, $p_1(Q)$ *and* $e(Q)$ *in* $\mathrm{H}^4(M; \mathbb{Z}) \cong \mathbb{Z}$. *There is no constraint on* $w_2(Q)$. *Let* $b \in H^2(M; \mathbb{Z})$ *be such that* $w_2(Q) = b \bmod 2$. *Then* $p_1(Q) = 2(p - q) + \beta$ *where* $b \cup b = \beta \in \mathbb{Z} \cong \mathrm{H}^4(M; \mathbb{Z})$, *and* $e(Q) = p + q$ *for some* $p, q \in \mathbb{Z}$. *Hence we can parametrize* $Q = Q_M(\beta, p, q)$, *where* β, p, q *are integers satisfying the constraints above.*

Let us examine the special case when $M = \mathbb{C}\mathrm{P}^2$, which is also studied in [14]. A principal $\mathrm{SO}(4)$-bundle Q over $\mathbb{C}\mathrm{P}^2 \setminus B(R, x_0)$, where $B(R, x_0)$ is a small open $4D$-ball of radius R centered at $x_0 \in \mathbb{C}\mathrm{P}^2$, is classified by its second Stiefel–Whitney class $w_2(Q) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2 \setminus B(R, x_0); \mathbb{Z}_2) \cong \mathrm{H}^2(\mathbb{C}\mathrm{P}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Now $\mathbb{C}\mathrm{P}^2 \setminus B(R, x_0)$ retracts onto $\mathbb{C}\mathrm{P}^1$, therefore Q is determined by a smooth map $f : S^1 \to \mathrm{SO}(4)$ whose homotopy class is $w_2(Q)$, which we will assume is the *nontrivial* class. The restriction of Q to the boundary $\partial(\mathbb{C}\mathrm{P}^2 \setminus B(R, x_0)) = S^3$ is determined by a map $S^2 \mapsto \mathrm{SO}(4)$. But any such map is null homotopic since $\pi_2(\mathrm{SO}(4)) \cong 0$, therefore the restriction of Q to the boundary S^3 is trivializable. Upon fixing trivializations over the boundary S^3 and also prescribing a map $S^3 \mapsto \mathrm{SO}(4)$, fixes the Pontryagin class p_1 and the Euler class e. Such a map corresponds to a pair of integers (p, q) and we denote the resulting $\mathrm{SO}(4)$-bundle over $\mathbb{C}\mathrm{P}^2$ by $Q_{\mathbb{C}\mathrm{P}^2}(1, p, q)$.

Lemma 2.4. *Any principal* $\mathrm{SO}(4)$*-bundle* Q *over* $\mathbb{C}\mathrm{P}^2$ *is classified by the invariants* $w_2(Q) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2, \mathbb{Z}_2) \cong \mathbb{Z}_2$, $p_1(Q)$ *and* $e(Q)$ *in* $\mathrm{H}^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{Z}$. *More precisely,*

1. *if* $w_2(Q) = 0$, *then in the notation of Lemma 2.3, for each pair of integers* (p, q), $Q \cong Q_{\mathbb{C}\mathrm{P}^2}(0, p, q)$ *whose Euler number* $e(Q_{\mathbb{C}\mathrm{P}^2}(0, p, q)) = p + q \in \mathbb{Z} \cong \mathrm{H}^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$ *and whose* 1*st Pontryagin number is* $p_1(Q_{\mathbb{C}\mathrm{P}^2}(0, p, q)) = 2(p - q) \in \mathbb{Z} \cong \mathrm{H}^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$.
2. *if* $w_2(Q) \neq 0$, *then for each pair of integers* (p, q), $Q \cong Q_{\mathbb{C}\mathrm{P}^2}(1, p, q)$ *as above, whose Euler number* $e(Q_{\mathbb{C}\mathrm{P}^2}(1, p, q)) = p + q \in \mathbb{Z} \cong \mathrm{H}^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$ *and whose* 1*st Pontryagin number is* $p_1(Q_{\mathbb{C}\mathrm{P}^2}(1, p, q)) = 2(p - q) + 1 \in \mathbb{Z} \cong \mathrm{H}^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$.

Consider the example of the oriented frame bundle,

$$\begin{array}{ccc} \mathrm{SO}(4) & \longrightarrow & \mathrm{SO}(\mathbb{C}\mathrm{P}^2) \\ & & \pi\downarrow \\ & & \mathbb{C}\mathrm{P}^2 \end{array} \tag{2.7}$$

over $\mathbb{C}\mathrm{P}^2$. Since $\mathbb{C}\mathrm{P}^2$ is not a spin manifold, it follows that $w_2(\mathrm{SO}(\mathbb{C}\mathrm{P}^2)) \neq 0$. One calculates the Euler number, $e(\mathbb{C}\mathrm{P}^2) = 3$ and the signature, $\mathrm{sign}(\mathbb{C}\mathrm{P}^2) = 1$. By the Atiyah–Singer index theorem, it follows that $p_1(\mathbb{C}\mathrm{P}^2) = 3$. By Lemma 2.4, $3 = p+q$ and $3 = 2(p-q)+1$. Therefore $p = 2$, $q = 1$ and $\mathrm{SO}(\mathbb{C}\mathrm{P}^2) \cong Q_{\mathbb{C}\mathrm{P}^2}(1, 2, 1)$.

We can use Lemma 2.4 to make Lemma 2.3 more explicit. Since M is a simply-connected, compact, oriented 4 dimensional manifold, one uses the universal coefficient theorem and Poincaré duality to see that $\mathrm{H}^2(M;\mathbb{Z})$ is torsion free and $\mathrm{H}^2(M;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^{b_2(M)}$. Given a principal $\mathrm{SO}(4)$-bundle Q over M with $w_2(Q) \neq 0$, then $w_2(Q) = b \bmod 2$ for some $b \in \mathrm{H}^2(M;\mathbb{Z}) \cong [M, \mathbb{C}\mathrm{P}^2]$. So $b : M \to \mathbb{C}\mathrm{P}^2$ and one can pullback the bundles in Lemma 2.4 via b, and show that $Q = b^*(Q_{\mathbb{C}\mathrm{P}^2}(1, p, q))$, for some p, q that are integers as in Lemma 2.3.

When $\dim(M) > 4$, the invariants in Lemma 2.3 do not completely classify principal $\mathrm{SO}(4)$-bundles. However, they do classify these bundles rationally.

3. Examples of oriented non-principal SU(2) bundles

This section contains examples of oriented non-principal $\mathrm{SU}(2)$-bundles that are obtained from principal $\mathrm{SO}(4)$-bundles via the associated bundle construction. Some of the examples can be found in [15], Section 10.6.

Example 1. For each pair of integers (p, q), define a nonprincipal $\mathrm{SU}(2)$-bundle $E(p, q)$ over S^4 as the associated bundle $E(p, q) = Q(p, q) \times_{\mathrm{SO}(4)} \mathrm{SU}(2)$, where $Q(p, q)$ is the principal $\mathrm{SO}(4)$-bundle constructed above Lemma 2.2. In particular, $E(1, 1) = \mathrm{SO}(5) \times_{\mathrm{SO}(4)} \mathrm{SU}(2)$. The Euler number of $E(p, q)$ is $e(E(p, q)) = p + q \in \mathbb{Z} \cong H^4(S^4)$ and the 1st Pontryagin number of $E(p, q)$ is $p_1(E(p, q)) = 2(p - q) \in \mathbb{Z} \cong H^4(S^4)$ [10].

Explicitly, let U_N be the open hemisphere in S^4 with the north pole as center and U_S be the open hemisphere in S^4 with the south pole as center. Consider the trivial bundles $U_N \times \mathrm{SU}(2)$ and $U_S \times \mathrm{SU}(2)$ with transition function $\tilde{g}(x, h) = (x, g(x)^p h g(x)^q)$ for all $(x, h) \in (U_N \cap U_S) \times \mathrm{SU}(2)$. Here $g : U_N \cap U_S \to \mathrm{SU}(2)$ is a degree 1 map, where we observe that $U_N \cap U_S$ is a retraction to S^3. Let $E(p, q)$ be the (a priori) non-principal $\mathrm{SU}(2)$-bundle obtained via the clutching construction. Then $E(p, q)$ is a principal $\mathrm{SU}(2)$-bundle if and only if either $p = 0$ or $q = 0$, $E(p, q)$ is homeomorphic to S^7 if and only if $p + q = \pm 1$. If $p + q = \pm 1$ and $(p - q)^2 - 1 \neq 0 \pmod 7$, then $E(p, q)$ is not diffeomorphic to S^7, and in fact not mutually diffeomorphic. See [12] for details on the homeomorphism and diffeomorphism statements.

Example 2. Let M be a compact, oriented, simply-connected, 4 dimensional manifold. Then for integers β, p, q satisfying the constraints in Lemma 2.3, we have a principal $\mathrm{SO}(4)$-bundle $Q_M(\beta, p, q)$ over M. Define the nonprincipal $\mathrm{SU}(2)$-bundle $E_M(\beta, p, q)$ over M to be the associated bundle $E_M(\beta, p, q) = Q_M(\beta, p, q) \times_{\mathrm{SO}(4)} \mathrm{SU}(2)$. Then the 2nd Stiefel–Whitney class, Euler class and Pontryagin class of $E_M(\beta, p, q)$ coincide with those of $Q_M(\beta, p, q)$ as in Lemma 2.3.

For examples that are possibly more directly relevant to physics, see Section 7.

4. Gysin sequence and construction of spherical T-duals for oriented non-principal SU(2)-bundles

The goals in this section are to state the relevant Gysin sequence for (oriented) non-principal $\mathrm{SU}(2)$-bundles, and then to construct spherical T-duals for oriented non-principal $\mathrm{SU}(2)$-bundles.

Given an (oriented) non-principal $\mathrm{SU}(2)$-bundle $E \xrightarrow{\pi} M$, it comes with a Gysin sequence, Proposition 14.33, [16],

$$\cdots \to \mathrm{H}^p(M;\mathbb{Z}) \xrightarrow{\cup e(E)} \mathrm{H}^{p+4}(M;\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^{p+4}(E;\mathbb{Z}) \xrightarrow{\pi_!} \mathrm{H}^{p+1}(M;\mathbb{Z}) \xrightarrow{\cup e(E)} \cdots \tag{4.1}$$

where $e(E)$ denotes the Euler class of E.

For each pair of integers (p, q), consider the nonprincipal $\mathrm{SU}(2)$-bundle $E(p, q)$ over S^4 with $w_2(E(p, q)) = 0, e(E(p, q)) = p + q$ and $p_1(E(p, q)) = 2(p - q)$. Let $H = h\,\mathrm{vol}$ be a 7-cocycle on $E(p, q)$

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & E(p, q) \\ & & \pi\downarrow \\ & & S^4 \end{array} \tag{4.2}$$

The orientation of S^4 and $\mathrm{SU}(2)$ together with the Gysin sequence in Eq. (4.1) imply that π_* is a canonical isomorphism $\mathrm{H}^7(E(p, q);\mathbb{Z}) \cong \mathrm{H}^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$. Consider the family of spherical T-dual bundles

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & E(\hat{p}, \hat{q}) \\ & & \hat{\pi}\downarrow \\ & & S^4 \end{array} \tag{4.3}$$

with the property that $e(E(\hat{p},\hat{q})) = \hat{p}+\hat{q} = h$ while the dual 7-cocycle $\widehat{H} = \hat{h}\,\mathrm{vol} \in \mathrm{H}^7(E(\hat{p},\hat{q});\mathbb{Z})$ satisfies $e(E(p,q)) = p+q = \hat{h}$ by the isomorphism $\widehat{\pi}_*\colon \mathrm{H}^7(E(\hat{p},\hat{q});\mathbb{Z}) \cong \mathrm{H}^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$.

Thus we see that for each integer $\hat{p}$, there is a spherical T-dual pair $(E(\hat{p}, h-\hat{p}), (p+q)\,\mathrm{vol})$ over S^4 to any fixed spherical pair $(E(p,q), h\,\mathrm{vol})$ over S^4.

More generally, let M be a compact, oriented, simply-connected, 4 dimensional manifold and consider for each triple of integers (β, p, q), consider the nonprincipal SU(2)-bundle

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & E_M(\beta,p,q) \\ & & \pi\big\downarrow \\ & & M \end{array} \tag{4.4}$$

with $w_2(E_M(\beta,p,q)) = b \bmod 2 \neq 0$, $e(E_M(\beta,p,q)) = p+q$ and $p_1(E_M(\beta,p,q)) = 2(p-q)+\beta$, where b, β are as in Lemma 2.3 and Example 2 in Section 3. Arguing as above, the family of spherical T-dual bundles

$$\begin{array}{ccc} \mathrm{SU}(2) & \longrightarrow & E_M(\beta,\hat{p},\hat{q}) \\ & & \widehat{\pi}\big\downarrow \\ & & M \end{array} \tag{4.5}$$

with the property that $e(E_M(\beta,\hat{p},\hat{q})) = \hat{p}+\hat{q} = h$, $p_1(E_M(\beta,\hat{p},\hat{q})) = 2(\hat{p}-\hat{q})+\beta$, $w_2(E_M(\beta,\hat{p},\hat{q})) = w_2(E_M(\beta,p,q))$ while the dual 7-cocycle $\widehat{H} = \hat{h}\,\mathrm{vol} \in \mathrm{H}^7(E_M(\beta,\hat{p},\hat{q});\mathbb{Z})$ satisfies $e(E_M(\beta,\hat{p},\hat{q})) = p+q = \hat{h}$.

Thus we see that for each integer $\hat{p}$, there is a spherical T-dual pair $(E_M(\beta,\hat{p},h-\hat{p}), (p+q)\,\mathrm{vol})$ over M to any fixed spherical pair $(E_M(\beta,p,q), h\,\mathrm{vol})$ over M.

In contrast, recall from [8] that in the case of spherical pairs $(P(q), h)$ where $P(q)$ is a *principal* SU(2)-bundle over a compact, oriented 4 dimensional manifold M with 2nd Chern class $c_2(P(q)) = q \in \mathbb{Z} \cong \mathrm{H}^4(M;\mathbb{Z})$ and $h \in \mathbb{Z} \cong \mathrm{H}^7(P(q);\mathbb{Z})$ an integral 7-cocycle on $P(q)$, there is a *unique* spherical T-dual pair $(P(h), q)$. The reason for the family of spherical T-duals in the case of non-principal SU(2)-bundles over M is that a non-principal SU(2)-bundle is no longer determined by just its Euler class (and its second Stiefel–Whitney class). In fact, as we have seen, there is a lattice of non-principal SU(2)-bundles over M having the same second Stiefel–Whitney class w_2 and Euler class e.

Example 4.1. The Aloff–Wallach space $W_{k,l}$ [17], defined as being the homogeneous space $\mathrm{SU}(3)/T_{k,l}$, where the circle subgroup $T_{k,l} = \mathrm{diag}(z^k, z^l, z^{-(k+l)})$, $|z| = 1$, is a non-principal S^3-bundle over $\mathbb{C}\mathrm{P}^2$ iff $|k+l| = 1$. We have $W_{p,1-p} \cong S_{-1,p(p-1)}$, where $S_{p,q} = E_{\mathbb{C}\mathrm{P}^2}(1,p,q)$, in our notation above, and $e(S_{a,b}) = a-b$, $p_1(S_{a,b}) = 2(a+b)+1$ [14]. Hence

$$e(W_{p,1-p}) = -(p^2-p+1), \qquad p_1(W_{p,1-p}) = 2p(p-1)-1, \qquad w_2(W_{p,1-p}) = 1. \tag{4.6}$$

Note that for $(k,l) = (p, 1-p)$:

$$k^2+l^2+kl = p^2+(p-1)^2-p(p-1) = p^2-p+1, \tag{4.7}$$

consistent with $H^4(W_{k,l},\mathbb{Z}) \cong Z_{|k^2+l^2+kl|}$ [17]. Note that $(W_{p,1-p}, h)$ is dual to any $S_{\hat{p},\hat{q}}$ with $h = \hat{p}-\hat{q}$. In particular, to be dual to an Aloff–Wallach space $W_{\hat{p},1-\hat{p}}$ we need $\hat{p}(\hat{p}-1)-(1+h) = 0$, i.e.

$$\hat{p} = \frac{1}{2}\left(1 \pm \sqrt{1-4(1+h)}\right). \tag{4.8}$$

After parametrizing these values of h by integers $\hat{p}$, we find the duality between $(W_{p,1-p}, h = -(\hat{p}^2-\hat{p}+1))$ and $(W_{\hat{p},1-\hat{p}}, \hat{h} = -(p^2-p+1))$. This gives examples of cases where 3-Sasakian manifolds are spherical T-dual to other 3-Sasakian manifolds. See also the last section for a further discussion of 3-Sasakian manifolds.

5. Isomorphism of integral 7-twisted cohomologies and K-theories for spherical T-dual pairs

When M is a compact, oriented, simply-connected, 4 dimensional manifold, we will prove in this section that the spherical T-duality map induces parity changing isomorphisms between the integral 7-twisted cohomologies of $(E_M(\beta,p,q), h\,\mathrm{vol})$ and $(E_M(\beta,\hat{p},h-\hat{p}), (p+q)\,\mathrm{vol})$ for each integer $\hat{p}$.

We begin by considering the case of S^4 and the non-principal SU(2)-bundle $E(p,q)$ over it. The 7-twisted cohomology $\mathrm{H}^{even/odd}(E(p,q), h\,\mathrm{vol};\mathbb{Z})$ is defined as the cohomology of the $\mathbb{Z}_2$-graded complex $\left(\mathrm{H}^{even/odd}(E(p,q);\mathbb{Z}), h\mathrm{vol}\right)$. Using the Gysin sequence in Eq. (4.1) to calculate the cohomology groups $\mathrm{H}^{even/odd}(E(p,q);\mathbb{Z})$, we obtain for $p+q \neq 0$

$$\begin{aligned} &\mathrm{H}^j(E(p,q);\mathbb{Z}) \cong 0, \quad j \neq 0,4,7, \\ &\mathrm{H}^4(E(p,q);\mathbb{Z}) \cong \mathbb{Z}_{p+q}, \\ &\mathrm{H}^7(E(p,q);\mathbb{Z}) \cong \mathbb{Z} \cong \mathrm{H}^0(E(p,q);\mathbb{Z}). \end{aligned} \tag{5.1}$$

It follows that for $p+q\neq 0$, $h\neq 0$ one has

$$\begin{aligned}&\mathrm{H}^{even}(E(p,q),h\mathrm{vol};\mathbb{Z})\cong\mathbb{Z}_{p+q},\\&\mathrm{H}^{odd}(E(p,q),h\mathrm{vol};\mathbb{Z})\cong\mathbb{Z}_{h}.\end{aligned}\tag{5.2}$$

Therefore, by our explicit computation above, for each integer $\hat{p}$, there exists an isomorphism of 7-twisted cohomology groups over the integers with a parity change,

$$\begin{aligned}&\mathrm{H}^{even}(E(p,q),h\mathrm{vol};\mathbb{Z})\cong\mathrm{H}^{odd}(E(\hat{p},h-\hat{p}),(p+q)\,\mathrm{vol};\mathbb{Z}),\\&\mathrm{H}^{odd}(E(p,q),h\mathrm{vol};\mathbb{Z})\cong\mathrm{H}^{even}(E(\hat{p},h-\hat{p}),(p+q)\,\mathrm{vol};\mathbb{Z}).\end{aligned}\tag{5.3}$$

A choice of transgression of the Euler class gives an explicit isomorphism, as outlined in the next section.

More generally, let M be a simply-connected compact, oriented, 4 dimensional manifold, and consider the non-principal $\mathrm{SU}(2)$-bundle $E_M(\beta,p,q)$ over M as in Lemma 2.3 and Example 2 in Section 3, together with the 7-cocycle $H=h\mathrm{vol}$ on $E_M(\beta,p,q)$. Again, use the Gysin sequence of Eq. (4.1) to calculate the cohomology groups $\mathrm{H}^{even/odd}(E_M(\beta,p,q);\mathbb{Z})$, we obtain for $p+q\neq 0$

$$\begin{aligned}&\mathrm{H}^{j}(E_M(\beta,p,q);\mathbb{Z})\cong\mathrm{H}^{4-j}(M;\mathbb{Z}),\quad j=0,1,2,3,\\&\mathrm{H}^{4}(E_M(\beta,p,q);\mathbb{Z})\cong\mathbb{Z}_{p+q}\oplus\mathrm{H}^{1}(M;\mathbb{Z}),\\&\mathrm{H}^{7-j}(E_M(\beta,p,q);\mathbb{Z})\cong\mathrm{H}^{4-j}(M;\mathbb{Z}),\quad j=0,1,2,3.\end{aligned}\tag{5.4}$$

It follows that for $p+q\neq 0$, $h\neq 0$ one has

$$\begin{aligned}&\mathrm{H}^{even}(E_M(\beta,p,q),h\mathrm{vol};\mathbb{Z})\cong\mathbb{Z}_{p+q}\oplus\mathrm{H}^{2}(M;\mathbb{Z})\oplus\mathrm{H}^{1}(M;\mathbb{Z})\oplus\mathrm{H}^{3}(M;\mathbb{Z}),\\&\mathrm{H}^{odd}(E_M(\beta,p,q),h\mathrm{vol};\mathbb{Z})\cong\mathbb{Z}_{h}\oplus\mathrm{H}^{2}(M;\mathbb{Z})\oplus\mathrm{H}^{1}(M;\mathbb{Z})\oplus\mathrm{H}^{3}(M;\mathbb{Z}).\end{aligned}\tag{5.5}$$

Therefore by our explicit computation above, for each integer $\hat{p}$, there is an isomorphism of 7-twisted cohomology groups over the integers with a parity change,

$$\begin{aligned}&\mathrm{H}^{even}(E_M(\beta,p,q),h\mathrm{vol};\mathbb{Z})\cong\mathrm{H}^{odd}(E_M(\beta,\hat{p},h-\hat{p}),(p+q)\,\mathrm{vol};\mathbb{Z}),\\&\mathrm{H}^{odd}(E_M(\beta,p,q),h\mathrm{vol};\mathbb{Z})\cong\mathrm{H}^{even}(E_M(\beta,\hat{p},h-\hat{p}),(p+q)\,\mathrm{vol};\mathbb{Z}).\end{aligned}\tag{5.6}$$

The proof of the isomorphism, up to an extension problem, between 7-twisted K-theory and twisted cohomology proceeds similarly to Section 6.1.2 in [8] in the case of spherical pairs $(E_M(\beta,p,q),h\mathrm{vol})$,

$$\begin{aligned}&\mathrm{H}^{even}(E_M(\beta,p,q),h\mathrm{vol};\mathbb{Z})\cong\mathrm{K}^{0}(E_M(\beta,p,q),h\mathrm{vol}),\\&\mathrm{H}^{odd}(E_M(\beta,p,q),h\mathrm{vol};\mathbb{Z})\cong\mathrm{K}^{1}(E_M(\beta,p,q),h\mathrm{vol}).\end{aligned}\tag{5.7}$$

Therefore we conclude that there is an isomorphism of 7-twisted K-theories for spherical T-dual pairs with a parity change,

$$\begin{aligned}&\mathrm{K}^{0}(E_M(\beta,p,q),h\mathrm{vol})\cong\mathrm{K}^{1}(E_M(\beta,\hat{p},h-\hat{p}),(p+q)\,\mathrm{vol}),\\&\mathrm{K}^{1}(E_M(\beta,p,q),h\mathrm{vol})\cong\mathrm{K}^{0}(E_M(\beta,\hat{p},h-\hat{p}),(p+q)\,\mathrm{vol}).\end{aligned}\tag{5.8}$$

We now give more details. For brevity we will set $E=E_M(\beta,p,q)$, the coefficient rings will always be $\mathbb{Z}$. The 7-twisted K-theory can be constructed using a two step spectral sequence with differentials $d_1=Sq^3$ and $d_2=h\mathrm{vol}\cup$. The first differential acts trivially on $\mathrm{H}^0(E)$, $\mathrm{H}^1(E)$ and $\mathrm{H}^2(E)$ as $Sq^3:\mathrm{H}^k(E)\to\mathrm{H}^{k+3}(E)$ annihilates classes of dimension less than 3. Similarly it annihilates $\mathrm{H}^5(E)$, $\mathrm{H}^6(E)$ and $\mathrm{H}^7(E)$ because there are no nontrivial classes of dimension greater than 7. The image of $Sq^3\mathrm{H}^4(E)$ is a $\mathbb{Z}_2$-torsion element of $\mathrm{H}^7(E)$ but E is oriented so there are no such nontrivial elements.

In [8] it was shown that $Sq^3\mathrm{H}^3(E)$ is trivial using the Gysin sequence for principal bundles and applying the Künneth theorem to the case in which the characteristic class c_2 vanishes. As the S^3-bundle is no longer necessarily principal in the present note, the characteristic class which appears in the Gysin sequence is the Euler class $p+q$ and when it vanishes the bundle is not necessarily trivial, therefore the Künneth theorem may not be applied. However in the present case we have assumed that M and therefore E is simply-connected. By Poincaré duality $\mathrm{H}^6(E)=0$ and so again $Sq^3\mathrm{H}^3(E)=0$. Therefore the first differential $d_1=Sq^3$ acts trivially on $\mathrm{H}^\bullet(E)$. The cohomology of the second differential $d_2=h\mathrm{vol}\cup$ then is, up to extension problem, isomorphic to the 7-twisted K-theory. Therefore, for brevity restricting our attention to the nontrivial case $h\neq 0$, the 7-twisted K-groups are

$$\begin{aligned}&K^0(E,h)\cong\mathrm{H}^2(E)\oplus\mathrm{H}^4(E)\oplus\mathrm{H}^6(E)\cong\mathrm{H}^2(M)\oplus\mathbb{Z}_{p+q}\oplus\mathrm{H}^3(M)\\&K^1(E,h)\cong\mathrm{H}^1(E)\oplus\mathrm{H}^3(E)\oplus\mathrm{H}^5(E)\oplus\mathbb{Z}_h\cong\mathrm{H}^3(M)\oplus\mathrm{H}^2(M)\oplus\mathbb{Z}_h.\end{aligned}\tag{5.9}$$

Therefore spherical T-duality interchanges $p+q$ and h and so exchanges K^0 and K^1 as claimed.

6. T-duality isomorphisms via differential geometry

6.1. Euler class and Pfaffian form

Let S be a nonprincipal S^3-bundle over a simply-connected compact, oriented, 4 dimensional manifold M. Then $S = S(E)$ is the unit sphere bundle of rank 4 oriented real Riemannian vector bundle E over M. Let ∇ be a connection on E preserving the Riemannian metric on E, having curvature Ω_∇. Since Ω_∇ is a 2-form on M with values in skew-symmetric endomorphisms, one can consider the Pfaffian, $\mathrm{Pfaff}(\Omega_\nabla)$, which is a degree 4 differential form on M. Then the Euler class $e(E) = e(S) \in \mathrm{H}^4(M;\mathbb{Z})$ is represented by $\frac{1}{4\pi^2}\mathrm{Pfaff}(\Omega_\nabla)$.

6.2. Transgression of the Euler class

The pullback of the Euler class $\pi^*(e(S)) = 0$ where $\pi : S \to M$ is the projection. The transgression of the Euler class is a degree 3 differential form $\tau(\nabla)$ on S such that $\pi^*(\mathrm{Pfaff}(\Omega_\nabla)) = d\tau(\nabla)$. S.S. Chern was the first to define such a form, although the explicit expression that he obtained is somewhat complicated [18,19]. An alternate expression for $\tau(\nabla)$ was obtained in Section 7 of [20] using the Gaussian shaped Chern–Weil representative of the Thom class defined there using superconnections. See also the discussion in Section 3.3 of [21].

In more detail, let $\mathcal{A}_t = \frac{t^2}{2}|x|^2 + t\nabla x - \pi^*(\Omega_\nabla)$ be defined on E, where x is the fiber variable. Let $\int^B$ denote the Berezin (or Fermionic) integral. When restricted to the sphere bundle S, one has

$$\pi^*(\mathrm{Pfaff}(\Omega_\nabla)) = d\tau(\nabla)\tau(\nabla) = \int_0^\infty dt \int^B \left(x\, e^{-\mathcal{A}_t}\right). \tag{6.1}$$

Also $\int_{S^3}\tau(\nabla) = 1$, so the transgression form $\tau(\nabla)$ is the analog of the Chern–Simons form used in [8].

6.3. T-duality isomorphisms via differential geometry

Given a pair of spherical T-dual pairs (S, H), and $(\widehat{S}, \widehat{H})$, we can decompose them as

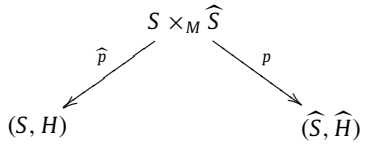
$$\begin{aligned} H &= \pi^* H_7 + \pi^* \mathrm{Pfaff}(\Omega_{\widehat{\nabla}}) \wedge \tau(\nabla), \\ \widehat{H} &= \widehat{\pi}^* H_7 + \widehat{\pi}^* \mathrm{Pfaff}(\Omega_\nabla) \wedge \tau(\widehat{\nabla}). \end{aligned} \tag{6.2}$$

Here $\widehat{\nabla}$ is the connection on the rank 4 oriented real Riemannian vector bundle $\widehat{E}$ over M such that $\widehat{S}$ is the unit sphere bundle of $\widehat{E}$. One checks easily that $dH = 0 = d\widehat{H}$.

Let ω be a d_H-closed form representing a class in $\mathrm{H}_H^{\mathrm{even/odd}}(S)$. Lifting to the correspondence space $S \times_M \widehat{S}$, applying the kernel $\exp(\tau(\nabla) \wedge \tau(\widehat{\nabla}))$, and integrating over the fiber, we define the *T-duality transform*

$$T_*(\omega) = \int_{\mathrm{SU}(2)} \exp(\tau(\nabla) \wedge \tau(\widehat{\nabla})) \wedge \widehat{p}^*\omega, \tag{6.3}$$

which one checks is a $d_{\widehat{H}}$-closed form. Here



Theorem 6.1. *The T-duality transform T_* induces an isomorphism of 7-twisted cohomology groups*

$$T_* : \mathrm{H}_H^{\mathrm{even/odd}}(S) \xrightarrow{\cong} \mathrm{H}_{\widehat{H}}^{\mathrm{odd/even}}(\widehat{S}). \tag{6.4}$$

Proof. Since $d(\tau(A) \wedge \tau(\widehat{A})) = -\widehat{p}^*H + p^*\widehat{H}$, we have

$$\begin{aligned} T_*(d_H\omega) &= \int_{\mathrm{SU}(2)} \exp(\tau(A) \wedge \tau(\widehat{A})) \wedge \widehat{p}^* d\omega - \int_{\mathrm{SU}(2)} \exp(\tau(A) \wedge \tau(\widehat{A})) \wedge \widehat{p}^*H \wedge \widehat{p}^*\omega \\ &= -d \int_{\mathrm{SU}(2)} \exp(\tau(A) \wedge \tau(\widehat{A})) \wedge \widehat{p}^*\omega - \int_{\mathrm{SU}(2)} \exp(\tau(A) \wedge \tau(\widehat{A})) \wedge (\widehat{p}^*H - \widehat{p}^*H + p^*\widehat{H}) \wedge \widehat{p}^*\omega \\ &= -d_{\widehat{H}} T_*(\omega), \end{aligned} \tag{6.5}$$

where in the last step we used the fact that $\int_{\mathrm{SU}(2)} = p_*$, together with the adjunction property of the pullback

$$p_*(\alpha \wedge p^*\beta) = (p_*(\alpha)) \wedge \beta. \tag{6.6}$$

Eq. (6.5) may be summarized by the statement $T \circ d_H = -d_{\widehat{H}} \circ T$. Therefore T takes d_H-exact (closed) forms on P to $d_{\widehat{H}}$-exact (closed) forms on $\widehat{P}$ and so it induces a well-defined homomorphism on the twisted cohomology groups. On verifies exactly as in Section 5 of [8] that it is an isomorphism. □

7. Examples: Sasakian manifolds

In the present context of non-principal bundles, again we have proven an isomorphism of twisted cohomology and so again spherical T-duality is guaranteed to provide an isomorphism of certain conserved charges in string theory. However, as we now no longer insist that the sphere bundles be principal, there are now a number of new examples available.

One class of compactifications which has been of great interest over the past 30 years is that of Sasakian manifolds. More concretely, type IIB supergravity comes with a 10-dimensional compactification manifold which may be $AdS_5 \times M^5$ and 11-dimensional supergravity comes with an 11-dimensional manifold which may be $AdS_4 \times M^7$. Here AdS is a Lorentzian hyperbolic space. If M is a Sasaki–Einstein or 3-Sasakian manifold, then the corresponding compactification preserves some supersymmetry, for example in the case of a 7-manifold M^7 it will preserve $\mathcal{N} = 2$ supersymmetry in the Sasaki–Einstein (and $\mathcal{N} = 3$ in the 3-Sasakian) case.

The examples of Sasaki–Einstein and 3-Sasakian manifolds [22] which have received the most attention in the physics literature are essentially all S^3 bundles. The oldest and simplest is the 7-sphere S^7, which is a principal S^3-bundle over S^4 with e =vol. Already in 5-dimensions the conifold $M^5 = T^{1,1}$ is usually described as an S^1-bundle over the product $S^2_a \times S^2_b$ of 2-copies of the 2-sphere. The Chern class is equal to the generator $a \in \mathrm{H}^2(S^2_a)$. Combining the fiber with S^2_a one obtains a trivial S^3 fibration over S^2_b. The slightly more complicated examples $Y^{p,q}$ [23] are circle bundles over $S^2_a \times S^2_b$ with Chern class $pa + qb$ where a and b generate $\mathrm{H}^2(S^2_a)$ and $\mathrm{H}^2(S^2_b)$ respectively. When p and q are relatively prime again these are homeomorphic to $S^3 \times S^2$ and so provide trivial S^3-bundles.

The 7-dimensional case is nontrivial in the current context because in these compactifications the volume of M^7 is a monotonic function of the 7-flux integrated over M^7 and in particular the M^7 only has nonvanishing volume when the 7-flux, which we have called $h$vol but is traditionally called $*G_4$, is nonvanishing. Therefore, even if the 7-dimensional S^3-bundle is trivial, the spherical T-dual will be nontrivial.

The two most popular Sasaki–Einstein examples of M^7 are $M^{1,1,1}$ [24] and $Q^{1,1,1}$ [25]. The first is a circle bundle over $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^2$ with Chern class $2a + 3b$ where a and b are the generators of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^1)$ and $\mathrm{H}^2(\mathbb{C}\mathrm{P}^2)$ respectively. The second is a circle bundle over 3 copies of $\mathbb{C}\mathrm{P}^1$ where the Chern class is the sum of the generators of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^1)$ of the three copies. In the case of $Q^{1,1,1}$, the circle fiber can be combined with any one of the $\mathbb{C}\mathrm{P}^1$'s on the base to create an S^3 fibered over the remain $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. The fibration will have a trivial Euler class. Nonetheless, as $h \neq 0$ in these compactifications, the spherical T-duals will necessarily have nontrivial Euler classes. As a result it seems likely that the spherical T-duals do not admit a Sasaki–Einstein metric and so will not manifest as much supersymmetry as the original compactifications. Nonetheless an isomorphism of those conserved charges classified by twisted cohomology is guaranteed by the isomorphism of the twisted cohomology.

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