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Classical and quantum monodromy via action–angle variables

Mark D. Hamilton

Department of Mathematics and Computer Science, Mount Allison University, 67 York St, Sackville, NB, E4L 1E6, Canada

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ABSTRACT

We give an elementary description of the relationship between the classical and quantum monodromy of a completely integrable system, from the point of view of geometric quantization, as a consequence of the construction of action–angle variables. We also describe the relation to Symington's notion of "affine monodromy".

1. Introduction

The celebrated Arnol'd–Liouville Theorem gives the structure of a completely integrable system near a regular level in terms of *action–angle coordinates.* Duistermaat addressed the question of the existence of *global* action–angle coordinates in his seminal paper [1]. One obstruction to the existence of global action–angle coordinates is the *monodromy,* which measures the failure of the torus bundle defined by the Liouville tori to be trivial.

There turns out to be a manifestation of monodromy in *quantum* systems as well. This was first observed by Cushman and Duistermaat for the spherical pendulum in [2], and since then "quantum monodromy" has become an active area of research, among physicists and molecular chemists as well as mathematicians. The relationship between the classical and quantum monodromy of a system was proved rigorously in the context of pseudodifferential quantization by Vu Ngoc in [3]. He explains:

> …when using appropriate tools, the link between classical and quantum monodromy becomes more or less trivial. Of course, there is a price to pay: these tools (Fourier Integral Operators, to name one of the most important) are actually quite delicate to define.

The purpose of this note is to show that these delicate tools are not necessary to show the relationship between the classical and quantum monodromy, and that the link can be easily explained in the context of geometric quantization.

This description is not really new; essentially the same argument is used by Sadovskii, Zhilinskii, and collaborators in the context of EBK quantization, for example in [4]. Similar ideas do underlie some of Vu Ngoc's arguments, for example in [3] and [5], but they can be difficult to isolate, and the essential ideas are so simple that I thought it worth laying them

E-mail address: mhamilton@mta.ca.

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out clearly in one place. I have also not seen them applied specifically to geometric quantization, and I thought it interesting that the connection between classical and quantum monodromy becomes so straightforward from this perspective. The argument is valid for any quantization scheme with a Bohr–Sommerfeld condition given by integer action variables, as in Section 3.1.

(Sansonetto and Spera also discuss monodromy in the framework of geometric quantization in [6], although their approach is somewhat different: They relate the classical monodromy to a choice of prequantum connection, and do not explicitly mention quantum monodromy, although there are references to the effect of the monodromy on quantum operators. Also, recently, in [7] Cushman and Śniatycki have taken a different approach to extending geometric quantization to systems with monodromy, as part of their program of "Bohr–Sommerfeld–Heisenberg quantization" as described in [8].)

Symington in [9] further clarifies the place of monodromy in the structure of the phase spaces, pointing out that "any topological monodromy of a regular Lagrangian fibration is reflected in the global geometry of the base". ([9], §2.3) She defines the concept of *affine monodromy* as distinct from "topological monodromy", and shows that the affine monodromy matrix is simply the inverse transpose of the topological monodromy matrix. We show that the affine monodromy is in fact the derivative of our formulation of quantum monodromy, giving an interpretation of the quantum monodromy as a reflection of the structure of the classical phase space.

The argument can be summarized in two paragraphs. The (classical) monodromy describes how a set of cycles forming a homology basis for H^1 of the fibre changes as we go around a loop in the base. Action coordinates can be computed by integrating the Liouville 1-form over exactly such a set of cycles, so the monodromy describes how the (locally defined) action coordinates change as we go around the loop.

In local action–angle coordinates near any regular leaf, the Bohr–Sommerfeld fibres are those all of whose action coordinates are integers. Even if the action–angle coordinates are only locally defined, the condition of "action coordinates are integers" makes sense globally. The "new" and "old" action coordinates give two bases of the local lattice of quantum states, the relation between which is the quantum monodromy. Since the action coordinates are related by the classical monodromy, the quantum monodromy is the same as the classical, modulo a transpose coming from the difference between vectors and their coordinate representation, and an inverse coming from solving a system of equations.

2. Integrable systems

Let (X, ω) be a symplectic manifold of dimension $2n$. A *completely integrable system* is a collection of n functions $f_1, \ldots, f_n$ which pairwise Poisson commute, and which are independent almost everywhere. Let $F = (f_1, \ldots, f_n): X \to \mathbb{R}^n$, and let $B = F(X)$. For $c \in B$, let F_c denote the c-level set of F, which we will generally assume to be compact and connected.

A canonical example of a completely integrable system is the energy–momentum map for the spherical pendulum, where $f_1 = E$ is the energy and $f_2 = L$ is the angular momentum. There are of course many other examples (a long list is given in §1.4 of [10]), and there is an extensive literature on integrable systems.

2.1. Arnol'd–Liouville and action–angle variables

The local structure of an integrable system is described by the Arnol'd–Liouville theorem, which gives a description of the system in terms of particularly simple coordinates called *action–angle coordinates.*

Theorem 1 (*Arnol'd–Liouville*)**.** *Let $c \in B$ be a regular value of an integrable system $F = (f_1, \ldots, f_n): X \to B$ and let $F_c = F^{-1}(c)$. Then F_c is a Lagrangian submanifold of X.*

Furthermore, assume F_c is compact and connected. Then there is a neighbourhood U of F_c in X and a diffeomorphism $(a, \alpha): U \to V \times T^n$, where V is an open subset of $\mathbb{R}^n$ and $T^n = (S^1)^n$ is a torus, such that (a, α) are symplectic coordinates, and F is a function of a only.

The coordinates (a, α) are called action–angle coordinates.

We can describe the conclusions of the theorem as follows (as in [11]):

1. The fibre F_c is diffeomorphic to a torus T^n, on which there are coordinates $\alpha_1, \ldots, \alpha_n$ in which the flow of the Hamiltonian vector fields of $f_1, \ldots, f_n$ are linear.
2. There is a complementary set of coordinates $a_1, \ldots, a_n$, called *action coordinates,* which Poisson commute with all f_j, such that the (a, α) form a symplectic chart.

See [11], Thm 18.12; [1], Thm 1.1; or [12], §§49–50. (Also discussed at length in [13], section II.2.) The above phrasing follows [1].

Action coordinates can be computed as follows.

Theorem 2 (*[1], Thm 1.2*)**.** *Let $c \in B$ be a regular value of F. Choose a neighbourhood $V \subset B$ of c consisting of regular values such that ω is exact on $F^{-1}(V)$, and let Θ be a primitive for ω. (Such a Θ exists by, for example, the Weinstein Lagrangian*

Neighbourhood Theorem.) By Arnold–Liouville the fibres F_q for points $q \in V$ are tori; choose loops $\gamma_1(q), \ldots, \gamma_n(q)$ depending smoothly on q whose homology classes form a basis of $H_1(F_q, \mathbb{Z})$ for each q. Define $a_j : V \to \mathbb{R}$ by

$$a_j(q) = \frac{1}{2\pi} \int_{\gamma_j(q)} \Theta.$$

Then the $\{a_j\}$ are action coordinates.[1]

The $\{a_j\}$ are defined up to addition of a constant (which comes from changing Θ) and multiplication by an element of $GL(n, \mathbb{Z})$ (from changing the homology basis $\{\gamma_j\}$). Thus, different choices of action coordinates will be related by an integral affine transformation.

2.2. Monodromy

By the Arnold–Liouville theorem, action–angle coordinates exist on a neighbourhood of any regular fibre.[2] Duistermaat addresses the question of the global existence of action–angle coordinates in [1]. He gives obstructions to the existence of global action–angle coordinates, one of which is the *monodromy*.

Let $F : X \to B$ be a completely integrable system, as above, and let $B_r \subseteq B$ be the set of regular values of F. The regular level sets of F are tori by the Arnol'd–Liouville theorem, and so $F\big|_U$ has the structure of a torus bundle over any open $U \subseteq B_r$. The monodromy captures the possible non-triviality of the bundle.

Intuitively, the idea is the following: Consider a loop $\ell : [0, 1] \to B_r$, beginning and ending at c, that is non-contractible in the space of regular values. Over $\ell\big((0, 1)\big)$, which is contractible, the torus bundle defined by F is trivial; to construct the bundle over the entire loop, glue the tori above 0 and 1 by some identification. This identification is the monodromy.

To state the definition more carefully, we follow Symington (see [9], §2.3). Let ℓ be a loop in B_r, and let X_ℓ denote the torus bundle defined by F over ℓ. Let ψ_ℓ denote a diffeomorphism $T^n \to T^n$ such that $X_\ell \cong I \times T^n / \sim$ where $(0, x) \sim \big(1, \psi_\ell(x)\big)$. Properly, then, monodromy is a map $\pi_1(B_r, c) \to \mathrm{Aut}(T^n)$, the mapping class group of the torus. Considering the induced action of ψ_ℓ on $H_1(T^n, \mathbb{Z}) \cong \mathbb{Z}^n$ identifies ψ_ℓ with a matrix $M \in GL(n, \mathbb{Z})$. If the monodromy is non-trivial, then the bundle is non-trivial and there cannot be action coordinates defined over all of ℓ.

Another way to visualize the monodromy is more explicitly in terms of the action on $H_1(T^n, \mathbb{Z})$, and it is often explained in this way (for example in [4]). Suppose $\gamma_1, \ldots, \gamma_n$ are loops forming a homology basis for $H_1(F_c, \mathbb{Z})$. We can "transport" these cycles around the loop ℓ using the trivialization as above. Upon returning to the starting point we obtain a new set of cycles $\delta_1, \ldots, \delta_n$, which may differ from the original cycles; the monodromy measures the relationship of the δs to the γs. To form the bundle we glue the end tori by identifying δ_j with γ_j. In terms of the preceding description, if the bundle X_ℓ has monodromy M with respect to a homology basis $\gamma_1, \ldots, \gamma_n$ of $H_1(F_c, \mathbb{Z})$, then $\delta = M^T \gamma$, in the sense that, symbolically,

$$\begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix} = M^T \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}. \tag{1}$$

Some authors use this as the definition, so that their monodromy is the transpose of ours.

Finally, the monodromy may be described in terms of the "period lattice" (which is the original description in [1]). In the proof of the Arnol'd–Liouville theorem, one considers vector fields having 1-periodic flows on each torus fibre and shows that these form a lattice in the tangent space to each fibre, called the *period lattice*. The collection of these gives a lattice bundle[3] in the vertical bundle of X, often also called the period lattice. In action–angle coordinates it is the $\mathbb{Z}$-span of the $\frac{\partial}{\partial \alpha_j}$ (although in the proof the construction goes the other way: the action and angle coordinates are defined from the period lattice). The monodromy is defined as the change in the period lattice after going around a loop in the base.

2.3. Affine monodromy

Several authors note that the monodromy is determined by the base manifold, more specifically by its integral affine structure. This is most clearly explained by Symington in [9], who defines "topological monodromy" to be the monodromy of the torus bundle, as described above, and *affine monodromy* as the monodromy of the integral affine structure on the base, as follows.[4]

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1 More precisely, the action coordinates are these functions a_j composed with F, since the a_j live on B, but we will blur this distinction and think of the a_j as functions on (a subset of) X.

2 More carefully, as Audin explains in §II.2.d of [13], if V is a contractible subset of the set of regular values of F, over which the fibres of F are compact, then there exist action–angle coordinates on $F^{-1}(V)$.

3 A *lattice bundle* Λ in a vector bundle E is a smooth sub-bundle whose fibre Λ_x at each point is a lattice in E_x.

4 Zung in [14] points out that the monodromy is determined by the affine structure on the base, though he blurs the distinction between the topological and affine monodromy. Vu Ngoc also mentions the affine monodromy in [15].

An *integral affine structure* on B is a set of charts for B such that the coordinate changes between charts are integral affine transformations of $\mathbb{R}^n$ (i.e. of the form $Ax+b$ for $A \in GL_n(\mathbb{Z})$ and $b \in \mathbb{R}^n$). Equivalently, it can be specified by a lattice bundle $\Lambda \subset TB$, in which case the affine charts are those which map each Λ_c to $\mathbb{Z}^n \subset \mathbb{R}^n = T_p\mathbb{R}^n$, whose coordinate changes must therefore be integral affine transformations.

A completely integrable system (indeed, any Lagrangian fibration) determines an integral affine structure on the set of regular values B_r (see [9], Theorem 2.6 and §2.2, as well as [16], Lemma 2.1). Essentially, a set of local action coordinates gives an affine chart; as noted in Section 2.1, different sets of action variables differ by an integral affine transformation. In local action–angle coordinates, the bundle Λ is the $\mathbb{Z}$-span of the vectors $\frac{\partial}{\partial a_j}$.

The *affine monodromy* is the monodromy of the bundle Λ, as follows. If $\ell\colon [0,\,1] \to B_r$ is a loop in the set of regular values, as in the previous section, let TB_ℓ and Λ_ℓ denote the restrictions of TB and Λ to ℓ. Then the affine monodromy around ℓ is the element ψ_ℓ of $\mathrm{Aut}(T_cB,\,\Lambda_c)$ such that $(TB_\ell,\,\Lambda_\ell) \cong I \times (\mathbb{R}^n,\,\mathbb{Z}^n)/\sim$ where $(0,x) \sim \big(1,\,\psi_\ell(x)\big)$. Choosing a suitable basis of T_cB identifies ψ_ℓ with a matrix $A \in GL(n,\mathbb{Z})$.

Symington shows in Prop 2.15 (actually, it is an exercise for the reader) that the affine and topological monodromy are related by $A = (M^T)^{-1}$, which (following [9]) we will denote M^{-T}. The topological monodromy is the monodromy of the period lattice $\mathcal{P}$ defined above (which is denoted by Λ_{vert} in [9]). The symplectic form gives a canonical pairing between the tangent space to a fibre and the cotangent space of the base, which determines a lattice bundle Λ^* in T^*B that has the same monodromy as Λ_{vert}. Then one can define the lattice bundle $\Lambda \subset TB$ as dual to Λ^*, by

$$\Lambda_c = \{u \in T_cB \mid \langle \xi, u\rangle \in \mathbb{Z} \quad \forall \xi \in \Lambda_c^*\}. \tag{2}$$

In action–angle coordinates, $\mathcal{P}$ is the $\mathbb{Z}$-span of the $\frac{\partial}{\partial \alpha_j}$, Λ^* is the $\mathbb{Z}$-span of the da_j, and Λ is the $\mathbb{Z}$-span of the $\frac{\partial}{\partial a_j}$, so Λ is the bundle defining the integral affine structure, whose monodromy is the affine monodromy. If ξ changes by M in (2) then u has to change by $(M^T)^{-1}$ to preserve the condition, and so the monodromy of Λ is M^{-T}.

3. Geometric quantization

The basic ingredient for the geometric quantization of a symplectic manifold $(X,\,\omega)$ is a complex Hermitian line bundle $L \to X$ with a connection ∇ whose curvature equals ω, called a *prequantum* line bundle and connection. One also requires a "polarization;" in our case we will take a *real polarization,* which consists of a foliation of X into Lagrangian submanifolds.[5] The "quantum space" associated with X is constructed from sections of L that are *leafwise flat,* namely, covariant constant with respect to ∇ in directions tangent to the leaves of the foliation.[6]

Such sections always exist locally, but do not generally exist over an entire leaf of the foliation. If a leaf $\mathcal{L}$ possesses a leafwise flat section defined on the entire leaf, it is called a *Bohr–Sommerfeld leaf.* A leaf $\mathcal{L}$ is a Bohr–Sommerfeld leaf iff the holonomy of the prequantum connection is trivial around every loop in $\mathcal{L}$; equivalently, iff $\mathrm{hol}_{\gamma_j} = 1$ for all γ_j in a homology basis for $H_1(\mathcal{L},\mathbb{Z})$. The set of Bohr–Sommerfeld leaves is typically discrete (in the space of leaves). (For example, if $X = \mathbb{R}\times S^1$, with trivial bundle and connection form $x\,d\theta$, foliated by $\{x\}\times S^1$, the Bohr–Sommerfeld leaves are those with $x \in \mathbb{Z}$.) This implies there are no globally defined (continuous) leafwise flat sections, and so the quantization is not simply the set of global leafwise flat sections.

One approach to defining the quantization in this case is to relax the requirement of continuity and use distributional sections, for example in the work of Nunes and collaborators (see [21] and references therein). Another approach, suggested by Kostant [22] and implemented by Śniatycki [23], is to use, rather than the space of leafwise flat sections, higher cohomology groups, as follows. Let $\mathcal{J}$ denote the sheaf over X of leafwise flat sections. Then the quantization of X is defined as

$$\mathcal{Q}(X) = \bigoplus_{k=0}^{\infty} H^k(X;\,\mathcal{J}).$$

(See [24, §2.4.1] for more details, including a discussion of different conventions in the Remark at the end of the section.)

The general picture is that the "quantum space" is (isomorphic to) the space of leafwise flat sections over all the Bohr–Sommerfeld leaves:

$$\mathcal{Q}(X) \cong \bigoplus_{x\in BS} \Gamma_\nabla(\mathcal{L}_x,\,L).$$

Since the value of such a section over the entire leaf is determined by its value at one point, the space $\Gamma_\nabla(\mathcal{L}_x)$ is one-dimensional, and so roughly speaking, "the [dimension of the] quantization is given by counting the Bohr–Sommerfeld fibres". For the sheaf-cohomological approach, this is made more precise by the following theorem of Śniatycki:

[5] For simplicity we do not give the (rather technical) general definition of a polarization, but direct the reader to any reference on geometric quantization (see next footnote). The other main type of polarization used is a *Kähler polarization,* which is given by a compatible complex structure on X. The "quantum space" is then the space of holomorphic sections of L.

[6] There are many references for geometric quantization, for the reader who wishes more than these very sketchy details, although few are at an introductory level. The books [17] and [18] are classic, if both rather technical; John Baez has a good brief introduction on the Web at [19]. [20] also has a brief introduction at the beginning of Chapter 6, and refers to numerous other sources.

Theorem ([23]). *Let X be a $2n$-dimensional symplectic manifold, with a prequantization line bundle L as above. Let P be a real polarization such that the projection map $\pi : X \to X/P$ is a fibration with compact fibres. Let $\mathcal{J}$ denote the sheaf of leafwise flat sections of L.*

Then $H^q(X; \mathcal{J}) = 0$ for all $q \neq n$. Furthermore, let $\Gamma_\nabla(BS, L)$ be the space of leafwise flat smooth sections of L along the union of Bohr–Sommerfeld leaves, and $C^\infty_P(X)$ be the ring of functions on X constant on leaves of π. Then, provided P satisfies an orientability condition, $H^n(X; \mathcal{J})$ is isomorphic to $\Gamma_\nabla(BS, L)$ as modules over $C^\infty_P(X)$.

As a vector space, $H^n(X; \mathcal{J})$ is isomorphic to the direct sum of copies of $\mathbb{C}$, with one copy for each Bohr–Sommerfeld leaf.

In the case of a completely integrable system, the regular fibres of the system are Lagrangian tori, although singular fibres may have a different form. We view the decomposition of X into fibres of F as a real polarization with singularities. The sheaf cohomology approach to the quantization of singular polarizations, including integrable systems, has been explored in [24–28]. It has not been satisfactorily computed for systems with singularities of "focus–focus" type, which are the known examples exhibiting monodromy.

One way around this difficulty is to restrict attention to the set of regular fibres, in which case we have an honest real polarization on an open manifold, to which Śniatycki's theorem applies. In this case we can simply take the "quantum states" to be the Bohr–Sommerfeld points in B_r, where we say that x is a *Bohr–Sommerfeld point* in B if F_x is a Bohr–Sommerfeld fibre, namely if the holonomy of the prequantum connection restricted to F_x is trivial. The contribution of the singular fibres to the sheaf cohomology is an open question, and is the subject of ongoing work of the author with Leah Duffett; for now, we will restrict our attention to the set of regular values.

Remark. It depends on one's perspective which is more fundamental: the sheaf cohomology or the count of Bohr–Sommerfeld fibres. One can view the sheaf-cohomology definition of the quantization as fundamental, by analogy to the index-theoretic definition of quantization using a Kähler polarization; in this case, Śniatycki's result becomes a theorem saying that the quantization can be computed by counting Bohr–Sommerfeld fibres. Several authors have subsequently taken this as motivation to *define* the (dimension of the) quantization to be the count of Bohr–Sommerfeld fibres, even in situations where the hypotheses of Śniatycki's theorem do not hold, such as Jeffrey–Weitsman in [29] and Guillemin–Sternberg in [30].

On the other hand, taking a perspective coming from physics, one can say that we know that the quantization of a system should be given by Bohr–Sommerfeld conditions, and so however we define it we should obtain the answer that the dimension of the quantization should equal the number of Bohr–Sommerfeld fibres. From this perspective, Śniatycki's result shows that the sheaf-cohomological approach suggested by Kostant "works" for computing the quantization in these situations. (This is the perspective from which Śniatycki wrote the original paper [23].) In this case the open problem mentioned above is less "determine the contributions of the focus–focus singular points to the quantization" than "find a definition of the quantization of a focus–focus system that gives the results we want".

3.1. Bohr–Sommerfeld and action–angles

There is a connection between Bohr–Sommerfeld points and action–angle variables, as Guillemin and Sternberg discuss in section 2 of [30]. Following [17, Section A.3], we describe a connection ∇ on the line bundle L in terms of its *potential one-form* Θ as follows: Given a local trivializing section s of L over some open set U, define Θ by $\nabla_Y s = -i\Theta(Y)s$ for all vectors Y. If s is unitary, then Θ will be real-valued; also, $d\Theta$ is the curvature of ∇, which for a prequantum connection is the symplectic form ω. In this description, the holonomy of ∇ around a loop γ in U is given by

$$\mathrm{hol}_\gamma = \exp i \int_\gamma \Theta. \tag{3}$$

Now suppose we have trivialized L over a neighbourhood $U = F^{-1}(V)$ of regular leaves,[7] and let Θ be the potential one-form in this trivialization. Since $d\Theta = \omega$, we can use Θ to construct action variables as described in Theorem 2; if $\gamma_1(q), \ldots, \gamma_n(q)$ are loops depending smoothly on q forming a homology basis of $H_1(F_q, \mathbb{Z})$ for each q, then it follows that $\mathrm{hol}_{\gamma_j(q)} = e^{2\pi i a_j(q)}$. Since F_q is a Bohr–Sommerfeld leaf iff $\mathrm{hol}_{\gamma_j} = 1$ for all γ_j, we see that *q is a Bohr–Sommerfeld point iff all $a_j(q)$ are integers.*

A different choice of trivializing section $s' = \psi s$ for $\psi : U \to \mathbb{C}^\times$ will lead to a different potential one-form Θ', related to the original one by $\Theta' = \Theta + i\frac{1}{\psi}d\psi$. This will change the integral in Theorem 2 by

$$\frac{1}{2\pi}\int_\gamma \frac{i}{\psi}d\psi = \frac{1}{2\pi} i \cdot 2\pi i k = -k \qquad \text{for some } k \in \mathbb{Z},$$

[7] This is always possible: we can trivialize L over a contractible neighbourhood transverse to the leaves, and then use the local T^n-action to "sweep out" the trivialization over the rest of U. Cf. Lemma 8.2.3 in [24].

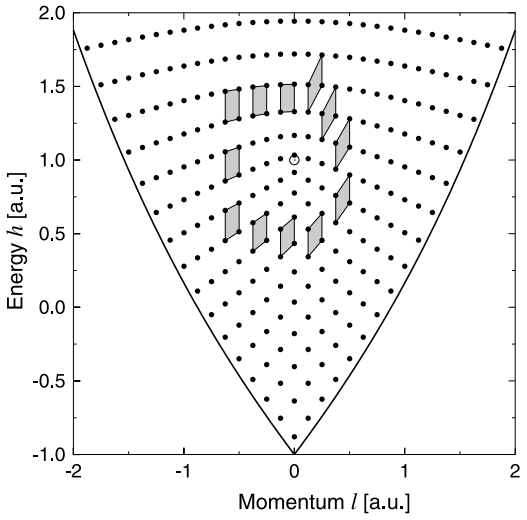

Fig. 1. A typical picture with quantum monodromy.
Source: From [32].

so the resulting action coordinate a_j' will differ from a_j by an integer. Thus the expression for the holonomy as $\mathrm{hol}_{\gamma_j(q)} = e^{2\pi i a_j'(q)}$ is still valid.

This construction is local. To go from local to global, any choice of action variables constructed using a potential one-form of the connection as the one-form in Theorem 2 will satisfy $\mathrm{hol}_{\gamma_j(q)} = e^{2\pi i a_j(q)}$, and so will take integer values on the Bohr–Sommerfeld set. As discussed in 2.1, any two choices of action variables differ by a constant and an element of $GL_n(\mathbb{Z})$; this choice of one-form implies the constant will always be an integer. Thus, the condition of "all action variables integers" is well-defined globally.

Remark. There is a subtlety about normalization not addressed in [30]. There, they define action variables by integrating an arbitrary primitive of ω, and then assume $a_j(p) = 0$ at a Bohr–Sommerfeld point p. This is equivalent to choosing a particular primitive of ω.

If we use the one-form coming from the connection, as in the above discussion, the resulting action coordinates are automatically "normalized" so that they take integer values on the Bohr–Sommerfeld set. Thus the prequantum connection gives a preferred choice of action coordinates, in which the Bohr–Sommerfeld points actually are those with integer action coordinates.

Moreover, in many singular cases, we lose the freedom of adding a constant, and the condition that $a_j \in \mathbb{Z}$ for all j is forced by the structure of the system. (See for example the discussion of "Bohr–Sommerfeld rigidity" in [24], §2.6.)

4. Quantum monodromy

The phenomenon of quantum monodromy was first noted by Cushman and Duistermaat in [2], who pointed out that it was impossible to consistently define a lattice structure on the spectrum of the quantum spherical pendulum. The phenomenon was further explored by a number of other authors, particularly Sadovskii, Zhilinskii, Cushman, Nekhoroshev, and collaborators. It also received attention from physicists because of its implications for the structure of molecules; see [31] for a review of the problem from this perspective and numerous references to the physics literature. On the mathematics side, San Vu Ngoc has a number of results about quantum monodromy, including in [3] where he proves the relationship between the classical and quantum monodromy, from the point of view of pseudodifferential quantization.

The idea of quantum monodromy is the following: The set of quantum states of a system "locally looks like" a lattice, but may not actually be a lattice globally: a basis at one point may not be consistently extendable to the entire space B. If we take a basis for the "lattice" at one point and follow it around a (non-contractible) loop, we may come back to a different basis, and the difference is expressed by the quantum monodromy. This is often visualized using a "Zhilinskii diagram" like Fig. 1. (Fig. 1 shows the spectrum for the spherical pendulum, prepared by Dmitrii Sadovskii using data by Igor Kozin, seen in [32] and [33]. Used with permission.)

We will think of the quantum monodromy as follows. The "set of quantum states" will be a subset of B. For example, in the case of the spherical pendulum, where X has dimension 4 and F is the energy–momentum map, this is the set of jointly quantized energy and angular momentum values. In geometric quantization, this is simply the Bohr–Sommerfeld set BS. A "local lattice chart" will be a diffeomorphism from an open set U in B to an open set V in $\mathbb{R}^n$, mapping $U \cap BS$ to $V \cap \mathbb{Z}^n$. A choice of local lattice chart is equivalent to a choice of "local quantum numbers" in physics terminology. (See [32] or [33, p. 285].)

The idea of "how the lattice changes as you travel around a loop" can be captured as follows. A local lattice chart centred at $c \in B$ gives a basis $\{u_1, \ldots, u_n\}$ for the local lattice of quantum states at c, by just taking the image of the standard basis of $\mathbb{Z}^n$. If ℓ is a loop through c, covering ℓ with overlapping local lattice charts provides a way to "parallel transport" this basis along the loop. After travelling around the loop and coming back to c, we get another basis $\{v_1, \ldots, v_n\}$ for the local lattice at c. We define the quantum monodromy to be the linear transformation mapping the "old basis" to the "new basis", which, given an integral basis of $\mathbb{Z}^n$, we can express as a matrix $M_q \in GL_n(\mathbb{Z})$ such that $v_j = M_q u_j$ for all j.

Remark. Different authors use different conventions about which exact map the quantum monodromy matrix represents, which are sometimes not entirely clear. In [33, §C.3] they say, "The map from the original cell to the final cell [defined by the vectors $\{u_j\}$ and $\{v_j\}$] is the monodromy map", which seems to agree with our definition. In [4], their quantum monodromy matrix satisfies

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = M_q \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is the matrix whose rows are the vectors u_1 and u_2. This means that their quantum monodromy is the transpose of ours. They also define the classical monodromy as discussed after (1) in Section 2.2, and so their classical and quantum monodromy are still related by an inverse transpose.

In [5] the quantum monodromy is defined as a Čech cocycle with coefficients in the group of affine transformations. Unravelling the definitions, it seems that their definition is consistent with ours, in that the linear part of their affine transformation is our quantum monodromy.

5. Classical and quantum monodromy

Theorem 3. *Let $F: X \to B \subset \mathbb{R}^n$ be a completely integrable system, and let B_r be the set of regular values of F. Let ℓ be a loop in B_r beginning and ending at a regular Bohr–Sommerfeld point c, and suppose the system has (classical) monodromy over ℓ, given by a matrix $M \in GL(n, \mathbb{Z})$. Then the Bohr–Sommerfeld set in B_r exhibits quantum monodromy around ℓ, as in Section 4, with matrix $M_q = M^{-T}$.*

Proof. Let ℓ be a loop in B_r from c to c. Let $\{a_j, \alpha_j\}$ be a set of local action–angle coordinates centred at c, defined by integrating a potential one-form Θ for the connection over a homology basis of cycles $\gamma_1, \ldots, \gamma_n$ as in Section 3.1, normalized so that $a(c) = 0$. If we "parallel transport" the cycles γ_j around ℓ, we will end up with a new set of cycles $\delta_1, \ldots, \delta_n$ in F_c, related to γ_j by $\delta = M^T \gamma$, as in Section 2.2. (Here $M = (M_{ij})$ is the matrix giving the monodromy around ℓ.)

Define a new set of action coordinates $\{b_j\}$ centred at c by

$$b_j = \int_{\delta_j} \Theta.$$

Since integration is linear, this implies that $b = M^T a$; more explicitly,

$$b_j = \int_{M_{1j}\gamma_1 + \cdots + M_{nj}\gamma_n} \Theta = M_{1j}a_1 + \cdots + M_{nj}a_n.$$

(This implies in particular that $b(c) = 0$ as well.)

The Bohr–Sommerfeld set is defined by the conditions $\{a_j \in \mathbb{Z}\}$, or $a \in \mathbb{Z}^n$. If $a(c) = 0$, then a primary cell in the local lattice at c is defined by the points $q_1, \ldots, q_n$ satisfying $a(q_j) = e_j$, where e_j is the jth standard basis vector of $\mathbb{Z}^n$. The image of the primary cell after transporting around the loop ℓ will be defined by the points p_j satisfying $b(p_j) = e_j$. Since $\{a \in \mathbb{Z}^n\}$ and $\{b \in \mathbb{Z}^n\}$ define the same lattice, $\{a(p_j)\}$ defines a new basis of $\mathbb{Z}^n$, which is related to the standard basis by the quantum monodromy: $a(p_j) = M_q e_j$. Since $b = M^T a$, $b(p_j) = M^T a(p_j) = e_j$, so $a(p_j) = (M^T)^{-1} e_j$. Thus $M_q = M^{-T}$.

A priori each monodromy is only defined up to conjugation, since each depends on a choice of basis, but each is defined starting from a basis of $H_1(F_c, \mathbb{Z})$. Using the same basis for each monodromy gives this exact relationship. □

5.1. Affine and quantum monodromy

As remarked in Section 2.3, an integrable system determines an integral affine structure on the set of regular values B_r in the base, essentially given by choices of local action coordinates. The prequantum structure gives a preferred set of affine charts, one for which the affine transition maps $Ax + b$ have $b \in \mathbb{Z}^n$ always.

The affine monodromy of B is the monodromy of the lattice bundle $\Lambda \subset TB$ defined in Section 2.3. In particular, given a loop ℓ beginning and ending at $c \in B_r$, the (affine) monodromy around ℓ is the element of $\mathrm{Aut}(T_c B, \Lambda_c) \cong \mathrm{Aut}(T^n)$ that expresses the change in the lattice Λ_c induced by transporting around the loop. This change in Λ_c is the composition of the derivatives of the affine coordinate change maps through a series of charts covering ℓ.

The quantum monodromy, on the other hand, is the change in action coordinates "downstairs" on B obtained by composing the coordinate change maps around ℓ. Thus the affine monodromy is the *derivative* of the quantum monodromy. Since the quantum monodromy is a linear transformation, the two are equal.

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