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Minimal models of compact symplectic semitoric manifolds

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ABSTRACT

A symplectic semitoric manifold is a symplectic 4-manifold endowed with a Hamiltonian $(S^1 \times \mathbb{R})$-action satisfying certain conditions. The goal of this paper is to construct a new symplectic invariant of symplectic semitoric manifolds, the helix, and give applications. The helix is a symplectic analogue of the fan of a nonsingular complete toric variety in algebraic geometry, that takes into account the effects of the monodromy near focus–focus singularities. We give two applications of the helix: first, we use it to give a classification of the minimal models of symplectic semitoric manifolds, where "minimal" is in the sense of not admitting any blowdowns. The second application is an extension to the compact case of a well known result of Vũ Ngọc about the constraints posed on a symplectic semitoric manifold by the existence of focus–focus singularities. The helix permits to translate a symplectic geometric problem into an algebraic problem, and the paper describes a method to solve this type of algebraic problem.

1. Introduction

The revolution in symplectic toric geometry started in the 1980s with the proof of the convexity of the image of the momentum map $F = (f_1, \ldots, f_k) : (M, \omega) \to \mathbb{R}^k$ associated to a compact symplectic $2n$-manifold acted upon by a k-dimensional compact connected abelian Lie group T (i.e. a k-dimensional torus $T = (S^1)^k$), due independently to Guillemin–Sternberg [1] and Atiyah [2]. In fact, $F(M)$ is the polytope Δ equal to the convex hull of the image under F of the fixed points of the T-action. In the case that $n = k$ such manifolds are called *symplectic toric*.

Shortly after, Delzant proved [3] that in the symplectic toric case the image Δ encodes all of the information about the manifold M, the form ω, and the ω-preserving T-action. That is, Δ is the only symplectic T-equivariant invariant of (M, ω, F). He moreover showed that any simple, rational, smooth polytope Δ arises as the image of a momentum map of a symplectic-toric manifold; following Guillemin these polytopes are now called *Delzant*.

The existence of this action poses restrictions on (M, ω) and F. For instance, F only has elliptic singularities; moreover, the fibers are tori of dimension 0 up to n (in particular, they are submanifolds of M).

Delzant's classification was extended in [4,5] to compact and noncompact symplectic 4-manifolds acted upon by the noncompact Lie group $S^1 \times \mathbb{R}$, under certain assumptions (the action must be Hamiltonian, all singularities must be non-degenerate, with none of hyperbolic type, the moment map of the S^1-action must be proper, and each fiber contains at most one isolated singularity) these manifolds are called *symplectic semitoric*, and so far are classified when M is 4-dimensional. In this case the momentum map of the $(S^1 \times \mathbb{R})$-action is $F = (f_1, f_2)$, where the Hamiltonian vector field associated to f_1 is periodic, but not necessarily the one associated to f_2. The main novelty with respect to symplectic toric manifolds is that F may have, in addition to elliptic singularities, another type of singularities known as focus–focus singularities. The fiber

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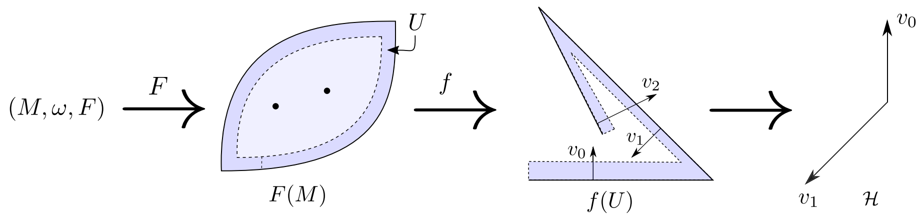


Fig. 1. The helix is intrinsically constructed by defining a toric momentum map on the preimage of U, a neighborhood of the boundary of the image of the momentum map minus a single cut, and collecting the inwards pointing normal vectors of the piecewise linear boundary of the resulting set in $\mathbb{R}^2$. Notice that v_2 does not occur in the representative of the helix we have drawn because after applying the monodromy it is equal to v_0.

containing a focus–focus singularity is not a submanifold, it is homeomorphic to a sphere with its south and north poles identified (i.e. a torus pinched at the focus–focus singularity). Symplectic semitoric manifolds are an example of almost toric manifolds, as introduced by Symington [6].

Symplectic semitoric manifolds are characterized by five invariants, one of which is a polygon P constructed from $F(M)$ according to Vũ Ngọc [7], by unfolding the singular affine structure induced by F on $F(M)$ as a subset of $\mathbb{R}^2$ (in fact $F(M)$ need not even be convex[1]). The other four invariants account for the effect of the focus–focus singularities and the monodromy around them (a fundamental phenomena studied by Duistermaat [8]). There are natural notions of blowdown in the symplectic toric and symplectic semitoric settings which we describe in Section 2.4.

Definition 1.1. A symplectic toric or symplectic semitoric manifold is *minimal* if it does not admit a blowdown.

For a symplectic toric manifold chopping off a corner of Δ corresponds to T-equivariantly blowing up M at a T-fixed point, and the inverse operation corresponds to blowing down. To Δ one can associate a *fan*, the one corresponding to (M, ω) when viewed as a nonsingular complete toric variety (the explicit relation appears in [9]). Because of this correspondence the search for their minimal model is reduced to an algebraic problem concerning fans associated to Delzant polytopes. If $2n \geqslant 6$ the problem is still too difficult but when $2n = 4$ the corresponding 2-dimensional fans have been classified; a proof of the following result, originally due to Oda in the 1970s, may also be found for instance in Fulton [10].

Theorem 1.2 (*Oda [11, Theorem 8.2]*). *The inequivalent minimal models of symplectic toric manifolds are $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and a Hirzebruch surface with parameter $k \neq 1$.*

The Delzant polytopes of the minimal models are: a simplex ($M = \mathbb{CP}^2$ with any multiple of the Fubini–Study form), a rectangle ($M = \mathbb{CP}^1 \times \mathbb{CP}^1$ with any product form), and a trapezoid (M a Hirzebruch surface, with one of its standard forms). The question is whether this classification can cover more cases.

Main Question. *What are the inequivalent minimal models of compact symplectic semitoric manifolds?*

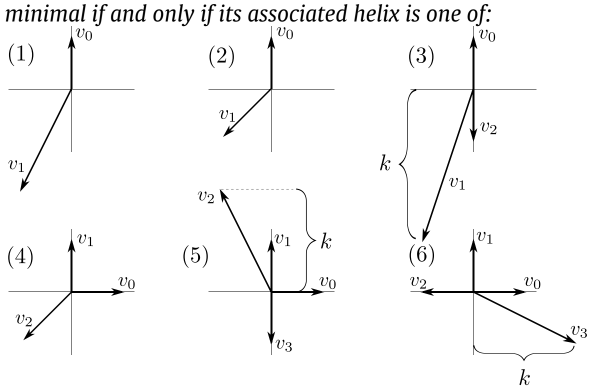
Even more interesting would be to know whether the question can be answered as an application of the known invariants. However, it is not clear what the effect of blowing up and down is on the known invariants we have just mentioned. The image $F(M)$ is no longer necessarily a polygon, or even a convex set. The polygon P is obtained as the image of a homeomorphism $\varphi : F(M) \to P \subset \mathbb{R}^2$ which unfolds the singular affine structure of $F(M)$ into P, taking into account the monodromy (the construction of φ is delicate, see [7]). The effect of blowing up or down on P depends on the position of the focus–focus values of F, and here is where a new invariant of compact symplectic semitoric manifolds comes into play, we call it the *semitoric helix* and denote it by $\mathcal{H}$. Like in the toric case, $\mathcal{H}$ is given by (an equivalence class of) vectors in $\mathbb{Z}^2$, plus some additional information which we describe later more precisely and which includes the information of focus–focus singularities and monodromy (this does not appear in the toric case).

Analogous to the way in which from a Delzant polygon one constructs a fan, from P one constructs the helix $\mathcal{H}$ (after making some corrections related to the focus–focus singular points), see Fig. 4, though the helix can also be constructed directly from M, bypassing the polygon, as in Fig. 1. We describe the construction of $\mathcal{H}$ in detail in Section 4.1. The helix $\mathcal{H}$ contains the information encoding blowing up and blowing down, information which appears to be very difficult to extract from known invariants. And $\mathcal{H}$ generalizes the fan while taking into account the effects of the monodromy around the focus–focus singularities.[2] Moreover, $\mathcal{H}$ can be studied with algebraic techniques, and can be applied to prove the following, which is the main theorem of this paper.

1 And in all important examples it is never a polygon, including the coupled spin-oscillator and the spin–orbit system.

2 The helix is also related to the notion of *semitoric fan* introduced in [12], though they are not equivalent, the precise relation is discussed in Section 3.5.

Theorem 1.3. *Let (M, ω, F) be a compact symplectic semitoric manifold with $c > 0$ focus–focus points and $d > 0$ elliptic–elliptic points. If $d < 5$ then (M, ω, F) is minimal if and only if its associated helix is one of:*



with parameters in each case given by (1) $c = 1$; (2) $c = 2$; (3) $k \neq \pm 2$, $c = 1$; (4) $c \neq 2$; (5) $k \neq \pm 1, 0$, $c \neq 1$; (6) $k \neq -1, 1 - c, c > 0$. If $d \geqslant 5$ then (M, ω, F) is minimal if and only if $d > 5$ and in this case the associated helix is completely determined by c and a positively oriented basis (v_0, v_1) of $\mathbb{Z}^2$.

After we have described the required background a more precise version of Theorem 1.3 appears as Theorem 3.6. An example of a helix with $d > 5$ is shown in Fig. 3 and discussed in Section 3.4. Theorem 1.3 and its applications can be used to study many integrable systems from classical mechanics such as the coupled angular momenta system [13,14] that we describe in Section 2.3, which is minimal of type (3) with $k = 1$. It has precisely one focus–focus singularity with monodromy, and is an example of a compact (nontoric) symplectic semitoric manifold.

Minimal models have also been studied in related contexts. For instance, Karshon classified 4-dimensional Hamiltonian S^1-spaces which are minimal with respect to S^1-equivariant blowups [15]. The blowups we study in the present paper are more restrictive since they are required to respect the structure of the semitoric manifold, not just the induced S^1-action, but it can be seen that the minimal models from Theorem 1.3 can be reduced to the minimal Hamiltonian S^1-spaces by performing S^1-equivariant blowups. Additionally, semitoric manifolds are an example of almost-toric manifolds [6] and minimal almost toric 4-manifolds are classified up to diffeomorphism by Leung–Symington [16]. The biggest difference between the present paper and that one is that by classifying the semitoric helices we include information about not just the diffeomorphism type but also the integrable system structure (and thus the singular Lagrangian fibration) of the manifolds. Related to this, the Leung–Symington classification does not differentiate between fixed points of elliptic–elliptic type and of focus–focus type, while our classification does. In fact, in [16, Section 6.4] the authors mention the problem of classifying almost toric manifolds up to fiber-preserving symplectomorphism, which is related to what is done in this paper since the helix encodes the fibration. In [16] they make use of what they call the *defining set* to classify almost toric manifolds, which is related to the semitoric helix, though they are not equivalent (see Remark 4.8).

In a different direction, and as an application of Theorem 2.6, in [17] some properties of the associated moduli spaces of manifolds were studied in detail; it would be interesting to use Theorem 1.3 to study a semitoric analogue. First steps towards this have been carried out by the second author in [18], where a natural topology on the space of symplectic semitoric manifolds is constructed. For other recent results studying the affine structure of integrable systems in dimension four, see [19,20].

Theorem 1.3 has the following consequence in the study of symplectic semitoric manifolds, which extends a theorem of Vũ Ngọc [7] from noncompact to compact symplectic semitoric manifolds.

Theorem 1.4. *If a compact symplectic semitoric manifold has at least two focus–focus singular points and the component of the momentum map with periodic flow achieves its maximum and minimum at a single point each, then the system must have exactly two focus–focus points and be a minimal symplectic semitoric manifold of type (2) from Theorem 1.3.*

Theorem 1.4 follows from Lemma 3.8 and is proven in Section 3.3. In [7, Theorem 3] Vũ Ngọc uses an argument related to the Duistermaat–Heckman measure on symplectic semitoric manifolds to prove that there do not exist noncompact symplectic semitoric manifolds for which the component of the momentum map with periodic flow achieves its maximum and minimum at a single point each and which have more than one focus–focus point. Recently, S. Sabatini drew to our attention that she had announced a version of Theorem 1.4 at a conference in 2013 and outlined a different proof from the one given in the present paper.

We conclude by briefly explaining the idea of the proof of Theorem 1.3. The proof of this theorem operates by translating the problem into algebraic language in which a number of things are easier to work with. The basic ideas behind this technique were already present in [12], but they are refined and developed so as to be useful in practical applications (see Section 3.5 for an explanation of the relationship between the present article and [12]). The algebraic correspondence works as follows. To any semitoric helix, as on the right hand side of Fig. 1, there is a natural way to associate a word of a particular form in $\mathrm{SL}_2(\mathbb{Z})$. The word is $\sigma = ST^{a_0} \ldots ST^{a_{d-1}}$, where the a_i are integers and $S, T \in \mathrm{SL}_2(\mathbb{Z})$ represent the specific matrices given in Eq. (2.1). Our attempt to classify helices will operate by attempting to understand the associated words. The vectors forming a helix must wrap only a single time around the origin before repeating, and in order to detect this from the word σ in $\mathrm{SL}_2(\mathbb{Z})$ we lift σ to the universal cover of $\mathrm{SL}_2(\mathbb{R})$. In the universal cover we define a function on words which agrees with the number of times the associated vectors circle the origin, which we call the winding number of that word. We let G denote the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$. We are then able to produce an exact correspondence between minimal semitoric helices and words of particular form in G that lie in one of a small number of conjugacy classes in G.

The key idea in our analysis now follows from the observation that the elements that we are looking for will necessarily have nearly the smallest possible winding number of any word representing the correct element of $\mathrm{SL}_2(\mathbb{Z})$. In order to properly analyze this, we show that each element of $\mathrm{PSL}_2(\mathbb{Z})$ has a unique minimal word associated to it with the smallest possible winding number. In fact, any representation of the given element can be reduced to the minimal one by means of a few simple reduction steps. The thrust of our argument is now to look at the minimal word associated to the element represented by our helix. Noting that the word corresponding to our helix reduces to this minimal word in only a few steps allows us to reduce ourselves to a small number of possibilities, which correspond to the minimal helices in Theorem 1.3. The main novelty of the paper is precisely this method of proof.

Structure of the article

In Section 2 we give the background required to state the main theorems of the paper and put them in context. In Section 3 we state all the main results of the paper, and leave some of the longer proofs to later sections. In Section 4 we describe the construction of the semitoric helix and prove Proposition 3.1. The remaining sections are devoted to the proof of one result each: in Section 5 we prove Proposition 3.3, in Section 6 we prove Theorem 3.5, and in Section 7 we prove the main result of the paper, Theorem 3.6.

2. Preliminaries

2.1. Minimal symplectic toric manifolds

Here we give a pedestrian exposition of toric manifolds and integrable systems from the point of view of symplectic geometry.

An *integrable system* is a triple (M, ω, F) where (M, ω) is a $2n$-dimensional symplectic manifold and $F = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ is a smooth map such that its components $f_1, \dots, f_n$ Poisson commute and are independent almost everywhere. That is, $\omega(\mathcal{X}_{f_i}, \mathcal{X}_{f_j}) = 0$ for all $i, j = 1, \dots, n$ and $(\mathcal{X}_{f_1})_p, \dots, (\mathcal{X}_{f_1})_p$ are linearly independent in T_pM for almost all $p \in M$, where $\mathcal{X}_{f_i}$ denotes the Hamiltonian vector field of f_i.

Definition 2.1. A *symplectic toric manifold* is an integrable system (M, ω, F) such that (M, ω) is a compact and connected $2n$-dimensional symplectic manifold, each $\mathcal{X}_{f_i}$ has 2π-periodic flow, and the $\mathbb{T}^n$-action produced by these flows is effective.

A convex, compact, rational polygon in $\mathbb{R}^2$ is a *Delzant polygon* if the collection of inwards-pointing integer normal vectors to the polygon of minimal length form what is known as a toric fan. For vectors $v, w \in \mathbb{Z}^2$ let $\det(v, w)$ denote the determinant of the matrix with columns v, w. For the purposes of this paper we use the following definition:

Definition 2.2. A *toric fan* of length $d \in \mathbb{Z}_{>0}$ is a collection of vectors $(v_0, \dots, v_{d-1}) \in (\mathbb{Z}^2)^d$ such that

1. $\det(v_i, v_{i+1}) = 1$ for $i = 0, \dots, d-1$ where $v_d := v_0$;
2. $v_0, \dots, v_{d-1}$ are arranged in counter-clockwise order.

Associated to each toric manifold is a toric fan, formed from the Delzant polygon $F(M)$ in this way.

Definition 2.3. If $(v_0, \dots, v_{d-1})$ is a toric fan of length d such that $v_i = v_{i-1} + v_{i+1}$ then a new toric fan of length $d-1$ can be produced by removing v_i. This operation is known as the *blowdown* and the inverse operation, inserting the sum of two adjacent vectors, is known as a *blowup*.

Definition 2.4. A toric fan is *minimal* if $v_i \neq v_{i-1} + v_{i+1}$ for $i = 0, \dots, d-1$.

Minimal toric fans are those on which a blowdown cannot be performed. A toric fan can be reduced to a minimal toric fan by performing blowdowns until no more are possible. On the other hand, this implies that any toric fan may be obtained from a minimal toric fan by a finite sequence of blowups. Minimal toric manifolds are those that do not admit a symplectic toric blowdown (see Section 2.4).

Proposition 2.5 (*[10]*)**.** *A blowup/down on a fan corresponds to a blowup/down on the associated toric manifold. In particular, a toric manifold is minimal if and only if its fan is minimal.*

Minimal toric fans were classified in [10], and this implies a classification of minimal toric manifolds. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on a toric fan by acting on each vector in the fan.

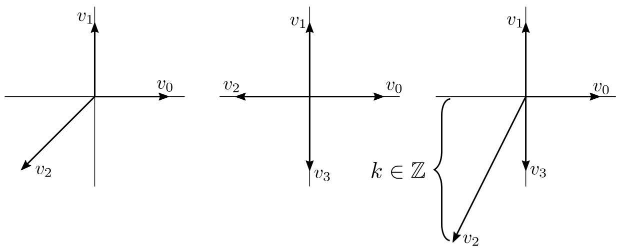


Fig. 2. The three possible minimal toric fans (up to $\mathrm{SL}_2(\mathbb{Z})$) listed in Theorem 2.6, where $k \in \mathbb{Z}$ is the parameter for the Hirzebruch trapezoid and in the figure we show the case of $k = -2$.

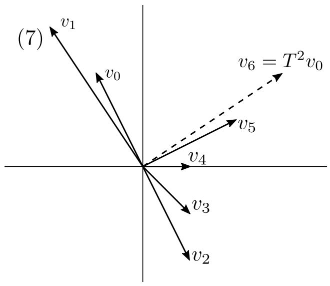


Fig. 3. A minimal semitoric helix of length 6 and complexity 2. In the classification from Theorem 3.6 this is a type (7) minimal semitoric helix with $A_0 = ST^2ST^2$.

Theorem 2.6 (*Fulton [10]*)**.** *A toric manifold is minimal if and only if its fan is one of the following up to the action of* $\mathrm{SL}_2(\mathbb{Z})$*:*

1. $v_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_1 = \begin{pmatrix}0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}-1\\-1\end{pmatrix}$;
2. $v_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_1 = \begin{pmatrix}0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}-1\\0\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\-1\end{pmatrix}$;
3. $v_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_1 = \begin{pmatrix}0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}-1\\k\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\-1\end{pmatrix}$ *for* $k \in \mathbb{Z}$, $k \neq 0, \pm 1$.

These fans are shown in Fig. 2. Respectively, these are known as the Delzant triangle, the square, and the Hirzebruch trapezoid named for the shapes of their associated Delzant polygons. They correspond, in order, to $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and a Hirzebruch surface.

2.2. Semitoric manifolds

Definition 2.7. A *symplectic semitoric manifold* is a connected 4-dimensional integrable system $(M, \omega, F = (J, H))$ such that:

1. J is proper, that is, if $K \subset \mathbb{R}$ is compact then $J^{-1}(K)$ is compact;
2. the Hamiltonian vector field $\mathcal{X}_J$ induced by J has periodic flow of period 2π and the S^1-action generated by this flow is effective;
3. all singularities of F are non-degenerate and contain no hyperbolic blocks.

Item (3) refers to the Williamson classification of singularities for integrable systems (see [21]). In [22] Eliasson extends the pointwise classification of singular points implied by Williamson's classification of Cartan subalgebras of $\mathfrak{sp}(2n)$ [23] to a local normal form for non-degenerate singular points. Since $\dim(M) = 4$, item (3) implies that any point $p \in M$ in a symplectic semitoric manifold is one of: completely regular; elliptic–regular; elliptic–elliptic; or focus–focus. In this article we assume that all symplectic semitoric manifolds are *simple*, which means there is at most one focus–focus point in each level set of J.

Definition 2.8. Given two symplectic semitoric manifolds (M, ω, F) and (M', ω', F') a symplectomorphism $\psi : (M, \omega) \to (M', \omega')$ is a *semitoric isomorphism* if $\psi^*(J', H') = (J, f(J', H'))$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map with $\frac{\partial}{\partial y} f > 0$ everywhere.

2.3. Coupled angular momenta

Here we present an example of a semitoric system which is not toric. Consider $\mathbb{S}^2 \times \mathbb{S}^2$ with coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$, where $\mathbb{S}^2$ is the 2-sphere. The Delzant polygon of $\mathbb{S}^2 \times \mathbb{S}^2$ endowed with the toric integrable system given by

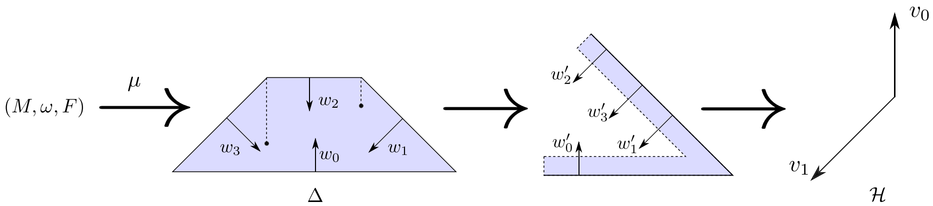


Fig. 4. The helix can be recovered from the semitoric polygon invariant by "unwinding" the polygon to correct for the effect of the focus–focus points and then removing the resulting repeated vectors.

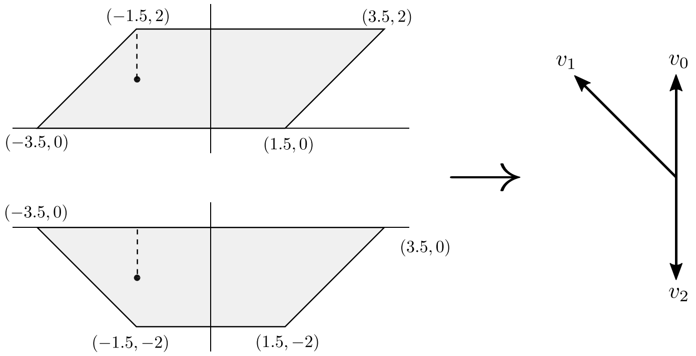


Fig. 5. Helix for the coupled spin system.

$F = (z_1, z_2)$ with any product symplectic form is a rectangle, where the lengths of the sides are determined by the symplectic area of each copy of $\mathbb{S}^2$. Its associated fan is formed by the normal vectors to the faces of the polygon, given by $(1, 0), (0, 1), (-1, 0), (0, -1)$. If instead we consider the symplectic semitoric manifold with $F = (J, H)$ on $\mathbb{S}^2 \times \mathbb{S}^2$ with the standard product symplectic form where

$$J = z_1 + \frac{5}{2} z_2$$
$$H = \frac{1}{2} z_1 + \frac{1}{2}(x_1 x_2 + y_1 y_2 + z_1 z_2)$$

then we obtain the coupled spin system. In [13] it is shown that this is indeed a symplectic semitoric manifold and the authors find the two representatives of the polygons associated to the system obtained by Vũ Ngọc's cutting procedure. One of these polygons has vertices $(-3.5, 0), (-1.5, 2), (3.5, 2), (1.5, 0)$, shown in Fig. 5. We construct the helix of this system in Section 4.5.

2.4. Toric blowups/downs for symplectic toric and semitoric manifolds

These operations are standard, and can be found for instance in [24, Chapter 7]. Let (M, ω, F) be a symplectic toric or symplectic semitoric 4-manifold with $p \in M$ an elliptic–elliptic point. Then there exists complex coordinates z_1, z_2 in an open chart $U \subset M$ centered at p such that the symplectic form is given by $\omega_0 = \frac{-\mathrm{i}}{2}\left(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z}_2\right)$ and $F(z_1, z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) + F(0, 0)$. Let $\phi : U \to \mathbb{C}^2$ denote the map $\phi = (z_1, z_2)$ and let $V = \phi(U)$. Let $\mathrm{B}^4(r) \subset \mathbb{C}^2$ denote the standard ball of radius $r > 0$. For any $\lambda > 0$ sufficiently small such that $\mathrm{B}^4(\lambda) \subset V$ we can define locally in this chart the toric blowup of weight λ. Since p is an elliptic–elliptic point this must be possible for some $\lambda > 0$.

Define $\widetilde{\mathbb{C}}^2 \subset \mathbb{C}^2 \times \mathbb{CP}^1$ to be those pairs (z, ℓ) such that $z \in \ell$. That is,

$$\widetilde{\mathbb{C}}^2 = \{(z_1, z_2; [w_0, w_1]) \mid w_j z_k = w_{k-1} z_{j+1} \text{ for } j = 0, 1 \text{ and } k = 1, 2\}$$

(the manifold $\widetilde{C}^2$ is the usual (non-symplectic) blowup of $\mathbb{C}^2$ at the origin). There are natural projections

$$\begin{tikzcd} & \widetilde{\mathbb{C}}^2 \arrow[dl, "\pi_{\mathbb{C}^2}"'] \arrow[dr, "\pi_{\mathbb{CP}^1}"] & \\ \mathbb{C}^2 & & \mathbb{CP}^1 \end{tikzcd}$$

and for each $r > 0$ define $L(r) = \pi_{\mathbb{C}^2}^{-1}(\mathrm{B}^4(r))$. For each $\lambda > 0$ define a symplectic form $\rho(\lambda)$ on $\widetilde{C}^2$ by $\rho(\lambda) = \pi_{\mathbb{C}^2}^* \omega_0 + \lambda^2 \pi_{\mathbb{CP}^1}^* \omega_{\mathrm{FS}}$ where ω_{FS} is the Fubini–Study form on $\mathbb{CP}^1$ and ω_0 is the standard symplectic form on $\mathbb{C}^2$. Finally, with λ and δ chosen small enough so that $\mathrm{B}^4(\sqrt{\lambda^2+\delta^2}) \subset V$, define $\widetilde{\mathbb{C}^2}_\lambda = \left(\mathbb{C}^2 \setminus B^4\left(\sqrt{\lambda^2+\delta^2}\right)\right) \cup L(\delta)$. Since $\rho(\lambda) = \omega_0$ outside of $\mathrm{B}^4(\sqrt{\lambda^2+\delta^2})$ there is no problem defining a symplectic structure on $\widetilde{M}(p,\lambda) = (M \setminus \phi^{-1}(B^4(\sqrt{\lambda^2+\delta^2}))) \cap L(\delta)$, which is known as the *symplectic toric blowup of M at p of size λ*. This is similar to the standard symplectic blowup except that the choice of chart forces the embedded ball used in this construction to be $\mathbb{R}^2$-equivariantly embedded, where the $\mathbb{R}^2$-action on M comes from the flow of $\mathcal{X}_{F_1}$ and $\mathcal{X}_{F_2}$, which descends to a $\mathbb{T}^2$-action for symplectic toric manifolds and an $(\mathbb{S}^1 \times \mathbb{R})$-action for symplectic semitoric manifolds (see [25] for an investigation of symplectic semitoric manifolds as symplectic $(S^1 \times \mathbb{R})$-manifolds).

The inverse of this operation is known as a *toric blowdown*. Performing a toric blowup or down on a toric manifold corresponds to performing a blowup or down on the associated toric fan. We will see that performing a toric blowup/down on a symplectic semitoric manifold corresponds to performing a combinatorial operation, which we call a blowup/down, on the associated semitoric helix (see Section 4). We will often simply call a toric blowup a *blowup* (and similar for a *blowdown*).

Definition 2.9. A symplectic semitoric manifold (M, ω, F) is *minimal* if it does not admit a blowdown.

That is, a symplectic semitoric manifold is minimal if there does not exist any symplectic semitoric manifold (M', ω', F') such that (M, ω, F) can be obtained from (M', ω', F') by a symplectic blowup.

For the present paper we will not be concerned with the size of the blowups since this will not change the associated helix and will not effect whether or not the resulting manifold is minimal. Thus, we will often say "the blowup of M at p" to really mean "one of the blowups of M at p" or even "the family of all manifolds which can be obtained by performing a blowup of some weight on M at p".

Remark 2.10. This definition of blowup/down can be extended to be used around any completely elliptic point of any integrable system of any dimension. $\diamond$

2.5. The algebraic technique

Let $S, T \in \mathrm{SL}_2(\mathbb{Z})$ be the standard generators given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

so $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid STS = T^{-1}ST^{-1}, S^4 = I\rangle$ and $\mathrm{PSL}_2(\mathbb{Z}) = \langle S, T \mid STS = T^{-1}ST^{-1}, S^2 = I\rangle$. We denote by $\mathbb{Z} * \mathbb{Z}$ the free group on letters S and T.

Notation: Since we consider several groups generated by S and T we use $=_H$ to denote equality in the group H. For instance, $S^4 =_{\mathrm{SL}_2(\mathbb{Z})} I$ but $S^4 \neq_{\mathbb{Z}*\mathbb{Z}} I$.

Given $v, w \in \mathbb{Z}^2$ we denote by $[v, w]$ the 2×2 matrix with first column v and second column w and denote by $\det(v, w)$ the determinant of $[v, w]$.

Definition 2.11. Let $G = \langle S, T \mid STS = T^{-1}ST^{-1}\rangle$.

The group G is isomorphic to the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$ [12, Proposition 3.7], as in the following diagram:

$$\begin{tikzcd} G \arrow[r, hook, "\rho"] \arrow[d] & \widetilde{\mathrm{SL}_2(\mathbb{R})} \arrow[d] \\ \mathrm{SL}_2(\mathbb{Z}) \arrow[r, hook, "i"] & \mathrm{SL}_2(\mathbb{R}) \end{tikzcd}$$

where $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ denotes the universal cover of $\mathrm{SL}_2(\mathbb{R})$, which has fundamental group $\mathbb{Z}$. Above $\rho : G \to \mathrm{SL}_2(\mathbb{R})$ denotes the map that takes G isomorphically to the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ given by

$$\rho(T) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}_{0\leqslant t\leqslant 1} \text{ and } \rho(S) = \begin{pmatrix} \cos\left(\dfrac{\pi t}{2}\right) & -\sin\left(\dfrac{\pi t}{2}\right) \\ \sin\left(\dfrac{\pi t}{2}\right) & \cos\left(\dfrac{\pi t}{2}\right) \end{pmatrix}_{0\leqslant t\leqslant 1} \tag{2.2}$$

(as in [12, Proposition 3.7]), $i : \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ is the inclusion map and the other two maps are the natural projections. Each element of the kernel of the natural projection from G to $\mathrm{SL}_2(\mathbb{Z})$, denoted $\ker(G \to \mathrm{SL}_2(\mathbb{Z}))$, represents a closed loop in $\mathrm{SL}_2(\mathbb{R})$.

Let $(\mathbb{R}^2)^* := \mathbb{R}^2 \setminus \{(0, 0)\}$.

Definition 2.12. Given any closed loop $\widetilde{\gamma} : [0, 1] \to (\mathbb{R}^2)^*$, $\widetilde{\gamma}(0) = \widetilde{\gamma}(1)$, we denote by $\mathrm{wind}(\widetilde{\gamma}) \in \mathbb{Z}$ the usual winding number of $\widetilde{\gamma}$ in $(\mathbb{R}^2)^*$.

Define $\mathrm{pr} : \mathrm{SL}_2(\mathbb{R}) \to (\mathbb{R}^2)^*$ by

$$\mathrm{pr} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}.$$

Since $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \pi_1((\mathbb{R}^2)^*) \cong \mathbb{Z}$ and pr sends a generator of $\pi_1(\mathrm{SL}_2(\mathbb{R}))$ to a generator of $\pi_1((\mathbb{R}^2)^*)$, pr induces an isomorphism at the level of fundamental groups.

Definition 2.13. Given any loop $\gamma : [0, 1] \to \mathrm{SL}_2(\mathbb{R})$, $\gamma(0) = \gamma(1)$, we define the *winding number of* γ, denoted $\mathrm{wind}(\gamma)$, by $\mathrm{wind}(\gamma) := \mathrm{wind}\big(\mathrm{pr}(\gamma)\big)$.

Next we extend the map $\mathrm{wind} \circ \rho : \ker(G \to \mathrm{SL}_2(\mathbb{Z})) \to \mathbb{Z}$ to all of G. Let $W : \mathbb{Z} * \mathbb{Z} \to \frac{1}{12}\mathbb{Z}$ be the homomorphism generated by $W(S) = \frac{1}{4}$ and $W(T) = \frac{-1}{12}$. Since $W(STS) = W(T^{-1}ST^{-1})$, W descends to a map on G which we also denote W. The map is known as the *winding number* [12] because if $\sigma \in \ker(G \to \mathrm{SL}_2(\mathbb{Z}))$ then $W(\sigma)$ agrees with $\mathrm{wind}(\rho(\sigma))$ as in Definition 2.13, where ρ is as in Eq. (2.2).

Lemma 2.14 (*[12]*). *Given* $\sigma \in \ker(G \to \mathrm{SL}_2(\mathbb{Z}))$, $W(\sigma) = \mathrm{wind}(\mathrm{pr} \circ \rho(\sigma))$.

Proof. The map W is a homomorphism and $W(S^4) = \mathrm{wind}(\rho(S^4))$. Since S^4 is a generator of $\ker(G \to \mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{Z}$ this uniquely defines it. □

2.6. *The semitoric helix and* $\mathrm{SL}_2(\mathbb{Z})$

Let $(\mathbb{Z}^2)^\infty$ be the set of sequences indexed by $\mathbb{Z}$ in $\mathbb{Z}^2$. For $\{v_i\}_{i\in\mathbb{Z}}, \{w_i\}_{i\in\mathbb{Z}} \in (\mathbb{Z}^2)^\infty$ let $\sim$ be the equivalence relation on $(\mathbb{Z}^2)^\infty$ given by $\{v_i\}_{i\in\mathbb{Z}} \sim \{w_i\}_{i\in\mathbb{Z}}$ if and only if there exists $k, \ell \in \mathbb{Z}$ such that

$$v_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k w_{i+\ell}$$

for all $i \in \mathbb{Z}$. Let $[\{v_i\}_{i\in\mathbb{Z}}] \in (\mathbb{Z}^2)^\infty/\sim$ denote the equivalence class of $\{v_i\}_{i\in\mathbb{Z}} \in (\mathbb{Z}^2)^\infty$.

Definition 2.15. A *semitoric helix* is a triple $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ where $d \in \mathbb{Z}_{>0}$, $c \in \mathbb{Z}_{\geqslant 0}$, and $[\{v_i\}_{i\in\mathbb{Z}}] \in (\mathbb{Z}^2)^\infty/\sim$ such that:

1. $\det(v_i, v_{i+1}) = 1$ for all $i \in \mathbb{Z}$;
2. $v_0, \ldots, v_{d-1}$ are arranged in counter-clockwise order;
3. $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} v_i = v_{i+d}$ for all $i \in \mathbb{Z}$.

We say that a semitoric helix $(d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ has *length* d and *complexity* c. It is *minimal* if

$$v_i \neq v_{i-1} + v_{i+1}$$

for all $i \in \mathbb{Z}$.

Lemma 2.16. *The minimality condition does not depend on the choice of representative of* $[\{v_i\}_{i\in\mathbb{Z}}]$.

Proof. Let $\{w_i\}_{i\in\mathbb{Z}} \in [\{v_i\}_{i\in\mathbb{Z}}]$ so there exist $k, \ell \in \mathbb{Z}$ such that $v_i = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} w_{i+\ell}$ for all $i \in \mathbb{Z}$. Thus $v_j = v_{j-1} + v_{j+1}$ if and only if $w_{j+\ell} = w_{j+\ell-1} + w_{j+\ell+1}$ by applying $\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$. □

A minimal semitoric helix is shown in Fig. 3. In light of item (3) of Definition 2.15, a semitoric helix of given complexity $c > 0$ and length d is determined by any d consecutive vectors in any representative. We take the helix to be an infinite list of vectors instead of a finite one because working with a finite set of vectors would lead to complicated behavior relating the first and last vector in the list.

Definition 2.17. Let $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ be a semitoric helix. The *blowup of* $\mathcal{H}$ *at* v_i is the helix $(d+1, c, [\{w_i\}_{i\in\mathbb{Z}}])$, where $\{w_i\}$ is formed from $\{v_i\}_{i\in\mathbb{Z}}$ by inserting $v_{i+kd} + v_{i+1+kd}$ between v_{i+kd} and v_{i+1+kd} for all $k \in \mathbb{Z}$. If $v_j = v_{j-1} + v_{j+1}$ for some $j \in \mathbb{Z}$ then the *blowdown of* $\mathcal{H}$ *at* v_i is the helix $(d-1, c, [\{u_i\}_{i\in\mathbb{Z}}])$ where $\{u_i\}$ is produced by removing $\{v_{j+nd}\}_{n\in\mathbb{Z}}$ from $\{v_i\}_{i\in\mathbb{Z}}$.

3. Results

In this section we state the results of this paper. The proofs of the four results stated in Section 3.1, Proposition 3.1, Proposition 3.3, Theorem 3.5, and Theorem 3.6, are the subject of the remaining four sections of the paper.

3.1. Statements of main results

Proposition 3.1. *Associated to each semitoric manifold there is a unique semitoric helix. Moreover, if two semitoric manifolds are isomorphic then they have the same semitoric helix and given any semitoric helix there exists a semitoric manifold to which it is associated.*

Proposition 3.1 is a combination of Lemmas 4.2, 4.3, and 4.9, each proven in Section 4, in which we also describe the construction of the semitoric helix. Proposition 3.1 allows us to make the following definition.

Definition 3.2. Let $\mathcal{S}_{\mathrm{ST}}$ denote the set of symplectic semitoric manifolds and $\mathcal{S}_{\mathrm{H}}$ denote the collection of semitoric helices. The map $\mathrm{hlx} : \mathcal{S}_{\mathrm{ST}} \to \mathcal{S}_{\mathrm{H}}$ assigns to each symplectic semitoric manifold (M, ω, F) a semitoric helix $\mathrm{hlx}(M, \omega, F) = \mathcal{H}$, where $\mathcal{H}$ is the semitoric helix associated to (M, ω, F).

We need some notation for the next result. For a semitoric helix $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ let $-\mathcal{H} = (d, c, [\{-v_i\}_{i\in\mathbb{Z}}])$. If $\mathcal{H} = \mathcal{H}'$ or $\mathcal{H} = -\mathcal{H}'$ we write $\mathcal{H} = \pm\mathcal{H}'$. A cyclic permutation of a list $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ of integers is given by

$$(a_{k \bmod d}, a_{k+1 \bmod d}, \ldots, a_{k+d-1 \bmod d})$$

for some $k \in \mathbb{Z}$. The motivation for the next result is to find a correspondence between helices up to isomorphism and lists of integers satisfying some conditions up to cyclic shift.

Proposition 3.3. *Given any semitoric helix $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ there exists a list of integers $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ such that*

$$a_{i \bmod d} v_{i+1} = v_i + v_{i+2} \tag{3.1}$$

for all $i \in \mathbb{Z}$, which we call the integers associated to $\mathcal{H}$. Furthermore, the following hold:

1. *$(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ are associated to a semitoric helix if and only if*
$$ST^{a_0} \ldots ST^{a_{d-1}} =_G S^4 X^{-1} T^c X$$
for some X in G;
2. *the integers associated to a given helix in this way are unique up to cyclic permutation;*
3. *semitoric helices $\mathcal{H}$ and $\mathcal{H}'$ have the same length, complexity, and associated integers if and only if $\mathcal{H} = \pm\mathcal{H}'$.*

Proposition 3.3 is proven in Section 5.

Definition 3.4. A word $\eta \in \mathbb{Z} * \mathbb{Z}$ is *S-positive* if it can be written using only non-negative powers of S, T, and T^{-1}.

To classify the minimal models of symplectic semitoric manifolds we will show that the associated word of a minimal helix (as in Proposition 3.3) is very close to the following standard form in $\mathrm{PSL}_2(\mathbb{Z})$. In particular, we will see that the word associated to a minimal helix has nearly the smallest possible winding number, which means it can be obtained from its standard form by only a small number of transformations.

Theorem 3.5 (*Standard form in* $\mathrm{PSL}_2(\mathbb{Z})$)**.** *If $X \in \mathrm{SL}_2(\mathbb{Z})$ there exists a unique string $\overline{X} \in \mathbb{Z} * \mathbb{Z}$ such that $X =_{\mathrm{PSL}_2(\mathbb{Z})} \overline{X}$ and*

$$\overline{X} =_{\mathbb{Z}*\mathbb{Z}} T^b ST^{a_0} \ldots ST^{a_{d-1}}$$

where $a_i > 1$ for $i = 0, \ldots, d-2$. Moreover, $W(\overline{X}) \leqslant W(\eta)$ for all S-positive $\eta \in \mathbb{Z} * \mathbb{Z}$ satisfying $\eta =_{\mathrm{PSL}_2(\mathbb{Z})} X$.

We call $\overline{X}$ the *standard form of X*. Theorem 3.5 is proven in Section 6.

Let

$$\mathcal{S} = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \overline{A} = ST^{a_0} \ldots ST^{a_{d-1}}, \text{for } d > 5, a_{d-1} \notin \{0, 1\} \right\}. \tag{3.2}$$

Recall a semitoric helix of length d is determined by specifying the complexity and any d consecutive vectors in any representative of the helix. Also recall that in [7] it is shown that any symplectic semitoric manifold has only finitely many focus–focus singular points.

Theorem 3.6. *Suppose that (M, ω, F) is a minimal compact symplectic semitoric manifold with $c > 0$ focus–focus points and associated semitoric helix $(d, c, [\{v_i\}_{i\in\mathbb{Z}}]) = \mathrm{hlx}(M, \omega, F)$. If $d < 5$ then the representative $\{v_i\}_{i\in\mathbb{Z}}$ can be chosen to be exactly one of the following:*

Type	Length	$v_0, \dots, v_{d-1}$		Complexity
(1)	$d = 2$	$\binom{0}{1}, \binom{-1}{-2}$		$c = 1$
(2)	$d = 2$	$\binom{0}{1}, \binom{-1}{-1}$		$c = 2$
(3)	$d = 3$	$\binom{0}{1}, \binom{-1}{k}, \binom{0}{-1}$	$k \neq \pm 2$	$c = 1$
(4)	$d = 3$	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{-1}$		$c \neq 2$
(5)	$d = 4$	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{k}, \binom{0}{-1}$	$k \neq \pm 1, 0$	$c \neq 1$
(6)	$d = 4$	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{0}, \binom{k}{-1}$	$k \neq -1$ $k \neq 1 - c$	$c > 0$

Otherwise, $d \geqslant 5$, in which case $d > 5$, and we say that the symplectic semitoric manifold and helix are minimal of type (7). There is a one-to-one correspondence between minimal helices of type (7) and the set $\mathcal{S} \times \mathbb{Z}_{>0}$. Given $c \in \mathbb{Z}_{>0}$ and a basis v_0, v_1 of $\mathbb{Z}^2$ satisfying $[v_0, v_1] \in \mathcal{S}$ then the corresponding minimal helix of type (7) is determined by the following procedure: Let $a_0, \dots, a_{d-1}, d \in \mathbb{Z}$ be the unique integers which satisfy

$$S^2\overline{[v_0, v_1]^{-1}}T^c\overline{[v_0, v_1]} =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \dots ST^{a_{d-1}}. \tag{3.3}$$

Then the recurrence relation

$$v_j = a_{j-2}v_{j-1} + v_{j-2}$$

for $j = 0, \dots, d-1$ and given v_0, v_1 determines the vectors $\{v_i\}_{0 \leqslant i < d}$ which, along with the complexity c, determine the helix, $\mathcal{H}$.

Types (1)–(6) are shown in Theorem 1.3 and a representative example of type (7) is shown in Fig. 3. Theorem 3.6 is a direct consequence of Lemma 7.6 and is proven in Section 7.3.

3.2. Idea of proof of Theorem 3.6

In the proof of Theorem 3.5, the standard form in $\mathrm{PSL}_2(\mathbb{Z})$, we use a reduction algorithm with four types of operations. Three of these steps reduce the winding number by 1/2 and the remaining step, which corresponds to a blowdown, does not change the winding number. We will see, by Lemmas 7.1 and 7.3, that if $a_0, \dots, a_{d-1}$ is associated to a semitoric helix then

$$W(ST^{a_0} \dots ST^{a_{d-1}}) - W(\overline{ST^{a_0} \dots ST^{a_{d-1}}}) = \begin{cases} 1, & X =_{\mathrm{PSL}_2(\mathbb{Z})} T^k \\ \dfrac{1}{2}, & \text{otherwise} \end{cases}$$

and thus we know that $ST^{a_0} \dots ST^{a_{d-1}}$ can be reduced to the standard form from Theorem 3.5 by using only one or two of the moves which reduce W along with any number of blowdowns. Assuming that the original helix is minimal means that the first step we take is not a blowdown, and thus the reduction is achieved by only one step (possibly followed by blowdowns) except for in the special case that $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^k$.

This observation allows us to prove Lemma 7.6, which classifies all minimal words satisfying Eq. (5.2). This implies Theorem 3.6, which is proven in Section 7.2. The method of the proof of Theorem 3.6 is carried out on a specific example in Section 3.4.

3.3. Consequences of Theorem 3.6

Corollary 3.7. *Suppose that $\mathcal{H}$ is a minimal helix of length $d > 4$. Then $\mathcal{H}$ is of type (7) from Theorem 3.6 and there exists a representative $\{v_i\}_{i\in\mathbb{Z}}$ such that $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ and the following hold:*

1. *$v_0 = -v_2$;*
2. *there exists $k \in \mathbb{Z}$ with $2 < k < d$ such that v_k is $\binom{1}{0}$ or its negative.*

Proof. From Eq. (3.3) in Theorem 3.6 we see that $a_0 = 0$ and so the given recurrence relation $v_j = a_{j-2}v_{j-1} + v_{j-2}$ with $j = 2$ gives $v_2 = -v_0$.

Suppose $\mathcal{H}$ has associated integers $a_0, \dots, a_{d-1}$ and let $A_0 = [v_0, v_1]$. Then

$$S^2\overline{A_0^{-1}}T^c\overline{A_0} =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \dots ST^{a_{d-1}}, \tag{3.4}$$

implies

$$S^2\overline{A_0^{-1}}T^c =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \dots ST^{a_{k+1}} \tag{3.5}$$

for some $k \in \mathbb{Z}$, since $\overline{A_0}$ starts with S. By the recurrence relation $a_i v_{i+1} = v_i + v_{i+2}$ we see

$$[v_{k+2}, v_{k+3}] =_{\mathrm{SL}_2(\mathbb{Z})} A_0 S T^{a_0} \dots S T^{a_{k+1}}. \tag{3.6}$$

Combining Eqs. (3.5) and (3.6) yields $[v_{k+2}, v_{k+3}] =_{\mathrm{PSL}_2(\mathbb{Z})} A_0 S^2 A_0^{-1} T^c$ which implies

$$[v_{k+2}, v_{k+3}] =_{\mathrm{PSL}_2(\mathbb{Z})} T^c =_{\mathrm{PSL}_2(\mathbb{Z})} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

so v_{k+2} is the required vector. □

We now have the tools to prove Theorem 1.4, but first we need the following lemma.

Lemma 3.8. *Any semitoric helix of complexity $c > 2$ includes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or its negative.*

Proof. We will show that this vector is in every minimal semitoric helix of complexity $c > 2$, and since every semitoric helix can be produced by a sequence of blowups on a minimal semitoric helix and blowups do not remove vectors from the helix or change the complexity, the result will follow.

Since $c > 2$ the only possible types for minimal models are types (4)–(7). By Theorem 3.6 we see that (4), (5), and (6) include the required vector. Helices of type (7) include the required vector by Corollary 3.7. □

Proof of Theorem 1.4. Suppose that $(M, \omega, F = (J, H))$ is a compact symplectic semitoric manifold with $c \geqslant 2$ focus–focus points and that J achieves its maximum and minimum at a single point each. This means that $F(M)$ does not include as its boundary a vertical line segment, which in turn, since a straightening map cannot produce a vertical wall, implies that $\mathrm{hlx}(M, \omega, F)$ does not include a vector on the x-axis.

Lemma 3.8 states that if $c > 2$ then $\mathrm{hlx}(M, \omega, F)$ includes a vector on the x-axis, so this case cannot occur. Otherwise, $c = 2$. Since blowups do not change the complexity, Theorem 3.6 implies that $\mathrm{hlx}(M, \omega, F)$ can be obtained from a minimal semitoric helix of either type (2) or type (7) by a sequence of blowups. By Corollary 3.7 any helix of type (7) includes a vector on the x-axis and thus any blowup of a helix of type (7) must also include such a vector. Thus, this case cannot occur.

Suppose that $\mathrm{hlx}(M, \omega, F)$ can be obtained from a minimal helix of type (2) via a nonzero number of blowups. There are two distinct blowups which can be performed on a minimal helix of type (2), adding either the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

or the vector

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In either case a vector on the x-axis is including in the resulting helix, so this case cannot occur either. The only remaining case is that (M, ω, F) is minimal of type (2). □

3.4. A representative example of Theorem 3.6

Suppose that $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ is a minimal semitoric helix of length $d > 4$. By Corollary 3.7, $\mathcal{H}$ is of type (7) and the representative $\{v_i\}_{i\in\mathbb{Z}}$ can be chosen to satisfy $v_0 = -v_2$. Then $\mathcal{H}$ is determined by its complexity c and the basis (v_0, v_1) of $\mathbb{Z}^2$.

Let $\mathcal{H}$ have complexity $c = 2$ and

$$v_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ and } v_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Next we compute $\mathcal{H}$. Define $A_0 := [v_0, v_1]$, which means that $A_0 = ST^2ST^2$ in terms of the generators S, T. Define $A_i = [v_i, v_{i+1}]$ for $i \in \mathbb{Z}$ and notice that $\det(v_i, v_{i+1}) = \det(v_{i+1}, v_{i+2}) = 1$ implies that there exists some $a_i \in \mathbb{Z}$ so that

$$[v_{i+1}, v_{i+2}] = [v_i, v_{i+1}] \begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix}$$

which means that $A_{i+1} = A_i S T^{a_i}$ for all $i \in \mathbb{Z}$. Then,

$$A_d =_{\mathrm{SL}_2(\mathbb{Z})} A_{d-1} S T^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} A_0 S T^{a_0} S T^{a_{d-2}} \dots S T^{a_{d-1}}$$

and since $\mathcal{H}$ is of length d and complexity 2, $A_d = T^2 A_0$. Thus $T^2 A_0 =_{\mathrm{SL}_2(\mathbb{Z})} A_0 S T^{a_0} \dots S T^{a_{d-1}}$ which implies

$$S T^{a_0} \dots S T^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} A_0^{-1} T^2 A_0. \tag{3.7}$$

Our goal is now to recover $a_0, \ldots, a_{d-1}$. Let $\sigma =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \ldots ST^{a_{d-1}}$. Lifting Eq. (3.7) to the group G yields

$$\sigma =_G S^{4k} A_0^{-1} T^2 A_0, \tag{3.8}$$

for some choice of $k \in \mathbb{Z}$. The fact that the vectors $v_0, \ldots, v_{d-1}$ in the semitoric helix are arranged in counter-clockwise order forces $k = 1$ by Proposition 3.3.

By substituting for A_0 and using the relations of G on Eq. (3.8) we see that

$$\sigma =_G S^4(ST^2ST^2)^{-1}T^2(ST^2ST^2) =_G S^2T^{-1}ST^2ST^3ST^2ST^2$$

and by taking the standard form of both sides of this equality we have

$$\overline{\sigma} =_{\mathbb{Z}*\mathbb{Z}} T^{-1}ST^2ST^3ST^2ST^2. \tag{3.9}$$

From this we deduce $W(\sigma) - W(\overline{\sigma}) = \frac{10}{12} - \frac{4}{12} = \frac{1}{2}$ which means that σ can be reduced to $\overline{\sigma}$ by a substitution of either

$$S^2 =_{\mathrm{PSL}_2(\mathbb{Z})} I \text{ or } ST^{-n}S =_{\mathrm{PSL}_2(\mathbb{Z})} (TST)^n$$

combined with several applications of the blowdown operation, $STS =_{\mathrm{PSL}_2(\mathbb{Z})} T^{-1}ST^{-1}$. This comes from observing that $S^2 =_{\mathrm{PSL}_2(\mathbb{Z})} I$ and $ST^{-n}S =_{\mathrm{PSL}_2(\mathbb{Z})} (TST)^n$ each reduce the winding number by $\frac{1}{2}$ and the proof of Theorem 3.5, which uses an algorithm consisting of these three reductions to put an element of $\mathrm{PSL}_2(\mathbb{Z})$ in standard form.

To finish, we recall that $\mathcal{H}$ is minimal, so it does not admit a blowdown. This means σ may be obtained from $\overline{\sigma}$ by a sequence of blowups followed by either the addition of a S^2 term or the replacement of $(TST)^n$ by $ST^{-n}S$ for some $n > 0$. By examining the form of $\overline{\sigma}$ in Eq. (3.9) we see the requirement that $a_0 = 0$ forces the addition of S^2 at the front of the word and the requirement that $a_i \neq 1$ for all i implies that no blowups may be performed before adding S^2 to the front which forces $\sigma =_{\mathbb{Z}*\mathbb{Z}} S^2T^{-1}ST^2ST^3ST^2ST^2$. Thus $d = 7$ and $a_0 = 0, a_1 = -1, a_2 = 2, a_3 = 2, a_4 = 3, a_5 = 2, a_6 = 2$. Since we were given v_0 and v_1 these values determine $\mathcal{H}$ by the recurrence relation Eq. (5.1). An image of the first seven vectors in this helix is given in Fig. 3.

3.5. Relationship to [12]

The present paper uses the results of the paper by the same authors [12] in Section 2.5 and the proof of Lemma 6.3. However, the tools from [12] are suited to study a different problem than the one the present paper concerns. Indeed, in [12] we introduce the *semitoric fan*, which is a set of vectors in $\mathbb{Z}^2$ associated to a given semitoric manifold, but it is not the same as the semitoric helix. The semitoric fan can be obtained as the inwards pointing normal vectors of the semitoric polygon (see Section 4.2). In [12] we discuss/introduce:

- The relationship between vectors in $\mathbb{Z}^2$ for which consecutive vectors form a basis to sequences of matrices in $\mathrm{SL}_2(\mathbb{Z})$;
- The presentation of the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$;
- The winding number;
- The algebraic formulation of a blowup in terms of the matrices S and T;
- A few basic tools such as [12, Lemma 3.8] (which we state in the present paper as Lemma 6.3).

The semitoric fan is not suited to study minimal models and blowups/downs for several reasons. For instance, performing a blowdown on an edge which passes over a focus–focus point affects the fan in a very complicated way, despite the fact that the presence of a distant focus–focus point does not affect the symplectic geometry of the blowdown. Also, the presence of focus–focus points affects the relationship between adjacent vectors, and if all of these effects are moved to the end of the list of vectors (as is done in [12]) there is an asymmetry in choosing which vector represents the end of the list, and changing the end vectors can be complicated. For these and related reasons we introduce the helix is the present paper, which changes in a simple, uniform way with respect to all blowups and blowdowns, and can be used to investigate the more delicate situation of finding minimal models. Additionally, while changing the end of a semitoric fan can be complicated, the analogous operation for a semitoric helix (changing which vector is v_0) is simple: it is just translating the indices on the $\{v_i\}$. In order to classify semitoric minimal models, we build upon the existing techniques from [12] but need to refine them in several ways:

1. We introduce the notion of a helix as a more symmetric geometric notion (Proposition 3.1);
2. We improve the correspondence to an exact correspondence between helices and certain algebraic structures (Proposition 3.3);
3. We introduce the notion of the minimal representation of an element of $\mathrm{SL}_2(\mathbb{Z})$ and prove Theorem 3.5, which states that such a representation always exists and is unique.

Items (1), (2), and (3) above are precisely what we need to classify minimal helices in Theorem 3.6. Items (1) and (2) allow us to translate questions about semitoric manifolds directly to questions about specific words in $\mathrm{SL}_2(\mathbb{Z})$, and item (3) helps us to understand the form of the words which correspond to minimal semitoric manifolds.

4. The semitoric helix

In this section we give the details of the construction of the semitoric helix and prove Proposition 3.1. To do this, we need the following result of Vũ Ngọc, adapted slightly to fit the present situation.

Theorem 4.1 (*Follows from [7, Theorem 3.8]*). *If (M, ω, F) is a symplectic semitoric manifold and $U \subset F(M)$ is simply connected, open as a subset of $F(M)$, and contains no values of focus–focus points of F then there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \circ F$ is a momentum map for a Hamiltonian $\mathbb{T}^2$-action on $F^{-1}(U)$ and f fixes the first component, i.e. there exists a function $f^{(2)} : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = (x, f^{(2)}(x, y))$.*

Such a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is known as a *straightening map* for the symplectic semitoric manifold (M, ω, F).

4.1. Intrinsic construction of the helix

Let $(M, \omega, F = (J, H))$ be a compact symplectic semitoric manifold; we will construct the associated semitoric helix, $\mathrm{hlx}(M, \omega, F)$. The images under F of the elliptic–regular and elliptic–elliptic singular points all lie in the boundary $\partial F(M)$ and there are finitely many focus–focus points, whose images lie in the interior $\mathrm{int}(F(M))$ (see [7]). Choose a set $U' \subset F(M)$ such that

1. U' is open as a subset of $F(M)$;
2. U' contains $\partial F(M)$;
3. U' does not contain the image of any focus–focus point;
4. U' has fundamental group $\mathbb{Z}$.

This is possible because $F(M)$ is simply connected [7, Theorem 3.4] and compact (by assumption). For instance, U' could be chosen to be the set of all points in $F(M)$ less than a distance of ε from the boundary for a sufficiently small $\varepsilon > 0$. Let $\ell \subset F(M)$ be any line segment starting from a point in $F(M) \setminus U'$ and ending outside $F(M)$ which intersects $\partial F(M)$ in exactly one connected component and does not include any singular points of maximal rank of $F(M)$. Let $U = U' \setminus \ell$. We call such a subset a *helix neighborhood* for (M, ω, F), see the first step of Fig. 1.

By Theorem 4.1 there exists a straightening map $f : \mathbb{R}^2 \to \mathbb{R}^2$ so that $\mu = f \circ F$ is the momentum map for a Hamiltonian $\mathbb{T}^2$-action on $F^{-1}(U)$. Thus, $f(\partial F(M) \cap U)$ is piecewise linear of finitely many segments each with rational slope, because it is the image of the elliptic–regular and elliptic–elliptic singular points of $(F^{-1}(U), \omega, \mu)$ and this system has only finitely many elliptic–elliptic fixed points. Let $d \in \mathbb{Z}$ be one less than the number of segments so that there are $d + 1$ segments in this piecewise linear curve and let $v_0, \ldots, v_d \in \mathbb{Z}^2$ be the consecutive primitive vectors normal to these segments facing towards the interior of $f(U)$, numbered so that $v_0, \ldots, v_{d-1}$ are arranged in counter-clockwise order, as shown in the last step of Fig. 1.

The relationship between v_0 and v_d is determined by the monodromy from the focus–focus points of the system. In [7] Vũ Ngọc studies the monodromy effect of focus–focus points on toric momentum maps defined on the preimage of the momentum map image minus a few "cuts" that remove the focus–focus points and keep the set simply connected. The proof holds for other simply connected sets, such as the set U, and in this case implies that $v_d = T^c v_0$ because the set U loops around all c focus–focus points of the system.

Finally, by Definition 2.15 part (3) $v_0, \ldots, v_d$ extend to a unique semitoric helix $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$. We say that $\mathcal{H}$ is associated to the given symplectic semitoric manifold (M, ω, F).

Now we must show that the semitoric helix constructed in this was is the unique one associated to M. That is, we show the helix does not depend on the choices of open set U', line segment ℓ, and straightening map f made during the construction.

Lemma 4.2. *There is precisely one semitoric helix associated to each symplectic semitoric manifold.*

Proof. Let (M, ω, F) be a symplectic semitoric manifold with d elliptic–elliptic points and c focus–focus points. Any semitoric helix produced from the above construction must have length d and complexity c. Now let $\mathcal{H}_j = (c, d, [\{v_i^j\}_{i\in\mathbb{Z}}])$ be a semitoric helix constructed from (M, ω, F) as above using a set U_j', line segment ℓ_j, and straightening map f_j for $j = 1, 2$. We will show $\mathcal{H}_1 = \mathcal{H}_2$.

We may assume that $U_1' = U_2'$ by replacing each with $U' = U_1' \cap U_2'$ and using the restricted straightening maps. Now $U_1 = U' \setminus \ell_1$ and $U_2 = U' \setminus \ell_2$ and, assuming $U \cap \ell_1 \neq U \cap \ell_2$, the set $U_1 \cap U_2$ has two connected components (if $U \cap \ell_1 = U \cap \ell_2$ the remainder of the proof simplifies). Denote these two components by A and B ordered so that $v_0^1, \ldots, v_k^1$ are the inwards pointing normal vectors of the boundary of $f_1(A)$ and $v_{k+1}^1, \ldots, v_d^1$ are the inwards pointing normal vectors of $f_1(B)$.

Since $A \subset U_j$ for $j = 1, 2$ we see $f_j|_A \circ F : F^{-1}(A) \to \mathbb{R}^2$ is a toric momentum map for $j = 1, 2$. Thus, by [7, Theorem 3.8] there exists $k_A \in \mathbb{Z}$ and $x_A \in \mathbb{R}^2$ such that

$$f_1|_A = T^{k_A} \circ f_2|_A + x_A \tag{4.1}$$

and similarly there exists $k_B \in \mathbb{Z}$ and $x_B \in \mathbb{R}^2$ such that

$$f_1|_B = T^{k_B} \circ f_2|_B + x_B. \tag{4.2}$$

Thus, $v_i^1 = T^{k_A} v_{i+d-k}^2$ for $i = 0, \dots, k$ and $v_i^1 = T^{k_B} v_{i-k-1}^2$ for $i = k+1, \dots, d$. Now, $\{v_i^2\}_{i\in\mathbb{Z}}$ is equivalent in $\mathbb{Z}^2/\sim$ to $\{\widetilde{v}_i^2\}_{i\in\mathbb{Z}}$ defined by $\widetilde{v}_i^2 = T^{k_A} v_{i+d-k}^2$ and thus $v_i^1 = \widetilde{v}_i^2$ for $i = 0, \dots, k$ and

$$v_i^1 = T^{k_B} v_{i-k}^2 = T^{k_B} T^{-c} v_{i-k+d}^2 = T^{k_B} T^{-c} T^{-k_A} \widetilde{v}_{(i-k+d)-d+k}^2 = T^{k_B - k_A - c} \widetilde{v}_i^2$$

for $i = k, \dots, d$ because $\mathcal{H}_2$ has complexity c. Thus, $v_i^1 = \widetilde{v}_i^2$ for all $i \in \mathbb{Z}$ if $k_B - k_A = c$, in which case the proof is complete.

By Eq. (4.1) f_1 and T^{k_A} differ by a translation on A so $f_1|_B = T^c(T^{k_A} \circ f_2)|_B + x_B'$ for some $x_B' \in \mathbb{R}^2$ because this is precisely the effect of the monodromy of the set $U_1 \cap U_2$ encircling all of the c focus–focus points of the system (see [7, Theorem 3.8]). Combining this with Eq. (4.2) we see that $T^c \circ T^{k_A} \circ f_2|_B = T^{k_B} \circ f_2|_B + x_B''$ for some $x_B'' \in \mathbb{R}^2$ and thus $k_B - k_A = c$ as desired. □

Now we can prove the first part of Proposition 3.1.

Lemma 4.3. *Suppose that (M, ω, F) and (M', ω', F') are isomorphic as symplectic semitoric manifolds. Then* $\mathrm{hlx}(M, \omega, F) = \mathrm{hlx}(M', \omega', F')$.

Proof. Let (M, ω, F) and (M', ω', F') be symplectic semitoric manifolds and let $\phi : M \to M'$ be a semitoric isomorphism, so there exists a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$ with $\partial f/\partial y \neq 0$ such that $\phi^* F' = (J, f(J, H))$. This implies they must each have the same number of focus–focus points and elliptic–elliptic points. Let $d \in \mathbb{Z}$ be the number of elliptic–elliptic points and let $c \in \mathbb{Z}$ be the number of focus–focus points. Let $U \subset F'(M')$ be a helix neighborhood for (M', ω', F'), which is to say it is an open subset that can be used to construct the helix associated to (M', ω', F') as is done above, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a straightening map for U. This means there exists some $g^{(2)} : \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y) = (x, g^{(2)}(x, y))$ and $g \circ F$ is a toric momentum map on $F^{-1}(U)$. The semitoric helix associated to (M', ω', F') is $\mathcal{H}' = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ where $v_0, \dots, v_{d-1}$ are the inwards pointing normal vectors of the piecewise linear boundary of $g(U)$ and v_i for $i < 0$ and $i \geqslant d$ is determined by Definition 2.15 part (3) from the other vectors and the complexity.

The map ϕ descends to the map $\widehat{\phi} : F(M) \to F'(M')$ given by $\widehat{\phi}(x, y) = (x, f(x, y))$ so the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "F"'] & M' \arrow[d, "F'"] \\ F(M) \arrow[r, "\widehat{\phi}"] & F'(M') \end{tikzcd}$$

The set $\widehat{\phi}^{-1}(U)$ is a helix neighborhood for (M, ω, F). Let $\widetilde{g} = g \circ \widehat{\phi}$ and notice that $\widetilde{g}$ is a straightening map for $\widehat{\phi}^{-1}(U) \subset F(M)$. Indeed, $\widetilde{g}$ clearly preserves the first component (it is the composition of maps which preserve the first component) and the second component of $\widetilde{g} \circ F : M \to \mathbb{R}$, which is $g^{(2)}(J, f(J, H))$, has 2π-periodic Hamiltonian flow because $g^{(2)}(J, f(J, H)) = \phi^*(g^{(2)}(J', H'))$, ϕ is a symplectomorphism, and g is a straightening map for (M', ω', F') so $g^{(2)}(J', H')$ has 2π-periodic flow. The inwards pointing normal vectors of the piecewise linear portion of the boundary of $g(\hat{\phi}^{-1}(U))$ generate the helix for (M, ω, F), which we denote $\mathcal{H}$. Thus, $\mathcal{H} = \mathcal{H}'$ because $g(\hat{\phi}^{-1}(U)) = g(U)$, and since the helix constructed in this way is unique by Lemma 4.2 the helix for (M, ω, F) agrees with the helix for (M', ω', F'). □

The following result shows that to classify minimal semitoric manifolds it is sufficient to classify minimal semitoric helices.

Lemma 4.4. *Let (M, ω, F) be a symplectic semitoric manifold with associated helix* $\mathrm{hlx}(M, \omega, F)$*. The symplectic semitoric manifold (M', ω', F') can be obtained from (M, ω, F) by a blowup if and only if the associated helix* $\mathrm{hlx}(M, \omega, F)$ *can be obtained from* $\mathrm{hlx}(M, \omega, F)$ *by a blowup of semitoric helices. Moreover, (M', ω', F') can be obtained from (M, ω, F) by a blowdown if and only if the associated helix* $\mathrm{hlx}(M', \omega', F')$ *can be obtained from* $\mathrm{hlx}(M, \omega, F)$ *by a blowdown of semitoric helices.*

Proof. The helix is obtained as the inwards pointing normal vectors on the image of a toric momentum map on a subset of M, and the blowups we have defined are those which produce toric blowups with respect to this momentum map. Thus, the correspondence between toric blowups/downs of toric manifolds and blowups/downs of toric fans implies the result. □

To prove the remaining part of Proposition 3.1, that each possible semitoric helix is associated to some symplectic semitoric manifold, we need to invoke the classification of symplectic semitoric manifolds, particularly the semitoric polygon invariant.

4.2. Delzant semitoric polygons

Here we quickly review the definition of a Delzant semitoric polygon from [5] so we can explain the relationship between semitoric polygons and the semitoric helix.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ denote projection onto the first component and for any $\lambda \in \mathbb{R}$ let $\ell_\lambda = \pi^{-1}(\lambda)$. A *weighted polygon of complexity* $c \in \mathbb{Z}_{\geqslant 0}$ is a triple $\Delta_w = (\Delta, (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j)_{j=1}^c)$ where

1. $\Delta \subset \mathbb{R}^2$ is a convex, closed (possibly non-compact), rational polygon;

2. $\epsilon_j \in \{\pm 1\}$ for $j = 1, \ldots, c$;
3. $\lambda_j \in \mathrm{int}(\pi(\Delta))$ for $j = 1, \ldots, c$;
4. $\lambda_1 < \lambda_2 < \cdots < \lambda_c$.

Let $G_c = \{\pm 1\}^c$ and $\mathcal{G} = \{(T^t)^k : k \in \mathbb{Z}\}$ where T^t is the transpose of the matrix T given in Eq. (2.1). Given $k \in \mathbb{Z}$ and any vertical line $\ell = \ell_\lambda, \lambda \in \mathbb{R}$, define $t_\ell^k : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$t_\ell^k(x, y) = \begin{cases} (x, y), & x \leqslant \lambda \\ (x, k(x - \lambda) + y), & x > \lambda \end{cases}$$

and for $\vec{u} = (u_j)_{j=1}^c \in \mathbb{Z}^c$ and $\vec{\lambda} = (\lambda_j)_{j-1}^c \in \mathbb{R}$ let $t_{\vec{u},\vec{\lambda}} = t_{\ell_{\lambda_1}}^{u_1} \circ \cdots \circ t_{\ell_{\lambda_c}}^{u_c}$. The group $G_c \times \mathcal{G}$ acts on a weighted polygon by

$$((\epsilon_j')_{j=1}^c, (T^t)^k) \cdot \left(\Delta, (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j)_{j=1}^c\right) = \left(t_{\vec{u},\vec{\lambda}} \circ (T^t)^k(\Delta), (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j'\epsilon_j)_{j=1}^c\right),$$

where $\vec{u} = \left({}^{(\epsilon_j - \epsilon_j\epsilon_j')}/{}_2\right)_{j=1}^c$. A weighted polygon is called *admissible* if this action of $G_c \times \mathcal{G}$ preserves its convexity. Let $\mathcal{W}\mathrm{Polyg}_c(\mathbb{R}^2)$ denote the set of all admissible weighted polygons of complexity $c \in \mathbb{Z}_{\geqslant 0}$. An element of $\mathcal{W}\mathrm{Polyg}_c(\mathbb{R}^2) / G_c \times \mathcal{G}$ is known as a *semitoric polygon*.

Let $\Delta_w = (\Delta, (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j)_{j=1}^c)$ be a weighted polygon of complexity $c \in \mathbb{Z}_{\geqslant 0}$. Let $p \in \Delta$ be a vertex and let $v, w \in \mathbb{Z}^2$ be the inwards pointing normal vectors to the edges adjacent to p of minimal length ordered so that $\det(v, w) > 0$. Such vectors exist because Δ is rational. The vertex p satisfies:

1. the *Delzant condition* if $\det(v, w) = 1$;
2. the *hidden Delzant condition* if $\det(Tv, w) = 1$;
3. the *fake condition* if $\det(Tv, w) = 0$.

Let

$$\partial^{\mathrm{top}}\Delta = \left\{(x, y) : x \in \pi(\Delta),\ y = \sup\{y_0 \in \mathbb{R} : (x, y_0) \in \Delta\}\right\}$$

denote the top boundary of Δ.

Definition 4.5 (*[5]*). Let $[\Delta_w] \in \mathcal{W}\mathrm{Polyg}_c(\mathbb{R}^2) / G_c \times \mathcal{G}$ be a semitoric polygon and suppose that Δ_w is a representative of the form $\Delta_w = (\Delta, (\ell_{\lambda_j})_{j=1}^c, (+1)_{j=1}^c)$. Then $[\Delta_w]$ is a *Delzant semitoric polygon* if

1. $\Delta \cap \ell_\lambda$ is either compact or empty for all $\lambda \in \mathbb{R}$;
2. each point in $\partial^{\mathrm{top}}\Delta \cap \ell_{\lambda_j}$ satisfies either the hidden Delzant or fake condition, and is hence known as a hidden or fake corner, respectively, for $j = 1, \ldots, c$;
3. all other vertices of Δ satisfy the Delzant condition and are known as Delzant corners.

We say $[\Delta_w]$ is compact whenever Δ is compact for one, and hence all, representatives.

Every symplectic semitoric manifold determines a Delzant semitoric polygon [5,7], which is compact if the manifold is compact.

4.3. Construction of the helix from the polygon invariant

The helix can also be constructed from the Vũ Ngọc polygon associated to the symplectic semitoric manifold [7]. Here we give a brief outline of that construction, shown in Fig. 4. Let (M, ω, F) be a compact symplectic semitoric manifold.

Step 1: **Construct polygon:** Associated to (M, ω, F) is a semitoric polygon

$$[\Delta_w] = [(\Delta, (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j)_{j=1}^c)]$$

as in [7, Theorem 3.9] and described above in Section 4.2.

Step 2: **Construct semitoric fan:** Δ is rational, so take the collection of inwards pointing integer normal vectors $w_0, \ldots, w_{m-1}$ of minimal length to its edges (this is known as a *semitoric fan*, see [12]). These can be chosen so that the corner between w_{m-1} and w_0 is not on any of the vertical lines ℓ_{λ_j};

Step 3: **Correct for monodromy effect:** Each consecutive pair of vectors (w_j, w_{j+1}) is labeled as either fake, hidden, or Delzant depending on the type of corner of Δ it corresponds to. For each j such that (w_j, w_{j+1}) is either hidden or fake replace $w_{j+1}, \ldots, w_{m-1}$ by

$$Tw_{j+1}, \ldots, Tw_{m-1}.$$

Label the new list of vectors $w_0', \ldots, w_{m-1}'$;

Step 4: **Remove repeated vectors:** Now each pair (w_i', w_{i+1}') either satisfies $\det(w_i', w_{i+1}') = 1$ or $w_i' = w_{i+1}'$. For each j such that $w_j' = w_{j+1}'$ remove w_{j+1}' from the list and when all repeated vectors are removed denote the remaining vectors by $v_0, \ldots, v_{d-1}$. Notice $\det(v_i, v_{i+1}) = 1$ for all $i = 0, \ldots, d-2$;

Step 5: **Extend to helix:** By condition 3. of Definition 2.15 there exists a unique helix of length d and complexity c (the number of focus–focus points of the original symplectic semitoric manifold) with the given $v_0, \dots, v_{d-1}$ from the previous step.

Remark 4.6. Let $[(\Delta, (\ell_{\lambda_j}, \epsilon_j, k_j)_{j=1}^c)]$ be a compact semitoric polygon which has no hidden corners and such that all of the fake corners are consecutive (while traversing the boundary of Δ) and let $v_0, \dots, v_{d-1}$ be the primitive integral inwards pointing normal vectors to every edge of Δ which is adjacent to at least one Delzant corner. The associated semitoric helix is the unique helix of length d and complexity c with the given $v_0, \dots, v_{d-1}$. ◇

Remark 4.7. Lemma 4.3, which states that the semitoric helix is a well-defined invariant of semitoric manifolds, also follows from the fact that the semitoric polygon invariant is an invariant of the semitoric isomorphism type and the above construction of the helix from the semitoric polygon, but we have chosen to prove Lemma 4.3 in a way which is independent of the existence of the semitoric polygon invariant. ◇

Remark 4.8. In [16] the authors make use of what they call the *defining set* to classify minimal almost toric 4-manifolds whose base is a disk. An almost toric 4-manifold is a symplectic manifold with a singular Lagrangian fibration for which all singularities are nondegenerate and of elliptic, elliptic–elliptic, or focus–focus type. In contrast to semitoric manifolds there is no assumption of a global S^1-action. To produce the defining set a *nodal trade* is performed on each elliptic–elliptic fixed point, which means that it is replaced by a focus–focus point by changing the fibration (but not the symplectomorphism type of the symplectic manifold). After removing all of the elliptic–elliptic points in this way an affine immersion is constructed from the base of the fibration minus a set of *branch curves* (curves that connect the images of the focus–focus points to the boundary) into $\mathbb{R}^2$ and the defining set is the inwards pointing normal vectors of the resulting image (see [16] for more details). This is similar to the construction of the semitoric helix but there are two main differences here. First, the defining set contains no information about the relative number of elliptic–elliptic and focus–focus singular points, since it does not distinguish between them, while the semitoric helix does. This is because in [16] they are only interested in classifying the diffeomorphism type of the manifold. Second, the defining set is only defined up to the action of $GL(2, \mathbb{Z})$, since there is no preference for any particular direction, unlike in the semitoric case. Thinking of a semitoric manifold as an example of an almost toric manifold the defining set can be recovered from the semitoric polygon by taking the inwards pointing normal vectors, and since a given semitoric helix is related to only a finite number of semitoric polygons the helix determines a set of possible related defining sets. ◇

4.4. Surjectivity of the helix map

Lemma 4.9. *Given any semitoric helix $\mathcal{H}$ there exists a symplectic semitoric manifold (M, ω, F) such that* $\mathrm{hlx}(M, \omega, F) = \mathcal{H}$.

Proof. Let $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ be a semitoric helix. Define a collection of vectors $w_0, \dots, w_{d+c}$ by $w_i = v_i$ for $i = 0, \dots, d-1$ and $w_i = T^{i-d+1}v_{d-1}$ for $i = d, \dots, d+c$. Then $\det(w_i, w_{i+1}) = 1$ for $i = 0, \dots, d-1$, $\det(Tw_i, w_{i+1}) = \det(Tw_i, Tw_i) = 0$ for $i = d, \dots, d+c-1$, and $w_0 = w_{d+c}$ by the periodicity requirement on the helix $\mathcal{H}$. The vectors $w_0, \dots, w_{d+c-1}$ are arranged counter-clockwise so there exists a polygon $\Delta \subset \mathbb{R}^2$ with $d+c$ edges which has these as inwards pointing normal vectors. The polygon Δ has d Delzant corners c fake corners, and since T does not change the y-value of a vector we see that either all of the fake corners are on the top boundary of Δ or all of the fake corners are on the bottom boundary of Δ. Let λ_i be the horizontal position of the ith fake corner and we may number these so that $\lambda_1 < \lambda_2 < \cdots < \lambda_c$ since each vertical line intersects the top and bottom boundaries at most once each. If the fake corners are on the top boundary let $\epsilon_j = +1$ for $j = 1, \dots, c$ and otherwise let $\epsilon_j = -1$ for $j = 1, \dots, c$. Then, $\Delta_w = [(\Delta, (\ell_{\lambda_j})_{j=1}^c, (\epsilon_j)_{j=1}^c)]$ is a Delzant semitoric polygon with associated semitoric helix $\mathcal{H}$. By [5, Theorem 4.6] there exists a symplectic semitoric manifold with $[\Delta_w]$ as its semitoric polygon. □

Remark 4.10. Lemma 4.9 shows that the map $\mathrm{hlx} : \mathcal{S}_{\mathrm{ST}} \to \mathcal{S}_{\mathrm{H}}$ is surjective by producing a right inverse, but this map is not injective. In terms of the Pelayo–Vũ Ngọc invariants this is because the helix does not encode any information about the Taylor series invariant, the volume invariant, the twisting index, the horizontal position of the focus–focus points, or the lengths of the edges of the semitoric polygon. ◇

4.5. Helix of the coupled angular momenta

The semitoric helix associated to the coupled angular momenta system (described in Section 2.3) can be recovered from the polygons as described in Section 4.3. The helix is represented by the three vectors

$$v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and is minimal of type (3) from Theorem 3.6 with $k = 1$. This is shown in Fig. 5. Since the system has only one focus–focus point, the helix can be obtained as the inwards pointing normal vectors of the semitoric polygon computed in [13], where v_0 is chosen to be the inwards pointing normal vector of an edge adjacent to the fake corner produced by the focus–focus point. This is because in systems with one focus–focus point the same toric momentum map that produces the semitoric polygon also works for the construction of the semitoric helix, as described in Section 4.1 (see Remark 4.6).

5. Semitoric helices and $\mathrm{SL}_2(\mathbb{Z})$

In this Section we prove Proposition 3.3, which is the tool we use to translate questions about semitoric helices into questions about words on letters S and T.

Lemma 5.1. *Given any semitoric helix $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ there exists a list of integers $(a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$ such that*

$$a_{i \bmod d} v_{i+1} = v_i + v_{i+2} \tag{5.1}$$

for all $i \in \mathbb{Z}$. Furthermore, given v_0, v_1, and $(a_0, \dots, a_{d-1})$ the helix can be recovered.

Proof. Let $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$. Let $A_i = [v_i, v_{i+1}]$ and write v_{i+2} in the (v_i, v_{i+1}) basis as $v_{i+2} = b_i v_i + a_i v_{i+1}$, for $a_i, b_i \in \mathbb{Z}$. Thus,

$$A_i \begin{pmatrix} 0 & b_i \\ 1 & a_i \end{pmatrix} = A_{i+1}$$

and since $A_i, A_{i+1} \in \mathrm{SL}_2(\mathbb{Z})$ we see the determinant of each side is 1 so $b_i = -1$ and $v_{i+2} + v_i = a_i v_{i+1}$ as desired. Conversely, given v_0, v_1 and $(a_0, \dots, a_{d-1})$ the helix can be recovered by using the recurrence relation Eq. (5.1). □

Definition 5.2. The $(a_0, \dots, a_{d-1}) \in \mathbb{Z}^2$ in Lemma 5.1 are the *associated integers* to $\mathcal{H}$.

Recall $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ denotes the universal cover of $\mathrm{SL}_2(\mathbb{R})$ with base point at the identity, so $\alpha \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ is a continuous map $\alpha : [0, 1] \to \mathrm{SL}_2(\mathbb{R})$ satisfying $\alpha(0) = I$. The group G (Definition 2.11) is isomorphic to the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ [12, Proposition 3.7] via the homomorphism $\rho : G \to \mathrm{SL}_2(\mathbb{Z})$ generated by its action on S and T given in Eq. (2.2). The operation in G is concatenation of paths. If $\alpha, \beta \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ then $\alpha, \beta : [0, 1] \to \mathrm{SL}_2(\mathbb{R})$ and we define $\alpha\beta \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ by

$$\alpha\beta(t) = \begin{cases} \alpha(2t), & 0 \leqslant t \leqslant 1/2 \\ \alpha(1)\beta(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

That is, the path $\alpha\beta$ is obtained by traveling first along the path α and then along the path produced by multiplying each element of the path β on the left by $\alpha(1)$. It turns out that the path produced by traveling first along β and then along α multiplied on the right by $\beta(1)$ is homotopic to $\alpha\beta$. The next result follows from the fact that the fundamental group of a topological group is abelian (see [26, Section 3.C, Exercise 5]), but we prove it here for completeness.

Lemma 5.3. *If $\alpha, \beta \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ then the paths in $\mathrm{SL}_2(\mathbb{R})$ from I to $\alpha(1)\beta(1)$ given by*

$$\gamma_0(t) = \begin{cases} \beta(2t), & 0 \leqslant t \leqslant 1/2 \\ \alpha(2t-1)\beta(1), & 1/2 < t \leqslant 1 \end{cases}$$

and

$$\gamma_1(t) = \begin{cases} \alpha(2t), & 0 \leqslant t \leqslant 1/2 \\ \alpha(1)\beta(2t-1), & 1/2 < t \leqslant 1 \end{cases}$$

are homotopic.

Proof. A continuous homotopy between them is given by

$$\gamma_s(t) = \begin{cases} \alpha(2t), & 0 \leqslant t \leqslant s/2 \\ \alpha(s)\beta(2t-s), & s/2 \leqslant t \leqslant \dfrac{1+s}{2} \\ \alpha(2t-1)\beta(1), & \dfrac{1+s}{2} \leqslant t \leqslant 1 \end{cases}$$

for $0 \leqslant s \leqslant 1$, which is shown in Fig. 6. Indeed, γ_s is continuous because $\gamma_s(s/2) = \alpha(s)$ since $\beta(0) = I$ and $\gamma_s((1+s)/2) = \alpha(s)\beta(1)$. It is left to the reader to check that it is a homotopy from γ_0 to γ_1. □

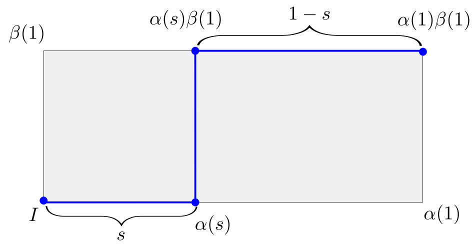


Fig. 6. The homotopy from the proof of Lemma 5.3. A point (x, y) in the above plane represents $\alpha(x)\beta(y) \in \mathrm{SL}_2(\mathbb{R})$.

Recall the map $\mathrm{pr} : \mathrm{SL}_2(\mathbb{R}) \to (\mathbb{R}^2)^*$, where $(\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{(0, 0)\}$, given by $\mathrm{pr}([v_1, v_2]) = v_1$. Since $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \pi_1((\mathbb{R}^2)^*) \cong \mathbb{Z}$ and

$$\mathrm{pr}\begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} = \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$$

for $0 \leqslant t \leqslant 1$ we see pr sends a generator of $\pi_1(\mathrm{SL}_2(\mathbb{R}))$ to a generator of $\pi_1((\mathbb{R}^2)^*)$, so $\mathrm{pr}^* : \pi_1(\mathrm{SL}_2(\mathbb{R})) \to \pi_1((\mathbb{R}^2)^*)$ is an isomorphism.

Definition 5.4. Let $\theta : (\mathbb{R}^2)^* \to [0, 2\pi)$ be the usual angle coordinate from polar coordinates on $\mathbb{R}^2$ and let $R_\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by the angle $\phi \in [0, 2\pi)$. We say a path $\gamma : [0, 1] \to (\mathbb{R}^2)^*$ *travels counter-clockwise at most one full rotation* if there exists some $\phi \in [0, 2\pi)$ such that $t \mapsto \theta(R_\phi(\gamma(t)))$ is an increasing function for $t \in (0, 1)$.

Lemma 5.5. *Let $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ be a semitoric helix with associated integers $(a_0, \dots, a_{d-1})$ and let $A_0 = [v_0, v_1]$. If $\sigma \in G$ is given by*

$$\sigma =_G ST^{a_0} \dots ST^{a_{d-1}}$$

then $\mathrm{pr}\big(A_0\, \rho(\sigma)\big)$ is homotopic to a path from v_0 to v_{d-1} which travels counter-clockwise at most one full rotation.

Proof. Let $A_i = [v_i, v_{i+1}]$ for $1 \leqslant i \leqslant d-1$ and recall $A_i = A_{i-1} ST^{a_{i-1}}$. Thus,

$$\mathrm{pr}\big(A_{i-1}\, \rho(ST^{a_{i-1}})\big)$$

is a path from v_i to v_{i+1} which is homotopic to

$$\gamma_i(t) = \mathrm{pr}\left(A_{i-1}\begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) + t a_{i-1}\cos\left(\frac{\pi t}{2}\right) \\ \sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) + t a_{i-a}\sin\left(\frac{\pi t}{2}\right) \end{pmatrix}\right) = \cos\left(\frac{\pi t}{2}\right) v_{i-1} + \sin\left(\frac{\pi t}{2}\right) v_i$$

for $0 \leqslant t \leqslant 1$. The path γ_i travels only counter-clockwise at most one full rotation from v_{i-1} to v_i so the composition of paths $\gamma_1, \dots, \gamma_{d-1}$ travels counter-clockwise from v_0 to v_{d-1}. The result follows because $v_0, \dots, v_{d-1}$ are arranged in counter-clockwise order. $\square$

Lemma 5.6. *If the integers $(a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$ are associated to a semitoric helix of complexity $c \geqslant 0$ then*

$$ST^{a_0} \dots ST^{a_{d-1}} =_G S^4 X^{-1} T^c X \tag{5.2}$$

for some $X \in G$. If $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ is a semitoric helix with associated integers $(a_0, \dots, a_{d-1})$ then $A_0 = [v_0, v_1]$ satisfies $X =_G A_0$.

Proof. Let $A_i = [v_i, v_{i+1}]$. By Lemma 5.1 and the fact that

$$\begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix} = ST^{a_i}$$

we find that $A_{i+1} =_{\mathrm{SL}_2(\mathbb{Z})} A_i ST^{a_i}$ for all $i \in \mathbb{Z}$. We conclude that

$$A_d =_{\mathrm{SL}_2(\mathbb{Z})} A_0 ST^{a_0} \dots ST^{a_{d-1}}$$

and since $T^c A_0 =_{\mathrm{SL}_2(\mathbb{Z})} A_d$ this implies that

$$ST^{a_0} \dots ST^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} A_0^{-1} T^c A_0.$$

Since S^4 generates the kernel of the projection $G \to \mathrm{SL}_2(\mathbb{Z})$ we have that

$$ST^{a_0}\dots ST^{a_{d-1}} =_G S^{4k}A_0^{-1}T^cA_0$$

for some $k \in \mathbb{Z}$. This is because S^2 is in the center of $\mathrm{SL}_2(\mathbb{Z})$ and when reducing an element of $\mathrm{SL}_2(\mathbb{Z})$ we can assume that the relation $S^4 =_{\mathrm{SL}_2(\mathbb{Z})} I$ is not used until the last step. Rearranging we have

$$A_0ST^{a_0}\dots ST^{a_{d-1}}A_0^{-1}T^{-c} =_G S^{4k}. \tag{5.3}$$

To complete the proof we must only show that $k = 1$ in Eq. (5.3).

Let $\sigma, \eta \in G$ be given by

$$\sigma =_G ST^{a_0}\dots ST^{a_{d-1}} \quad\text{and}\quad \eta =_G A_0\sigma A_0^{-1}T^{-c}$$

so Eq. (5.3) becomes $\eta =_G S^{4k}$. Since $W(S^{4k}) = k$, it is sufficient to show that $W(\eta) = 1$. Recall $\pi_1(\mathrm{SL}_2(\mathbb{R}))$ is abelian so the class of a loop is well-defined without fixing the basepoint. By Lemma 2.14, $\eta =_{\mathrm{SL}_2(\mathbb{Z})} I$ implies that $W(\eta) = \mathrm{wind}(\rho(\eta))$. By Lemma 5.3 $\rho(\eta)$ is homotopic to

$$\big(\rho'(\eta)\big)(t) = \begin{cases} \rho(A_0^{-1})(4t), & 0 \leqslant t \leqslant 1/4 \\ \Big(\big(\rho(\sigma)\big)(4t-1)\Big)A_0^{-1}, & 1/4 \leqslant t \leqslant 1/2 \\ \sigma A_0^{-1}\Big(\big(\rho(T^{-c})\big)(4t-2)\Big), & 1/2 \leqslant t \leqslant 3/4 \\ \Big(\big(\rho(A_0)\big)(4t-3)\Big)\sigma A_0^{-1}T^{-c}, & 3/4 \leqslant t \leqslant 1. \end{cases}$$

Let $\gamma_0 : [0, 1] \to \mathrm{SL}_2(\mathbb{R})$ be the path from A_0^{-1} to itself given by

$$\gamma_0(t) = \begin{cases} \Big(\big(\rho(\sigma)\big)(2t)\Big)A_0^{-1}, & 0 \leqslant t \leqslant 1/2 \\ \sigma A_0^{-1}\Big(\big(\rho(T^{-c})\big)(2t-1)\Big), & 1/2 \leqslant t \leqslant 1. \end{cases} \tag{5.4}$$

The paths γ_0 and $\rho'(\eta)$ are homotopic via the homotopy

$$\big(\rho'_s(\eta)\big)(t) = \begin{cases} \rho(A_0^{-1})(\frac{4t}{s}), & 0 \leqslant t \leqslant s/4 \\ \Big(\big(\rho(\sigma)\big)(\frac{4t-s}{2-s})\Big)A_0^{-1}, & 2/4 \leqslant t \leqslant 1/2 \\ \sigma A_0^{-1}\Big(\big(\rho(T^{-c})\big)(\frac{4t-2}{2-s})\Big), & 1/2 \leqslant t \leqslant \frac{4-s}{4} \\ \Big(\big(\rho(A_0)\big)(\frac{4t+s-4}{s})\Big)\sigma A_0^{-1}T^{-c}, & \frac{4-s}{4} \leqslant t \leqslant 1 \end{cases}$$

for $0 < s \leqslant 1$ where γ_0 is defined as above. Thus, to complete the proof we only must show $\mathrm{wind}(\gamma_0) = 1$ where γ_0 is as in Eq. (5.4). By Lemma 5.5, the path

$$\mathrm{pr}\Big(\rho(\sigma)(2(\cdot))\Big) : [0, 1/2] \to (\mathbb{R}^2)^*$$

is homotopic to a path which travels counter-clockwise at most one full rotation. The path

$$\mathrm{pr}\Big(\sigma A_0^{-1}\,\rho(T^{-c})(2(\cdot)-1)\Big) : [1/2, 1] \to (\mathbb{R}^2)^*,$$

travels only counter-clockwise and cannot cross the line $\{y = 0\}$, so it completes at most one half-rotation. Since $\sigma A_0^{-1}T^{-c} =_{\mathrm{SL}_2(\mathbb{Z})} A_0^{-1}$, the path γ_0 thus circles the origin an integer number of times, so we conclude that $\mathrm{wind}(\mathrm{pr}(\gamma_0)) = \mathrm{wind}(\gamma_0) = 1$. This completes the proof. (See Fig. 7) □

5.1. Correspondence between helices and lists of integers

Proof (*Proof of Proposition 3.3*)**.** Let $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ be a semitoric helix. The existence of associated integers $(a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$ as in Definition 5.2 is guaranteed by Lemma 5.1.

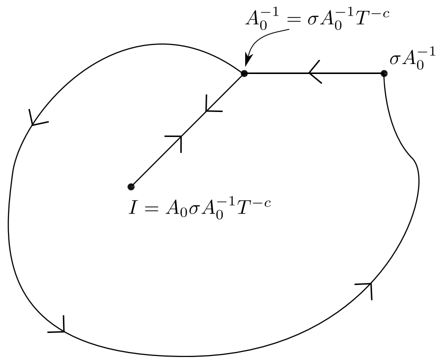


Fig. 7. The loop $\rho(\eta)$ from the proof of Lemma 5.6, which has winding number 1.

One direction of part (1) follows from Lemma 5.6. To prove the other direction suppose $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ is a list of integers satisfying

$$ST^{a_0} \ldots ST^{a_{d-1}} =_G S^4 X^{-1} T^{-c} X$$

for some $c \in \mathbb{Z}_{>0}$. Let $A_0 \in \mathrm{SL}_2(\mathbb{Z})$ be any matrix satisfying $X =_{\mathrm{SL}_2(\mathbb{Z})} A_0$ and define $v_0, v_1 \in \mathbb{Z}^2$ so that $A_0 = [v_0, v_1]$. Then define $v_2, \ldots, v_{d-1}$ by $v_i = a_{i-2} v_{i-1} - v_{i-2}$ for $i = 2, \ldots, d-1$. Use the relationship $v_{i+d} = T^c v_i$ to extend $v_0, \ldots, v_{d-1}$ to $\{v_i\}_{i \in \mathbb{Z}}$. Since $W(ST^{a_0} \ldots ST^{a_{d-1}}) = 1$, the vectors $v_0, \ldots, v_{d-1}$ are in counter-clockwise order and by construction $\det(v_i, v_{i+1}) = 1$ for all $i \in \mathbb{Z}$, so $[\{v_i\}_{i \in \mathbb{Z}}]$ is a semitoric helix with the prescribed associated integers.

Next we prove part (2). Let $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$ be associated integers for $\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])$ defined so that $a_i v_{i+1} = v_i + v_{i+2}$ and let $\{w_i\}_{i \in \mathbb{Z}} \in [\{v_i\}_{i \in \mathbb{Z}}]$. By Definition 2.15 there exists some $k, \ell \in \mathbb{Z}$ such that $v_i = T^k w_{i+\ell}$ for all $i \in \mathbb{Z}$ so $a_i v_{i+1} = v_i + v_{i+2}$ implies that $a_i w_{i+1+\ell} = w_{i+\ell} + w_{i+2+\ell}$ and, denoting $a_j := a_{j \bmod d}$, this implies that $a_{i-\ell} w_{i+1} = w_i + w_{i+2}$. Thus, the associated integers for $\{w_i\}_{i \in \mathbb{Z}}$ are given by $(a_{-\ell}, a_{1-\ell}, \ldots, a_{d-1-\ell})$ which agrees with those integers for $\{v_i\}_{i \in \mathbb{Z}}$ up to cyclic permutation, as desired.

Finally, we prove part (3). If $\mathcal{H}$ and $\mathcal{H}'$ satisfy $\mathcal{H} = \pm \mathcal{H}'$ then they have the same associated integers since those integers are defined by a linear equation, Eq. (5.1), which is invariant under the action of $-I$.

Conversely, suppose that $\mathcal{H} = (d, c, [\{v_i\}_{i \in \mathbb{Z}}])$ and $\mathcal{H}' = (d, c, [\{v_i'\}_{i \in \mathbb{Z}}])$ are semitoric helices of the same length, complexity, and associated integers. Let $A_0 = [v_0, v_1]$ and $A_0' = [v_0', v_1']$ and let $a_0, \ldots, a_{d-1}$ be the common associated integers. Then

$$ST^{a_0} \ldots ST^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} A_0^{-1} T^c A_0 \quad \text{and} \quad ST^{a_0} \ldots ST^{a_{d-1}} =_{\mathrm{SL}_2(\mathbb{Z})} (A_0')^{-1} T^c (A_0')$$

so $A_0^{-1} T^c A_0 =_{\mathrm{SL}_2(\mathbb{Z})} (A_0')^{-1} T^c (A_0')$. Thus $A_0' A_0^{-1}$ commutes with T^c, so $A_0' A_0^{-1} =_{\mathrm{SL}_2(\mathbb{Z})} \pm T^k$ for some $k \in \mathbb{Z}$, and so we may assume $A_0 =_{\mathrm{SL}_2(\mathbb{Z})} \pm A_0'$ because T^k is already included in the equivalence relation on helices. Finally, $v_0 = \pm v_0'$ and $v_1 = \pm v_1'$ implies $v_i = \pm v_i'$ by Eq. (5.1). $\square$

6. Standard form in $\mathrm{PSL}_2(\mathbb{Z})$ and the winding number

This section is devoted to proving Theorem 3.5. We start with several lemmas.

Lemma 6.1. *If $\sigma \in \mathbb{Z} * \mathbb{Z}$ is S-positive and $\sigma =_{\mathrm{PSL}_2(\mathbb{Z})} I$ then $W(\sigma) \geqslant 0$ where $W(\sigma) = 0$ if and only if σ is the empty word.*

Proof. If σ is the empty word then $W(\sigma) = 0$ and the claim holds. Assume σ is not the empty word. Since σ is S-positive up to conjugation by T, which does not change $W(\sigma)$, we may write it as $\sigma =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \ldots ST^{a_{d-1}}$ for some $a_0, \ldots, a_{d-1} \in \mathbb{Z}$. We define a sequence of vectors $v_0, \ldots, v_{d-1} \in \mathbb{Z}^2$ by choosing any $v_0, v_1 \in \mathbb{Z}^2$ with $\det(v_0, v_1) = 1$ and defining $v_2, \ldots, v_{d-1}$ by $v_{i+2} = -v_i + a_i v_{i+1}$ for $i = 0, \ldots, d-3$. Let $\gamma : [0, 1] \to (\mathbb{R}^2)^*$ be a path which connects $v_0, \ldots, v_{d-1}$ in order and travels only counter-clockwise. Then, $W(\sigma) = \mathrm{wind}(\gamma)$ and $\mathrm{wind}(\gamma) > 0$ because γ must travel at least once around the origin to move only counter-clockwise and return to $\gamma(0)$. $\square$

Lemma 6.2. *If $X \in \mathrm{PSL}_2(\mathbb{Z})$ then there exists some $q \in \frac{1}{12}\mathbb{Z}$ such that $w(\sigma) \geqslant q$ for all $\sigma \in \mathbb{Z} * \mathbb{Z}$ which are S-positive and satisfy $\sigma =_{\mathrm{PSL}_2(\mathbb{Z})} X$.*

Proof. Since $S =_{\mathrm{PSL}_2(\mathbb{Z})} S^{-1}$ every element of $\mathrm{PSL}_2(\mathbb{Z})$ has an S-positive representation. Fix some S-positive $\eta \in \mathbb{Z} * \mathbb{Z}$ such that $\eta =_{\mathrm{PSL}_2(\mathbb{Z})} X^{-1}$ and let $q = -W(\eta)$. Let σ be any S-positive element of $\mathbb{Z} * \mathbb{Z}$ such that $\sigma =_{\mathrm{PSL}_2(\mathbb{Z})} X$. Now $\sigma\eta =_{\mathrm{SL}_2(\mathbb{Z})} I$, so $W(\sigma\eta) \geqslant 0$ by Lemma 6.1. This means $W(\sigma) + W(\eta) \geqslant 0$ so $W(\sigma) \geqslant q$ and the result follows because q does not depend on the choice of σ. $\square$

The following is a special case of [12, Lemma 3.8], but for the sake of being self-contained and because of the increased clarity of the argument in this special case, we include the proof here.

Lemma 6.3. *Suppose $d > 0$ and $b, a_0, \dots, a_{d-1} \in \mathbb{Z}$ are such that*

$$T^b ST^{a_0} \dots ST^{a_{d-1}} =_{\mathrm{PSL}_2(\mathbb{Z})} I. \tag{6.1}$$

Then $d > 1$. If $d = 2$ then $a_0 = 0$ and $a_1 = -b$. If $d > 2$ then $a_i \in \{0, \pm 1\}$ for some $0 \leqslant i < d - 2$.

Proof. The group $\mathrm{PSL}_2(\mathbb{Z})$ acts faithfully on the extended real line $\mathbb{R} \cup \{\infty\}$ by $T(x) = x + 1$, $S(x) = \frac{-1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$, $T(0) = 0$, $S(0) = \infty$, $T(\infty) = \infty$, and $S(\infty) = 0$.

If $d = 1$, then Eq. (6.1) states that $T^b ST^{a_0} =_{\mathrm{PSL}_2(\mathbb{Z})} I$ which is impossible because $T^b ST^{a_0}(\infty) = b \neq \infty$.

If $d = 2$, then, after conjugating by ST^{a_1}, Eq. (6.1) states that $ST^{a_1+b}ST^{a_0} =_{\mathrm{PSL}_2(\mathbb{Z})} I$ and evaluation of both sides at infinity gives $S(a_1 + b) = \infty$ which implies $a_1 = -b$. Thus, $S^2T^{a_0} =_{\mathrm{PSL}_2(\mathbb{Z})} I$ so $a_0 = 0$.

Finally, suppose $d > 2$ and $a_i \notin \{0, \pm 1\}$ for all $i = 0, \dots, d - 2$. Conjugate Eq. (6.1) by $ST^{a_{d-1}}$ to produce

$$ST^{a_{d-1}+b}ST^{a_0} \dots ST^{a_{d-2}} =_{\mathrm{PSL}_2(\mathbb{Z})} I. \tag{6.2}$$

Let $y = ST^{a_0} \dots ST^{a_{d-2}}(\infty)$ so Eq. (6.2) implies $ST^{a_{d-1}+b}(y) = \infty$. On the other hand, $ST^{a_{d-2}}(\infty) = 0$ and if $x \in \mathbb{R}$ with $|x| < 1$ then $0 < \left|ST^k(x)\right| < 1$ for any integer $k \notin \{\pm 1, 0\}$, so

$$|y| = \left|ST^{a_0} \dots ST^{a_{d-3}}(0)\right| \in (0, 1),$$

and thus

$$ST^{a_{d-1}+b}(y) = \frac{1}{y + a_{d-1} + b} \neq \infty,$$

forming a contradiction. □

Lemma 6.4. *$ST^{-n}S =_{\mathrm{PSL}_2(\mathbb{Z})} (TST)^n$ for $n \geqslant 0$.*

Proof. First $STS =_{\mathrm{PSL}_2(\mathbb{Z})} T^{-1}ST^{-1}$ implies $S =_{\mathrm{PSL}_2(\mathbb{Z})} TSTST$ so $ST^{-1}S =_{\mathrm{PSL}_2(\mathbb{Z})} TST$ since $S =_{\mathrm{PSL}_2(\mathbb{Z})} S^{-1}$. Now,

$$ST^{-n}S =_{\mathrm{PSL}_2(\mathbb{Z})} (ST^{-1}S)^n =_{\mathrm{PSL}_2(\mathbb{Z})} (TST)^n$$

for $n > 0$, and if $n = 0$ the claim reduces to $S^2 =_{\mathrm{PSL}_2(\mathbb{Z})} I$. □

6.1. Standard form for elements of $\mathrm{PSL}_2(\mathbb{Z})$

Proof of Theorem 3.5. Let $\sigma \in \mathbb{Z} * \mathbb{Z}$ any S-positive word with $\sigma =_{\mathrm{PSL}_2(\mathbb{Z})} X$. There are three steps to the reduction algorithm we will use on σ, where Reduction 2 holds by Lemma 6.4. The reductions are:

Reduction 1 replace S^2 with I;
Reduction 2 replace $ST^{-n}S$ with $(TST)^n$, for some $n > 0$;
Reduction 3 replace STS with $T^{-1}ST^{-1}$;

To reduce the word we iteratively apply Reduction 1, Reduction 2, and Reduction 3 until no more are possible. Each of these reductions preserves the value of σ in $\mathrm{PSL}_2(\mathbb{Z})$ and recall that the winding number cannot decrease indefinitely by Lemma 6.2. Reduction 1 and Reduction 2 reduce the winding number while Reduction 3 preserves the winding number but reduces the number of times S appears in the word, which is bounded below by zero. Thus, this process must terminate and after the reduction the word will be of the required form.

Now we will show uniqueness. Suppose that $\sigma, \eta \in \mathbb{Z} * \mathbb{Z}$ with $\sigma =_{\mathrm{PSL}_2(\mathbb{Z})} \eta$ and $\sigma =_{\mathbb{Z}*\mathbb{Z}} T^b ST^{a_0} \dots ST^{a_{d-1}}$, $\eta =_{\mathbb{Z}*\mathbb{Z}} T^{b'} ST^{a'_0} \dots ST^{a'_{d'-1}}$, where $a_i, a'_j > 1$ for $i = 0, \dots, d - 2$ and $j = 0, \dots, d' - 2$. First assume $\min(d, d') \leqslant 1$, and in this case assume $d \geqslant d'$.

If $d' = 0$ then $T^{b-b'} ST^{a_0} \dots ST^{a_{d-1}} =_{\mathrm{PSL}_2(\mathbb{Z})} I$ which contradicts Lemma 6.3 unless $d = 0$, in which case $T^{b-b'} =_{\mathrm{PSL}_2(\mathbb{Z})} I$ so $b = b'$. If $d' = 1$ then

$$T^{b-b'} ST^{a_0} \dots ST^{a_{d-1}-a'_0} ST^0 =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

so $a_{d-1} - a'_0 \in \{0, \pm 1\}$ by Lemma 6.3. Consider the cases if $d > 1$. If $a_{d-1} - a'_0 = 0$ then

$$T^{b-b'} ST^{a_0} \dots ST^{a_{d-2}} S^2 =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

which contradicts Lemma 6.3 after replacing S^2 by I. If $a_{d-1} - a'_0 = -1$ then

$$T^{b-b'} ST^{a_0} \dots ST^{a_{d-2}} ST^{-1} S =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

which contradicts Lemma 6.3 after replacing $ST^{-1}S$ by TST. Finally, if $a_{d-1} - a'_0 = 1$, then

$$T^{b-b'}ST^{a_0}\dots ST^{a_{d-2}-1}ST^{-1} =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

which contradicts Lemma 6.3 unless $a_{d-2} = 2$. This process is repeated to conclude that $a_0 = \cdots = a_{d-2} = 2$ so $T^{b-b'}(ST^2)^{d-1}STS =_{\mathrm{PSL}_2(\mathbb{Z})} I$ which implies $T^{b-b'-1}ST^{-d} =_{\mathrm{PSL}_2(\mathbb{Z})} I$. By Lemma 6.3 this cannot hold. Thus, $d = 1$, in which case $T^{b-b'}ST^{a_0-a'_0}S =_{\mathrm{PSL}_2(\mathbb{Z})} I$, so $b - b' = 0$ and $a_0 - a'_0 = 0$ by Lemma 6.3.

Finally, assume $d, d' > 1$ and assume that $a_{d-1} \neq a'_{d'-1}$, otherwise cancel $ST^{a_{d-1}}$ from both sides. In this case we see that $\sigma\eta^{-1} =_{\mathrm{PSL}_2(\mathbb{Z})} I$ implies

$$T^bST^{a_0}\dots ST^{a_{d-1}-a'_{d'-1}}ST^{-a'_{d'-2}}\dots ST^{-a'_0}S =_{\mathrm{PSL}_2(\mathbb{Z})} I$$

and since some power of T must be in $\{0, \pm 1\}$ by Lemma 6.3, but $a_i, a'_j > 1$ for $i = 0, \dots, d-2, j = 0, \dots, d'-2$ since σ and η are in standard form, we conclude $a_{d-1} - a'_{d'-1} \in \{0, \pm 1\}$. We have assumed $a_{d-1} \neq a'_{d'-1}$ so $a_{d-1} - a'_{d'-1} = \pm 1$, and furthermore we can assume $a_{d-1} - a'_{d'-1} = 1$, otherwise exchange σ and η. Then choose maximal $k \in \mathbb{Z}_{\geqslant 0}$ such that $a_{d-2} = a_{d-3} = \cdots = a_{d-2-(k-1)} = 2$. where $k = 0$ if $a_{d-2} \neq 2$. If $k < d-1$ then

$$\begin{aligned}\sigma\eta^{-1} &=_{\mathrm{PSL}_2(\mathbb{Z})} T^bST^{a_0}\dots ST^{a_{d-2-k}}(ST^2)^k(STS)T^{-a'_{d'-2}}\dots ST^{-a'_0}ST^{-b'}\\ &=_{\mathrm{PSL}_2(\mathbb{Z})} T^bST^{a_0}\dots ST^{a_{d-2-k}-1}(TST)^kST^{-a'_{d'-2}-1}\dots ST^{-a'_0}ST^{-b'}\\ &=_{\mathrm{PSL}_2(\mathbb{Z})} T^bST^{a_0}\dots ST^{a_{d-2-k}-1}ST^{-a'_{d'-2}-k-1}\dots ST^{-a'_0}ST^{-b'}.\end{aligned}$$

Since $a_{d-2-k} - 1 > 1$ and $-a'_{d'-2} - k - 1 < -1$ this expression cannot evaluate to the identity in $\mathrm{PSL}_2(\mathbb{Z})$ by Lemma 6.3.

Otherwise, $k = d-1$, in which case

$$\begin{aligned}\sigma\eta^{-1} &=_{\mathrm{PSL}_2(\mathbb{Z})} T^b(ST^2)^{d-1}(STS)T^{-a'_{d'-2}}\dots ST^{-a'_0}ST^{-b'}\\ &=_{\mathrm{PSL}_2(\mathbb{Z})} T^{b-1}(TST)^{d-1}ST^{-a'_{d'-2}-1}\dots ST^{-a'_0}ST^{-b'}\\ &=_{\mathrm{PSL}_2(\mathbb{Z})} T^{b-1}ST^{-a'_{d'-2}-d}\dots ST^{-a'_0}ST^{-b'}.\end{aligned}$$

which again cannot evaluate to the identity in $\mathrm{PSL}_2(\mathbb{Z})$ by Lemma 6.3. This completes the proof of uniqueness.

Lastly, we will show the standard form has minimal winding number. Let $X \in \mathrm{PSL}_2(\mathbb{Z})$ and suppose $\eta \in \mathbb{Z}*\mathbb{Z}$ is S-positive with $\eta =_{\mathrm{PSL}_2(\mathbb{Z})} X$. Then η can be reduced to the standard form of X, denoted $\overline{X} \in \mathbb{Z}*\mathbb{Z}$, by following the reduction algorithm at the beginning of the proof. Since each of Reduction 1–Reduction 3 in the algorithm either preserves or reduces the winding number, $W(\overline{X}) \leqslant W(\eta)$. □

7. Minimal models of semitoric helices

The purpose of this section is to prove Theorem 3.6.

7.1. Standard forms and the winding number

Recall that given any $X \in \mathrm{PSL}_2(\mathbb{Z})$ we denote by $\overline{X} \in \mathbb{Z} * \mathbb{Z}$ the standard form of X, as given in Theorem 3.5.

Lemma 7.1. *If $X \in \mathrm{PSL}_2(\mathbb{Z}) \setminus \{T^k\}_{k\in\mathbb{Z}}$ then*

$$W\left(\overline{X}\right) + W\left(\overline{X^{-1}}\right) = \frac{1}{2}.$$

Proof. Write $\overline{X} = T^bST^{a_0}\dots ST^{a_{d-1}}$ and since $X \neq T^k$ for any $k \in \mathbb{Z}$, $d > 0$. Now, $W\left(\left(\overline{X}\right)^{-1}\right) = -W\left(\overline{X}\right)$ where

$$\left(\overline{X}\right)^{-1} = S^{-1}T^{-a_{d-1}}\dots S^{-1}T^{-a_0}S^{-1}T^{-b}.$$

We will reduce $\left(\overline{X}\right)^{-1}$ to standard form using the reduction steps in the proof of Theorem 3.5 and keep track of the winding number. Replacing each S^{-1} by S increases the winding number by $d/2$. Now replace each $ST^{-a_i}S$ with $(TST)^{a_i}$ for each even index i which at most increases the odd indexed powers of T by 2. Since each $a_i \geqslant 2$ for $i = 0, \dots, d-2$ we do the replacement $ST^{-a_i+2}S = (TST)^{a_i-2}$ for odd $0 < i < d-3$ and the replacement $ST^{-a_i+1}S = (TST)^{a_i-1}$ for $i = 1$ and the highest odd $i \leqslant d-2$. Thus we have now used $ST^{-n}S = (TST)^n$, for varying values of $n > 0$, a total of $d-1$ times decreasing W by $1/2$ each time. The word produced in this way is now in standard form so it is equal to $\overline{X^{-1}}$ and

$$W\left(\overline{X^{-1}}\right) = -W\left(\overline{X}\right) + \frac{d}{2} - \frac{d-1}{2} = -W\left(\overline{X}\right) + \frac{1}{2}$$

as desired. □

We can now prove that in many cases the first power of T in $\overline{X}$ and the last power of T in $\overline{X^{-1}}$ must sum to 1.

Lemma 7.2. *For $X \in \mathrm{PSL}_2(\mathbb{Z})$ write*

$$\overline{X} =_{\mathbb{Z}*\mathbb{Z}} T^{b}ST^{a_0}\dots ST^{a_{d-1}} \text{ and } \overline{X^{-1}} =_{\mathbb{Z}*\mathbb{Z}} T^{b'}ST^{a'_0}\dots ST^{a'_{d'-1}}.$$

Then

$$a_{d-1} + b' = a'_{d'-1} + b = 0$$

if $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^kST^a$ or $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ for some $k, a \in \mathbb{Z}$, and

$$a_{d-1} + b' = a'_{d'-1} + b = 1$$

otherwise.

Proof. The cases of $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^kST^a$ and $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ are easily checked. Suppose X is not of that form. Since $\overline{X^{-1}}\,\overline{X} =_{\mathrm{SL}_2(\mathbb{Z})} I$ by Lemma 6.3 some power of T that is not at the front or end of the word must be -1, 1, or 0. Since $\overline{X}$ and $\overline{X^{-1}}$ are in standard form, $X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^kST^a$, and $X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ this means that $a'_{d'-1} + b \in \{\pm 1, 0\}$.

If $a'_{d'-1} + b = 0$ then S^2 is a subword of $\overline{X^{-1}}\,\overline{X}$ which can be replaced by I and if $a'_{d'-1} + b = -1$ then $ST^{-1}S$ is a subword of $\overline{X^{-1}}\,\overline{X}$ which can be replaced by TST. In either case this means that $W\left(\overline{X^{-1}X}\right) \leqslant W\left(\overline{X^{-1}}\right) + W\left(\overline{X}\right) - \frac{1}{2} = 0$ where the last equality is by Lemma 7.1. By Lemma 6.1 $W(\overline{X^{-1}X}) \geqslant 0$ with equality only when $X = I$. Since $X \neq I$ we must have $a'_{d'-1} + b = 1$. The same analysis on $\overline{X}\,\overline{X^{-1}}$ implies that $a_{d-1} + b' = 1$. $\square$

Lemma 7.3. *Let $X \in \mathrm{PSL}_2(\mathbb{Z})$ and $c \in \mathbb{Z}_{>0}$. Then $\overline{X^{-1}T^cX} =_G \overline{X^{-1}}T^c\overline{X}$ and in particular $W(\overline{X^{-1}T^cX}) = W(\overline{X^{-1}}T^c\overline{X})$.*

Proof. If $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ for some $k \in Z$ then $\overline{X^{-1}T^cX} =_{\mathbb{Z}*\mathbb{Z}} \overline{X^{-1}}T^c\overline{X} =_{\mathbb{Z}*\mathbb{Z}} T^c$ so the result holds. If $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^kST^a$ for some $k, a \in \mathbb{Z}$ then there are two cases. If $c > 1$ then $\overline{X^{-1}T^cX} =_{\mathbb{Z}*\mathbb{Z}} \overline{X^{-1}}T^c\overline{X} =_{\mathbb{Z}*\mathbb{Z}} T^kST^cST^{-k}$ so the result holds. If $c = 1$ then $\overline{X^{-1}T^cX} =_{\mathbb{Z}*\mathbb{Z}} T^{k-1}ST^{-k-1}$ while $\overline{X^{-1}}T^c\overline{X} =_{\mathbb{Z}*\mathbb{Z}} T^kSTST^{-k}$ and the result still holds.

If $X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ and $X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^kST^a$ for all $k, a \in \mathbb{Z}$, then $\overline{X^{-1}}T^c\overline{X}$ is already in standard form for any $c > 0$ by Lemma 7.2, so $\overline{X^{-1}T^cX} =_{\mathbb{Z}*\mathbb{Z}} \overline{X^{-1}}T^c\overline{X}$. $\square$

7.2. Minimal words

Definition 7.4. An S-positive word with no leading T, $ST^{a_0}\dots ST^{a_{d-1}} \in \mathbb{Z}*\mathbb{Z}$, is *minimal* if and only if $a_0, \dots, a_{d-1} \neq 1$.

Minimal words are those associated to minimal helices.

Lemma 7.5. *Suppose $\sigma = ST^{a_0}\dots ST^{a_{d-1}} \in \mathbb{Z}*\mathbb{Z}$ is minimal and there exists $X \in G \setminus \{S^{2\ell}T^k\}_{\ell,k\in\mathbb{Z}}$ such that*

$$\sigma =_G S^4X^{-1}T^cX.$$

Then, after cyclically reordering $a_0, \dots, a_{d-1}$ if necessary, $a_0 \leqslant 0$ and one of the following hold:

(i) $a_0 = 0$ *and* $\overline{\sigma} =_{\mathbb{Z}*\mathbb{Z}} T^{a_1}ST^{a_2}\dots ST^{a_{d-1}}$;
(ii) $a_0 < 0$ *and* $\overline{\sigma} =_{\mathbb{Z}*\mathbb{Z}} (TST)^{-a_0}T^{a_1}ST^{a_2}\dots ST^{a_{d-1}}$.

Proof. Notice $W(\sigma) = W(S^4X^{-1}T^cX) = 1 - \frac{c}{12}$ while

$$\begin{aligned} W(\overline{\sigma}) &= W(\overline{X^{-1}T^cX}) = W(\overline{X^{-1}}T^c\overline{X}) \\ &= W(\overline{X^{-1}}) + W(\overline{X}) - \frac{c}{12} = \frac{1}{2} - \frac{c}{12} \end{aligned}$$

by Lemmas 7.3 and 7.1 since $X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ for any $k \in \mathbb{Z}$. Thus, $W(\sigma) \neq W(\overline{\sigma})$ so σ is not in standard form. This means that $a_j \leqslant 1$ for some fixed $j \in \{0, \dots, d-2\}$ and since σ is minimal this implies $a_j \leqslant 0$.

If $a_j = 0$ for some $j \in \{0, \dots, d-2\}$ then reorder so $a_0 = 0$ and $\sigma = S^2T^{a_1}ST^{a_2}\dots ST^{a_{d-1}}$. Notice that $\eta = T^{a_1}ST^{a_2}\dots ST^{a_{d-1}}$ satisfies $\eta =_{\mathrm{PSL}_2(\mathbb{Z})} \sigma$ so $\overline{\eta} =_{\mathbb{Z}*\mathbb{Z}} \overline{\sigma}$ and also notice $W(\eta) = W(\sigma) - 1/2 = W(\overline{\sigma})$. All steps in the reduction algorithm in the proof of Theorem 3.5 reduce the winding number, except for the blowdown $STS \to T^{-1}ST^{-1}$, so the only possible step to reduce η into standard form is a blowdown. For a blowdown to be possible we must have $a_j = 1$ for some $j \in \{1, \dots, d-1\}$, contradicting the minimality of σ. Thus, $\eta =_{\mathbb{Z}*\mathbb{Z}} \overline{\eta}$ so $\overline{\sigma} =_{\mathbb{Z}*\mathbb{Z}} \eta$.

Otherwise, $a_j \neq 0$ for all $j \in \{0, \dots, d-1\}$ so, after cyclically reordering, we may assume $a_0 < 0$. In this case let

$$\eta' = (TST)^{-a_0}T^{a_1}ST^{a_2}\dots ST^{a_1}$$

and notice $\eta' =_{\mathrm{PSL}_2(\mathbb{Z})} \sigma$ so $\overline{\eta'} =_{\mathbb{Z}*\mathbb{Z}} \overline{\sigma}$. Again, $W(\eta') = W(\overline{\sigma})$ so the only possible reduction move would be a blowdown, but if a blowdown could be performed on η' that would contradict the minimality of σ, except in the case that $a_1 = 0$, which we have assumed does not occur. Thus, $\overline{\sigma} = \eta'$. □

Here we classify all words associated to minimal semitoric helices. Recall $\mathcal{S}$ from Eq. (3.2).

Lemma 7.6 (*Classification of Minimal Words*). *Let $\mathcal{H}$ be a helix with complexity $c > 0$. If $\mathcal{H}$ is minimal then there is an associated word $\sigma \in \mathbb{Z} * \mathbb{Z}$ which is exactly one of the following, where $A_0 = [v_0, v_1]$ for some $\{v_i\}_{i\in\mathbb{Z}}$ such that $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$.*

Type	$\sigma \in \mathbb{Z} * \mathbb{Z}$	c	A_0		
(1)	$\sigma = ST^{-1}ST^{-4}$	$c = 1$	ST^{-2}	$=$	$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$
(2)	$\sigma = ST^{-2}ST^{-2}$	$c = 2$	ST^{-1}	$=$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
(3)	$\sigma = S^2T^aST^{-a-2}$, $a \neq 1, -3$	$c = 1$	ST^{-a-1}	$=$	$\begin{pmatrix} 0 & -1 \\ 1 & -a-1 \end{pmatrix}$
(4)	$\sigma = ST^{-1}ST^{-1}ST^{c-1}$	$c \neq 2$	I	$=$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(5)	$\sigma = S^2T^aST^cST^{-a}$, $a \neq \pm 1$	$c \neq 1$	ST^{-a}	$=$	$\begin{pmatrix} 0 & -1 \\ 1 & -a \end{pmatrix}$
(6)	$\sigma = S^2T^aS^2T^{c-a}$, $a \neq 1, c-1$	$c > 0$	I	$=$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(7)	$\sigma = S^2\overline{A_0^{-1}}T^c\overline{A_0}$	$c > 0$	A_0	$\in$	$\mathcal{S}$

where $a \in \mathbb{Z}$ is a parameter.

Proof. Suppose that $\sigma =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \ldots ST^{a_{d-1}}$ is minimal and associated to a semitoric helix $\mathcal{H}$ of length d and complexity $c > 0$. By Lemma 5.6 there exists some $X \in G$ such that

$$\sigma =_G S^4X^{-1}T^cX. \tag{7.1}$$

We will proceed by cases on X, and show that in each case that σ is one of type (1)–(7) in the statement of the Lemma.

Case I: $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ *for some* $k \in \mathbb{Z}$. This implies that

$$ST^{a_0} \ldots ST^{a_{d-1}-c} =_G S^4,$$

and so $(a_0, \ldots, a_{d-2}, a_{d-1} - c) \in \mathbb{Z}^d$ are associated to a minimal toric fan. Such words are completely classified in [12, Lemma 4.8] and we conclude either $d = 3$ and $a_0 = a_1 = a_2 - c = -1$, which is minimal only when $c \neq 2$, or $d = 4$ and $a_0 = a_2 = 0$, $a_3 = c - a_1$, which is minimal only when $a \neq 1, c - 1$. Thus σ is either of type (4) or (6).

Case II: $X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^k$ *for all* $k \in \mathbb{Z}$. In light of Eq. (7.1) apply Lemma 7.5 to σ and conclude that, after passing to an equivalent helix by cyclically permuting,

$$\sigma =_{\mathbb{Z}*\mathbb{Z}} ST^{a_0} \ldots ST^{a_{d-1}}$$

satisfies either

1. $a_0 = 0$; or
2. $a_j \neq 0$ for all $j = 0, \ldots, d-1$ and $a_0 < 0$.

If $a_0 = 0$ then

$$\overline{\sigma} = T^{a_1}ST^{a_2} \ldots ST^{a_{d-1}}$$

and otherwise

$$\overline{\sigma} = (TST)^{-a_0}T^{a_1}ST^{a_2} \ldots ST^{a_{d-1}}.$$

By cyclically permuting the a_i, which corresponds to conjugating Eq. (7.1) by ST^{a_i} for various i, we change X, but $X \neq T^k$ still holds. We now have three further cases on X.

Case IIa: $X =_{\mathrm{PSL}_2(\mathbb{Z})} T^kST^a$ *for* $k, a \in \mathbb{Z}$. First assume $a_0 = 0$. If $c = 1$ then

$$\overline{X^{-1}T^cX} =_{\mathbb{Z}*\mathbb{Z}} T^{-a-1}ST^{a-1}$$

so $\sigma = S^2T^{-a-1}ST^{a-1}$ which is minimal for $a \neq \pm 2$ and is of type (3). If $c \neq 1$ then

$$\overline{X^{-1}T^cX} = T^{-a}ST^cST^a$$

so $\sigma = S^2T^{-a}ST^cST^a$ which is minimal if $a \neq \pm 1$ and is of type (5).

Now suppose $a_0 \neq 0$. Then

$$\overline{\sigma} = (TST)^{-a_0}T^{a_1}ST^{a_2}\dots ST^{d-1}$$

and $a_0 < 0$. If $c = 1$ then $\overline{\sigma} = T^{-a-1}ST^{a-1}$ so $a = -2$ and thus $\overline{\sigma} = (TST)T^{-4}$ so $\sigma = ST^{-1}ST^{-4}$, which is of type (1). If $c = 2$, then $\overline{\sigma} = T^{-a}ST^2ST^a$ which means $a = -1$ and $a_0 = -2$ so $\sigma = ST^{-2}ST^{-2}$, which is of type 2. If $c > 2$, then $\overline{\sigma} = T^{-a}ST^cST^a$ which means $a = -1$ and $a_0 = -1$ so $\sigma = ST^{-1}ST^{c-1}ST^{-1}$, which is of type (4).

Case IIb: *$X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^k, T^kST^a$ for all $k, a \in \mathbb{Z}$ and $a_i \neq 0$ for all $i = 0, \dots, d-1$*. In this case $a_0 < 0$. If $d = 2$, then $\sigma = ST^{a_0}ST^{a_1}$ and $\overline{\sigma} = (TST)^{-a_0}T^{a_1}$ which means $(TST)^{-a_0}T^{a_1} =_{\mathbb{Z}*\mathbb{Z}} \overline{X^{-1}}T^c\overline{X}$. Since $\overline{X^{-1}}T^c\overline{X}$ starts with TS it must end with S by Lemma 7.2 so $a_1 = -1$. Now $W(\sigma) = 1 - {}^c/_{12}$ from Eq. (7.1) and

$$W(\sigma) = W(ST^{-n}ST^{-1}) = \frac{1}{12}(7+n)$$

so $n = 5 - c$ and we have

$$(TST)^{5-c}T^{-1} =_{\mathbb{Z}*\mathbb{Z}} \overline{X^{-1}}T^c\overline{X}. \tag{7.2}$$

The right side of Eq. (7.2) contains T^{c+1} while the highest power of T on the left side is T^2, so $c > 0$ implies $c = 1$. Thus we obtain $\sigma = ST^{-4}ST^{-1}$ and have type (3).

If $d > 2$ then

$$(TST)^{-a_0}T^{a_1}ST^{a_2}\dots ST^{a_{d-1}} =_{\mathbb{Z}*\mathbb{Z}} \overline{X^{-1}}T^c\overline{X}$$

implies that $\overline{X^{-1}}T^c\overline{X}$ must end with S by Lemma 7.2, so $a_{d-1} = 0$ which contradicts our assumption in this case.

Case IIc: *$X \neq_{\mathrm{PSL}_2(\mathbb{Z})} T^k, T^kST^a$ for all $k, a \in \mathbb{Z}$ and $a_i = 0$ for some $i \in \{0, \dots, d-1\}$*. We have already used Lemma 7.5 to establish that either $a_0 = 0$ or $a_j \neq 0$ for all $j = 0, \dots, d-1$. Thus, in this case, $a_0 = 0$ so

$$\sigma = S^2\overline{\sigma} = S^2\overline{X^{-1}}T^c\overline{X}$$

which is minimal if $\overline{X}$ does not end with ST and $\overline{X^{-1}}$ does not begin with TS, and is of type (7). □

7.3. *Proof of the main result*

We now have all of the tools to prove the main result of the paper.

Proof of Theorem 3.6. Let $\mathcal{H} = (d, c, [\{v_i\}_{i\in\mathbb{Z}}])$ be a minimal semitoric helix with associated integers $(a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$. Then $\sigma = ST^{a_0}\dots ST^{a_{d-1}}$ is a minimal word and, passing to an equivalent helix if necessary, we conclude σ must be of some type (1)–(7) in Lemma 7.6. Types (1)–(6) for σ in Lemma 7.6 correspond to types (1)–(6) for $\mathcal{H}$ in Theorem 3.6. Notice these each have length $d < 5$.

Otherwise, σ must be of type (7), which means there exists some $X =_G A_0$, where $A_0 = [v_0, v_1] \in \mathcal{S}$ and

$$\sigma =_{\mathbb{Z}*\mathbb{Z}} S^2\overline{X^{-1}}T^c\overline{X}.$$

Since $A_0 \in \mathcal{S}$ notice that $A_0 = ST^{a_0}\dots ST^{a_{\ell-1}}$ with $\ell \geqslant 2$, which implies that σ has at least six occurrences of S, so the length d of $\mathcal{H}$ satisfies $d \geqslant 6$.

By Lemma 4.4 minimal helices are exactly those which correspond to minimal semitoric manifolds, so the proof is complete. □

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