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SYMMETRIES AND CONSERVATION LAWS OF A NONLINEAR SIGMA MODEL WITH GRAVITINO

JÜRGEN JOST, ENNO KESSLER, JÜRGEN TOLKSDORF, RUIJUN WU, MIAOMIAO ZHU

ABSTRACT. We study the symmetries and invariances of a version of the action functional of the nonlinear sigma model with gravitino, as considered in [12]. The action is invariant under rescaled conformal transformations, super Weyl transformations and diffeomorphisms. In particular cases the functional possesses a degenerate supersymmetry. The corresponding conservation laws lead to a geometric interpretation of the energy-momentum tensor and supercurrent as holomorphic sections of appropriate bundles.

1. INTRODUCTION

The main motivation for the introduction of the two-dimensional supersymmetric nonlinear sigma model in quantum field theory, or more specifically supergravity and superstring theory, are its symmetries, see for instance [2, 5, 6, 10]. Furthermore, as argued in [14], the functional is determined by its symmetries together with suitable bounds on the order of its Euler–Lagrange equations. While supersymmetric models are usually formulated using anticommuting variables, in [12] an analogue of the two-dimensional nonlinear supersymmetric sigma model using only commuting variables was introduced. Here we would like to give a detailed geometric account of the symmetries of this purely commutative model.

We briefly recall the two-dimensional nonlinear sigma model constructed in [12]. Let M be a Riemann surface and let (N, h) be a Riemannian manifold. In the classical nonlinear sigma model, the action functional is given by the Dirichlet energy functional which is defined for a map $\phi: M \rightarrow N$ and a Riemannian metric g on M . In our model we need to take also their superpartners into consideration. These superpartners are geometrically formulated via suitable spinor fields. To be more precise, given the Riemannian metric g , we fix a spin structure $\xi: P_{\text{Spin}}(M, g) \rightarrow P_{\text{SO}}(M, g)$. An irreducible representation of the Clifford algebra $\text{Cl}_{0,2}$ induces the real spin representation $\mu: \text{Spin}(2) \rightarrow \text{GL}(V)$, where V is a representation space of real dimension four. The associated spinor bundle $S_g := P_{\text{Spin}}(M, g) \times_{\mu} V$ is equipped with the canonical spinor metric g_s and the spin connection ∇^s induced by the Levi-Civita connection on TM . Choosing an isomorphism between the vector spaces V and $\text{Cl}_{0,2}$ we get a Clifford map $\gamma: TM \rightarrow \text{End}(S_g)$ which satisfies the Clifford relation

$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Sections of S_g will be referred to as (pure) spinors, which describe matter fields with half-integer spins in physics. The spin Dirac operator $\not{D}_g := \gamma(e_{\alpha})\nabla_{e_{\alpha}}^s: \Gamma(S_g) \rightarrow \Gamma(S_g)$ is a first-order elliptic differential operator, which is self-adjoint if M is closed.

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The superpartner of the map ϕ is given by a section $\psi \in \Gamma(S_g \otimes \phi^*TN)$ known as a vector spinor, while the superpartner of the Riemannian metric g is given by a section $\chi \in \Gamma(S_g \otimes TM)$ called a gravitino. Note that on the twisted vector bundle $S_g \otimes \phi^*TN$ there is an induced metric $g_s \otimes \phi^*h$ and an induced metric connection denoted by $\tilde{\nabla}$. Moreover, it also admits a Clifford multiplication by tangent vectors acting on the spinorial part. Then $S_g \otimes \phi^*TN$ is turned into a Dirac bundle by the canonically defined Dirac operator \not{D}_g . The action functional of our two-dimensional nonlinear sigma model with gravitino then is

$$(1) \quad \begin{aligned} \mathbb{A}(\phi, \psi; g, \chi) := & \int_M |d\phi|_{g^\vee \otimes \phi^*h}^2 + \langle \psi, \not{D}_g \psi \rangle_{g_s \otimes \phi^*h} \\ & - 4\langle (\mathbb{1} \otimes \phi_*)(Q\chi), \psi \rangle_{g_s \otimes \phi^*h} - |Q\chi|_{g_s \otimes g}^2 |\psi|_{g_s \otimes \phi^*h}^2 - \frac{1}{6} R^N(\psi) \, \text{dvol}_g, \end{aligned}$$

where g^\vee stands for the dual metric on the cotangent bundle T^*M , the operator $Q: S_g \otimes TM \rightarrow S_g \otimes TM$ is a projection and $R^N(\psi)$ is an abbreviation for

$$R_{ijkl}^N(\phi) \langle \psi^i, \psi^k \rangle_{g_s} \langle \psi^j, \psi^l \rangle_{g_s}$$

in local coordinates $\{y^i\}$ of N , with $\psi = \psi^i \otimes \phi^*(\partial_{y^i})$. Here and in the sequel, we use the Einstein summation convention. In the action functional (1), the first term is the Dirichlet energy for the map ϕ and the second term is the Dirac action for a spinor field ψ along the map. Both of them involve the metric g . The next two terms bring in the gravitino χ and couple it with the other fields. The last term is a contraction of the vector spinor with the curvature tensor R^N of the target manifold N .

Let (e_α) be a local g -orthonormal frame on M . Then the Euler–Lagrange equations for the map and the vector spinor are

$$(2a) \quad \begin{aligned} 0 = EL(\phi) = & \tau(\phi) - \frac{1}{2} R^{\phi^*TN}(\psi, \gamma(e_\alpha)\psi) \phi_* e_\alpha + \frac{1}{12} S \nabla R^N(\psi) \\ & + \left(\left\langle \nabla_{e_\beta}^s (\gamma(e_\alpha) \gamma(e_\beta) \chi^\alpha), \psi \right\rangle_{g_s} + \left\langle \gamma(e_\alpha) \gamma(e_\beta) \chi^\alpha, \tilde{\nabla}_{e_\beta} \psi \right\rangle_{g_s} \right), \end{aligned}$$

$$(2b) \quad 0 = EL(\psi) = \not{D}_g \psi - |Q\chi|^2 \psi - \frac{1}{3} S R(\psi) - 2(\mathbb{1} \otimes \phi_*) Q\chi.$$

Here $\tau(\phi)$ is the tension field of the map ϕ and $SR(\psi)$ is a term involving the curvature of the target and is of third order in ψ . The term $S \nabla R(\psi)$ involves derivatives of R^N and is of fourth order in ψ . For the precise form of the curvature terms we refer to [12].

As symmetries play a very important role in applications, we want to study them in detail in this paper. Analogously to the models studied in physics, we expect conformal invariance, super Weyl invariance (an analogue of conformal invariance affecting the gravitino) and diffeomorphism invariance. We will show that this is indeed the case. The major difficulty in the study of those symmetries arises from the metric dependence of the spinors. In order to express the variation of the spinors in terms of the variation of the metric we employ the methods developed in [1]. In contrast to the original physical model, which involves anticommuting variables, here we stay in the context of commuting variables and therefore, the functional studied here possesses only a “degenerate supersymmetry” in special cases. Nevertheless, the origin of the action functional (1) in quantum field theory will strongly inspire our mathematical treatment. In turn, we hope our mathematical approach can also shed new light on some structure of quantum field theory.

By Noether’s principle, a differentiable family of symmetries yields a conservation law of the system. Here the conservation laws corresponding to those four symmetries are expressed in terms of the energy-momentum tensor T and the supercurrent J , which are the variation

of the action with respect to the Riemannian metric g and the gravitino χ respectively. We show that rescaled conformal invariance and super Weyl invariance lead to algebraic constraints on T and J , while diffeomorphism invariance and degenerate supersymmetry yield equations for the divergences of T and J . All four symmetries together allow for an identification of T and J with holomorphic sections of appropriate bundles. This is completely analogous to the supergeometric setting, where T and J constitute tangent vectors to the moduli space of the so-called super Riemann surfaces, see [11, 13].

The article is organized as follows: In Section 2, we study the behavior of the functional \mathbb{A} under rescalings of the metric and certain transformations of the gravitino. In Section 3, we derive the energy-momentum tensor T and the supercurrent J of \mathbb{A} and show that Weyl invariance and super Weyl invariance yield algebraic constraints for T and J . Diffeomorphism invariance of \mathbb{A} , studied in Section 4, yields a coupled divergence equation for T and J . This coupled divergence equation is uncoupled in the case that \mathbb{A} possesses degenerate supersymmetry as is shown in Section 5. The appendix gathers several formulas concerning the g -divergence and its analogue, the χ -divergence.

2. WEYL AND SUPER WEYL INVARIANCE

Recall that the Dirichlet energy defined for maps $\phi: M \rightarrow N$ by

$$E(\phi; g) := \int_M |\mathrm{d}\phi|_g^2 \mathrm{dvol}_g$$

stays invariant under conformal transformations: for any $u \in C^\infty(M)$,

$$E(\phi; g) = E(\phi; e^{2u}g).$$

This is known as the conformal invariance of the Dirichlet energy functional in dimension two. Hence the Dirichlet energy depends only on the conformal class of g and the functional E can thus be used to study the Teichmüller space, see [17, 18, 9].

However, when studying the Weyl invariance of the action \mathbb{A} , we observe that one cannot keep the spinorial fields ψ and χ fixed while varying the metric, because the spin structure depends on the metric and so do the spinor bundle S_g and the Dirac operator \not{D}_g . Fortunately they can be transformed covariantly such that the action functional (1) stays invariant.

We first recall some preliminaries from [1, 16, 8]. For any two metrics g and g' on M , there is a unique self-adjoint automorphism $H \in \mathrm{Aut}(TM)$ such that $g'(\cdot, \cdot) = g(H\cdot, \cdot)$. The automorphism

$$b \equiv b_g^g := H^{-1/2} \in \mathrm{Aut}(TM)$$

is then an isometry from (TM, g) to (TM, g') ; that is, $g'(b(\cdot), b(\cdot)) = g(\cdot, \cdot)$. Since b is $\mathrm{SO}(2)$ -equivariant, it defines an isomorphism of principle bundles $b: P_{\mathrm{SO}}(M, g) \rightarrow P_{\mathrm{SO}}(M, g')$ which can be lifted to an isomorphism $\tilde{b}: P_{\mathrm{Spin}}(M, g) \rightarrow P_{\mathrm{Spin}}(M, g')$. Here it is important to assume that $P_{\mathrm{Spin}}(M, g')$ corresponds to the same topological spin structure as $P_{\mathrm{Spin}}(M, g)$. Let S_g and $S_{g'}$ be spinor bundles associated to $P_{\mathrm{Spin}}(M, g)$ and $P_{\mathrm{Spin}}(M, g')$ respectively via the same representation $\mu: \mathrm{Spin}(2) \rightarrow \mathrm{SO}(V)$. Being $\mathrm{Spin}(2)$ -equivariant, \tilde{b} induces an isometry between the spinor bundles as Riemannian vector bundles which is denoted by

$$\beta \equiv \beta_{g'}^g: S_g \rightarrow S_{g'},$$

i.e., $g_s(\cdot, \cdot) = g'_s(\beta(\cdot), \beta(\cdot))$. Furthermore, note that for a vector $v \in TM$ and a spinor $\sigma \in S_g$ over the same basepoint, we have

$$\beta(\gamma(v)\sigma) = \gamma'(b(v))\beta(\sigma),$$

where γ' denotes the Clifford map with respect to the metric g' .

In particular, consider $g' = e^{2u}g$ for some $u \in C^\infty(M)$. Note that in general the conformal invariance does not hold:

$$(3) \quad \mathbb{A}(\phi, (\beta \otimes \mathbf{1})\psi; g', (\beta \otimes b)\chi) \neq \mathbb{A}(\phi, \psi; g, \chi).$$

Instead the Weyl invariance takes another form as stated below. Recall from [8, Prop. 1.3.10], with notation as above, that for any pure spinor field $\sigma \in \Gamma(S_g)$,

$$\not{D}_{g'}\beta(\sigma) = e^{-u}\beta\left(\not{D}_g\sigma + \frac{m-1}{2}\gamma(\text{grad}(u))\sigma\right).$$

Moreover, if we rescale the spinor by $e^{-\frac{1}{2}u}$, then the Dirac operator transforms homogeneously:

$$\not{D}_{g'}\beta(e^{-\frac{1}{2}u}\sigma) = e^{-\frac{3}{2}u}\beta(\not{D}_g\sigma).$$

Proposition 1. *The model has the following rescaled conformal invariance: for any $u \in C^\infty(M)$,*

$$(4) \quad \mathbb{A}\left(\phi, e^{-\frac{1}{2}u}(\beta \otimes \mathbf{1})\psi; e^{2u}g, e^{-\frac{1}{2}u}(\beta \otimes b)\chi\right) = \mathbb{A}(\phi, \psi; g, \chi).$$

Proof. Under the conformal transformation $g' = e^{2u}g$, the volume form transforms as $d\text{vol}_{g'} = e^{2u}d\text{vol}_g$. We check that the integrands will give rise to a factor e^{-2u} to cancel the scaling from the volume form. From the theory of harmonic maps we know that the energy density for the map has the expected scaling. So we start with the Dirac term:

$$\begin{aligned} & \not{D}_{e^{2u}g}\left(e^{-\frac{1}{2}u}(\beta \otimes \mathbf{1})\psi\right) \\ &= \not{D}_{g'}\left(e^{-\frac{1}{2}u}\beta(\psi^j)\right) \otimes \phi^*\left(\frac{\partial}{\partial y^j}\right) + \gamma'(b(e_\alpha))(e^{-\frac{1}{2}u}\beta\psi^j) \otimes \phi^*\left(\nabla_{b(e_\alpha)}^{\phi^*TN} \frac{\partial}{\partial y^j}\right). \end{aligned}$$

Note that here the map b is nothing but a scaling by e^{-u} . Therefore,

$$\begin{aligned} & \left\langle e^{-\frac{1}{2}u}(\beta \otimes \mathbf{1})\psi, \not{D}_{e^{2u}g}\left(e^{-\frac{1}{2}u}(\beta \otimes \mathbf{1})\psi\right) \right\rangle_{g'_s \otimes \phi^*h} \\ &= g'_s\left(e^{-\frac{1}{2}u}\beta(\psi^i), \not{D}_{g'}\left(e^{-\frac{1}{2}u}\beta(\psi^j)\right)\right) h_{ij}(\phi) \\ & \quad + g'_s\left(e^{-\frac{1}{2}u}\beta(\psi^i), \gamma'(b(e_\alpha))(e^{-\frac{1}{2}u}\beta\psi^j)\right) h(\phi) \left(\frac{\partial}{\partial y^i}, \nabla_{b(e_\alpha)}^{\phi^*TN} \frac{\partial}{\partial y^j}\right) \\ &= g'_s\left(e^{-\frac{1}{2}u}\beta(\psi^i), e^{-\frac{3}{2}u}\beta(\not{D}_g(\psi^j))\right) h_{ij}(\phi) \\ & \quad + e^{-u}g'_s\left(\beta\psi^i, \beta(\gamma(e_\alpha)\psi^j)\right) h(\phi) \left(\frac{\partial}{\partial y^i}, \nabla_{e^{-u}e_\alpha}^{\phi^*TN} \frac{\partial}{\partial y^j}\right) \\ &= e^{-2u}g_s(\psi^i, \not{D}_g\psi^j)h_{ij}(\phi) + e^{-2u}g_s(\psi^i, \gamma(e_\alpha)\psi^j)h(\phi) \left(\frac{\partial}{\partial y^i}, \nabla_{e_\alpha}^{\phi^*TN} \frac{\partial}{\partial y^j}\right) \\ &= e^{-2u}\langle \psi, \not{D}_g\psi \rangle_{g_s \otimes \phi^*h}. \end{aligned}$$

For the third term it holds:

$$\begin{aligned}
 & -4\langle (\mathbb{1} \otimes \phi_*) Q' e^{-\frac{1}{2}u} (\beta \otimes b) \chi, e^{-\frac{1}{2}u} (\beta \otimes \mathbb{1}) \psi \rangle_{g'_s \otimes \phi^* h} \\
 & = 2e^{-u} \left\langle \gamma'(b(e_\beta)) \gamma'(b(e_\alpha)) \beta(\chi^\beta) \otimes b(e_\alpha), \beta(\psi^j) \otimes \phi^* \left(\frac{\partial}{\partial y^j} \right) \right\rangle_{g'_s \otimes \phi^* h} \\
 & = 2e^{-u} \left\langle \beta(\gamma(e_\beta) \gamma(e_\alpha) \chi^\beta) \otimes (e^{-u} e_\alpha), \beta(\psi^j) \otimes \phi^* \left(\frac{\partial}{\partial y^j} \right) \right\rangle_{g'_s \otimes \phi^* h} \\
 & = 2e^{-2u} \langle \gamma(e_\beta) \gamma(e_\alpha) \chi^\beta \otimes e_\alpha, \psi \rangle_{g_s \otimes \phi^* h} \\
 & = -4e^{-2u} \langle (\mathbb{1} \otimes \phi_*) Q \chi, \psi \rangle_{g_s \otimes \phi^* h}.
 \end{aligned}$$

Here Q' denotes the projection operator corresponding to the rescaled metric g' . The remaining terms can be checked in a similar way, using the fact that b and β are isometries. \square

In contrast with the rescaled conformal invariance, super Weyl invariance affects only the gravitino. Let (e_α) be a local g -orthonormal frame of M and write $\chi = \chi^\alpha \otimes e_\alpha$, then the Q -projection operator and its orthogonal complement $P = \text{Id} - Q$ can be expressed as

$$Q\chi = -\frac{1}{2}\gamma(e_\alpha)\gamma(e_\beta)\chi^\alpha \otimes e_\beta, \quad P\chi = -\frac{1}{2}\gamma(e_\beta)\gamma(e_\alpha)\chi^\alpha \otimes e_\beta.$$

As only $Q\chi$ enters into the action functional, we have

$$(5) \quad \mathbb{A}(\phi, \psi; g, \chi + \zeta) = \mathbb{A}(\phi, \psi; g, \chi)$$

for any $\zeta \in \Gamma(S_g \otimes TM)$ with $\zeta \in \ker Q$. The property (5) is known as super Weyl invariance of the model.

We now take the opportunity to recall some properties of the bundle $S_g \otimes TM$ and of the projections P and Q , which have been presented in detail in [12]. The Riemann surface M possesses an almost complex structure J_M such that $g(J_M X, Y) = \text{dvol}_g(X, Y)$ for all vector fields X and Y . Left multiplication of spinors by the negative volume form $-\omega = -e_1 \cdot e_2 \in \text{Cl}(M, -g)$ yields an almost complex structure on S_g that is compatible with J_M , that is $\gamma(J_M X)s = -\gamma(X)\omega s$ for all spinors s . The decomposition of S_g in the even and the odd part yields $S_g = \Sigma \oplus \Sigma$, where both summands are isomorphic as associated vector bundles to $P_{\text{Spin}}(M, g)$. The almost complex structure ω restricts to Σ and the restriction will be denoted by J_Σ . For the complex line bundle $W = (\Sigma, J_\Sigma)$ it holds that $W \otimes_{\mathbb{C}} W = T^*M$ and W is actually holomorphic. With this preparation we can now identify the summands in $S_g \otimes TM = \ker Q \oplus \text{Im } Q$ to be

$$\ker Q = S_g = W \oplus W \quad \text{Im } Q = S_g \otimes_{\mathbb{C}} TM = (W \otimes_{\mathbb{C}} W^* \otimes_{\mathbb{C}} W^*) \oplus (W \otimes_{\mathbb{C}} W^* \otimes_{\mathbb{C}} W)$$

Hence both $\ker Q$ and $\text{Im } Q$ are naturally holomorphic vector bundles. Applying metric dualization, we obtain

$$(6) \quad S_g \otimes T^*M = W \oplus W \oplus (W \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W) \oplus (W \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} W).$$

3. THE CONSERVED QUANTITIES: SUPERCURRENT AND ENERGY-MOMENTUM TENSOR

The supercurrent is the variation of the action with respect to the gravitino. As the gravitino enters the action only algebraically, computation of the supercurrent is straightforward. Fix (ϕ, ψ)

as well as the metric g and vary the gravitino via $X(t) = X^\alpha(t) \otimes e_\alpha \in \Gamma(S_g \otimes TM)$ with $X(0) = \chi$ and

$$\left. \frac{d}{dt} \right|_{t=0} X^\alpha(t) = \zeta^\alpha.$$

Then

$$\left. \frac{d}{dt} \right|_{t=0} \mathbb{A}(\phi, \psi; g, X(t)) = \left. \frac{d}{dt} \right|_{t=0} \int_M -4 \langle (\mathbb{1} \otimes \phi_*)(QX(t)), \psi \rangle_{g_s \otimes \phi^* h} - |QX(t)|_{g_s \otimes g}^2 |\psi|_{g_s \otimes \phi^* h}^2 dvol_g.$$

This can be computed as follows.

$$\begin{aligned} & \left. \frac{d}{dt} \right|_{t=0} \int_M 2 \langle \gamma(e_\alpha) \gamma(e_\beta) X^\alpha \otimes \phi_* e_\beta, \psi \rangle_{g_s \otimes \phi^* h} dvol_g \\ &= \int_M 2 \left\langle \gamma(e_\alpha) \gamma(e_\beta) \left(\frac{d}{dt} X^\alpha \right) \otimes \phi_* e_\beta, \psi \right\rangle_{g_s \otimes \phi^* h} dvol_g \Big|_{t=0} \\ &= \int_M 2 \left\langle \left(\frac{d}{dt} X^\alpha \right), \langle \phi_* e_\beta, \gamma(e_\beta) \gamma(e_\alpha) \psi \rangle_{\phi^* h} \right\rangle_{g_s} dvol_g \Big|_{t=0} \\ &= \int_M 2 \langle \zeta^\alpha, \langle \phi_* e_\beta, \gamma(e_\beta) \gamma(e_\alpha) \psi \rangle_{\phi^* h} \rangle_{g_s} dvol_g, \\ & \left. \frac{d}{dt} \right|_{t=0} \int_M \frac{1}{2} \langle X^\beta, \gamma(e_\alpha) \gamma(e_\beta) X^\alpha \rangle_{g_s} |\psi|_{g_s \otimes \phi^* h}^2 dvol_g \\ &= \int_M \left\langle \left(\frac{d}{dt} X^\alpha \right), \gamma(e_\alpha) \gamma(e_\beta) X^\alpha \right\rangle_{g_s} |\psi|_{g_s \otimes \phi^* h}^2 dvol_g \Big|_{t=0} \\ &= \int_M \langle \zeta^\beta, \gamma(e_\alpha) \gamma(e_\beta) \chi^\alpha \rangle_{g_s} |\psi|_{g_s \otimes \phi^* h}^2 dvol_g. \end{aligned}$$

Hence setting the *supercurrent* to be $J = J^\alpha \otimes e_\alpha \in \Gamma(S_g \otimes TM)$ with

$$J^\alpha = 2 \langle \phi_* e_\beta, \gamma(e_\beta) \gamma(e_\alpha) \psi \rangle_{\phi^* h} + |\psi|^2 \gamma(e_\beta) \gamma(e_\alpha) \chi^\beta,$$

we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \mathbb{A}(\phi, \psi; g, X(t)) = \int_M \langle \zeta, J \rangle_{g_s \otimes g} dvol_g.$$

The conservation law associated to the super Weyl symmetry is obtained as follows. For any $\zeta \in \Gamma(S_g \otimes TM)$, we have $QP\zeta = 0$ and hence

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mathbb{A}(\phi, \psi; g, \chi + tP\zeta) = \int_M \langle P\zeta, J \rangle = \int_M \langle \zeta, PJ \rangle_{g_s \otimes g} dvol_g.$$

Since ζ can be arbitrary, we conclude that J satisfies

$$PJ = 0,$$

and hence $J \in \Gamma(\text{Im } Q)$.

Remark. Note that for a section $\zeta = \zeta^\alpha \otimes e_\alpha \in \Gamma(S_g \otimes TM)$, the following equations are equivalent:

$$P\zeta = 0 \quad \Leftrightarrow \quad \gamma(e_\alpha) \zeta^\alpha = 0 \quad \Leftrightarrow \quad \zeta^1 = \gamma(e_1) \gamma(e_2) \zeta^2 \quad \Leftrightarrow \quad \zeta^2 = -\gamma(e_1) \gamma(e_2) \zeta^1,$$

which in turn are equivalent to $\zeta = \zeta^\alpha \otimes e_\alpha$ lying in $\text{Im } Q = (S_g, J_\Sigma \oplus J_\Sigma) \otimes_{\mathbb{C}} TM$, where J_Σ is the complex structure on Σ . Note that the bundle $\text{Im } Q$ is a holomorphic vector bundle over the Riemann surface M .

We will also calculate the variation formula of the action functional with respect to the Riemannian metric. To this end we need a parametrized version of the metric dependence of the spinors. Let $(g_t)_t$ be a smooth family of Riemannian metrics parametrized by t in a small neighborhood of zero in \mathbb{R} such that $g_0 = g$. The t -derivative at $t = 0$ is a smooth symmetric 2-form, say $k \in \Gamma(\text{Sym}(T^*M \otimes T^*M))$. As before, there is a unique family of self-adjoint endomorphisms $(H_t)_t \subset \text{End}(TM)$ such that $g_t(\cdot, \cdot) = g(H_t \cdot, \cdot)$. We put $b_t \equiv b_{g_t}^g$ and $\beta_t \equiv \beta_{g_t}^g: S_g \rightarrow S_{g_t}$. Notice that

$$\left. \frac{d}{dt} \right|_{t=0} b_t = -\frac{1}{2}K \in \text{End}(TM)$$

where K is the endomorphism associated to k by metric dualization, which is also the t -derivative of H_t at $t = 0$.

Transport the Dirac operator $\not{D}_{g_t}: \Gamma(S_{g_t}) \rightarrow \Gamma(S_{g_t})$ via the isometry β_t to obtain a differential operator on $\Gamma(S_g)$:

$$\bar{\not{D}}_{g_t} := \beta_t^{-1} \circ \not{D}_{g_t} \circ \beta_t: \Gamma(S_g) \rightarrow \Gamma(S_g).$$

It is shown in [1] that

$$\left. \frac{d}{dt} \right|_{t=0} \bar{\not{D}}_{g_t} = -\frac{1}{2}\gamma(e_\alpha)\nabla_{K(e_\alpha)}^s + \frac{1}{4}\gamma\left(\text{grad}(\text{Tr}_g T) - \text{div}_g(k)_\sharp\right).$$

With the transformation behavior of the Dirac operator at hand, it is straightforward to calculate that

$$\left. \frac{d}{dt} \right|_{t=0} \mathbb{A}(\phi, (\beta_t \otimes \mathbb{1})\psi; g_t, (\beta_t \otimes b_t)\chi) = -\frac{1}{2} \int_M \left\langle \left. \frac{\partial g_t}{\partial t} \right|_{t=0}, T \right\rangle \text{dvol}_g,$$

where the inner product under the integral is the induced one on symmetric covariant two-tensors and $T = T_{\alpha\beta}e^\alpha \otimes e^\beta$ with

$$\begin{aligned} T_{\alpha\beta} = & 2\langle \phi_*e_\alpha, \phi_*e_\beta \rangle_{\phi^*h} + \frac{1}{2} \left\langle \psi, \gamma(e_\alpha)\tilde{\nabla}_{e_\beta}\psi + \gamma(e_\beta)\tilde{\nabla}_{e_\alpha}\psi \right\rangle_{g_s \otimes \phi^*h} \\ & + \langle \gamma(e_\eta)\gamma(e_\alpha)\chi^\eta \otimes \phi_*e_\beta + \gamma(e_\eta)\gamma(e_\beta)\chi^\eta \otimes \phi_*e_\alpha, \psi \rangle_{g_s \otimes \phi^*h} \\ & - \left(|\text{d}\phi|_{g^\vee \otimes \phi^*h}^2 + \langle \psi, \not{D}_g \psi \rangle - 4\langle (\mathbb{1} \otimes \phi_*)Q\chi, \psi \rangle - |Q\chi|^2|\psi|^2 - \frac{1}{6}R^N(\psi) \right) g_{\alpha\beta}. \end{aligned}$$

Arising from the variation of a symmetric two-tensor, T is clearly symmetric.

The energy-momentum tensor is in general not traceless, due to the fact that the action functional is not invariant under conformal transformations on g , see (3), but only invariant under rescaled conformal transformations (4). Furthermore, the conservation law corresponding to the rescaled conformal invariance prescribes the trace of the energy-momentum tensor. Indeed,

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \mathbb{A}(\phi, e^{-\frac{1}{2}tu}(\beta_t \otimes \mathbb{1})\psi; e^{2tu}g, e^{-\frac{1}{2}tu}(\beta_t \otimes b_t)\chi) \\ &= \int_M 2 \left\langle -\frac{1}{2}u\psi, EL(\psi) \right\rangle - \frac{1}{2} \langle 2ug, T \rangle + \left\langle -\frac{1}{2}u\chi, J \right\rangle \text{dvol}_g \\ &= - \int_M u \left(\text{Tr}_g(T) + \langle \psi, EL(\psi) \rangle + \frac{1}{2} \langle \chi, J \rangle \right) \text{dvol}_g. \end{aligned}$$

As the integral has to vanish for all functions u , we conclude

$$\mathrm{Tr}_g(T) = -\langle \psi, EL(\psi) \rangle - \frac{1}{2} \langle \chi, J \rangle,$$

where $EL(\psi)$ denotes the Euler–Lagrange equation for ψ , given in (2b). The right hand side is due to the additional rescaling on the spinorial fields compared to only conformal transformations. Notice, if the Euler–Lagrange equation for ψ is satisfied and either if χ or J vanishes, then T is actually traceless. In that case T can be identified with a smooth section of $T^*M \otimes_{\mathbb{C}} T^*M$, that is, a quadratic differential.

4. DIFFEOMORPHISM INVARIANCE

While diffeomorphism invariance of the model may seem obvious at first glance, its precise geometric formulation is not entirely trivial due the presence of spinors. Here we want to indicate how spinors transform under diffeomorphisms and obtain the resulting invariance of the action \mathbb{A} for closed manifolds and the corresponding conservation law.

Let f be a smooth diffeomorphism of M . The pullback metric g_f on TM and the differential map Tf fit into the following commutative diagram:

$$\begin{array}{ccc} (TM, g_f) & \xrightarrow{Tf} & (TM, g) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

This induces a map on the orthonormal frame bundles, which is also denoted by Tf . As Tf is $\mathrm{SO}(2)$ -equivariant, there exists a unique spin structure $P_{\mathrm{Spin}}(M, g_f) \rightarrow P_{\mathrm{SO}}(M, g_f)$ such that Tf lifts to the corresponding principal $\mathrm{Spin}(2)$ -bundles. Indeed, as explained in [15, Theorem II.1.7], the spin structures on M are in one-to-one correspondence to elements of $H^1(M, \mathbb{Z}_2)$ and $P_{\mathrm{Spin}}(M, g_f)$ is given by the pullback of the cohomology class corresponding to $P_{\mathrm{Spin}}(M, g)$. Hence, in general, the topological spin structures corresponding to $P_{\mathrm{Spin}}(M, g)$ and $P_{\mathrm{Spin}}(M, g_f)$ do not need to coincide. The situation is summarized the following commutative diagram on the left:

$$\begin{array}{ccc} P_{\mathrm{Spin}}(M, g_f) & \xrightarrow{\widetilde{Tf}} & P_{\mathrm{Spin}}(M, g) \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}}(M, g_f) & \xrightarrow{Tf} & P_{\mathrm{SO}}(M, g) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array} \quad \begin{array}{ccc} (S_{g_f}, g_s^f) & \xrightarrow{F} & (S_g, g_s) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

An irreducible spin representation $\mu: \mathrm{Spin}(2) \rightarrow \mathrm{SO}(V)$ will give rise to spinor bundles associated to the above principal $\mathrm{Spin}(2)$ -bundles and the isomorphism \widetilde{Tf} induces an isometry F of the corresponding spinor bundles, as shown in the commutative diagram above on the right.

In particular, since F is an isometry we have for any $y \in M$ with $f(y) \equiv x \in M$ and for any $\sigma_1, \sigma_2 \in (S_g)_x$,

$$g_{s|x}(\sigma_1, \sigma_2) = g_{s|y}^f \left(F_{|y}^{-1} \sigma_1, F_{|y}^{-1} \sigma_2 \right).$$

As a result, the Dirac operators \not{D}_g on S_g and \not{D}_{g_f} on S_{g_f} are the “same” up to the isometry F in the sense that

$$(7) \quad (\not{D}_g)_x = F|_y \circ (\not{D}_{g_f})_y \circ F|_y^{-1}.$$

Remark. The formula (7) holds because F is induced by the isometry $f: (M, g_f) \rightarrow (M, g)$. Then Tf preserves the Levi-Civita connection and hence F preserves the spin connection. For comparison, although the morphism β constructed in the previous section is also an isometry, it will not preserve the Dirac operator. Indeed, the map b preserves the metric, but not the Lie bracket (this can be seen already in the case of a conformal perturbation of the Riemannian metric), hence b does not preserve the Levi-Civita connection and consequently β does not necessarily preserve the spin connection.

Proposition 2. *Let $f \in \text{Diff}(M)$. With the notation as above, consider the following diffeomorphism transformation:*

$$(8) \quad \begin{aligned} \phi &\mapsto \phi' := \phi \circ f, \\ \psi = \psi^j \otimes \phi^* \left(\frac{\partial}{\partial y^j} \right) &\mapsto \psi' := F^{-1} \circ \psi^j \circ f \otimes (\phi \circ f)^* \left(\frac{\partial}{\partial y^j} \right), \\ g &\mapsto g' := g_f, \\ \chi = \chi^\alpha \otimes e_\alpha &\mapsto \chi' := F^{-1} \circ \chi^\alpha \circ f \otimes (Tf)^{-1} e_\alpha. \end{aligned}$$

The model possesses diffeomorphism invariance in the sense that

$$\mathbb{A}(\phi', \psi'; g', \chi') = \mathbb{A}(\phi, \psi; g, \chi).$$

Proof. Suppose that under the diffeomorphism f we have $y \mapsto x = f(y)$. Then as in the harmonic map case

$$|\text{d}\phi'|_{g'^\vee \otimes \phi'^* h}(y) = |\text{d}\phi|_{g^\vee \otimes \phi^* h}(x).$$

For those terms involving spinors, we note that for any spinor $\sigma \in \Gamma(S)$,

$$(9) \quad F^{-1}(\gamma(e_\alpha)\sigma)_{f(y)} = \gamma'((Tf)_y^{-1} e_\alpha) F^{-1}(\sigma)_{f(y)}$$

where γ' denotes the Clifford multiplications with respect to the metric $g' = g_f$. From this we will see that the other terms are also invariant:

(i) First consider the Dirac term

$$\begin{aligned} \not{D}_{g'} \psi'(y) &= (\not{D}_{g'})_y F|_y^{-1} (\psi^k \circ f)_y \otimes \phi'^* \left(\frac{\partial}{\partial y^k} \right) \\ &\quad + \gamma'(Tf^{-1}(e_\alpha)) F|_y^{-1} (\psi^k \circ f)_y \otimes \nabla_{(T\phi')(Tf)^{-1} e_\alpha}^{TN} \left(\frac{\partial}{\partial y^k} \right) \\ &= F|_y^{-1} (\not{D}_g)_x \psi^k(x) \otimes \phi'^* \left(\frac{\partial}{\partial y^k} \right) + F|_y^{-1} (\gamma(e_\alpha) \psi^k(x)) \otimes \nabla_{T\phi(e_\alpha)}^{TN} \left(\frac{\partial}{\partial y^k} \right). \end{aligned}$$

Thus

$$\begin{aligned}
 & \langle \psi', \mathcal{D}_g \psi' \rangle(y) \\
 &= g_{s|y}^f \left(F_{|y}^{-1}(\psi^j \circ f)_y, F_{|y}^{-1}(\mathcal{D}_g)_x \psi^k(x) \right) h(\phi'(y))_{jk} \\
 &+ g_{s|y}^f \left(F_{|y}^{-1}(\psi^j \circ f)_y, F_{|y}^{-1}(\gamma(e_\alpha) \psi^k(x)) \right) h_{|\phi'(y)} \left(\frac{\partial}{\partial y^j}, \nabla_{T\phi(e_\alpha)}^{TN} \frac{\partial}{\partial y^k} \right) \\
 &= g_{s|x}(\psi^j(x), (\mathcal{D}_g)_x \psi^k(x)) h(\phi(x))_{jk} \\
 &+ g_{s|x}(\psi^j(x), \gamma(e_\alpha) \psi^k(x)) h_{|\phi(x)} \left(\frac{\partial}{\partial y^j}, \nabla_{T\phi(e_\alpha)}^{TN} \left(\frac{\partial}{\partial y^k} \right) \right) \\
 &= \langle \psi, \mathcal{D}_g \psi \rangle(x).
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 |\psi'(y)|_{g_{s \otimes \phi'^* h}}^2 &= g_{s|y}^f \left(F_{|y}^{-1}(\psi^j \circ f)_y, F_{|y}^{-1}(\psi^k \circ f)_y \right) h(\phi'(y))_{jk} \\
 &= g_{s|x}(\psi^j(x), \psi^k(x)) h(\phi(x))_{jk} \\
 &= |\psi(x)|_{g_s \otimes \phi^* h}^2
 \end{aligned}$$

and

$$\begin{aligned}
 R^N(\psi')|_y &= R_{ijkl}^N(\phi'(y)) g_{s|y}^f \left(F_{|y}^{-1}(\psi^i \circ f)_y, F_{|y}^{-1}(\psi^k \circ f)_y \right) \\
 &\quad \times g_{s|y}^f \left(F_{|y}^{-1}(\psi^j \circ f)_y, F_{|y}^{-1}(\psi^l \circ f)_y \right) \\
 &= R^N(\phi(x))_{ijkl} g_{s|x}(\psi^i(x), (\mathcal{D}_g)_x \psi^k(x)) g_{s|x}(\psi^j(x), (\mathcal{D}_g)_x \psi^l(x)) \\
 &= R^N(\psi)|_x.
 \end{aligned}$$

(ii) For the gravitino, from (9) it follows that

$$Q' \chi'(y) = F_{|y}^{-1}(\gamma(e_\alpha) \gamma(e_\beta) \chi^\alpha)_{|f(y)} \otimes (Tf)^{-1} e_\beta.$$

Hence we have

$$|Q' \chi'(y)|_{g_{s \otimes g_t}^f}^2 = |Q \chi(x)|_{g_s \otimes g}^2.$$

For the mixed term, note that

$$\begin{aligned}
 (\mathbb{1} \otimes \phi'_*) Q' \chi'(y) &= \gamma'((Tf)_y^{-1} e_\alpha) \gamma'((Tf)_y^{-1} e_\beta) F_{|y}^{-1} \chi^\alpha(f(y)) \otimes \phi'_*(Tf)^{-1} e_\beta \\
 &= F_{|f(y)}^{-1}(\gamma(e_\alpha) \gamma(e_\beta) \chi^\alpha)_{f(y)} \otimes \phi_* e_\beta.
 \end{aligned}$$

Then it is immediate that

$$\langle (\mathbb{1} \otimes \phi'_*) Q' \chi'(y), \psi'(y) \rangle_{g_{s \otimes \phi'^* h}} = \langle (\mathbb{1} \otimes \phi_*) Q \chi(x), \psi(x) \rangle_{g_s \otimes \phi^* h}.$$

Therefore, by the change of variables formula, the diffeomorphism invariance of \mathbb{A} is confirmed. \square

To obtain the corresponding conservation law, we take a (local) one-parameter group of diffeomorphisms (f_t) of M with $f_0 = \text{Id}_M$ and

$$\left. \frac{d}{dt} \right|_{t=0} f_t = X \in \Gamma(TM).$$

For example, the flow generated by X is such a family; the flow is actually global since M is assumed to be compact. Write $M_t = f_t^{-1}(M) = f_{-t}(M)$ and denote the pullback metrics on

M by $g_t \equiv g_{f_t}$. The differential Tf_t is again an isometry and hence can be viewed as a map between the principal $\text{SO}(2)$ -bundles. Note that $M_t = M$ and hence g is also a Riemannian metric on TM_t . These two metrics can be related by an isometry $b_t \equiv b_{g_t}^g : (TM_t, g) \rightarrow (TM_t, g_t)$ as before and can be lifted to the corresponding principal $\text{Spin}(2)$ -bundles. In order to construct the lift, we notice that a diffeomorphism homotopic to the identity preserves the topological spin structure. Therefore we have the following commutative diagram:

$$\begin{array}{ccccc}
 P_{\text{Spin}}(M, g) & \xrightarrow{\tilde{b}_t} & P_{\text{Spin}}(M_t, g_t) & \xrightarrow{\tilde{T}f_t} & P_{\text{Spin}}(M, g) \\
 \downarrow & & \downarrow & & \downarrow \\
 P_{\text{SO}}(M, g) & \xrightarrow{b_t} & P_{\text{SO}}(M_t, g_t) & \xrightarrow{Tf_t} & P_{\text{SO}}(M, g) \\
 \downarrow & & \downarrow & & \downarrow \\
 M & \xrightarrow{\text{Id}} & M_t & \xrightarrow{f_t} & M \\
 \\
 y & \longmapsto & \text{Id}(y) = y & \longmapsto & x = f_t(y)
 \end{array}$$

The bottom line exhibits the pointwise behavior of the maps on the base manifolds. Note that in this diagram all the horizontal maps are diffeomorphisms or isomorphisms. The associated commutative diagram of spinor bundles is given by

$$\begin{array}{ccccc}
 (S_g, g_s) & \xrightarrow{\beta_t} & (S_{g_t}, g_s(t)) & \xrightarrow{F_t} & (S_g, g_s) \\
 \downarrow & & \downarrow & & \downarrow \\
 M & \xrightarrow{\text{Id}} & M_t & \xrightarrow{f_t} & M
 \end{array}$$

The fields (ϕ, ψ, g, χ) are transformed to $(\phi_t, \psi_t, g_t, \chi_t)$ by the Equations (8) induced by f_t . Then

$$\mathbb{A}(\phi_t, \psi_t; g_t, \chi_t) = \mathbb{A}(\phi, \psi; g, \chi)$$

for each t . Thus,

$$\begin{aligned}
 (10) \quad 0 &= \left. \frac{d}{dt} \right|_{t=0} \mathbb{A}(\phi_t, \psi_t; g_t, \chi_t) \\
 &= \int_M -2 \left\langle \left. \frac{d}{dt} \right|_{t=0} \phi_t, EL(\phi) \right\rangle + 2 \left\langle \nabla_{\partial_t}^{S_g \otimes \phi_t^* TM} ((\beta_t^{-1} \otimes \mathbb{1})\psi_t) \Big|_{t=0}, EL(\psi) \right\rangle d\text{vol}_g \\
 &\quad + \int_M -\frac{1}{2} \left\langle \left. \frac{d}{dt} \right|_{t=0} g_t, T \right\rangle + \left\langle \left. \frac{d}{dt} \right|_{t=0} \chi_t, J \right\rangle d\text{vol}_g \\
 &= \int_M -2 \langle \phi_*(X), EL(\phi) \rangle + 2 \langle \delta\psi(X), EL(\psi) \rangle - \frac{1}{2} \langle \mathcal{L}_X g, T \rangle + \langle \mathcal{L}_X^{S_g \otimes TM} \chi, J \rangle d\text{vol}_g.
 \end{aligned}$$

Note that for the vector spinor we have to take the covariant derivative to obtain the variation field which is abbreviated as $\delta\psi(X)$ in the last formula, which is the same as the approach taken in [12]. Here the Lie derivative on $S_g \otimes TM$ is the one defined in [1]; that is, under a local orthonormal frame (e_α) , for $\chi = \chi^\alpha \otimes e_\alpha$,

$$\mathcal{L}_X^{S_g \otimes TM} \chi := \left. \frac{d}{dt} \right|_{t=0} \beta_t^{-1} F_t^{-1} (\chi^\alpha \circ f_t) \otimes b_t^{-1} (Tf_t)^{-1} (e_\alpha \circ f_t).$$

Notice that on the vector part of χ the above t -derivative will not result in the ordinary Lie derivative on tangent vectors, see also [1]. Using the formal divergence operator defined in Section 6 one knows that

$$\int_M \langle \mathcal{L}_X g, T \rangle \, d\text{vol}_g = -2 \int_M \langle X, \text{div}_g(T) \rangle \, d\text{vol}_g$$

and

$$\int_M \langle \mathcal{L}_X^{S_g \otimes TM} \chi, J \rangle \, d\text{vol}_g = \int_M \langle X, \text{div}_\chi(J) \rangle \, d\text{vol}_g.$$

Therefore, along solutions of the Euler–Lagrange equations (2), the following identity holds:

$$\text{div}_g(T) + \text{div}_\chi(J) = 0.$$

Remark. If the gravitino vanishes, then from (10) we know that, along solutions of the Euler–Lagrange equations,

$$0 = \int_M -\frac{1}{2} \langle \mathcal{L}_X g, T \rangle \, d\text{vol}_g = \int_M \langle X, \text{div}_g T \rangle \, d\text{vol}_g.$$

This tells us that T is divergence-free and hence the energy-momentum tensor corresponds to a holomorphic quadratic differential.

5. DEGENERATE SUPERSYMMETRY

The action functional (1) is motivated by the action functional of two-dimensional supersymmetric sigma models [2] also called the superconformal action functional in [11]. The major reason for the introduction of those models was supersymmetry. As was argued in [14] supersymmetry requires anticommuting variables. Hence a full supersymmetry cannot be expected for the action functional (1). Surprisingly, the following special case of supersymmetry, which we will call degenerate supersymmetry, persists:

Proposition 3. *Let q be a section of S_g . Assume $\chi = 0$ and the following holds:*

$$(11) \quad \int_M 6 \left\langle \psi, R^N \left(\langle q, \psi \rangle_{g_s}, \phi_* e_\alpha \right) \gamma(e_\alpha) \psi \right\rangle + \left\langle S \nabla R(\psi), \langle q, \psi \rangle_{g_s} \right\rangle - 4 \langle SR(\psi), \gamma(\text{grad } \phi) q \rangle \, d\text{vol}_g = 0.$$

Then the action functional (1) is invariant under the following infinitesimal transformations

$$\begin{aligned} \delta \phi &= \langle q, \psi \rangle_{g_s} & \delta \psi &= -\gamma(\text{grad } \phi) q \\ \delta g &= 0 & \delta \chi &= (\nabla^s q)_\# \end{aligned}$$

where $(\nabla^s q)_\# \equiv \nabla_{e_\alpha}^s q \otimes e_\alpha \in \Gamma(S_g \otimes TM)$.

The condition (11) is in particular fulfilled if the target manifold N is flat.

Remark. In the special case at hand the degenerate supersymmetry transformations of Proposition 3 coincide up to the sign of $\delta \psi$ and $\delta \chi$ with the supersymmetry transformations given in [13] for the superconformal action functional. The sign difference is necessary to compensate for the use of the opposite Clifford algebra in this work. Again, in contrast to [13], all variables, including the supersymmetry parameter q , are usual, commuting variables.

Remark. We use here the Lagrange formalism common in the calculus of variations and physics, see [7]. For example the expression $\delta\phi$ is to be read as $\frac{d}{dt}\big|_{t=0}\phi$ where ϕ depends implicitly on an additional parameter t . Similarly, $\delta\mathbb{A}(\phi, \psi; g, \chi)$ is to be read as the derivative with respect to t at the point $t = 0$ and can be evaluated using the chain rule. Hence, $\delta\mathbb{A}(\phi, \psi; g, \chi)$ is a linear functional in $\delta\phi$, $\delta\psi$, δg and $\delta\chi$.

Proof. The variation of $\mathbb{A}(\phi, \psi; g, \chi)$ is given by

$$\begin{aligned} \delta\mathbb{A}(\phi, \psi; g, \chi) = & \int_M 2g^{\alpha\beta} \phi^* h \left(\nabla_{e_\alpha}^{\phi^* TN} \langle q, \psi \rangle, \phi_* e_\beta \right) - \langle \gamma(e_\alpha) q \otimes \phi_* e_\alpha, \mathbb{D}\psi \rangle \\ & - \langle \psi, \mathbb{D}_g (\gamma(e_\alpha) q \otimes \phi_* e_\alpha) + R^N \left(\langle q, \psi \rangle_{g_s}, \phi_* e_\alpha \right) \gamma(e_\alpha) \psi \rangle \\ & + 2 \langle \gamma(e_\alpha) \gamma(e_\beta) \nabla_{e_\alpha}^s q \otimes \phi_* e_\beta, \psi \rangle \\ & - \frac{1}{6} \left(\langle S \nabla R(\psi), \langle q, \psi \rangle_{g_s} \rangle - 4 \langle SR(\psi), \gamma(\text{grad } \phi) q \rangle \right) \text{dvol}_g. \end{aligned}$$

Here we have used

$$\begin{aligned} \delta(\mathbb{D}_g \psi) &= \mathbb{D}_g(\delta\psi) + R^N(\delta\phi, \phi_* e_\alpha) \gamma(e_\alpha) \psi, \\ \delta(R^N(\psi)) &= \langle S \nabla R(\psi), \delta\phi \rangle + 4 \langle SR(\psi), \delta\psi \rangle, \end{aligned}$$

compare [12, Section 4.1, (2) and (5)]. If we now use (11), the variation of the action reduces to:

$$\begin{aligned} \delta\mathbb{A}(\phi, \psi; g, \chi) &= \int_M 2g^{\alpha\beta} \phi^* h \left(\nabla_{e_\alpha} \langle q, \psi \rangle, \phi_* e_\beta \right) - \langle \gamma(e_\alpha) q \otimes \phi_* e_\alpha, \mathbb{D}_g \psi \rangle \\ &\quad - \langle \psi, \mathbb{D}_g (\gamma(e_\alpha) q \otimes \phi_* e_\alpha) \rangle + 2 \langle \gamma(e_\alpha) \gamma(e_\beta) \nabla_{e_\alpha}^s q \otimes \phi_* e_\beta, \psi \rangle \text{dvol}_g \\ &= \int_M 2g^{\alpha\beta} \left(\langle \nabla_{e_\alpha}^s q \otimes \phi_* e_\beta, \psi \rangle + \langle q \otimes \phi_* e_\beta, \tilde{\nabla}_{e_\alpha} \psi \rangle \right) + \langle \gamma(e_\beta) \gamma(e_\alpha) q \otimes \phi_* e_\alpha, \tilde{\nabla}_{e_\beta} \psi \rangle \\ &\quad - \langle \gamma(e_\beta) \left(\gamma(\nabla_{e_\beta} e_\alpha) q \otimes \phi_* e_\alpha + \gamma(e_\alpha) \nabla_{e_\beta}^s q \otimes \phi_* e_\alpha + \gamma(e_\alpha) q \otimes \nabla_{e_\beta} \phi_* e_\alpha \right), \psi \rangle \\ &\quad + 2 \langle \gamma(e_\alpha) \gamma(e_\beta) \nabla_{e_\alpha}^s q \otimes \phi_* e_\beta, \psi \rangle \text{dvol}_g \\ &= \int_M - \langle \gamma(e_\beta) \gamma(e_\alpha) \nabla_{e_\alpha}^s q \otimes \phi_* e_\beta, \psi \rangle - \langle \gamma(e_\beta) \gamma(e_\alpha) q \otimes \nabla_{e_\alpha} \phi_* e_\beta, \psi \rangle \\ &\quad - \langle \gamma(e_\beta) \gamma(e_\alpha) q \otimes \phi_* e_\beta, \tilde{\nabla}_{e_\alpha} \psi \rangle - \langle \gamma(e_\beta) \gamma(\nabla_{e_\beta} e_\alpha) q \otimes \phi_* e_\beta, \psi \rangle \\ &\quad + \langle \gamma(e_\beta) \gamma(e_\alpha) q \otimes (\nabla_{e_\alpha} \phi_* e_\beta - \nabla_{e_\beta} \phi_* e_\alpha), \psi \rangle \text{dvol}_g \\ &= \int_M -e_\alpha \langle \gamma(e_\beta) \gamma(e_\alpha) q \otimes \phi_* e_\beta, \psi \rangle + \langle (\gamma(\nabla_{e_\alpha} e_\beta) \gamma(e_\alpha) + \gamma(e_\beta) \gamma(\nabla_{e_\alpha} e_\alpha)) q \otimes \phi_* e_\beta, \psi \rangle \\ &\quad - \langle \gamma(e_\beta) \gamma(\nabla_{e_\beta} e_\alpha) q \otimes \phi_* e_\beta, \psi \rangle + \langle \gamma(e_\beta) \gamma(e_\alpha) q \otimes (\nabla_{e_\alpha} \phi_* e_\beta - \nabla_{e_\beta} \phi_* e_\alpha), \psi \rangle \text{dvol}_g \\ &= 0. \end{aligned}$$

Here in the last step we have used the torsion-freeness of the connection ∇ . \square

Theorem 1. Suppose as in Proposition 3 that $\chi = 0$ and condition (11) is fulfilled. In addition we assume that ϕ and ψ fulfill the Euler–Lagrange equations (2). Then $\text{div}_g J = 0$. In particular, the supercurrent J can be identified with a holomorphic section of $S_g \otimes_{\mathbb{C}} T^*M$ by metric duality.

Proof. In the particular case that ϕ and ψ fulfill the Euler–Lagrange equations, we know that for all spinors q :

$$0 = \delta\mathbb{A}(\phi, \psi; g, \chi) = \int_M \langle \delta\chi, J \rangle \, d\text{vol} = \int_M \langle (\nabla^s q)_\sharp, J \rangle \, d\text{vol} = - \int_M \langle q, \text{div}_g J \rangle \, d\text{vol}.$$

With respect to the oriented orthonormal basis e_α , the super current can be written as $J = J^\alpha \otimes e_\alpha$. Let $\tilde{J} = J^\alpha \otimes e^\alpha$ be its metric dual. From $PJ = 0$ it follows that \tilde{J} is a section of $S_g \otimes T^*M$, see Equation (6) and $\text{div}_g J = 0$ translates into holomorphicity of \tilde{J} as explained in Section 6. \square

In the remainder of this subsection we give an application of the degenerate supersymmetry to the action functional of Dirac-harmonic maps, with or without curvature term. Recall that the functional of Dirac-harmonic maps with curvature term in [4] can be obtained from the functional (1) by setting the gravitino to zero. It is a corollary of Proposition 3 that the functional of Dirac-harmonic maps with or without curvature term may also have a degenerate supersymmetry.

Corollary 1 (Degenerate supersymmetry of the functional of Dirac-harmonic maps with curvature term). *Let $q \in \Gamma(S_g)$ be a twistor spinor. The functional of Dirac-harmonic maps with curvature term is invariant under the following infinitesimal transformations*

$$\delta\phi = \langle q, \psi \rangle_{g_s} \quad \delta\psi = -\gamma(\text{grad } \phi)q$$

provided that the curvature condition (11) holds.

Proof. The calculation proceeds as in the proof of Proposition 3 with the additional condition that the term that describes the variation of the gravitino needs to be zero. The term of $\delta\mathbb{A}(\phi, \psi; g, \chi)$ that arises from the variation of the gravitino is

$$\int_M 2 \langle \gamma(e_\alpha) \gamma(e_\beta) \nabla_{e_\alpha}^s q \otimes \phi_* e_\beta, \psi \rangle \, d\text{vol}_g = - \int_M 2 \langle (2\nabla_{e_\beta}^s q + \gamma(e_\beta) \not\partial_g q) \otimes \phi_* e_\beta, \psi \rangle \, d\text{vol}_g.$$

This term vanishes if q is a twistor spinor, i.e. for all vector fields X it holds

$$\nabla_X^s q + \frac{1}{2} \gamma(X) \not\partial_g q = 0. \quad \square$$

Remark. Recall that we have equipped the bundle S_g with the complex structure $J_\Sigma \oplus J_\Sigma$ arising from the Clifford multiplication by $-\omega$. One can check that twistor spinors are holomorphic sections of S_g . Hence, Corollary 1 does not come as a surprise, as also for super Riemann surfaces a supersymmetry with holomorphic parameter q leaves the $\frac{3}{2}$ -part of the gravitino invariant, see [13, Chapter 11.1]. On the other hand, it is shown in [8, Proposition A.2.1 and Note A.2.2] that the only surfaces admitting twistor spinors are the sphere and the torus.

Recall that the Dirac-harmonic map functional in [3] does not include the curvature term of the target manifold. But note that a curvature term will arise when taking variations.

Corollary 2 (Degenerate supersymmetry of the Dirac-harmonic map functional). *Let $q \in \Gamma(S_g)$ be a twistor spinor. The functional of Dirac-harmonic maps is invariant under the following infinitesimal transformations*

$$\delta\phi = \langle q, \psi \rangle_{g_s} \quad \delta\psi = -\gamma(\text{grad } \phi)q$$

provided that the following curvature condition

$$(12) \quad \int_M \langle \psi, R(\langle q, \psi \rangle_{g_s}, \phi_* e_\alpha) \gamma(e_\alpha) \psi \rangle_{g_s \otimes \phi^* h} \, d\text{vol}_g = 0.$$

holds.

In [3] two seemingly unrelated Dirac-harmonic maps have been constructed: The trivial solution $(\phi, 0)$, where $\phi: M \rightarrow N$ is a harmonic map and the twistor spinor solution (ϕ, ψ) where $\psi = -\gamma(e_\alpha)q \otimes \phi_* e_\alpha$ is constructed from a twistor spinor $q \in \Gamma(S_g)$ and the harmonic map ϕ . While the construction of the trivial solution and the twistor spinor solution was presented in [3] only for $M = S^2$ it can be carried out for every Riemann surface M admitting twistor spinors and arbitrary target N . We will now show that those two solutions are related via degenerate supersymmetry.

Consider the family defined on the time interval $[0, 1]$ given by

$$\phi_t = \phi, \quad \psi_t = -t\gamma(e_\alpha)q \otimes \phi_* e_\alpha.$$

The family (ϕ_t, ψ_t) interpolates between the trivial and the twistor spinor solution via a family of degenerate supersymmetries. Indeed for every $t = \tau$ we have

$$\left. \frac{d}{dt} \right|_{t=\tau} \phi_t = 0 = \langle q, \psi_\tau \rangle, \quad \left. \frac{d}{dt} \right|_{t=\tau} \psi_t = -\gamma(e_\alpha)q \otimes \phi_* e_\alpha = -\gamma(\text{grad } \phi)q.$$

The condition (12) is fulfilled along this family and hence we conclude that (ϕ_t, ψ_t) is critical for all $t \in [0, 1]$. Consequently, we have a critical family of degenerate supersymmetries and the twistor spinor solution should be considered equivalent to the trivial solution.

We expect that more non-trivial critical Dirac-harmonic maps can be constructed with the help of supersymmetry. The difficulty lies in the construction of suitable families (ϕ_t, ψ_t) .

6. APPENDIX

As we frequently use the divergence operators on various fields, we review them in this appendix and explain their relation to Lie derivatives.

Recall that on a Riemannian manifold (M, g) the divergence of a vector field X is a function on M that can be expressed with the help of the Levi-Civita covariant derivative as

$$\text{div}_g(X) = \text{Tr}_g(\nabla X) \in C^\infty(M).$$

Furthermore, the divergence operator is the negative L^2 -adjoint operator of the gradient operator: for any $f \in C^\infty(M)$ and any $X \in \Gamma(TM)$,

$$\int_M \langle X, \text{grad}(f) \rangle \, \text{dvol}_g = \int_M \langle -\text{div}_g X, f \rangle \, \text{dvol}_g.$$

The divergence operator on symmetric two-tensors is defined in a similar way. Let $k \in \Gamma(\text{Sym}(T^*M \otimes T^*M))$, the divergence operator on k is the one-form given by

$$\text{div}_g k = \text{Tr}_g(\nabla k) := \sum_{\alpha=1}^{\dim M} (\nabla_{e_\alpha} k)(e_\alpha, \cdot) \in \Gamma(T^*M),$$

where (e_α) is a local orthonormal frame. Then, for any vector field $X \in \Gamma(TM)$,

$$\int_M \langle \mathcal{L}_X g, k \rangle \, \text{dvol}_g = -2 \int_M (\text{div}_g k)(X) \, \text{dvol}_g.$$

Note that a symmetric, traceless and divergence-free two-tensor corresponds to a holomorphic quadratic differential on a Riemann surface, see e.g. [18, Section 2.4].

Now we consider the divergence operators on spinor fields. In [1], the Lie derivative of a spinor $\sigma \in \Gamma(S_g)$ in the direction $X \in \Gamma(TM)$, is defined by

$$\mathcal{L}_X^S \sigma := \frac{d}{dt} \Big|_{t=0} \beta_t^{-1} F_t^{-1} (\sigma \circ f_t),$$

using the notation from the Section 4. This spinor Lie derivative \mathcal{L}^S is related to the spin connection ∇^s via

$$\mathcal{L}_X^S \sigma = \nabla_X^s \sigma - \frac{1}{4} \gamma(dX^\flat) \sigma,$$

where X^\flat denotes the dual one-form of the vector field X and the 2-form dX^\flat acts via Clifford multiplication. From now on consider an oriented closed Riemannian surface (M, g) , with almost complex structure $J_M \in \text{Aut}(TM)$. For a spinor field $\rho \in \Gamma(S_g)$, define the divergence of ρ with respect to σ by

$$(13) \quad \text{div}_\sigma(\rho) = \langle \nabla^s \sigma, \rho \rangle_\# + \frac{1}{4} J_M \text{grad}(\langle \gamma(\omega) \sigma, \rho \rangle) \in \Gamma(TM),$$

where ω stands for the volume element in the Clifford bundle.

Lemma 1. *With notation as above, for any vector field $X \in \Gamma(TM)$, it holds that*

$$\int_M \langle \mathcal{L}_X^S \sigma, \rho \rangle \, d\text{vol}_g = \int_M \langle X, \text{div}_\sigma(\rho) \rangle \, d\text{vol}_g.$$

Proof. Take local isothermal coordinates (x^α) and write $X = X^\alpha \frac{\partial}{\partial x^\alpha}$. Then $X^\flat = X^\alpha g_{\alpha\beta} dx^\beta \equiv X_\beta dx^\beta$. Note that

$$\begin{aligned} \int_M \langle \gamma(dX^\flat) \sigma, \rho \rangle \, d\text{vol}_g &= \int_M \left\langle \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2} \right) \frac{1}{\sqrt{\det g}} \gamma(\omega) \sigma, \rho \right\rangle \sqrt{\det g} \, dx \\ &= \int_M \frac{\partial X_2}{\partial x^1} \langle \gamma(\omega) \sigma, \rho \rangle - \frac{\partial X_1}{\partial x^2} \langle \gamma(\omega) \sigma, \rho \rangle \, dx \\ &= \int_M -X_2 \frac{\partial}{\partial x^1} (\langle \gamma(\omega) \sigma, \rho \rangle) + X_1 \frac{\partial}{\partial x^2} (\langle \gamma(\omega) \sigma, \rho \rangle) \, dx \\ &= \int_M \langle J_M X, \text{grad}(\langle \gamma(\omega) \sigma, \rho \rangle) \rangle \, d\text{vol}_g \\ &= \int_M \langle X, -J_M \text{grad}(\langle \gamma(\omega) \sigma, \rho \rangle) \rangle \, d\text{vol}_g. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_M \langle \mathcal{L}_X^S \sigma, \rho \rangle \, d\text{vol}_g &= \int_M \langle \nabla_X^s \sigma, \rho \rangle - \frac{1}{4} \langle \gamma(dX^\flat) \sigma, \rho \rangle \, d\text{vol}_g \\ &= \int_M \langle X, \langle \nabla^s \sigma, \rho \rangle_\# + \frac{1}{4} J_M \text{grad}(\langle \gamma(\omega) \sigma, \rho \rangle) \rangle \, d\text{vol}_g. \end{aligned} \quad \square$$

Next consider the divergence operators defined on $S_g \otimes TM$. Let $\varphi = \varphi^\alpha \otimes e_\alpha \in \Gamma(S_g \otimes TM)$ be a gravitino. In a local orthonormal frame (e_α) , the g -divergence operator is

$$\text{div}_g(\varphi) := \text{Tr}_g(\widehat{\nabla} \varphi) = \sum_\alpha \langle \widehat{\nabla}_{e_\alpha} \varphi, e_\alpha \rangle \in \Gamma(S_g),$$

where we use $\widehat{\nabla}$ to denote the connection of $S_g \otimes TM$. For any spinor field $q \in \Gamma(S_g)$, using integration by parts,

$$\begin{aligned} \int_M \langle q, \operatorname{div}_g \varphi \rangle \operatorname{dvol}_g &= \int_M \langle q, \langle \widehat{\nabla}_{e_\alpha} \varphi, e_\alpha \rangle \rangle \operatorname{dvol}_g = \int_M \langle q \otimes e_\alpha, \widehat{\nabla}_{e_\alpha} \varphi \rangle \operatorname{dvol}_g \\ &= - \int_M \langle \nabla_{e_\alpha}^s q \otimes e_\alpha, \varphi \rangle \operatorname{dvol}_g = - \int_M \langle (\nabla^s q)_\sharp, \varphi \rangle \operatorname{dvol}_g. \end{aligned}$$

Recall that $P\varphi = 0$ implies that φ is a section of the complex vector bundle $S_g \otimes_{\mathbb{C}} TM$. If in addition $\operatorname{div}_g \varphi = 0$, then $\tilde{\varphi} \equiv \varphi^\alpha \otimes e^\alpha \in \Gamma(S_g \otimes_{\mathbb{C}} T^*M)$ is a holomorphic section. Indeed, holomorphicity can be verified by a local computation and for both summands of $S_g \otimes_{\mathbb{C}} T^*M = (W \oplus W) \otimes_{\mathbb{C}} T^*M$ separately. We can assume that $e^\beta = e^u dx^\beta$ for some coordinates x^1, x^2 which are the real and imaginary parts of a holomorphic coordinate $z = x^1 + ix^2$. Furthermore, let $s^a = e^{u/2} t^a$ be a local oriented orthogonal frame for W and $t = t^1 + it^2$ a holomorphic frame, such that $t \otimes t = dz$. Using those frames, the condition $P\varphi = 0$ for some $\varphi \in W \otimes_{\mathbb{C}} TM$ is given in terms of $\tilde{\varphi} = \varphi_{a\beta} t^a \otimes dx^\beta$ by $\varphi_{11} = -\varphi_{22}$ and $\varphi_{12} = \varphi_{21}$. Consequently, $\tilde{\varphi}$ can be identified with $\tilde{\varphi} = \frac{1}{2}(\varphi_{11} - i\varphi_{12}) t \otimes dz$. The divergence equation $\operatorname{div}_g \varphi = 0$ reads in that basis

$$0 = t^1 (\partial_{x^1} \varphi_{11} + \partial_{x^2} \varphi_{12}) + t^2 (\partial_{x^1} \varphi_{12} - \partial_{x^2} \varphi_{11}).$$

Hence the divergence equation yields the Cauchy–Riemann equations for $\tilde{\varphi}$.

Given a gravitino χ , we define the χ -divergence of φ to be

$$\operatorname{div}_\chi(\varphi) = \langle \nabla^{S \otimes TM} \chi, \varphi \rangle_\# + \frac{1}{4} J_M \operatorname{grad}((\gamma(\omega) \otimes \mathbb{1}_{TM})\chi, \varphi) + \frac{1}{2} J_M \operatorname{grad}((\mathbb{1}_S \otimes \gamma(\omega))\chi, \varphi).$$

Lemma 2. *For any $\chi, \varphi \in \Gamma(S_g \otimes TM)$ and any $X \in \Gamma(TM)$, it holds that*

$$\int_M \langle \mathcal{L}_X^{S_g \otimes TM} \chi, \varphi \rangle \operatorname{dvol}_g = \int_M \langle X, \operatorname{div}_\chi \varphi \rangle \operatorname{dvol}_g,$$

Proof. With respect to a local orthonormal frame (e_α) , write $\chi = \chi^\alpha \otimes e_\alpha$, $\varphi = \varphi^\beta \otimes e_\beta$. Then for any $X \in \Gamma(TM)$, by the Leibniz rule we have

$$\mathcal{L}_X^{S \otimes TM} \chi = (\mathcal{L}_X^S \chi^\alpha) \otimes e_\alpha + \chi^\alpha \otimes (\mathcal{L}_X^{TM} e_\alpha),$$

and hence

$$\int_M \langle \mathcal{L}_X^{S \otimes TM} \chi, \varphi \rangle \operatorname{dvol}_g = \int_M \langle \mathcal{L}_X^S \chi^\alpha, \varphi^\alpha \rangle + \langle \chi^\alpha, \varphi^\beta \rangle \langle \mathcal{L}_X^{TM} e_\alpha, e_\beta \rangle \operatorname{dvol}_g.$$

The metric Lie derivative \mathcal{L}^{TM} differs from the classical Lie derivative \mathcal{L} by

$$\mathcal{L}_X^{TM} Y = \mathcal{L}_X Y + \frac{1}{2} (\mathcal{L}_X g)_\# Y, \quad \forall X, Y \in \Gamma(TM),$$

see [1, Proposition 14]. Expressing the Lie derivative in terms of the Levi-Civita covariant derivative one obtains

$$\langle \mathcal{L}_X^{TM} Y, Z \rangle = \langle \mathcal{L}_X Y, Z \rangle + \frac{1}{2} (\mathcal{L}_X g)(Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} (g(Y, \nabla_Z X) - g(\nabla_Y X, Z))$$

for all vector fields X, Y and Z on M . The term $(g(Y, \nabla_Z X) - g(\nabla_Y X, Z))$ is tensorial and anti-symmetric in Y and Z . Therefore we can write it as

$$(g(Y, \nabla_Z X) - g(\nabla_Y X, Z)) =: \kappa \operatorname{dvol}_g(Y, Z) = \kappa g(J_M Y, Z)$$

for some $\kappa \in C^\infty(M)$ because $\dim M = 2$. Letting (e_α) be a local orthonormal frame as before, we have for $Y = e_1 + e_2$ and $Z = J_M Y$

$$\begin{aligned} 2\kappa &= \kappa g(J_M(e_1 + e_2), -e_1 + e_2) \\ &= g(e_1 + e_2, \nabla_{-e_1+e_2} X) - g(\nabla_{e_1+e_2} X, -e_1 + e_2) \\ &= g(J_M(e_1 + e_2), J_M(\nabla_{-e_1+e_2} X)) - g(J_M(\nabla_{e_1+e_2} X), J_M(-e_1 + e_2)) \\ &= g(-e_1 + e_2, \nabla_{-e_1+e_2}(J_M X)) + g(\nabla_{e_1+e_2}(J_M X), e_1 + e_2) \\ &= 2 \operatorname{Tr}_g(\nabla(J_M X)) = 2 \operatorname{div}_g(J_M X). \end{aligned}$$

Therefore

$$\begin{aligned} \int_M \langle \chi^\alpha, \varphi^\beta \rangle \langle \mathcal{L}_X^{TM} e_\alpha, e_\beta \rangle \operatorname{dvol}_g &= \int_M \langle \chi^\alpha, \varphi^\beta \rangle \left(\langle \nabla_X e_\alpha, e_\beta \rangle + \frac{1}{2} g(J_M e_\alpha, e_\beta) \operatorname{div}_g(J_M X) \right) \operatorname{dvol}_g \\ &= \int_M \left\langle X, \langle \chi^\alpha, \varphi^\beta \rangle g(\nabla e_\alpha, e_\beta)_\# + \frac{1}{2} J_M \operatorname{grad}(\langle \chi^\alpha, \varphi^\beta \rangle g(J_M e_\alpha, e_\beta)) \right\rangle \operatorname{dvol}_g. \end{aligned}$$

Combining this with (13) we get the conclusion. \square

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