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Symmetries in 4-dimensional manifolds with metric of neutral signature

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Abstract

This paper studies some general principles of metric (Killing) symmetry on a 4−dimensional, connected manifold M admitting a metric of neutral signature. First a general survey of neutral geometry is presented and this is followed by a discussion of the generalised distribution, integral manifolds and orbits which arise from a general Lie algebra of global smooth vector fields on M. Next, this theory is applied to the finite-dimensional Lie algebra of global Killing vector fields on M and gives a decomposition of M into its various Killing orbits. The zeros of Killing vector fields and the consequent isotropies are then introduced, the latter being described in terms of subalgebras of $o(2,2)$. The concepts of *stability* and *dimensional stability* for orbits is described and theorems are proved which relate the dimension of the Killing algebra to the orbit and isotropy dimensions. Certain special subalgebras are then introduced and related to isotropies in dimensionally stable orbits. Several examples of metrics with symmetries are then presented and which reveal many of the above features. Finally, a theorem describing the consequences for the Ricci and Weyl tensors under Killing isotropy is given in terms of algebraic classifications of these tensors. A table is provided which gives full details of the above results.

1 Introduction

This paper is intended as a contribution to the theory of metric symmetry on a 4-dimensional, connected manifold with metric g of neutral signature $(+,+,-,-)$. Section 2 will be devoted to notation and to the conventions used and to a brief survey of the neutral geometry of (M,g). In section 3 the interrelated concepts of distribution, maximal integral manifold, orbit and leaf associated with a Lie algebra of global vector fields J on M will be summarised. In sections $4-6$ symmetry is finally introduced through a discussion of the existence of global Killing vector fields on (M,g), the associated Killing algebra and orbits and

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the isotropy subalgebras at each $m \in M$. The concept of dimensional and full stability for orbits is described and a topological breakdown of M with respect to these orbits is investigated. A theorem relating the dimensions of the Killing algebra, its orbits and its isotropies is given. In section 7 examples of metrics and their associated Killing algebras are given and which reflect many of the structures arising in section $4-6$. In section 8 a review of the algebraic classification of the Ricci and Weyl tensors on (M, g) is given and the restrictions on these algebraic types at those points of M where non-trivial isotropies occur are investigated.

2 Neutral Geometry

In this paper M denotes a $4-$dimensional, smooth, connected, Hausdorff, paracompact manifold with smooth metric g of neutral signature $(+,+,-,-)$ labelled collectively as (M, g). The Levi-Civita connection arising from g is denoted by ∇, the associated curvature tensor by *Riem* with components $R^a{}_{bcd}$, the corresponding Ricci tensor by *Ricc* with components $R_{ab} \equiv R^c{}_{acb}$ and the Weyl conformal tensor by C with components $C^a{}_{bcd}$. The tangent space at $m \in M$ is denoted by T_mM and the $6-$dimensional vector space of $2-$forms (*bivectors*) at m by Λ_mM. The distinction between the tangent and cotangent spaces will sometimes be ignored, as will the index placing on tensors, because of the existence of the metric. For $u, v \in T_mM$, the symbol $u.v$ denotes their inner product, $g(m)(u, v)$, at m. A non-zero member $u \in T_mM$ is called *spacelike* if $u.u > 0$, *timelike* if $u.u < 0$ and *null* if $u.u = 0$ and the $1-$dimensional subspaces (*directions*) that each span are called, respectively, *spacelike*, *timelike* and *null*. The symbol $*$ denotes the usual Hodge duality (linear) operator on Λ_mM. Since g has neutral signature one may choose a pseudo-orthonormal basis x, y, s, t at $m \in M$ with $x.x = y.y = -s.s = -t.t = 1$ and an associated *null* basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l.n = L.N = 1$ and all other such inner products are zero. A $2-$dimensional subspace ($2-$space) V of T_mM is called *spacelike* if each non-zero member of V is spacelike, or each non-zero member of V is timelike, *timelike* if V contains exactly two, distinct, null directions, *null* if V contains exactly one null direction and *totally null* if each non-zero member of V is null. Thus a totally null $2-$space consists, apart from the zero vector, of null vectors any two of which are orthogonal. This list is mutually exclusive and exhaustive for $2-$spaces. A $3-$dimensional subspace of T_mM is *spacelike* (respectively, *timelike, null*) if its normal is spacelike (respectively, timelike, null). Hence a $3-$dimensional spacelike or timelike subspace of T_mM has an induced metric from $g(m)$ of Lorentz signature. [It is remarked that this nomenclature is different from that used in $4-$dimensional Lorentz manifolds and is justified only by "convention".] A bivector E at m with components $E^{ab} (= -E^{ba})$ necessarily has even matrix rank. If this rank is 2, E is called *simple* and if 4, it is called *non-simple*. If E is simple it may be written $E^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_mM$ and the $2-$space spanned by u and v is uniquely determined by E and called the *blade* of E (and then, unless more precision is required, E or its blade is written $u \wedge v$). A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) if its blade is *spacelike* (respectively, *timelike*, *null* or *totally null*).

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For each $E \in \Lambda_m M$ its dual is defined by $\overset{*}{E}_{ab} = \frac{1}{2}\epsilon_{abcd}E^{cd}$ where $\epsilon_{abcd} \equiv \sqrt{detg}\delta_{abcd}$ with δ denoting the usual alternating symbol and $detg$ the determinant of g. For neutral signature $\overset{**}{E} = E$ for all bivectors E and so the only eigenvalues of the linear map $*$ are ± 1. Now define the subspaces $\overset{+}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = E\}$ and $\overset{-}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = -E\}$ and also the subset $\widetilde{S}_m \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$, of $\Lambda_m M$. Then $\overset{+}{S}_m \cap \overset{-}{S}_m = \{0\}$ and any $E \in \Lambda_m M$ may be written uniquely as $E = \overset{+}{E} + \overset{-}{E}$ with $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$. In terms of the above null basis, $\overset{+}{S}_m$ is spanned by $l \wedge n - L \wedge N$, $l \wedge N$ and $n \wedge L$ and $\overset{-}{S}_m$ by $l \wedge n + L \wedge N$, $l \wedge L$ and $n \wedge N$. At this point it is convenient to introduce a bivector metric on $\Lambda_m M$ denoted by $|\,|$ and given for $A, B \in \Lambda_m M$ by $|A, B| = A^{ab}B_{ab}$ and which induces a Lorentz metric of signature $(-,-,+)$ on each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. Then if $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, one has $|\overset{+}{E}, \overset{-}{E}| = 0$ and $[\overset{+}{E}, \overset{-}{E}] = 0$ where $[\,]$ denotes matrix commutation. Now each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is a Lie algebra under $[\,]$, Lie isomorphic to the Lie algebra $o(1,2)$ and so $\Lambda_m M$ is the Lie algebra product $\overset{+}{S}_m \oplus \overset{-}{S}_m$. The only $2-$dimensional subalgebra of $\overset{+}{S}_m$, up to isomorphism, is spanned by $l \wedge n - L \wedge N$ and $l \wedge N$ and the bracket of these two bivectors equals $2(l \wedge N)$ (and similarly for $\overset{-}{S}_m$). Such subalgebras are denoted by $\overset{+}{B}$ ($\overset{-}{B}$).

If E and E' are independent and totally null and both are in $\overset{+}{S}_m$ or both are in $\overset{-}{S}_m$ their blades intersect in only the zero vector whereas if $E \in \overset{+}{S}_m$ and $E' \in \overset{-}{S}_m$ their blades intersect in a unique null direction. It follows that if $k \in T_m M$ is null there are exactly two totally null $2-$spaces containing k, one in $\overset{+}{S}_m$ and one in $\overset{-}{S}_m$. Clearly $E \in \Lambda_m M$ is simple if and only if $\overset{*}{E}$ is. It is sometimes useful to note that for a general non-zero member $E \in \Lambda_m M$ the statements that (i) E is simple, (ii) $E_{ab}\overset{*}{E}^{bc} = 0$ and (iii) $|E, \overset{*}{E}| = 0$ are equivalent. It can also be checked that for $E \in \widetilde{S}_m$, the statements that $|E, E| = 0$, that E is simple and that E is totally null are equivalent. A bivector E *not in* $\widetilde{S}_m$ is null if and only if it is simple with a null vector l in its blade satisfying $E_{ab}l^b = 0$. More details on some of these matters may be found in [1, 2, 3, 4, 5].

3 Vector Fields, Generalised Distributions, Integral Manifolds and Orbits

The work in this section is a summary of the relevant aspects of the treatment of generalised (sometimes called singular) distributions given, for example, in [6, 7, 8, 9] (but note that the concept of a zero-dimensional manifold is required and this was not used in [9]). Let J denote a (real) *finite-dimensional* Lie algebra of smooth, global vector fields on M under the binary relation of the Lie bracket. For $m \in M$ define the subspace J_m of $T_m M$ by $J_m = \{X(m) : X \in J\}$. The map $D : m \to J_m$ is then a *generalised distribution* on M arising from J (here,

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generalised means that $\dim D(m)$ is not necessarily constant on M and so D is not necessarily a distribution in the sense of Frobenius). A submanifold N of M with (smooth) inclusion $i : N \to M$ is then an *integral manifold* of D (or J) if for each $m \in N$ the differential (*push forward*) of i at m, $i_{*(m)}$, satisfies $i_*(T_mN) = J_m$ so that J_m is the subspace of T_mM tangent to N at m. Since J is finite-dimensional it follows that J (or D) is *integrable* [6], that is, through each $m \in M$ there is a unique, connected, *maximal* (by inclusion) integral manifold of D. [A problem arises here in that a submanifold N of M may not be a *regular* submanifold [10], that is, the natural manifold topology of N may not equal its induced topology from the natural manifold topology of M. Thus, for example, if $f : M \to M$ is smooth and $f(M) \subset N$, the restriction $f : M \to N$ may not be smooth (as it would be if N were regular [10]). However, the integral manifolds encountered here, although not necessarily regular submanifolds, satisfy the extra condition that they are *leaves* of M and if N is such a leaf the restriction of the map f given above is smooth [8]].

These constructions can be described in another way and which is more natural for present purposes. Given $X \in J$ and $m_o \in M$ there exists a *flow box* given by a triple (U, I, ϕ) where U is an open subset of M containing m_0, I is an interval in $\mathbb{R}$ containing zero and ϕ is a smooth map $U \times I \to M$ such that for each $m \in U$, the smooth map $c_m : I \to M$ given by $c_m(t) = \phi(m, t)$ is an integral curve of X starting from m (that is, $c_m(0) = m$). The map $\phi_t : U \to M$ given by $\phi_t(m) = \phi(m, t)$ is a smooth diffeomorphism between U and $\phi_t(U)$. Flow boxes are unique in the sense that if (U', I', ϕ') is another such flow box, ϕ and ϕ' agree on the intersection of their domains [11]. Each of the maps ϕ_t is called a *local flow* of X. Returning to the Lie algebra J let k be a positive integer and let $X_1, ..., X_k \in J$ with associated local flows $\phi^1_t, ..., \phi^k_t$. Consider the collection G of smooth maps (where defined) each on some open subset of M and given by

$$m \to \phi^1_{t_1}(\phi^2_{t_2}(...\phi^k_{t_k}(m)...)) \qquad (m \in M) \tag{1}$$

for all choices of k, $X_1, ..., X_k$ and $(t_1, ..., t_k) \in \mathbb{R}^k$. Now define an equivalence relation $\sim$ on M by $m_1 \sim m_2$ if and only if for some $f \in G$, $f(m_1) = m_2$. The resulting equivalence classes in M are called the *orbits* associated with J. Recalling that J is finite-dimensional, it follows that if O_m is such an orbit containing m, O_m can be given the structure of a leaf of M and coincides with the (connected) maximal integral manifold of J through m (and $\dim O_m = \dim J_{m'}$ for any $m' \in O_m$). In addition, J is *invariant* for G in the sense that, for each $m \in M$ and $f \in G$, $f_*(J_m) \subset J_{f(m)}$ [6, 7, 8].

4 Killing Symmetry on M

For the geometry of (M, g) let X be a smooth, *global* vector field on M each of whose local flows ϕ_t is, on its range, a *local isometry* for g, that is, its *pull back* satisfies $\phi^*_t(g) = g$. Then, using $\mathcal{L}$ to denote a Lie derivative, one has the equivalent condition $\mathcal{L}_X g = 0$ on M and X is referred to as a *Killing vector field* on M. The collection of all Killing vector fields on M is a finite-dimensional Lie algebra under the Lie bracket operation (since $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$) denoted by $K(M)$ and this will play the rôle of J in the above discussion. [It is stressed here that, by definition, all Killing vector fields are *global* vector fields

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on M in the sense that any local Killing vector field defined on some open subset U of M and with respect to the induced metric on U from g can be smoothly extended to the whole of M as a Killing vector field for g on M and is then included in $K(M)$. Thus the definitions (given below) of orbit, isotropy, etc, apply to these vector fields only. (In the event of the existence of non-extendible to M, local Killing vector fields, one could appeal to the work of [12, 13]). It is pointed out that, in the examples given in section 7, M is always simply connected and no non-extendible local Killing vector fields exist [12, 13].]

For work in local coordinates, and using a semi-colon to denote a covariant derivative with respect to the Levi Civita connection ∇ of g, X satisfies

$$X_{a;b} + X_{b;a} = 0 \,(\Leftrightarrow X_{a;b} = F_{ab} = -F_{ba}), \qquad X^a{}_{;bc} = F^a{}_{b;c} = R^a{}_{bcd}X^d \quad (2)$$

where F is the *Killing bivector (field)* of X. With the geometry (M, g) given, (2) gives rise along any curve in M to a first order system of differential equations in the 4 components of X and 6 of F and hence, since being a connected manifold, M is path connected, $\dim K(M) \leq 10$. Thus if $X(m) = 0$ and $F(m) = 0$ it follows that X is identically zero on M and that, for $X, Y \in K(M)$, if X vanishes on some non-empty open subset of M it vanishes on M (and hence if $X = Y$ on some non-empty open subset of M, $X = Y$ on M). It is also clear from (2) that the *nature* (spacelike, timelike or null) of $X \in K(M)$ is the same along any integral curve of X since $X.X$ is constant along this curve. The Lie algebra $K(M)$ gives rise to orbits, as described above, using (1), and which are then leaves of M called *Killing orbits* of $K(M)$. Any such orbit is a maximal integral manifold of $K(M)$ and the orbit through $m \in M$ is sometimes denoted O_m. The local flows ϕ_t arising from members of $K(M)$ are smooth maps $M \to M$, hence, by inclusion, smooth maps $O \to M$ for some orbit O and with range in O and hence, since each O is a leaf of M, smooth maps $O \to O$. As described above they also have the invariance property $\phi_{t*}(J_m) \subset J_{\phi_t(m)}$ (where, henceforth, $J = K(M)$) for each $m \in O$.

For $K(M)$ and $m \in M$ define the subspace $I_m = \{X \in K(M) : X(m) = 0\}$ of $K(M)$. This is easily checked to be a subalgebra of $K(M)$ called the *isotropy subalgebra* at m. Any local flow ϕ_t arising from some member of I_m satisfies $\phi_t(m) = m$ (so m is a *zero* of X and a *fixed point* of ϕ_t) and the associated linear map $\phi_{t*} : T_mM \to T_mM$ is a member of the group $O(2, 2)$ with respect to $g(m)$. Noting that the collection $\{F^a{}_b(m)\}$ where $F(m)$ is the Killing bivector of some $X \in I_m$ is, under matrix commutation, a Lie subalgebra of $o(2, 2)$ ($g_{ac}F^c{}_b + g_{bc}F^c{}_a = 0$) the association of $X \in I_m$ with its Killing bivector at m is a Lie algebra isomorphism (recalling that for $X \in I_m$, $F(m) = 0 \Rightarrow X \equiv 0$ on M). The collection of all such maps ϕ_{t*} arising from members of I_m is a subset of $O(2, 2)$ but may not be a subgroup of it. However, each is the exponential of the bivector of some member of I_m and it is this latter subalgebra which is important here. If m, m' are points on the same orbit, I_m and $I_{m'}$ are isomorphic. Finally, for any $m \in M$, consider the map $f : K(M) \to T_mM$ given by $X \to X(m)$. This is a linear map with range space J_m and kernel I_m and so $\dim K(M) = \dim O_m + \dim I_m$ where O_m is the orbit through m. It is noted that the dimensions of O_m and I_m in general depend on m but $\dim K(M)$ is fixed for (M, g).

A Killing orbit O is called *proper* if $1 \leq \dim O \leq 3$. If $u \in T_mM$ is tangent to O at m, $\phi_{t*}(u)$ is tangent to O at $\phi_t(m)$ for any local flow ϕ_t of $X \in K(M)$ (by

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invariance) and since each ϕ_t is a local isometry and O is connected, the *nature* (spacelike, timelike, null or totally null, with the last of these applying only in the case when O is 2−dimensional) of J_m is the same at all points of O. Thus one can speak of a spacelike, timelike, null or totally null orbit. If O is proper and not null or totally null, it carries an induced metric $h = i^*g$ from g where i is the inclusion map $O \to M$. For any orbit O, if $X \in K(M)$ it is tangent to O and projects naturally onto a unique global smooth vector field $\widetilde{X}$ on O such that $i_*\widetilde{X} = X$. The local flow ϕ_t of X and the local flow $\widetilde{\phi}_t$ of $\widetilde{X}$ in O satisfy $i \circ \widetilde{\phi}_t = \phi_t \circ i$. If O admits an induced metric h it is easily checked that $\widetilde{\phi}_t^{\,*} h = h$ and so, since $i_*[\widetilde{X}, \widetilde{Y}] = [X, Y]$, the associated map $K(M) \to K(O)$ given by $X \to \widetilde{X}$ is a Lie algebra homomorphism into the Lie algebra of Killing vector fields on O, $K(O)$. However, as will be seen later, this map need not be either injective or surjective. Henceforth it is assumed that $J = K(M)$ is not trivial.

To characterise the nature and dimension of the orbits in more detail define, for $i = 0, 1, 2, 3, 4$ and for $p = 1, 2, 3$, the following subsets of M

$$\begin{aligned} V_i &= \{m \in M : \dim\ J_m = i\} \\ A_p &= \{m \in M : \dim J_m = p \ \text{ and } J_m \text{ spacelike}\} \\ B_p &= \{m \in M : \dim J_m = p \ \text{ and } J_m \text{ timelike}\} \\ C_p &= \{m \in M : \dim J_m = p \ \text{ and } J_m \text{ null}\} \\ D_2 &= \{m \in M : \dim J_m = 2 \ \text{ and } J_m \text{ totally null}\}. \end{aligned} \tag{3}$$

So $M = \bigcup_{i=0}^{4} V_i$. Then one can *disjointly* decompose M in the following ways (cf [14])

$$M = V_4 \cup \bigcup_{p=1}^{3} \mathrm{int}A_p \cup \bigcup_{p=1}^{3} \mathrm{int}B_p \cup \bigcup_{p=1}^{3} \mathrm{int}C_p \cup \mathrm{int}D_2 \cup \mathrm{int}V_0 \cup Z. \tag{4}$$

$$M = \bigcup_{i=0}^{4} \mathrm{int}V_i \cup Z' = V_4 \cup \bigcup_{i=0}^{3} \mathrm{int}V_i \cup Z'. \tag{5}$$

Here, int denotes the interior operator in the manifold topology on M and Z and Z' are defined by the disjointness of the decompositions and are closed. A consideration of rank shows that the subsets $\bigcup_{i=n}^{4} V_i$ $(0 \leq n \leq 4)$ are open in M. In particular, V_4 is open in M, hence $\mathrm{int}V_4 = V_4$ (and, by a remark above, $\mathrm{int}V_0 = \emptyset$ since $K(M)$ is non-trivial). Now if $\emptyset \neq U \subset Z'$ is open (and since by disjointness of (5), $U \cap V_4 = \emptyset$) one gets $U \cap (V_4 \cup V_3) = U \cap V_3$ is open which, if the latter is not empty, gives the contradiction that $U \cap \mathrm{int}V_3 \neq \emptyset$ and so $U \cap V_3 = \emptyset$. One similarly shows that $U \cap V_2 = U \cap V_1 = U \cap V_0 = \emptyset$ and so $U = \emptyset$ and $\mathrm{int}Z' = \emptyset$.

Similarly if $\emptyset \neq U \subset Z$ is open (so that $U \cap V_4 = \emptyset$), $U \cap V_3 = U \cap (A_3 \cup B_3 \cup C_3)$ is open. If this latter set is not empty, containing say the point m, suppose $m \in A_3$. Then there exists an open set W with $m \in W \subset U$ with $W \subset A_3$ (since the normal to J_m is spacelike, by definition of A_3, and may be chosen smoothly on W—see section 5) and so $U \cap \mathrm{int}A_3 \neq \emptyset$. This contradicts the disjointness of the decomposition (4) and so $U \cap A_3 = \emptyset$. Similarly $U \cap B_3 = \emptyset$ and so $U \cap C_3$ is open. If this latter set is non-empty the contradiction $U \cap \mathrm{int}C_3 \neq \emptyset$ follows. Thus $U \cap V_3 = \emptyset$. Similarly one shows that $U \cap V_2 = U \cap V_1 = \emptyset$. (For V_2 one

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arrives at $U \cap (C_2 \cup D_2)$ is open and then one appeals to a similar, previous argument using the fact that if $m \in C_2$, J_m contains a non-null member.) Thus $U = \emptyset$ and so $\text{int}Z = \emptyset$. The following lemma is thus established.

Lemma 4.1 *In the decompositions (4) and (5) above, intZ =intZ′ = ∅.*

It follows from [15] that the orbits of $K(M)$ through any point of V_4 are the components of V_4. If $M = V_4$ then, since M is connected, there is a single 4−dimensional orbit, M, and $K(M)$ is called *transitive* and (M, g) is called *homogeneous*.

Let O be a proper orbit of $K(M)$. Then either $O \subset A_p$, $O \subset B_p$, or $O \subset C_p$ for some p $(1 \leq p \leq 3)$, or $O \subset D_2$. A *proper* orbit O will be called *stable* (respectively, *dimensionally stable*) if, for each $m \in O$, there exists an open neighbourhood U (in M) of m such that each orbit intersecting U non-trivially has the same nature *and* dimension as O (respectively, has the same dimension as O) [9, 14]. It then follows that a proper orbit O is stable if and only if O is contained in $\text{int}A_p$, $\text{int}B_p$ or $\text{int}C_p$ $(1 \leq p \leq 3)$ or $\text{int}D_2$ and, of course, it is then necessarily dimensionally stable. A proper orbit O is dimensionally stable if and only if for some i $(1 \leq i \leq 3)$, $O \subset \text{int}V_i$. Using the local symmetries arising from Killing vector fields spanning such orbits it follows, for example, that if an orbit O satisfies $O \cap \text{int}A_p \neq \emptyset$ then $O \subset \text{int}A_p$ and similarly for B_p, C_p and D_2. Thus the properties of being stable (and similarly dimensionally stable) can be determined locally because of the isometric nature of the action of the local flows of members of $K(M)$ on the orbits. Of course, since V_4 is open, any 4−dimensional orbit O has the property that $m \in O$ implies the existence of an open neighbourhood of m through each point of which the orbit is 4−dimensional but the terminology "dimensionally stable" will be reserved for proper orbits. The open dense subset $M \setminus Z$ in (4) is the union of V_4 together with all stable orbits of $K(M)$ whilst Z is the union of all proper non-stable orbits of $K(M)$ and those points which are zeros of *all* $X \in K(M)$ (that is, V_0). Similarly $\bigcup_{i=1}^{4} \text{int}V_i = M \setminus Z'$ is the union of V_4 with all (proper) dimensionally stable orbits of $K(M)$ and is open and dense in M whilst Z' is the union of all (proper) non-dimensionally stable orbits of $K(M)$ and V_0.

A simple consideration of rank shows that any proper orbit of *maximum* dimension on M (in an obvious sense) is dimensionally stable (but not necessarily stable as a later example will show).

5 Zeros of Killing Vector Fields and Isotropy Subalgebras

Let X be a non-trivial member of $K(M)$, let $m \in M$ be a zero of X and let $F(m) \neq 0$ be the Killing bivector of X at m. Then on choosing normal coordinates x^a about m the components of X in the domain U of these coordinates satisfy $X^a = F^a{}_b x^b$ where F is the Killing bivector of X *evaluated at* m and hence yields a constant matrix on U. Thus if $\text{rank}F = 4$, m is an isolated zero of X (in the sense that it is the only zero of X in some neighbourhood of m). If $\text{rank}F = 2$ the kernel of (the linear map represented by) F in U give rise to a 2−dimensional submanifold of M containing zeros of X (see, e.g.[14, 9]). It now follows that if O is a 3−dimensional timelike or spacelike orbit of $K(M)$

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the Lie algebra homomorphism $K(M) \to K(O)$ given in the last section by $X \to \widetilde{X}$ is injective (but this result fails if O is a (spacelike or timelike) $1-$ or $2-$dimensional orbit as will be seen later). In other words, independent Killing vector fields on M may not project, under this homomorphism, to independent Killing vector fields on the orbits.

Let $\dim K(M) = n\,(\leq 10)$ and let O be a proper orbit of dimension n'. Suppose $n > n'$. Then for $m \in O$ $\dim I_m = n - n' > 0$. Let $X_1, ..., X_{n'}$ be independent members of $K(M)$ which span O at m. Then there exists a coordinate neighbourhood U of m at each point of which $X_1, ..., X_{n'}$ give independent tangent vectors and a smooth vector field k on U which is orthogonal to each of $X_1, ..., X_{n'}$ on U and $X_1, ..., X_{n'}$ span the tangent space to O at each point of $O \cap U$. Let $X \in I_m$ be non-trivial so that $X(m) = 0$ and X is tangent to O and orthogonal to k on $O \cap U$. If $c(t)$ is a curve through m lying in $O \cap U$ with tangent $p^a(t)$, $(X^a k_a)_{;b} p^b = 0$ along c and evaluating this, using (2), at m shows that the Killing bivector F of X satisfies $F_{ab} p^a k^b = 0$ at m for any such $p \in T_m M$. If O is, in addition, dimensionally stable one can, with the above notation, note that with a possible reduction of U, $X_1, ..., X_{n'}$ span the orbits on U. Thus one gets the stronger result $(X^a k_a)_{;b} = 0$ on U and evaluating at m then gives $F_{ab} k^b = 0$ [14]. Depending on the dimension of O there may be more than one independent choice for k and the consequences of the above restrictions on $F(m)$ also depend on the nature of the orbit and whether it is dimensionally stable or not. Thus, if $\dim O = 3$ and O is null but not dimensionally stable (and so k is null on $O \cap U$) the condition $F_{ab} p^a k^b = 0$ at m for any $p \in T_m M$ which is orthogonal to k at m shows that k is a (null) eigenvector of $F(m)$. Introducing a null tetrad l, n, L, N at m with $k = l$ (and using $< >$ to denote a spanning set) one sees that $F(m)$ is in $< l \wedge N, l \wedge L, l \wedge n, L \wedge N >$ and so $\dim I_m \leq 4$. If $\dim O = 3$ and O is null *and* dimensionally stable then again with the choice $k = l$ one has $F_{ab} l^b = 0$ and $I_m \subset < l \wedge N, l \wedge L, L \wedge N >$ and so $\dim I_m \leq 3$. If O is spacelike and with $k = x$ in the usual basis, one has, whether O is dimensionally stable or not, $F_{ab} x^b = 0$, $I_m \subset < t \wedge y, t \wedge s, y \wedge s >$ and $\dim I_m \leq 3$. Similarly if O is timelike with $k = t$, (and O dimensionally stable or not), $F_{ab} t^b = 0$, $I_m \subset < x \wedge y, x \wedge s, y \wedge s >$ and $\dim I_m \leq 3$.

If $\dim O = 2$ and O is null or totally null and not dimensionally stable choose independent $Y, Z \in K(M)$ which span O at m and an open coordinate neighbourhood U, as above, on which independent vector fields P and Q are defined and which are orthogonal to Y and Z on U. The technique above then shows that if $X \in I_m$ is non-trivial then X is tangent to O and orthogonal to P and Q on $O \cap U$ and so the Killing bivector $F(m)$ of X at m satisfies $F_{ab} P^a Y^b = F_{ab} P^a Z^b = F_{ab} Q^a Y^b = F_{ab} Q^a Z^b = 0$. If O is null take $P = Y = l$, $Z = y$ and $Q = s$ so that $I_m \subset < l \wedge y, l \wedge s, l \wedge n >$ and $\dim I_m \leq 3$. If O is totally null take $Y = P = l$ and $Q = Z = N$ and then $I_m \subset < l \wedge N, l \wedge L, l \wedge n, L \wedge N, n \wedge N >$ and $\dim I_m \leq 5$. If O is spacelike one may choose a tetrad x, y, s, t at m with $P = s$, $Q = t$, $Y = x$ and $Z = y$ to see that there are two independent solutions for $F(m)$ given by $x \wedge y$ and $s \wedge t$. Similarly if O is timelike, one may choose a tetrad l, n, L, N at m with $P = L$, $Q = N$, $Y = l$ and $Z = n$ and the independent solutions for $F(m)$ are $l \wedge n$ and $L \wedge N$. Thus in each case $\dim I_m \leq 2$.

If $\dim O = 2$ and O is dimensionally stable then in all cases one requires $F_{ab} P^b = F_{ab} Q^b = 0$ and so there is only one independent solution for $F(m)$ which is a simple bivector whose blade spans the tangent space to O at m.

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Thus, in all such dimensionally stable cases, $\dim I_m \leq 1$.

The situation for a $1-$dimensional orbit O is computed in a similar way. For a non-dimensionally stable orbit O and $m \in O$, $\dim I_m \leq 3$ if O is spacelike or timelike and $\dim I_m \leq 4$ if O is null (and with obvious choice of bases, I_m satisfies, respectively, $I_m \subset< t \wedge y, t \wedge s, s \wedge y > (F^a{}_b x^b = 0)$, $I_m \subset< x \wedge y, x \wedge s, y \wedge s > (F^a{}_b t^b = 0)$ and $I_m \subset< l \wedge n, l \wedge L, l \wedge N, L \wedge N >$). If O is dimensionally stable, $\dim I_m = 0$ in all cases.

Now suppose there exists a $3-$dimensional *spacelike* orbit O associated with $K(M)$ and that $m \in O$. Let X, Y, Z be independent members of $K(M)$ and U an open neighbourhood of m in M such that X, Y, Z span the tangent space to O at each point of $U \cap O$ and give independent members of $T_m M$ on U. Then a smooth vector field T exists on U which is orthogonal to X, Y and Z on U and whose integral curves are geodesics. This latter result follows from Killing's equations for X, Y, Z, since $(T^a X_a)_{;b} T^b = T^a{}_{;b} T^b X_a = 0$ on U (and similarly for Y and Z). Now if $W \in K(M)$ it is tangent to O and so orthogonal to T on $U \cap O$. Hence, by a classical result [16], W is orthogonal to T on U and the orbit dimension on U is ≤ 3. Since, by a rank argument, the orbit dimension in some neighbourhood of m is ≥ 3, one sees that, in this case (and in the similar case when $\dim O = 3$ and O is timelike), *O is necessarily dimensionally stable.*

One also has the (trivial) result that *all $1-$ and $2-$ dimensional, dimensionally stable, spacelike and timelike orbits are necessarily stable* and that *all $3-$dimensional, spacelike and timelike orbits are necessarily stable.* Thus, in the decomposition (4), the subsets A_3 and B_3 are open subsets of M and so $\mathrm{int} A_3 = A_3$ and $\mathrm{int} B_3 = B_3$.

These results together with the equation linking $\dim K(M)$, the dimension of an orbit through m and $\dim I_m$ give the following theorem.

Theorem 5.1 *Let M be a $4-$dimensional manifold with metric of neutral signature. Then the following hold for the orbits of $K(M)$ where $K(M)$ is assumed non-trivial.*

(i) If $K(M)$ admits a (proper) dimensionally stable orbit O with $m \in O$ and $dim I_m \geq 1$, then the non-zero Killing bivectors in I_m are each simple and have a common annihilator (that is, there exists $k \in T_m M$ such that $F_{ab} k^b = 0$ for each $F \in I_m$). This vector k is normal to the orbit at m.

(ii) If there exists a 3-dimensional, null orbit, $3 \leq dim K(M) \leq 7$. If there exists a $3-$dimensional, null, dimensionally stable orbit or a $3-$dimensional timelike or spacelike orbit, $3 \leq dim K(M) \leq 6$.

(iii) If there exists a $2-$dimensional, totally null orbit, $2 \leq dim K(M) \leq 7$, if there exists a $2-$dimensional, null orbit, $2 \leq dim K(M) \leq 5$ and if there exists a $2-$dimensional, spacelike or timelike orbit, $2 \leq dim K(M) \leq 4$. If there exists a $2-$dimensional, dimensionally stable orbit, $2 \leq dim K(M) \leq 3$.

(iv) If there exists a $1-$dimensional, null orbit, $1 \leq dim K(M) \leq 5$, if there exists a $1-$dimensional, spacelike or timelike orbit, $1 \leq dim K(M) \leq 4$ and if there exists a $1-$dimensional, dimensionally stable orbit, $dim K(M) = 1$.

It is also remarked that, under the conditions of part (i), $\dim I_m \leq 3$. But, in fact, quite generally, if B is a subspace of $\Lambda_m M$ such that all its members are simple it follows that $\dim B \leq 3$. To see this consider the subspace K of $\Lambda_m M$ spanned by (section 2) independent bivectors F, G, H where $F, G \in \overset{+}{S}_m$

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and $H \in \bar{S}_m$, where $|F,F| < 0$, $|G,G| < 0$, $|F,G| = 0$ (as is easily arranged) and $|H,H| > 0$. Then if F' is any non-trivial member of K, hence a linear combination of F, G and H, $|F', \overset{*}{F'}| \neq 0$ and so no member of K is simple (section 2). So if $\dim B \geq 4$, $\dim(B \cap K) \geq 1$ contradicting the previous sentence. This result is true for all signatures for this dimension. However, it is remarked that the assumptions about B, again for each signature, do not force B to have a common annihilator, This will be seen later.

It is not claimed that each of the possibilities arising in theorem 5.1 can occur. But many can and examples will be given later to illustrate this.

6 Special Subalgebras of the Subalgebras of $o(2,2)$

In view of theorem $5.1(i)$ above it is convenient for this section to define a subalgebra of $o(2,2)$ to be *special* if its spanning members possess a common annihilator (and hence each of it members is simple and its dimension is ≤ 3). This is useful generally and is particularly important in section 8 on the Weyl and Ricci tensors on (M,g). For each subalgebra of $o(2,2)$ each of its special subalgebras can be found (up to isomorphism) and the results are listed in the fifth column of table 1 in which $\overset{+}{S}$ denotes $< l \wedge n - L \wedge N, l \wedge N, n \wedge L >$ and $\bar{S}$ denotes $< l \wedge n + L \wedge N, l \wedge L, n \wedge N >$. In the first column of this table, each subalgebra of $o(2,2)$ is labelled with an integer (its dimension) and a Latin letter to distinguish between non-isomorphic subalgebras of the same dimension, whilst in the second column a basis is given for that subalgebra. In all cases $\alpha, \beta \in \mathbb{R}$ and, additionally, for the subalgebras $1(e)$ and $1(f)$, $0 \neq \alpha \neq \pm\beta \neq 0$. For subalgebras $1(g)$, $1(h)$, $1(l)$, $2(j)$ and $2(l)$, $\alpha\beta \neq 0$ and for $2(h)$ and $3(d)$, $\alpha \neq \pm\beta$. The third and fourth columns will be described later.

To obtain the results in the fifth column of table 1 it is noted that for the $1-$ and $2-$dimensional subalgebra cases they are straightforward and that $2(g)$, $2(h)(\alpha = 0)$, $2(h)(\beta = 0)$ and $2(k)$ are the only $2-$dimensional special subalgebras (and $2(h)(\alpha = 0)$ and $2(h)(\beta = 0)$ are isomorphic). It is noted that type $1(c)$ is a subalgebra of type $2(g)$. A few examples of proofs can now be given. In the $3-$dimensional cases type $3(a)$ clearly contains $1-$dimensional subalgebras of type $1(a)$ and $1(d)$ and it is easily checked that any linear combination F of its members satisfies $|F,F| \leq 0$ with $|F,F| = 0$ if and only if $F = l \wedge N$. Hence it does not contain subalgebras of type $1(b)$, $1(c)$ or $2(g)$ but it does contain special subalgebras of type $2(h)(\alpha\beta = 0)$. Similarly type $3(b)$ contains special subalgebras of the type $1(c)$ and $1(d)$ and, since any linear combination F of its members satisfies $|F,F| = 0$ if and only if F is a linear combination of $l \wedge N$ and $l \wedge L$, it contains type $2(g)$ but not types $1(a)$ or $1(b)$ and hence not types $2(h)(\alpha\beta = 0)$ or $2(k)$. Type $3(c)$ is itself special and contains special subalgebras of type $1(a)$, $1(b)$, $1(c)$ but not $1(d)$ (and hence not types $2(g)$ or $2(h)(\alpha\beta = 0)$) but it does contain one of type $2(k)$ and, of course, type $3(c)$. Type $3(d)(\alpha = 0)$ is itself special and contains special subalgebras $1(a)$, $1(c)$, $1(d)$, $2(g)$ and $2(h)(\alpha\beta = 0)$ whilst $3(d)(\beta = 0)$ contains $1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)(\alpha\beta = 0)$ and $2(k)$. It is remarked that type $3(d)(\beta = 0)$ has only simple members but no common annihilator (cf the remark at the end of section 5). For type $3(d)(\alpha\beta \neq 0)$ any member F of this subalgebra satisfies $|F, \overset{*}{F}| = 0$ if

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and only if F is a linear combination of $l\wedge L$ and $l\wedge N$ and hence contains special subalgebras of types $1(c)$, $1(d)$ and $2(g)$. Subalgebra $3(e)$ is clear and contains only the special subalgebra $1(d)$ whilst the only simple members of type $3(f)$ are of the form $l\wedge N$ and hence $1(d)$ is the only special subalgebra here also.

For type $4(a)$ let F be a simple member of this subalgebra and write $F = \overset{+}{F} + \overset{-}{F}$ with $\overset{+}{F} \in \overset{+}{S}$ and $\overset{-}{F} \in \overset{-}{S}$. Then $[F, l\wedge n + L\wedge N] = [\overset{-}{F}, l\wedge n + L\wedge N]$ is a member of the intersection of $4(a)$ and $\overset{-}{S}$, that is, it is a multiple of $l\wedge n+L\wedge N$. Thus $\overset{-}{F}$ and $l\wedge n + L\wedge N$ span a subalgebra of $\overset{-}{S}$ and, recalling the remarks on such subalgebras in section 2 and the definition of the type $4(a)$, this multiple is zero. So either $\overset{-}{F}$ is zero (which implies F is totally null) or $[\overset{-}{F}, l\wedge n+L\wedge N] = 0$, that is, $\overset{-}{F}$ is a multiple of $l\wedge n+L\wedge N$ and so $|\overset{-}{F}, \overset{-}{F}| \leq 0$. This shows that F (is simple and) has a timelike blade and generates a subalgebra of type $1(a)$. So this type contains only the special subalgebras of type $1(a)$, $1(d)$ and $2(h)(\alpha\beta = 0)$. Similar ideas apply to case $4(b)$ which has special subalgebras of type $1(b)$ and $1(d)$. For type $4(c)$ one clearly has special subalgebras of type $1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)(\alpha\beta = 0)$, $2(k)$ and $3(d)(\alpha = 0)$. To see that type $1(b)$ is not possible suppose F is simple with a spacelike blade and write, as above, $F = \overset{+}{F} + \overset{-}{F}$. Then $[F, \overset{+}{B}] = [\overset{+}{F}, \overset{+}{B}]$ cannot be zero for all members of $\overset{+}{B}$ but is always in $\overset{+}{S}$ and hence in $\overset{+}{B}$. But then $\overset{+}{F}$ (which is not in $\overset{+}{B}$) together with the members of $\overset{+}{B}$ constitute a basis for $\overset{+}{S}$ but gives a $2-$dimensional commutator subalgebra and hence a contradiction. The computation for type $4(d)$ is similar to that for $4(a)$ and gives special subalgebras $1(c)$, $1(d)$ and $2(g)$.

Finally, for the $5-$dimensional subalgebra with spanning members $l\wedge N$, $l\wedge L$, $n\wedge L$, $l\wedge n$ and $L\wedge N$, it is clear that types $1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)(\alpha\beta = 0)$, $2(k)$ and $3(d)(\alpha = 0)$ are contained. The argument given in the $4-$dimensional cases shows that $1(b)$ (and hence $3(c)$) are not contained. These results lead to the following theorem and the relevant subalgebras are listed in table 1, column 5.

Theorem 6.1 *The special subalgebras of $o(2,2)$ are those isomorphic to types $1(a)$, $1(b)$, $1(c)$, $1(d)$, $2(g)$, $2(h)(\alpha\beta = 0)$, $2(k)$, $3(c)$ and $3(d)(\alpha = 0)$.*

In the case when an orbit of $K(M)$ is proper but not dimensionally stable the analysis preceding theorem 5.1 suggests that I_m need not be special (examples will be given in the next section to confirm this suggestion) and allows the possibilities for I_m to be computed. In fact if O is such an orbit with $\dim O = 3$, so that O is necessarily null with normal, say, $l \in T_mM$, then a basis may be chosen at m so that any Killing bivector associated with I_m is a member of the subalgebra spanned by $l\wedge N$, $l\wedge L$, $L\wedge N$ and $l\wedge n$ and is thus a subalgebra of type $4(c)$. If, however, $\dim O = 2$ then with O spacelike (respectively, timelike) one similarly finds that I_m is a subalgebra of $2(e)$ (respectively, $2(b)$) whereas if O is null (respectively totally null) I_m is a subalgebra of $3(d)(\beta = 0)$ (respectively, of the $5-$dimensional subalgebra of $o(2,2)$). If $\dim O = 1$ then for O spacelike (respectively, timelike and null) I_m is a subalgebra of $3(c)$ (respectively, $3(c)$ and $4(c)$).

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Table 1: Subalgebras of $o(2,2)$

Type	Basis	Ricci Type	Weyl Type	Special Subalgebras
$1(a)$	$l \wedge n$	$\{1(111)\}$, $\{(11)(11)\}$, $\{11(11)\}$, $\{(11)z\bar{z}\}$, $\{2(11)\}$, $\{(211)\}$	$(\mathbf{D}_1, \mathbf{D}_1)$, $(\mathbf{D}_1, \mathbf{O})^\star$	$1(a)$
$1(b)$	$x \wedge y$	$\{1(111)\}$, $\{(11)(11)\}$, $\{11(11)\}$	$(\mathbf{D}_2, \mathbf{D}_2)$, $(\mathbf{D}_2, \mathbf{O})^\star$	$1(b)$
$1(c)$	$l \wedge y$ or $l \wedge s$	$\{1(111)\}$, $\{(211)\}$, $\{(21)1\}$, $\{(31)\}$	$(\mathbf{N}, \mathbf{N})$, $(\mathbf{N}, \mathbf{O})^\star$	$1(c)$
$1(d)$	$l \wedge N$	$\{(211)\}$, $\{(22)\}$ (over $\mathbb{R}$)	$(\mathbf{N}, \mathbf{A})$, $(\mathbf{N}, \mathbf{D}_1)$, $(\mathbf{N}, \mathbf{D}_2)$, $(\mathbf{N}, \mathbf{N})$ $(\mathbf{O}, \mathbf{A})$, $(\mathbf{O}, \mathbf{D}_1)$, $(\mathbf{O}, \mathbf{D}_2)$, $(\mathbf{N}, \mathbf{O})^\star$	$1(d)$
$1(e)$	$\alpha(x \wedge y) + \beta(s \wedge t)$	$\{(11)(11)\}$	$(\mathbf{D}_2, \mathbf{D}_2)$, $(\mathbf{D}_2, \mathbf{O})^\star$	None
$1(f)$	$\alpha(l \wedge n) + \beta(L \wedge N)$	$\{(11)(11)\}$	$(\mathbf{D}_1, \mathbf{D}_1)$, $(\mathbf{D}_1, \mathbf{O})^\star$	None
$1(g)$	$\alpha(l \wedge n + L \wedge N) + \beta(l \wedge N + n \wedge L)$	$\{(zz)(\bar{z}\bar{z})\}$	$(\mathbf{D}_2, \mathbf{D}_1)$, $(\mathbf{O}, \mathbf{D}_1)$, $(\mathbf{D}_2, \mathbf{O})$	None
$1(h)$	$\alpha(l \wedge N + n \wedge L) + \beta(l \wedge L)$	$\{(22)\}$ (over $\mathbb{R}$)	$(\mathbf{D}_2, \mathbf{N})$, $(\mathbf{D}_2, \mathbf{O})$, $(\mathbf{O}, \mathbf{N})$	None
$1(j)$	$l \wedge N + n \wedge L (= x \wedge y + s \wedge t)$	$\{(11)(11)\}$, $\{(22)\}$ (over $\mathbb{R}$), $\{(zz)(\bar{z}\bar{z})\}$	$(\mathbf{D}_2, \mathbf{A})$, $(\mathbf{D}_2, \mathbf{D}_2)$, $(\mathbf{D}_2, \mathbf{D}_1)$, $(\mathbf{D}_2, \mathbf{N})$ $(\mathbf{O}, \mathbf{A})$, $(\mathbf{O}, \mathbf{D}_1)$, $(\mathbf{D}_2, \mathbf{O})^\star$, $(\mathbf{O}, \mathbf{N})$	None
$1(k)$	$l \wedge n - L \wedge N$	$\{(11)(11)\}$, $\{22\}$ (over $\mathbb{R}$), $\{(22)\}$ (over $\mathbb{R}$), $\{(zz)(\bar{z}\bar{z})\}$	$(\mathbf{D}_1, \mathbf{A})$, $(\mathbf{D}_1, \mathbf{D}_1)$, $(\mathbf{D}_1, \mathbf{D}_2)$, $(\mathbf{D}_1, \mathbf{N})$ $(\mathbf{O}, \mathbf{A})$, $(\mathbf{D}_1, \mathbf{O})^\star$, $(\mathbf{O}, \mathbf{D}_2)$, $(\mathbf{O}, \mathbf{N})$	None
$1(l)$	$\alpha(l \wedge n + L \wedge N) + \beta(l \wedge N)$	$\{(11)(11)\}$, $\{22\}$ (over $\mathbb{R}$), $\{(22)\}$ (over $\mathbb{R}$)	$(\mathbf{N}, \mathbf{D}_1)$, $(\mathbf{O}, \mathbf{D}_1)$, $(\mathbf{N}, \mathbf{O})$	None
$2(a)$	$l \wedge n - L \wedge N, l \wedge N (= \overset{+}{B})$	—	$(\mathbf{O}, \mathbf{A})$, $(\mathbf{O}, \mathbf{D}_1)$, $(\mathbf{O}, \mathbf{D}_2)$, $(\mathbf{O}, \mathbf{N})$	$1(d)$
$2(b)$	$l \wedge n$, $L \wedge N$	$\{(11)(11)\}$	$(\mathbf{D}_1, \mathbf{D}_1)$, $(\mathbf{D}_1, \mathbf{O})^\star$	$1(a)$
$2(c)$	$l \wedge n - L \wedge N$, $l \wedge L + n \wedge N$	$\{(zz)(\bar{z}\bar{z})\}$	$(\mathbf{D}_1, \mathbf{D}_2)$, $(\mathbf{O}, \mathbf{D}_2)$, $(\mathbf{D}_1, \mathbf{O})$	None
$2(d)$	$l \wedge n - L \wedge N$, $l \wedge L$	$\{(22)\}$ (over $\mathbb{R}$)	$(\mathbf{D}_1, \mathbf{N})$, $(\mathbf{D}_1, \mathbf{O})$, $(\mathbf{O}, \mathbf{N})$	$1(d)$
$2(e)$	$x \wedge y$, $s \wedge t$	$\{(11)(11)\}$	$(\mathbf{D}_2, \mathbf{D}_2)$, $(\mathbf{D}_2, \mathbf{O})^\star$	$1(b)$
$2(f)$	$l \wedge N + n \wedge L$, $l \wedge L$	$\{(22)\}$ (over $\mathbb{R}$)	$(\mathbf{D}_2, \mathbf{N})$, $(\mathbf{D}_2, \mathbf{O})$, $(\mathbf{O}, \mathbf{N})$	$1(d)$
$2(g)$	$l \wedge N$, $l \wedge L$	$\{(211)\}$	$(\mathbf{N}, \mathbf{N})$, $(\mathbf{N}, \mathbf{O})^\star$	$1(c)$, $1(d)$, $2(g)$
$2(h)$	$l \wedge N$, $\alpha(l \wedge n) + \beta(L \wedge N)$ $\alpha\beta = 0$	$\{(211)\}$	$(\mathbf{O}, \mathbf{D}_1)$	$1(a)$, $1(d)$, $2(h)$ $(\alpha\beta = 0)$
	$\alpha\beta \neq 0$	—	$(\mathbf{O}, \mathbf{D}_1)$	$1(d)$
$2(j)$	$l \wedge N$, $\alpha(l \wedge n - L \wedge N) + \beta(l \wedge L)$	—	$(\mathbf{O}, \mathbf{N})$	$1(d)$
$2(k)$	$l \wedge y$, $l \wedge n$ or $l \wedge s$, $l \wedge n$	$\{1(111)\}$	—	$1(a)$, $1(c)$, $2(k)$
$2(l)$	$l \wedge N$, $\alpha(l \wedge n - L \wedge N) + \beta(l \wedge L + n \wedge N)$	—	$(\mathbf{O}, \mathbf{D}_2)$	$1(d)$
$3(a)$	$l \wedge n$, $l \wedge N$, $L \wedge N$	—	$(\mathbf{O}, \mathbf{D}_1)$	$1(a)$, $1(d)$, $2(h)$ $(\alpha\beta = 0)$
$3(b)$	$l \wedge n - L \wedge N$, $l \wedge N$, $l \wedge L$	—	$(\mathbf{O}, \mathbf{N})$	$1(c)$, $1(d)$, $2(g)$
$3(c)$	$x \wedge y$, $x \wedge t$, $y \wedge t$ or $x \wedge s$, $x \wedge t$, $s \wedge t$	$\{1(111)\}$	—	$1(a)$, $1(b)$, $1(c)$, $2(k)$, $3(c)$
$3(d)$	$l \wedge N$, $l \wedge L$, $\alpha(l \wedge n) + \beta(L \wedge N)$ $\alpha = 0$	$\{(211)\}$	—	$1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)$ $(\alpha\beta = 0)$ $3(d)$ $(\alpha = 0)$
	$\beta = 0$	—	—	$1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)$ $(\alpha\beta = 0)$, $2(k)$
	$\alpha\beta \neq 0$	—	—	$1(c)$, $1(d)$, $2(g)$
$3(e)$	$\overset{+}{S}$	—	$(\mathbf{O}, \mathbf{A})$, $(\mathbf{O}, \mathbf{D}_1)$, $(\mathbf{O}, \mathbf{D}_2)$, $(\mathbf{O}, \mathbf{N})$	$1(d)$
$3(f)$	$\overset{+}{B}$, $l \wedge L + n \wedge N$	—	$(\mathbf{O}, \mathbf{D}_2)$	$1(d)$
$4(a)$	$\overset{+}{S}$, $l \wedge n + L \wedge N$	—	$(\mathbf{O}, \mathbf{D}_1)$	$1(a)$, $1(d)$, $2(h)$ $(\alpha\beta = 0)$
$4(b)$	$\overset{+}{S}$, $l \wedge L + n \wedge N$	—	$(\mathbf{O}, \mathbf{D}_2)$	$1(b)$, $1(d)$
$4(c)$	$\overset{+}{B}$, $\bar{B} =< l \wedge L, l \wedge N, l \wedge n, L \wedge N >$	—	—	$1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)$ $(\alpha\beta = 0)$, $2(k)$, $3(d)$ $(\alpha = 0)$
$4(d)$	$\overset{+}{S}$, $l \wedge L$	—	$(\mathbf{O}, \mathbf{N})$	$1(c)$, $1(d)$, $2(g)$
5	$\overset{+}{S}$, $\bar{B}$	—	—	$1(a)$, $1(c)$ $1(d)$, $2(g)$, $2(h)$ $(\alpha\beta = 0)$, $2(k)$, $3(d)$ $(\alpha = 0)$
6	$o(2,2)$	—	—	All special subalgebras

7 Examples of Symmetries for Neutral Metrics

In this section several examples will be given which complement the theory given so far. Some of them are adapted from the work in [9, 14] (which dealt with Lorentz signature) in order to convert them to neutral signature, whilst others are believed to be new.

Example 1

Let $M = \mathbb{R}^4$ with global coordinate system labelled x, y, s, t and consider the metric

$$-dt^2 + e^{(x^2+y^2-s^2)}(dx^2 + dy^2 - ds^2) \tag{6}$$

on M. Here $K(M)$ is spanned by the vector fields $(0,0,0,1)$, $(y,-x,0,0)$, $(s,0,x,0)$ and $(0,s,y,0)$ and so $\dim K(M) = 4$. The $1-$dimensional submanifold $x = y = s = 0$ of M is a timelike $1-$dimensional orbit O (and this gives a simple example to show that the homomorphism $K(M) \to K(O)$ mentioned in section 4 ($\dim O = 1$) need not be injective). The Killing vector field $(y,-x,0,0)$ vanishes on O and its Killing bivector at any $m_0 \in O$ is, in an obvious notation, $\partial X^a/\partial x^b$ and is simple and spacelike, giving a contribution $x \wedge y$ to I_{m_0}. Similarly the Killing vector fields $(s,0,x,0)$ and $(0,s,y,0)$ vanish at each $m_0 \in O$ and give contributions $x \wedge s$ and $y \wedge s$ to I_{m_0} so that I_{m_0} is (isomorphic to) the Lie algebra $3(c)$ which is special. On O the Weyl conformal tensor can be

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shown to vanish and the Ricci tensor R_{ab} to be a multiple of $g_{ab} + t_a t_b$ where t^a is the vector $\partial/\partial t$. Since all other orbits are clearly 3−dimensional (and dimensionally stable) O is not dimensionally stable. The other orbits are given by $x^2 + y^2 - s^2 = k$ (t arbitrary) where k is a constant and the normal to this orbit is proportional to $(x, y, -s, 0)$ being spacelike (respectively, timelike, null) if $k > 0$ (respectively $k < 0$, $k = 0$). The general member $X \in K(M)$ is, in the above coordinate system, $X = (ay + bs, -ax + cs, bx + cy, d)$ for $a, b, c, d \in \mathbb{R}$ and if it vanishes at $m = (x, y, s, t)$ not on O, $a^2y^2 = b^2s^2$, $a^2x^2 = c^2s^2$, $b^2x^2 = c^2y^2$ and $d = 0$ and one can compute the Killing bivector F at m to see that $|F, F| = 2(a^2 - b^2 - c^2)$. The conditions above for X vanishing at m show that F is (simple) and spacelike (respectively, timelike or null) if $k < 0$ (respectively, $k > 0$ or $k = 0$). So I_m is special and isomorphic to $1(b)$ if $k < 0$, to $1(a)$ if $k > 0$ and to $1(c)$ if $k = 0$. Each of these 3−dimensional orbits is dimensionally stable but only the spacelike and timelike ones are stable.

Example 2
With $M = \mathbb{R}^4$ and with global coordinate system x, y, s, t consider the metric

$$e^{(x^2-t^2)}(dx^2 - dt^2) + e^{(y^2-s^2)}(dy^2 - ds^2) \tag{7}$$

on M. Here, $K(M)$ is 2−dimensional, being spanned by the *orthogonal* Killing vector fields $X \equiv (t, 0, 0, x)$ and $Y \equiv (0, s, y, 0)$ for which $[X, Y] = 0$. It is noted that the origin is a zero for each member of $K(M)$ and I_m at the origin is spanned by $x \wedge t$ and $y \wedge s$ and is of type $2(b)$ which is *not* special. (The Killing vector field $X + Y$ has an isolated zero at the origin and its bivector there is non-simple.) All proper orbits are either 1− or 2− dimensional. By an appropriate choice of $m \in M$ one can choose $X(m)$ spacelike, timelike or null, and similarly for $Y(m)$. Thus the 1− dimensional orbits (passing through points $(t, 0, 0, x)$ or $(0, s, y, 0)$ with $x^2 + t^2 \neq 0 \neq s^2 + y^2$) can be spacelike, timelike or null, are not dimensionally stable and at any point of these orbits I_m is of the special type $1(a)$. The 2−dimensional orbits may be spacelike, timelike, null or totally null and are dimensionally stable but only the spacelike and timelike ones are stable, the others being of maximum dimension, dimensionally stable but *not* stable. The 2−dimensional orbit through the point $(1, 1, 1, 1)$ is totally null being spanned by null, orthogonal Killing vector fields. If one replaces the second (y, s) group of terms in (7) by 2−dimensional Minkowski space, the resulting metric has $\dim K(M) = 4$ and the orbit O through the origin is 2−dimensional, timelike and not dimensionally stable with non-special isotropy subalgebra of type $2(b)$ at each point. All other orbits are 3−dimensional and dimensionally stable but could be spacelike, timelike or null with, in either case, I_m of type $1(a)$ at each point and only the non-null orbits are stable. (Since $\dim K(O) = 3$ this gives an example where the homomorphism $K(M) \to K(O)$ mentioned in section 4 ($\dim O = 2$) is not injective.)

Example 3
Consider $M = \mathbb{R}^4$ with global coordinate system u, v, x, y with $u > 0$. Then consider the metric

$$u(x^2 + y^2)du^2 + 2dudv + dx^2 - dy^2. \tag{8}$$

This metric is an example of the neutral signature equivalent of one of the well-known plane wave metrics in the Lorentz signature of general relativity

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theory [17, 18]. It can be checked that it is Ricci flat, $Ricc \equiv 0$ on M, and that $l \equiv \partial/\partial v$ is a parallel, null (Killing) vector field. It can be shown that $\dim K(M) = 5$ and that the orbits are 3−dimensional, null and stable being of the form $u =$constant. At each point I_m is a 2−dimensional subalgebra of the type $3(d)(\alpha = 0)$ the latter being spanned (section 5) by $l \wedge L$, $l \wedge N$ and $L \wedge N$, where L and N are global null vector fields in the span of $\partial/\partial x$ and $\partial/\partial y$. A consideration of the Weyl tensor (see the next section) shows that I_m is actually of the special type $2(g)$. If the coefficient of du^2 in (8) is changed to $u(x^2 - y^2)$, l is still parallel and the resulting metric has $\dim K(M) = 6$ with the same orbits as before. One can check that the Ricci tensor satisfies $R_{ab} = -2ul_a l_b$ and that $C \equiv 0$ on M, so that (M, g) is conformally flat. I_m, now 3−dimensional, is of the special type $3(d)(\alpha = 0)$ at each point (see the next section).

Example 4

Consider $M = \mathbb{R}^4$ with global coordinate system u, v, x, y (with $u > 0$). Then consider the metric

$$ue^{(x+y)}du^2 + 2dudv + dx^2 - dy^2. \tag{9}$$

This metric is a special case of one given in [4], satisfies $Ricc \equiv 0$ and admits a 4−dimensional Killing algebra spanned by vector fields $X \equiv (0, -(x+y), u, -u)$, $Y \equiv (-\frac{u}{3}, \frac{v}{3}, 1, 0)$, $Z \equiv (0, 0, 1, -1)$ and $W = (0, 1, 0, 0)$. However, $X = -(x + y)W + uZ$ and since this is the only such relationship between the Killing vector fields, the orbits are everywhere 3−dimensional and dimensionally stable. The vector field $P \equiv \frac{u}{3}Z + W$ is everywhere orthogonal to these orbits and is null. [In fact, $P^a = (0, 1, \frac{u}{3}, -\frac{u}{3})$ and so $P_a \equiv (1, 0, \frac{u}{3}, \frac{u}{3}) = u$ times the gradient of $(\log u + \frac{1}{3}(x + y))$ and so the orbits are given by $\log u + \frac{1}{3}(x + y) =$constant and are everywhere null and stable.] A general member of $K(M)$ is $Q \equiv aX + bY + cZ + dW$ $(a, b, c, d \in \mathbb{R})$ with components $Q^a = (\frac{-bu}{3}, -a(x + y) + \frac{bv}{3} + d, au + b + c, -au - c)$ and for this to vanish at some (any) point $m \in M$ one must have $b = 0$. Then the associated Killing bivector at m is $\partial Q^a/\partial x^b$ and is found to be proportional to $W \wedge Z$ at m and is totally null. Thus, at each $m \in M$, I_m is of type $1(d)$. [This metric can be simplified by an obvious coordinate change which transforms it into the metric $ue^r du^2 + 2dudv + drds$ where $r = x + y$ and $s = x - y$.] The vector fields Z and W are parallel (null and orthogonal) vector fields.

Example 5

Consider $M = \mathbb{R}^4$ with global coordinate system x, y, s, t, $(s > 0 < t)$, and metric

$$2se^{-ay}dxds + 2te^{-ay}dxdt + 2bdyds + 2cdydt + ds^2 + dt^2 \tag{10}$$

where a, b, c are positive constants and where M is restricted by $bt < cs$ (so that $detg > 0$). Then $X \equiv \partial/\partial x$ and $Y \equiv \partial/\partial y$ are null and orthogonal and so this metric certainly has neutral signature. One finds that $K(M)$ is spanned by the null, orthogonal Killing vector fields X and $Z \equiv xX + a^{-1}Y$ and so $\dim K(M) = 2$. Thus the Killing orbits are spanned everywhere by X and Z and are hence 2−dimensional, totally null and stable. Of course, the isotropy subalgebra in this case is trivial at each point.

Example 6

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Consider $M = \mathbb{R}^4$ with global coordinate system u, v, r, s and metric

$$rsdu^2 + 2dudv + 2drds. \tag{11}$$

In this example, $\dim K(M)$ is spanned by the seven independent Killing vector fields $(0, 0, -r, s)$, $(0, -kre^{ku}, 0, e^{ku})$, $(0, kre^{-ku}, 0, e^{-ku})$, $(0, -kse^{ku}, e^{ku}, 0)$, $(0, kse^{-ku}, e^{-ku}, 0)$, $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$, where $k = 2^{-\frac{1}{2}}$ and so $\dim K(M) = 7$ and (M, g) is homogeneous. The Weyl tensor can be checked to be zero on M and the Ricci tensor to satisfy $Ricc = -dudu$. To compute I_m it is noted that, since $\dim I_m = 3$, table 1 (see the next section) shows that I_m is isomorphic to $3(d)(\alpha = 0)$.

Example 7
Consider the metric on $\mathbb{R}^4$ with coordinates t, x, y, s $(t > 0)$ given by

$$-dt^2 + tdx^2 + t^2dy^2 - t^3ds^2. \tag{12}$$

In this example, $\dim K(M) = 3$ with independent Killing vector fields $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial s$. The orbits are the submanifolds of constant t and are $3-$dimensional and timelike. Any such orbit O is flat and so $\dim K(O) = 6$. This shows that the map $K(M) \to K(O)$ mentioned in section 4 is not necessarily surjective (onto).

8 Classification and Symmetries of The Weyl and Ricci Tensors

If $K(M)$ is non-trivial and, in particular, admits a non-trivial Killing isotropy at $m \in M$, restrictions are imposed on the Weyl conformal tensor C and Ricci tensor $Ricc$ at m and which are, perhaps, best explained in terms of an algebraic classification of these tensors. The classification of $Ricc(m)$ (in fact, any symmetric, second order tensor at m) has been given for this signature in [19] and is based on an algebraic Jordan form-Segre type classification of the linear map $T_mM \to T_mM$ given for $k \in T_mM$ by $k^a \to R^a{}_b k^b$. Then if the isotropy subalgebra I_m is not trivial the algebraic type of $Ricc(m)$ is restricted in the following way. Let $X \in I_m$ so that $X(m) = 0$ and let ϕ_t be a corresponding local flow for X $(\phi_t(m) = m)$. Since $X \in K(M)$, $\phi_t^* Ricc = Ricc$ and so $\mathcal{L}_X Ricc = 0$. Thus for $u, v \in T_mM$ one gets the algebraic restriction $Ricc(u, v) = Ricc(\phi_{t*}(u), \phi_{t*}(v))$ at m. More directly, the equivalent condition $\mathcal{L}_X Ricc = 0$ applied at m gives

$$R_{ac}F^c{}_b + R_{cb}F^c{}_a = 0 \tag{13}$$

where F is the Killing bivector of X. Since I_m is a subalgebra (or by direct calculation) if F and G satisfy (13) for a fixed $Ricc(m)$, so does $[F, G]$. Thus the members of I_m (that is, the Killing bivectors at m for each $X \in I_m$), collectively restrict $Ricc(m)$ through (13) and the (restricted) algebraic types for $Ricc(m)$ can be inferred in each case from the subalgebra I_m. In the general case, when m lies on a proper, dimensionally stable orbit of $K(M)$, I_m is a special subalgebra but this may not necessarily be the case if the orbit is not proper or dimensionally stable (section 6). The Segre types which result for

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Ricc for *each* subalgebra for I_m are listed in table 1, column 3. To see this one first notes that the calculation of the allowed Segre types for *Ricc* from (13) was effectively calculated (for different reasons) in [20], using the classification scheme in [19], but restricted to those subalgebras which are possible *holonomy* algebras since this is all that was needed there. However, this restriction does not apply here and the additional subalgebras, which are $1(e)$,.....,$1(l)$, $2(l)$, $3(e)$, $3(f)$ and $4(d)$, must also be investigated. It is remarked that the four $1-$dimensional subalgebras $1(a)$,...,$1(d)$ are the only possibilities if the spanning member F is simple whilst the seven $1-$dimensional subalgebra types $1(e)$,...,$1(l)$ are the only possibilities when F is not simple and arise either from geometrical considerations [3, 2] or from the six different algebraic Segre types permitted for F by regarding F as a linear map $T_mM \to T_mM$ given by $k^a \to F^a{}_b k^b$ and which are $\{z\bar{z}w\bar{w}\}$, $\{1111\}$, $\{22\}$ (over $\mathbb{C}$), $\{(zz)(\bar{z}\bar{z})\}$, $\{(11)(11)\}$ and $\{22\}$ (over $\mathbb{R}$), noting that the first of these types can occur in two algebraically distinct ways, explaining the seventh type [19].

To calculate the Segre types for $Ricc(m)$ from (13) the following lemma is useful. Parts $(i)-(iii)$ can be found in [9, 2] and part (iv) is easily checked.

Lemma 8.1 *Suppose (13) holds at m. Then in the usual bases at m*

(i) If F is simple its blade is an eigenspace of Ricc.

(ii) If $F = \alpha(l \wedge n) + \beta(L \wedge N)$ with $0 \neq \alpha \neq \pm\beta \neq 0$, $l \wedge n$ and $L \wedge N$ are eigenspaces for Ricc whilst if $0 \neq \alpha = -\beta$, $l \wedge N$ and $n \wedge L$ are invariant spaces for Ricc (and similarly, if $0 \neq \alpha = \beta$, with L and N interchanged).

(iii) If $F = \alpha(x \wedge y) + \beta(s \wedge t)$ with $0 \neq \alpha \neq \pm\beta \neq 0$, $x \wedge y$ and $s \wedge t$ are eigenspaces for Ricc whilst if $\alpha = \beta \neq 0$, one may choose an orthonormal tetrad x', y', s', t' such that $F = (x' \wedge y') + (s' \wedge t')$ and $x' \wedge t'$ and $s' \wedge y'$ are invariant spaces for Ricc (and similarly, if $\alpha = -\beta \neq 0$, with s and t interchanged).

(iv) If γ is a (real or complex) eigenvalue for F the $\gamma-$eigenspace for F is an invariant subspace for Ricc. Hence if γ is a non-degenerate eigenvalue for F, the corresponding eigenvector of F is an eigenvector of Ricc. The same result holds if F and Ricc are interchanged.

With these results the possible Segre types for *Ricc* can, with a little linear algebra, be deduced from (13) for each subalgebra of $o(2,2)$ by using the Segre type classification in [19]. It is important to note that what is being done here is a study of the restrictions on *Ricc* imposed by (13) for each subalgebra of $o(2,2)$ with the latter playing the formal role of the isotropy subalgebra I_m at some $m \in M$. But only the restrictions on *Ricc* arising from special subalgebras apply if m lies in a proper, dimensionally stable orbit (see theorem 8.1(A)). It is also noted that, whatever the type of orbit on which m lies, if $\dim I_m \geq 3$ and I_m is not of type $3(c)$ or $3(d)(\alpha = 0)$ (13) shows that T_mM is an eigenspace of $Ricc(m)$ and hence $Ricc(m)$ must be a multiple of $g(m)$ with Segre type $\{(1111)\}$. Since this type is always a solution of (13), it is omitted from column 3 of table 1.

In the case of the Weyl conformal tensor C a classification of it which is convenient for the present purposes has been given in [5]. In that work, $C(m)$ is decomposed uniquely into two parts, one part $\overset{+}{W}$ "lying in $\overset{+}{S}_m$" and the other part $\overset{-}{W}$ "lying in $\overset{-}{S}_m$" in a well-defined way as

$$C_{abcd} = \overset{+}{W}_{abcd} + \overset{-}{W}_{abcd} \tag{14}$$

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where, in terms of the left dual ${}^*C$ of C, $\overset{+}{W} \equiv \frac{1}{2}(C + {}^*C)$ and $\overset{-}{W} \equiv \frac{1}{2}(C - {}^*C)$. Thus $\overset{+*}{W} = \overset{+}{W}$ and $\overset{-*}{W} = -\overset{-}{W}$. In fact, if $C(m)$ is regarded as a linear map $f : \Lambda_m M \to \Lambda_m M$ according to $F^{ab} \to C^{ab}{}_{cd}F^{cd}$ there are analogous maps $\overset{+}{f}$ and $\overset{-}{f}$ arising in the same way from $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ and which, respectively, have $\overset{+}{S}_m$ and $\overset{-}{S}_m$ as invariant subspaces and, respectively, act trivially on $\overset{-}{S}_m$ and $\overset{+}{S}_m$. Then each of $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ may be conveniently, algebraically classified according to the Segre type of the maps $\overset{+}{f}$ and $\overset{-}{f}$ (since each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$, together with the bivector metric $||$ introduced in section 2, is isometric to $\mathbb{R}^3$ with the 3−dimensional Minkowski metric of signature $(-,-,+)$ and the algebraic classification of symmetric tensors in such spaces is well-known (see, e.g.[5])). This enables $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ to be written in a canonical form (type) in terms of bases for $\overset{+}{S}_m$ and $\overset{-}{S}_m$ given in section 2 which, in turn, are written in terms of a (canonical) basis for $T_m M$. The types arising, independently for each of $\overset{+}{W}$ and $\overset{-}{W}$, from their individual Segre types, their possible eigenvalue degeneracies and, in one case, the nature of a certain eigenspace, are labelled (essentially following that used in the case of 4−dimensions and Lorentz signature by Petrov [21], as modified by Pirani [22] and others—see e.g.[18]) **I**, **II**, **III**, $\mathbf{D_1}$, $\mathbf{D_2}$, **N** and **O** (together with some subtypes) for each of $\overset{+}{W}$ and $\overset{-}{W}$, the last type being the trivial case when $\overset{+}{W}$ (or $\overset{-}{W}$) vanishes. The possible types for C are then listed as label-pairs $(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X}$ (respectively, $\mathbf{Y}$) referring to the type of $\overset{+}{W}$, $(\overset{-}{W})$.

Also considered in [5] was the uniqueness of this canonical basis in $T_m M$ for each type for C. Degeneracies in either $\overset{+}{W}$ or $\overset{-}{W}$ result in multiple canonical bases linked by subgroups of $O(2,2)$ and the relationship between the type of C and this basis ambiguity was given there. Now if $X \in K(M)$ one has $\phi_t^* C = C$ and for non-trivial $X \in I_m$ the resulting maps ϕ_{t*} at m change the canonical basis at m but preserve (in an obvious way) $C(m)$. Thus the listing of canonical bases in [5], which is accompanied by a list of subalgebras which may play the formal role of I_m for each type $(\mathbf{X}, \mathbf{Y})$, is that which is required. Thus if a subalgebra in the complete list in table 1 is a subalgebra of one (or more) of those in the list in [5] for a certain algebraic type (or types) for C then it is a possible candidate for I_m when $C(m)$ has that type (or one of those types) and only for that type (or types).

Of course, for those points m lying on proper, dimensionally stable orbits only the corresponding special subalgebras are required and this leads to a considerable simplification. However, the restrictions on $C(m)$ arising from *each* possible subalgebra of $o(2,2)$ are listed in column four of table 1 where the symbol **A** denotes any of the non-degenerate types **I**, **II** and **III** for $\overset{+}{W}$ or $\overset{-}{W}$. It should be remembered, first, that there is an obvious isomorphism between the Weyl types $(\mathbf{X}, \mathbf{Y})$ and $(\mathbf{Y}, \mathbf{X})$ for any types $\mathbf{X}$ and $\mathbf{Y}$ and, second, that the listing of subalgebras in table 1 is up to isomorphism. Thus, for example, type $2(d)$ which is $< l \wedge n - L \wedge N, l \wedge L >$, where $l \wedge n - L \wedge N \in \overset{+}{S}$ and $l \wedge L \in \overset{-}{S}$, is

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isomorphic to (the unlisted) $< l \wedge N, l \wedge n + L \wedge N >$, where $l \wedge n + L \wedge N \in \bar{S}$ and $l \wedge N \in \overset{+}{S}$. Because of these isomorphisms care is needed when pairing off a Weyl type with its associated isotropy algebra both in table 1 and in theorem 8.1 below. Thus, in theorem 8.1(B) a given representation for the subalgebra is chosen once and for all (and as given in table 1) and the Weyl type is then paired with this choice of subalgebra. In table 1, column 4, the notation $(\mathbf{X}, \mathbf{Y})^*$ means $(\mathbf{X}, \mathbf{Y})$ *and* $(\mathbf{Y}, \mathbf{X})$. The type $(\mathbf{O}, \mathbf{O})$, which is always possible in column 4, is omitted. Since the Ricci types are given in terms of Segre symbols no such similar problems arise for this tensor.

It is remarked that, analogously to (13), the equation $\mathcal{L}_X C = 0$ evaluated at m gives

$$C^a{}_{bce}F^e{}_d + C^a{}_{bed}F^e{}_c + C^a{}_{ecd}F^e{}_b - C^e{}_{bcd}F^a{}_e = 0. \tag{15}$$

This equation is useful as a check on the above results but will not be needed in what is to follow. One thus has the following theorem (for both *Ricc* and C) which, for convenience, is restricted to the situation for special subalgebras of $o(2,2)$, the complete listing being given in table 1. It is remarked that for *any* $m \in M$, if $\dim I_m \geq 3$, the type of C is of the form $(\mathbf{X}, \mathbf{Y})$ with at least one of $\mathbf{X}$ and $\mathbf{Y}$ equal to $\mathbf{O}$.

Theorem 8.1 *Let $m \in O$ where O is a proper, dimensionally stable orbit of $K(M)$. The relationships between (the special subalgebra) I_m and the algebraic type of Ricc and C (up to isomorphism, using the subalgebra listing in table* 1 *and theorem* 6.1 *and with the possibilities that Ricc is a multiple of g and C is of type $(\mathbf{O}, \mathbf{O})$ being understood and hence omitted) is given below. For any other (special) subalgebra not listed under (B), $C(m) = 0$.*

(A) ***Ricci tensor****;*
$1(a)$*:* $\{1(111)\}$, $\{(11)(11)\}$, $\{11(11)\}$, $\{(11)z\bar{z}\}$, $\{2(11)\}$, $\{(211)\}$.
$1(b)$*:* $\{1(111)\}$, $\{(11)(11)\}$, $\{11(11)\}$.
$1(c)$*:* $\{1(111)\}$, $\{(211)\}$, $\{(21)1\}$, $\{(31)\}$.
$1(d)$*:* $\{(211)\}$, $\{(22)\}$ *(over* $\mathbb{R}$*).*
$2(g)$, $2(h)(\alpha\beta = 0)$ *and* $3(d)(\alpha = 0)$*:* $\{(211)\}$.
$2(k)$ *and* $3(c)$*:* $\{1(111)\}$.
(B) ***Weyl tensor****;*
$1(a)$*:* $(\mathbf{D_1}, \mathbf{D_1})$, $(\mathbf{D_1}, \mathbf{O})^*$.
$1(b)$, $(\mathbf{D_2}, \mathbf{D_2})$, $(\mathbf{D_2}, \mathbf{O})^*$
$1(c)$, $2(g)$*:* $(\mathbf{N}, \mathbf{N})$, $(\mathbf{N}, \mathbf{O})^*$.
$1(d)$*:* $(\mathbf{N}, \mathbf{A})$, $(\mathbf{N}, \mathbf{D_1})$, $(\mathbf{N}, \mathbf{D_2})$, $(\mathbf{N}, \mathbf{N})$, $(\mathbf{O}, \mathbf{A})$, $(\mathbf{O}, \mathbf{D_1})$, $(\mathbf{O}, \mathbf{D_2})$, $(\mathbf{N}, \mathbf{O})^*$.
$2(h)(\alpha\beta = 0)$*:* $(\mathbf{O}, \mathbf{D_1})$.

An alternative way of looking at this classification of $C(m)$ is through certain "canonical" null directions at m (given in [5] and inspired by the work of Luis Bel in the 4−dimensional Lorentz case in [23]). Let $0 \neq k \in T_m M$ and consider the restrictions for k and $C(m)$, the latter not of type $(\mathbf{O}, \mathbf{O})$, given by the equations

$$(a)\, C_{abcd}k^a k^c = \alpha k_b k_d \qquad (b)\, C_{abcd}k^a k^c = k_b p_d + p_b k_d. \tag{16}$$

In (a), $\alpha \in \mathbb{R}$ and in (b), p is a 1−form at m uniquely determined by k and $C(m)$ and which is assumed to be *non-zero and* to be such that p_a is *not*

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proportional to k_a. In (a), if $\alpha \neq 0$, k is necessarily null whilst if $\alpha = 0$ k is *assumed to be null*. In (b) k is necessarily null and orthogonal to p. If (a) holds (with k null) then k is called a *repeated principal null direction* (a *repeated pnd*) for $C(m)$ whereas if (b) holds k is called a *principal null direction* (*pnd*) for $C(m)$. (The reasons for the term "repeated" are discussed in [5] and it is clear that similar definitions for repeated pnds and pnds may be made for the tensors $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ separately.) There is a close relationship between the type $(\mathbf{X}, \mathbf{Y})$ for $C(m)$ and the number of repeated pnds and pnds for $C(m)$ [5]. Then for any algebraic type for $C(m)$ the members of the subgroup of $O(2,2)$ which preserve the canonical form of $C(m)$ must map the set of repeated pnds into itself and the set of pnds into itself. This is easily checked for each subalgebra of $o(2,2)$ using the actual repeated pnds or pnds for each type for $C(m)$ given in [5] and a few examples will now be described to clarify this.

For Weyl type $(\mathbf{D_1}, \mathbf{D_1})$ at m one may choose a canonical null basis l, n, L, N at m and totally null $2-$spaces $l \wedge N$ and $n \wedge L$ in $\overset{+}{S}_m$ such that the repeated pnds for $\overset{+}{W}$ are precisely the non-zero members of these two $2-$spaces. Similarly the totally null $2-$spaces $l \wedge L$ and $n \wedge N$ in $\overset{-}{S}_m$ give the repeated pnds for $\overset{-}{W}$. The pnds of $C(m)$ are given by the intersections of these $2-$spaces [5], are all repeated and are spanned by l, n, L, N. The canonical form for $C(m)$ is preserved by each of the subalgebras $1(a)$, $1(f)$, $1(k)$ and $2(b)$ (table 1) [5]. Each of these subalgebras admits each of l, n, L, N as common eigenvectors for its spanning bivector(s) and hence, on exponentiation, each of these null directions is preserved by the corresponding members of $O(2,2)$.

If the Weyl type is $(\mathbf{D_1}, \mathbf{III})$ at m there exists a canonical null basis l, n, L, N at m such that l and L span the only two repeated pnds, and n and N span the only two non-repeated pnds, of $C(m)$. This type is preserved only by the subalgebra $1(k)$ [5] and which admits l, n, L, N as eigenvectors of its spanning member. Hence each of these null directions is preserved.

If the Weyl type is $(\mathbf{O}, \mathbf{III})$ at m a canonical null basis l, n, L, N may be chosen at m such that the non-zero members of the totally null $2-$space $l \wedge L$ give precisely the repeated pnds for $\overset{-}{W}$ (and hence for $C(m)$) and similarly $n \wedge N$ gives precisely the non-repeated pnds for $\overset{-}{W}$ (and hence for $C(m)$). The canonical form in this case is preserved by the transformations arising from the subalgebra $\overset{+}{S}$ ($3(e)$ in table 1), together with its subalgebras (types $1(d)$, $1(j)$, $1(k)$ and $2(a)$) and although the resulting subgroups do not always preserve the repeated pnds and the pnds of $C(m)$ they do preserve the $2-$spaces $l \wedge L$ and $n \wedge N$, as required.

One may now give some simple links and consistency checks between these remarks on the tensors *Ricc* and C, table 1 and the examples in section 7. In example 1, and along the orbit O, I_m is of type $3(c)$ whilst it can be checked that the Ricci tensor at these points is of Segre type $\{1(111)\}$ and the Weyl tensor vanishes. In example 2, (as given in (7)), the origin is a zero for each member of $K(M)$ and at this point I_m is of type $2(b)$ and the Weyl tensor can be shown to be of type $(\mathbf{D_1}, \mathbf{D_1})$. (M, g) turns out to be, in fact, an Einstein space. In example 3 (as given in (8)) the Weyl tensor can be checked to be of type $(\mathbf{N}, \mathbf{N})$ everywhere and hence, from table 1, I_m is of type $2(g)$ everywhere.

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After the change in the coefficient of du^2, the orbits are still 3−dimensional, null and stable and the Ricci tensor is of type $\{(211)\}$ (and hence, from table 1, I_m has type $3(d)(\alpha = 0)$ at each $m \in M$). In example 4 the Weyl tensor can be checked to be of type (**N**, **O**) at each $m \in M$. (It is remarked here that the Weyl tensor, considered as a 6×6 matrix in the usual bivector notation, has rank equal to 1 everywhere (see, e.g. [4]). This cannot happen in Lorentz signature (where this rank is always an even integer) and reflects the fact that, in Lorentz signature, a bivector and its dual are always independent in $\Lambda_m M$ (see, e.g. [9])). In example 6, for each $m \in M$, I_m is of type $3(d)(\alpha = 0)$, *Ricc* is of Segre type $\{(211)\}$ and the Weyl tensor vanishes.

One can add the following theorem.

Theorem 8.2 *(i) If $m \in M$ lies on a proper, dimensionally stable orbit and $dim I_m \geq 3$ then $C(m) = 0$. For any $m \in M$ if $dim I_m \geq 4$, $Ricc(m)$ is a multiple of $g(m)$.*

(ii) Suppose $dim K(M) = 7$. Then V_4 is open and dense in M and so $dim I_m \geq 3$ at each $m \in M$. Thus at each $m \in M$ the Ricci tensor has type $\{1(111)\}$ or $\{(211)\}$ (and in each of these cases $C(m)$ vanishes) or is proportional to g and the possible Weyl tensor types may be read off from table 1 and are of the form $(\mathbf{X}, \mathbf{O})$ or $(\mathbf{O}, \mathbf{Y})$ for types $\mathbf{X}$ and $\mathbf{Y}$. Further, either $M = V_4$ or, if $m \in M \setminus V_4$, the only possible orbits through m are non-dimensionally stable and are either 2−dimensional and totally null (and on which $dim I_m = 5$ and $C \equiv O$) or 3−dimensional and null (and on which I_m is of type $4(c)$ and hence $C \equiv 0$). The Einstein space condition is satisfied at any point of each of these orbits.

(iii) If $dim K(M) \geq 8$ then necessarily $M = V_4$ and (M, g) is an Einstein space. Additionally, if $dim K(M) \geq 9$, (M, g) is conformally flat and is hence of constant curvature. In this case (M, g) admits, locally in some neighbourhood of any $m \in M$, a 10−dimensional Lie algebra of (local) Killing vector fields which, if each is extendible to the whole of M (as would be the case if, for example, M were simply connected [12, 13]), would give $dim K(M) = 10$.

Proof

Part (i) is clear from table 1 or theorem 8.1. For part (ii) if $U \subset M \setminus V_4$ with U open and non-empty let O be the orbit of maximum dimension (≤ 3) intersecting U non-trivially. If $m \in O \cap U$ a consideration of rank shows that there exist a neighbourhood $W \subset U$ of m in which the orbit dimension is that of O and so O is dimensionally stable. Thus, from theorem 5.1, $\dim K(M) \leq 6$, which is a contradiction. So $U = \emptyset$ and V_4 is dense in M. It follows that if $M \neq V_4$ any orbit intersecting a point of $M \setminus V_4$ is not dimensionally stable (otherwise $M \setminus V_4$ would contain a non-empty open set) and, given $\dim K(M) = 7$, the rest of part (ii) follows from theorem 5.1 and table 1. For part (iii) if $m \in M$, $\dim I_m \geq 4$ and the result follows from Part (i) above. □

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