



Semi-symmetric connection formalism for unification of gravity and electromagnetism

Gh. Fasihi-Ramandi

Faculty of Science, Department of Mathematics, Imam Khomeini International University, Qazvin, Iran



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ABSTRACT

In this paper, the notion of semi-symmetric connections provides us a good apparatus to establish a unified field theory. The field equations are derived from an action principle which is formed naturally by the scalar curvature of the semi-symmetric connection. The derived equations contain the Einstein and Maxwell equations in vacuum, simultaneously.

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1. Introduction

Attempts at the unification of gravitation and electromagnetism have been made ever since the advent of general relativity by Einstein in 1915. Most of these attempts share the idea that Einstein's original theory must in some way be generalized so that some part of geometry describes electromagnetism. This topic is also considered to be of importance today, due to the fact that any unified field theory can throw light on the nature of the forces.

Early attempts in unification of fields have been done by Weyl [15], Eddington [3], Schouten [13], Klein (1926), Infeld [11], and Einstein and Mayer [4]. (For more details, see [10].)

This problem has been investigated through many structures so far. For example, affine metrics and Lie Algebroid structures are geometric structures and theoretical schemes which are proposed to unify the gravitational and electromagnetic fields. (See [5,6] and [7].) This paper is another attempt in the unification of gravity and electromagnetism. The main idea of this paper is to replace the Levi-Civita connection by a semi-symmetric metric connection on the space-time manifold and so our idea is very similar to Schouten's theory. (See [13] and [14] for more details on Schouten's theory.) In fact, Schouten criticized Einstein's argument for using a symmetric connection and used the notion of semi-symmetric connections in his approach. Schouten's point of departure for the field equations was Einstein's first Lagrangian $\mathcal{L} = \sqrt{\det K_{ij}}$ and, consequently, his field equations were the same as Einstein's apart from additional terms. (Note that K_{ij} is a contraction of the Riemannian curvature tensor associated to a semi-symmetric metric connection.) Also, Schouten's definition of some of the observables was different; for example, his proposed electromagnetic field tensor is different from existing electromagnetic field tensors. (For more details on Schouten's Lagrangian and his electromagnetic field tensor you can consult [10], in Section 5, around Eqs. (132) and (133).) But, here we use a natural Lagrangian which is formed by the scalar curvature of the semi-symmetric connection and the calculus of variations to obtain field equations. Also, our electromagnetic field tensor is different from Schouten's one. The derived equations contain the Einstein and Maxwell equations in vacuum, simultaneously.

This paper is organized as follows. In Section 2, we present the necessary notions and results which will be used in the next section. Section 3 is devoted to present our main results. In this section, using calculus of variations and the least action principle we establish a unified field theory.

E-mail address: fasihi@sci.ikiu.ac.ir.

2. Preliminaries

In this section, we summarize some basic concepts that are needed in this paper.

Suppose that $\bar{\nabla}$ is a linear connection on a smooth manifold M of class C^∞ . The *torsion tensor* of the connection $\bar{\nabla}$ is denoted by T and defined as follows.

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y].$$

The $\bar{\nabla}$ is said to be *symmetric* if its torsion tensor vanishes identically, otherwise it is non-symmetric. A linear connection $\bar{\nabla}$ on M is said to be a *semi-symmetric connection* [9] if its torsion tensor T satisfies

$$T(X, Y) = \omega(Y)X - \omega(X)Y,$$

where ω is a 1-form defined by

$$\omega(X) = g(X, U),$$

and U is a vector field on M . The 1-form ω and the vector field U are called the associated 1-form and vector field of the semi-symmetric connection $\bar{\nabla}$, respectively.

A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a *metric connection* if there is a semi-Riemannian metric g on M such that $\bar{\nabla}g = 0$. Pak showed that the semi-symmetric connections are metric connections [12].

Let (M, g) be an n -dimensional Riemannian manifold with the metric tensor g and let ∇ be the Levi-Civita connection of (M, g) . Also, suppose that $\bar{\nabla}$ is a metric semi-symmetric connection on (M, g) . The relation between $\bar{\nabla}$ and ∇ is given by [17]

$$\bar{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)U.$$

In particular, the metric semi-symmetric connection reduces to the Levi-Civita connection if and only if the 1-form ω vanishes identically. Geometric properties of a semi-symmetric metric connection are investigated in [16].

Let R and \bar{R} denote the Riemannian curvature tensors related to the connections ∇ and $\bar{\nabla}$, respectively, then by [16] we have

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - \pi(Y, Z)X + \pi(X, Z)Y \\ &\quad - g(Y, Z)AX + g(X, Z)AY, \end{aligned}$$

where π is a $(0, 2)$ -tensor field defined by

$$\pi(X, Y) = (\nabla_X \omega)(Y) - \omega(X)\omega(Y) + \frac{1}{2}\omega(U)g(X, Y)$$

and A is $(1, 1)$ tensor field associated to π . In fact, for any vector fields X and Y , we have

$$g(AX, Y) = \pi(X, Y).$$

Also, the scalar curvature is given by [16]

$$\bar{R} = R - 2(n-1)|\pi|^2,$$

where R and \bar{R} stand for the scalar curvatures of ∇ and $\bar{\nabla}$, respectively.

3. Unified field equations

In this section, M is an arbitrary oriented connected manifold whose dimension is n and $2 \geq n$. The manifold M can be regarded as a space-time manifold. Semi-symmetric metric connections on M are determined by a semi-Riemannian metric on M and a 1-form $\omega \in \Omega^1(M)$. The semi-Riemannian metric on M will play the role of potential for gravity, and the 1-form ω is capable of describing the electromagnetic field. In fact, the $(1, 1)$ -type tensor field corresponding to $\nabla\omega$ can play the role of the electromagnetic tensor field. The aim of this section is to find a proper field equation which contains the Einstein and Maxwell field equations simultaneously.

Fix a semi-Riemannian metric g on M and denote its associated Levi-Civita connection by ∇ . Let \mathcal{M} be the set of all semi-symmetric metric connections on M . As mentioned before, for a semi-symmetric metric connection $\bar{\nabla}$ with associated 1-form ω and for all $X, Y \in \mathcal{X}(M)$ (note that $\mathcal{X}(M)$ stands for the Lie algebra of smooth vector fields on M), we have the following relation.

$$\bar{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)U,$$

where U is the associated vector field of the connection $\bar{\nabla}$. This shows that \mathcal{M} can be written as the set of some pairs (g, ω) , where g is a semi-Riemannian metric on M and ω is a 1-form. In fact, \mathcal{M} can be considered as an open subset of a Fréchet space with respect to the Whitney topology. For example, the Levi-Civita connection ∇ for g corresponds to $(g, 0)$, where 0 is the vanishing 1-form on M .

Denote the canonical volume form of a metric g' on the oriented manifold M by $dV_{g'}$. Also, denote the scalar curvature of the semi-symmetric connection determined by (g', ω) on M by \bar{R} . The Hilbert action \mathcal{L} on \mathcal{M} is defined as follows.

$$\mathcal{L}(g', \omega) = \int_M \bar{R} dV_{g'}.$$

To be more precise, we must assume M is compact or we must integrate on open subset W of M such that the closure \bar{W} is compact.

For a symmetric 2-covariant tensor s on M , and a 1-form δ , set

$$\begin{aligned}\tilde{g}(t) &= g + ts, \\ \tilde{\omega}(t) &= \omega + t\delta.\end{aligned}$$

For sufficiently small t , $(\tilde{g}(t), \tilde{\omega}(t))$ is a semi-symmetric connection on M and is a variation of (g, ω) . The semi-symmetric connection (g, ω) is a critical semi-symmetric connection for Hilbert action if and only if for any pair (s, δ) :

$$\frac{d}{dt} \mathcal{L}(\tilde{g}(t), \tilde{\omega}(t))|_{t=0} = \frac{d}{dt} \left(\int_M \bar{R}(t) dV_{g+ts} \right)|_{t=0} = 0, \quad (1)$$

where $\bar{R}(t)$ is the scalar curvature of $(\tilde{g}(t), \tilde{\omega}(t))$ and

$$\bar{R}(t) = R(t) - 2(n-1)|\pi(t)|^2.$$

To find derivative in (1), we must compute derivatives of $R(t)$, $|\pi(t)|^2$ and dV_{g+ts} for $t = 0$. In [2], it is shown:

$$R'(0) = -\langle s, \text{Ric} \rangle + \text{div}(X), \quad \text{for some } X \in \mathcal{X}(M), \quad (2)$$

$$(dV_{g+ts})'(0) = \frac{1}{2} \langle g, s \rangle dV_g. \quad (3)$$

By local computations, we find the derivative of $|\pi(t)|^2$ at $t = 0$. By means of a local coordinate system on M with local frame ∂_i , set $\pi(\partial_i, \partial_j) = \pi_{ij}$. Consequently,

$$\pi_{ij} = \nabla_i \omega_j - \omega_i \omega_j + \frac{1}{2} g_{ij} |\omega|^2$$

where, $\omega(\partial_i) = \omega_i$. We have

$$|\pi|^2 = g^{ij} \pi_{ij} = \text{div}(\omega) + \left(\frac{n}{2} - 1\right) |\omega|^2.$$

Now, we compute the derivative of $\text{div}(\tilde{\omega})$ and $|\tilde{\omega}|^2$ at $t = 0$. Local computation of divergence of a 1-form ω is as follows.

$$\text{div}(\omega) = g^{ij} \left(\frac{\partial \omega_j}{\partial x^i} - \Gamma_{ij}^k \omega_k \right).$$

So,

$$\begin{aligned}[\text{div}(\tilde{\omega})]'(0) &= \left(\tilde{g}^{ij}(t) \left[\frac{\partial(\omega_j + t\delta_j)}{\partial x^i} - \tilde{\Gamma}_{ij}^k(t)(\omega_k + t\delta_k) \right] \right)'(0) \\ &= -s^{ij} \left(\frac{\partial \omega_j}{\partial x^i} - \Gamma_{ij}^k \omega_k \right) + g^{ij} \left(\frac{\partial \delta_j}{\partial x^i} - \Gamma_{ij}^k \delta_k - (\tilde{\Gamma}_{ij}^k)'(0) \omega_k \right) \\ &= \langle -s, \nabla \omega \rangle + \text{div}(\delta) - (\tilde{\Gamma}_{ij}^k)'(0) \omega_k.\end{aligned}$$

We can write,

$$\begin{aligned}(\tilde{\Gamma}_{ij}^k)'(0) &= \frac{1}{2} g^{ij} g^{kl} (\nabla_i s_{jl} + \nabla_j s_{il} - \nabla_l s_{ij}) \\ &= \frac{1}{2} g^{kl} (g^{ij} \nabla_i s_{jl} + g^{ij} \nabla_j s_{il} - g^{ij} \nabla_l s_{ij}) \\ &= \frac{1}{2} g^{kl} (\text{div}(s)_l + \text{div}(s)_l - \nabla_l \text{tr}(s)) = \text{div}(s)^k - \frac{1}{2} (\nabla \text{tr}(s))^k.\end{aligned}$$

Consequently,

$$(\tilde{\Gamma}_{ij}^k)'(0) \omega_k = \text{div}(s)^k \omega_k - \frac{1}{2} (\nabla \text{tr}(s))^k \omega_k = \text{div}(s)(U) - \frac{1}{2} \langle \vec{\nabla} \text{tr}(s), U \rangle,$$

where $\vec{\nabla}$ denotes the gradient of the smooth function $\text{tr}(s)$ in the Riemannian manifold (M, g) and U is the vector field associated to the semi-symmetric connection (g, ω) . In the following, whenever it is convenient, we consider s as a $(1, 1)$ self-adjoint operator. One can easily check that for an arbitrary vector field Z , we have

$$\text{div}(s)(Z) = \text{div}(s(Z)) - \langle s, \nabla Z \rangle.$$

Hence,

$$[\operatorname{div}(\tilde{\omega})]'(0) = \langle -s, \nabla \omega \rangle + \operatorname{div}(\delta) + \operatorname{div}(s(U)) - \langle s, \nabla U \rangle - \frac{1}{2} \langle \vec{\nabla} \operatorname{tr}(s), U \rangle.$$

Now, we compute the derivative of $|\tilde{\omega}|^2$ at $t = 0$:

$$(|\tilde{\omega}|^2)'(0) = (\tilde{g}^{ij}(t)\tilde{\omega}_i(t)\tilde{\omega}_j(t))'(0) = -s^{ij}\omega_i\omega_j + 2g^{ij}\omega_i\delta_j = \langle \omega \otimes \omega, -s \rangle + 2\delta(U).$$

Consequently,

$$\begin{aligned} (|\pi(t)|^2)'(0) &= \langle \nabla \omega + (\frac{n}{2} - 1)\omega \otimes \omega, -s \rangle + \operatorname{div} \delta + (n - 2)\delta(U) \\ &\quad + \operatorname{div}(s(U)) - \langle s, \nabla U \rangle - \frac{1}{2} \langle \vec{\nabla} \operatorname{tr}(s), U \rangle. \end{aligned}$$

For every $T \in A^p(M)$ and $S \in A^{p-1}(M)$, we have [1]

$$\int_M \langle dS, T \rangle dV_g = \int_M \langle S, -\operatorname{div}(T) \rangle dV_g.$$

Also note that the integral of the divergence of every vector field on M is zero. Now, we are ready to compute the derivative of $\mathcal{L}(\tilde{g}(t), \tilde{\omega}(t))$ for $t = 0$. We can write

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(\tilde{g}(t), \tilde{\omega}(t))|_{t=0} &= \frac{d}{dt} \left(\int_M \tilde{R}(t) dV_{g+ts} \right)|_{t=0} = \int_M (\tilde{R}(t) dV_{g+ts})'(0) \\ &= \int_M [(\tilde{R}(t) - 2(n-1)|\pi(t)|^2) dV_{g+ts}]'(0) \\ &= \int_M [R(t) - 2(n-1)|\pi(t)|^2]'(0) dV_g \\ &\quad + \int_M (R - 2(n-1)|\pi|^2) [dV_{g+ts}]'(0) \\ &= \int_M \left(-\langle s, \operatorname{Ric} \rangle + \operatorname{div}(X) - 2(n-1) \left[\langle \nabla \omega + (\frac{n}{2} - 1)\omega \otimes \omega, -s \rangle \right. \right. \\ &\quad \left. \left. + \operatorname{div} \delta + (n-2)\delta(U) + \operatorname{div}(s(U)) - \langle s, \nabla U \rangle - \frac{1}{2} \langle \vec{\nabla} \operatorname{tr}(s), U \rangle \right] \right) dV_g \\ &\quad + \int_M (R - 2(n-1)|\pi|^2) \frac{1}{2} \langle g, s \rangle dV_g \\ &= \int_M \langle (-\operatorname{Ric} + \frac{1}{2}Rg + 2(n-1)\nabla \omega + (n-1)(n-2)\omega \otimes \omega - (n-1)|\pi|^2 g), s \rangle dV_g \\ &\quad + 2(n-1) \int_M (\langle s, \nabla U \rangle + \frac{1}{2} \langle \vec{\nabla} \operatorname{tr}(s), U \rangle) dV_g \\ &\quad - 2(n-1)(n-2) \int_M \delta(U) dV_g. \end{aligned}$$

Since the above expression must vanish for all pairs (s, δ) , if it is not the case that $n = 2$, then in the last above expression, whenever $s = 0$, one can easily see that $U = 0$, hence $\omega = 0$. This case in our formalism means that in such space-time solutions the electromagnetic field vanishes and the theory describes a space-time without electromagnetic fields. Therefore, we should assume $n = 2$ to be able to establish a unified field theory. Consequently, for every s , we have

$$\int_M \langle (-\operatorname{Ric} + \frac{R}{2}g + 2\nabla \omega - \operatorname{div}(\omega)g), s \rangle dV_g + 2 \int_M (\langle s, \nabla U \rangle + \frac{1}{2} \langle \vec{\nabla} \operatorname{tr}(s), U \rangle) dV_g = 0.$$

But, it is easily seen that

$$\langle \nabla \omega, s \rangle = \langle \nabla U, s \rangle, \quad \int_M \langle \vec{\nabla} \operatorname{tr}(s), U \rangle dV_g = \int_M \operatorname{tr}(s) \operatorname{div}(\omega) dV_g.$$

So, we have

$$\int_M \langle (-\operatorname{Ric} + \frac{R}{2}g + 4\nabla \omega), s \rangle dV_g = 0.$$

If this vanishes for all variations s , then

$$\operatorname{Ric} - \frac{1}{2}Rg = 4\operatorname{Sym}(\nabla \omega). \quad (4)$$

As mentioned, the case in which $n = 2$ is a sufficient condition for Eq. (4) to determine a critical semi-symmetric connection. In two dimensions, we know that the Ricci curvature can be written in terms of the scalar curvature as $\text{Ric} = \frac{1}{2}Rg$. So, we can write

$$\text{Ric} - \frac{1}{2}Rg = 0, \quad (5)$$

$$\text{Sym}(\nabla\omega) = 0. \quad (6)$$

These equations are the field equations for (g, ω) and determine critical semi-symmetric connections for the Hilbert action. The first equation (5) is the Einstein field equation in vacuum and the second one (6) can be interpreted as the Maxwell equation and this equation shows that $d\omega = \nabla\omega$ can play the role of a potential for electromagnetic force in this space-time. Also, this equation gives us an internal relation between g the potential of gravity and ω the potential of electromagnetic force. In fact, $d\omega = \nabla\omega$ indicates that $L_U g = 0$, meaning that the vector field U has to be a Killing vector field with respect to g .

Our proof leads us to the case $\dim(M) = 2$. In [8], various curious features of general relativity, and relativistic field theory in two space-time dimensions, are considered and what constrains of our understanding of physics in different dimensions is discussed. In [8], the authors have remarked that the magnetic field is undefined in two dimensions but, here we propose a mathematical framework for describing electromagnetic force in two dimensional spaces.

Denote by dU the $(1, 1)$ -tensor field equivalent to $d\omega$ and set $\square U = \text{div}(dU)$. It is well-known in general relativity that $\square U$ has to be zero or timelike.

Now, we present examples of space-times which satisfy our equations. The following example is a trivial example of such space-times.

Example 1. For any space-time (M, g) which satisfies Einstein field equation, the semi-symmetric connection $(\nabla, 0)$ satisfies our obtained field equations and represents space-times without matter and electromagnetic fields. We should remind ourselves that ∇ stands for Levi-Civita connection of g .

To establish a non trivial example of space-time which satisfies our derived equations, note that Schwarzschild space-time presents a vacuum solution of the Einstein field equation, there exist no charge and matter and electromagnetism. There exists no nonzero Killing vector field U such that $\square U$ be timelike in de-Sitter space-time, so that model cannot represent electromagnetism. Also, most space-time models are not able to represent electromagnetism. So, describing electromagnetism, needs some modification of existing ordinary models.

Example 2. Let $M = \mathbb{R} \times \mathbb{R}^+$ and consider the following metric on it.

$$g = dx \otimes dx - e^{2(\phi(t)+\psi(x))} dt \otimes dt.$$

A simple computation shows that the only nonzero Christoffel symbols are as follows.

$$\Gamma_{22}^1 = \psi'(x)e^{2(\phi(t)+\psi(x))}, \quad \Gamma_{12}^2 = \psi'(x), \quad \Gamma_{22}^2 = \phi'(t).$$

The vector field $U = e^{-\phi(t)} \frac{\partial}{\partial t}$ is a Killing vector field and by suitable choices for $\phi(t)$ and $\psi(t)$, the vector field $\square U$ is timelike. One can check that (g, U^b) satisfies our obtained field equations.

4. Conclusion

In this paper, we presented a new formalism to unify gravitational and electromagnetic fields in two-dimensional space-time manifolds. This unification was established in the context of semi-Riemannian geometry equipped with a semi-symmetric metric connection. This theory is simpler than existing theories and does not add any extra dimensions to space-time. Our derived equations contain the Einstein and the Maxwell equations in vacuum simultaneously. Regardless of that our defined Hilbert action was naturally formed by the scalar curvature of the semi-symmetric connection, leading us to conclude that $\dim M = 2$. Of course, this theory does not contain quantum effects, and so quantization of these equations would be a good project.

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