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Ricci curvature on warped product submanifolds in spheres with geometric applications

Akram Ali [a,b], Pişcoran Laurian-Ioan [c,*], Ali H. Alkhaldi [a]

[a] Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia
[b] Departamento de Matematica, Instituto de Ciéncias Exatas, Universidade Federal do Amazonas, Av. General Rodrigo Octavio, 69080-900 Manaus, Amazonas, Brazil
[c] Department of Mathematics and Computer Science Victoriei 76, North University Center of Baia Mare, Technical University of Cluj Napoca, 430122, Baia Mare, Romania

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ABSTRACT

The goal of this paper is to construct a fundamental theorem for the Ricci curvature inequality via partially minimal isometric warped product immersions into an m-dimensional unit sphere $\mathbb{S}^m$, involving the Laplacian of a well defined warping function, the squared norm of a warping function and the squared norm of the mean curvature. Moreover, the equality cases are discussed in detail and some applications are also derived due to involvement of the warping function. As applications, we provide sufficient condition that the base N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature $c = \frac{\lambda_1}{p}$. The obtained results in the paper give the partial solution of Ricci curvature conjecture, also known as Chen-Ricci inequality obtained by Chen (1999).

1. Introduction

The relationship between Ricci curvature and space forms has been one of the most popular and highly developed topics in Riemannian geometry. In this area, a central issue of concern is that of determining global warped product structures from local metric properties. Of particular interest to us is the so-called Ricci curvature bounds problem and related theorems in geometry. If a Riemannian manifold M^n is immersed as a submanifold in higher dimension Riemannian manifold $\widetilde{M}^m$, then various extrinsic curvature invariants are discovered for a submanifold M^n in $\widetilde{M}^m$. For example, the scalar-valued extrinsic curvature function $\ell : M^n \longrightarrow \mathcal{R}$ and the squared mean curvature have been studied effectively in a lot of papers (see [1,17,20,24,28,29,40]). One of the most important problems in submanifold theory is to establish the connection between the intrinsic invariant quantities and extrinsic invariant quantities of submanifolds (see [2–8,14,15,17–20,23,30] and references therein). One important step in the study of the Riemannian manifolds was the appearance of the Nash embedding theorem in [34], which states that every Riemannian manifold admits an isometric immersion into a Euclidean space of sufficient high codimension. This concept becomes very useful for submanifold theory

* Corresponding author.
E-mail addresses: akali@kku.edu.sa, akramali@ufam.edu.br (A. Ali), plaurian@yahoo.com (P. Laurian-Ioan), ahalkhaldii@kku.edu.sa (A.H. Alkhaldi).

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that, every Riemannian manifold could always be regarded as Riemannian submanifold of Euclidean spaces. Inspired by these ideas, S. Nolker [35] classifies the isometric immersion of warped product decompositions of the standard spaces. Motivated by these results, B. Y. Chen started a new path of research in order to study immersibility and non-immersibility of warped product submanifolds in Riemannian manifolds, especially in Riemannian space forms whose sectional curvatures are constant in different situations (see [13,16–18]). Recently, a number of solutions of such problems were provided by various mathematicians (see [18,24,25,28], and their reference). In 1999, Chen [15] established a new relation between intrinsic invariant (Ricci curvature invariant) and extrinsic invariant (squared mean curvature invariant) as follows:

Theorem 1.1. *Let $\ell : M^n \longrightarrow \widetilde{M}^m(c)$ be an isometric immersion of a Riemannian n-manifold M^n into Riemannian space form $\widetilde{M}^m(c)$.*

(i) For each unit tangent vector $X \in T_xM^n$, we have

$$\|\mathbb{H}\|^2(x) \geq \frac{4}{n^2}\left\{\mathcal{R}ic(X) - (n-1)c\right\},$$

where $\|\mathbb{H}\|^2(x)$ is the squared mean curvature and $\mathcal{R}ic(X)$ the Ricci curvature of M^n at X.

(ii) If $H(x) = 0$, then the unit tangent vector X at x satisfies the equality case of (i) if and only if X lies in the relative null space $\mathcal{N}_x$ at x.

(iii) The equality case holds identically for all unit tangent vector at x if and only if either x is a totally geodesic point or $n = 2$ and x is a totally umbilical point.

It is quite difficult to obtain *Chen–Ricci inequality* for the Ricci curvature and its relation to the warping functions of warped products and is always hard to derive such relation for the product of two Riemannian manifolds. That is, Theorem 1.1 remains an open problem and was not proven for any warped products of two Riemannian manifolds in spheres. Therefore, the class of minimal isometric immersions became a key tool for the study of such type of results. That is, an isometric immersion $\ell : M^n \longrightarrow \widetilde{M}^m$ of a Riemannian submanifold M^n into a Riemannian manifold $\widetilde{M}^m$ is called *minimal* if its mean curvature vector field $\mathbb{H}$ is identically zero everywhere on M^n. The study of minimal surfaces is one of the oldest subjects in differential geometry, having its origins in the works of Euler and Lagrange. In the last century, a series of works have been developed for the study of the properties of minimal immersions, whose ambient space has constant sectional curvature (see [12,25,32,40]). In particular, minimal immersions in the sphere $\mathbb{S}^m$ play an important role in this theory. In this respect, we have, for example, the famous paper of J. Simons [38] and also for some other examples please see [30,31,40]. Motivated by the previous studies, a fundamental question arises in the context of the product of two Riemannian manifolds:

Open problem:- *Is it possible to derive Ricci curvature inequality theorem for warped product submanifolds to the case of minimal isometric immersion and ambient space form, to be a sphere with constant curvature one? What are the relationships (equations) between Ricci curvature, the main extrinsic invariants and the main intrinsic invariants of a warped product submanifold?*

Therefore, we define a partial minimal isometric immersion from a special type product Riemannian manifolds (warped product manifolds) into another Riemannian manifold. An isometric immersion $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \widetilde{M}^m$ is said to be partially minimal isometric immersion from a warped product manifold $N_1^q \times_f N_2^q$ into a Riemannian manifold $\widetilde{M}^m$ if at least one of the mean curvature vector fields $\mathbb{H}_1$ and $\mathbb{H}_2$ with respect to N_1^p and N_2^q, vanishes. In this work, we will present optimal general solutions to these fundamental problems by imposing minimality on warped product submanifolds. A lot of interesting applications of these optimal general solutions will be presented in the present paper where the main extrinsic invariant is the squared mean curvature and the intrinsic invariant contains Ricci curvature and also the squared norm of the well-defined warping function and Laplacian of the warping function.

The theory of warped product manifolds naturally considered in Riemannian geometry and their applications are very important to be studied. For example, the Riemannian manifold $\mathbb{S}^m \backslash \mathbb{S}^{m-2}$, that is, the standard sphere with a codimension two totally geodesic subsphere removed, is isometric to the warped product $\mathbb{S}_+^{n-1} \times_f \mathbb{S}^1$ of an open hemisphere and a circle, for warping function $f \in \mathbb{C}^\infty(\mathbb{S}_+^{n-1})$. This was the important constituent in a potential result of Bruce Solomon [39] about a harmonic map from a compact Riemannian manifold into a sphere $\mathbb{S}^m$. Significantly, for the study of Einstein manifolds, warped products are very important. An Einstein manifold is a Riemannian manifold (M, g) whose Ricci tensor satisfies $Ric = \lambda g$ for some function $\lambda \in \mathbb{C}^\infty(M)$ and such classes of manifolds come into sight in the framework of regular surfaces. Actually, a surface of revolution is a warped product and any regular surfaces is an Einstein manifold.

From this point of view, the following optimal result provides a solution to an open problem, for the family of warped product manifolds endowed with their warped product structure which is isometrically partially minimal immersed in a unit sphere $\mathbb{S}^m$. We give now the following main result:

Theorem 1.2. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$ be a $\mathcal{D}_i$-minimal isometric immersion from a n-dimensional warped product submanifold M^n into a m-dimensional unit sphere $\mathbb{S}^m$, then for each unit vector field $\xi \in T_xN_i$, with $i = 1$ or $i = 2$, we have* *the following Ricci curvature inequality*

$$\mathcal{R}ic_M(\xi) \leq \frac{n^2}{4}\|\mathbb{H}\|^2 + q\left(\|\nabla(\ln f)\|^2 + p + 1 + \frac{p-1}{q} - \Delta(\ln f)\right). \tag{1.1}$$

1. *If $H(x) = 0$, then at each $x \in M^n$ there is a unit vector u which satisfies the equality case of (1.1) if and only if M^n is mixed totally geodesic and u lies in the relative null space $\mathcal{N}_x$ at x.*
2. *If M^n is N_1^p-minimal, then*
   - (a) *The equality case of (1.1) holds identically for all unit tangent vectors to N_1^p at each $x \in M^n$ if and only if M^n is totally geodesic and N_1^p-totally geodesic warped product submanifold into a unit sphere $\mathbb{S}^m$.*
   - (b) *The equality case of (1.1) holds identically for all unit tangent vectors to N_2^q at each $x \in M^n$ if and only if M^n is totally geodesic manifold and either a N_2^q-totally geodesic warped product submanifold, or M^n is N_2^q-totally umbilical warped product submanifold in $\mathbb{S}^m$ at x with $\dim N_2^q = 2$.*
3. *The equality case of (1.1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is a totally geodesic warped product submanifold, or M^n is a mixed totally geodesic, totally umbilical and N_1^p-totally geodesic warped product submanifold with $\dim N_2^q = 2$.*

It is interesting to notice that the second variation operator of a minimal submanifold from a Riemannian manifold, carries information about the stability properties of the submanifold. When the ambient Riemannian manifold is a sphere, Simons [38] characterized the totally geodesic submanifolds as the minimal submanifolds of sphere $\mathbb{S}^m$ either with the lowest index or lowest nullity. Our results should be considered as an extension of such variational problems from submanifolds to warped product submanifolds which include a positive differentiable function too. It is known that the Ricci curvature plays an important role in general relativity, where it is the key term in the Einstein field equations [9]. Ricci curvature also appears in the Ricci flow equation, where a time-dependent Riemannian metric is deformed in the direction of minus its Ricci curvature. This system of partial differential equations is a non-linear analog of the heat equation and was first introduced by Richard S. Hamilton in [27] the early 1980s. Since heat tends to spread through a solid until the body reaches an equilibrium state of constant temperature, Ricci flow may be hoped to produce an equilibrium geometry for a manifold for which the Ricci curvature is constant. There are global results concerning manifolds on Ricci curvature bounds. Myers' theorem in [33] states that if the Ricci curvature is bounded from below on a complete Riemannian manifold by $(n-1)k > 0$ then the manifold is necessarily compact and has diameter $\leq \frac{\pi}{\sqrt{k}}$. These results show that Ricci curvature bounds have strong topological consequences which find possible applications in physics. Therefore, our result becomes a special case for upper bounds of Ricci curvature which include the Laplacian and the squared norm of the warping functions, and open a new research path.

2. Preliminaries and notations

Let $\mathbb{S}^m$ denotes the sphere with constant sectional curvature $c = 1 > 0$ and dimension (m). We use the fact that $\mathbb{S}^m$ admits a canonical isometric embedding in $\mathbb{R}^{m+1}$ as

$$\mathbb{S}^m = \{X \in \mathbb{R}^{m+1} \ : \|X\|^2 = 1\}.$$

Thus, the Riemannian curvature tensor $\widetilde{R}$ of sphere $\mathbb{S}^m$ satisfies the following

$$\widetilde{R}(U, V, Z, W) = g(U, W)g(V, Z) - g(V, W)g(U, Z), \tag{2.1}$$

for any $U, V, Z, W \in \mathfrak{X}(\mathbb{S}^m)$. This means that a unit sphere $\mathbb{S}^m$ is a manifold of constant sectional curvature one. Let M be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian $\widetilde{M}^m$ with induced metric g and ∇, $\nabla^\perp$ are the induced connections on the tangent bundle $\mathfrak{X}(M)$ and normal bundle $\mathfrak{X}(M^\perp)$ of M^n, respectively. Then the Gauss and Weingarten formulas are defined as $\widetilde{\nabla}_U V = \nabla_U V + \sigma(U, V)$, $\quad \widetilde{\nabla}_U \xi = -A_\xi U + \nabla_U^\perp \xi$, respectively for each $U, V \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}(M^\perp)$, where σ and A_ξ are the second fundamental form and shape operator (corresponding to the normal vector field N) respectively for the immersion of M^n into $\widetilde{M}$, and they are related as $g(\sigma(U, V), N) = g(A_N U, V)$. Similarly, the equations of Gauss and Codazzi are, respectively, given by

$$\begin{aligned} (i)\ R(X, Y, Z, W) =& \widetilde{R}(X, Y, Z, W) + g\big(\sigma(X, W), \sigma(Y, Z)\big) \\ &- g\big(\sigma(X, Z), \sigma(Y, W)\big) \\ (ii)\ \big(\widetilde{R}(X, Y)Z\big)^\perp =& (\widetilde{\nabla}_X \sigma)(Y, Z) - (\widetilde{\nabla}_Y \sigma)(X, Z). \end{aligned} \tag{2.2}$$

for all $X, Y, Z, W \in \mathfrak{X}(\widetilde{M})$, where R and $\widetilde{R}$ are the curvature tensor of $\widetilde{M}^m$ and M^n, respectively. The mean curvature H of the Riemannian submanifold M^n is given by $\mathbb{H} = \frac{1}{n} trace(\sigma)$. A submanifold M^n of the Riemannian manifold $\widetilde{M}^m$ is said to be *totally umbilical* and *totally geodesic* if for any $U, V \in \mathfrak{X}(TM)$, we have: $\sigma(U, V) = g(U, V)\mathbb{H}$ and $\sigma(U, V) = 0$, respectively, where $\mathbb{H}$ is the mean curvature vector of M^n. Moreover, the related null space or kernel of the second fundamental form of M^n at x is defined by

$$\mathcal{N}_x = \Big\{X \in T_xM \ \ : \sigma(X,Y) = 0, \textit{for all } Y \in M\Big\}. \tag{2.3}$$

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\widetilde{M}^m$, and denoted at $\widetilde{\tau}(\widetilde{M}^m)$, which, at some x in $\widetilde{M}^m$, is given by

$$\widetilde{\tau}(\widetilde{M}^m) = \sum_{1\le\alpha<\beta\le m} \widetilde{K}_{\alpha\beta}, \tag{2.4}$$

where $\widetilde{K}_{\alpha\beta} = \widetilde{K}\big(e_\alpha \wedge e_\beta\big)$. It is clear that, the first equality (2.4) is congruent to the following equation which will be frequently used in subsequent proof

$$2\widetilde{\tau}(\widetilde{M}^m) = \sum_{1\le\alpha<\beta\le m} \widetilde{K}_{\alpha\beta}, \quad 1 \le \alpha, \beta \le n. \tag{2.5}$$

Similarly, the scalar curvature $\widetilde{\tau}(L_x)$ of an L-plane is given by

$$\widetilde{\tau}(L_x) = \sum_{1\le\alpha<\beta\le m} \widetilde{K}_{\alpha\beta}, \tag{2.6}$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_xM and if $e_r = (e_{n+1}, \dots, e_m)$ belong to an orthonormal basis of the normal space $T^\perp M$, then we have

$$\sigma^r_{\alpha\beta} = g(\sigma(e_\alpha, e_\beta), e_r) \ \textit{ and } \ \|\sigma\|^2 = \sum_{\alpha,\beta=1}^{n} g\big(\sigma(e_\alpha, e_\beta), \sigma(e_\alpha, e_\beta)\big) \tag{2.7}$$

Let $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ be the sectional curvatures of the plane section spanned and e_α at x in the submanifold M^n and in the Riemannian space form $\widetilde{M}^m(c)$, respectively. Thus $K_{\alpha\beta}$ and $\widetilde{K}_{\alpha\beta}$ are the intrinsic and extrinsic sectional curvatures of the span $\{e_\alpha, e_\beta\}$ at x. From the Gauss equation (2.2)(i), we have

$$K_{\alpha\beta} = \widetilde{K}_{\alpha\beta} + \sum_{r=n+1}^{m} \Big(\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta} - (\sigma^r_{\alpha\beta})^2\Big). \tag{2.8}$$

Further, we will assume that a local field of orthonormal frame $\{e_1, \dots e_n\}$ on M^n, global tensor is defined as

$$\widetilde{S}(X,Y) = \sum_{i=1}^{m} \Big\{\widetilde{g}(\widetilde{R}(e_\alpha, Y)Y, e_\alpha)\Big\}, \quad X, Y \in \ T_xM^m. \tag{2.9}$$

This tensor is called Ricci tensor. If we fix a distinct integer from $\{e_1, \dots e_n\}$ on M^n, which is governed by ξ, then the Ricci curvature is defined as:

$$Ric(\xi) = \sum_{\substack{\alpha=1\\ \alpha\neq\xi}}^{n} K(e_\alpha \wedge e_u) \tag{2.10}$$

Now we define an important Riemannian intrinsic invariant called the scalar curvature of M^m and it is denoted by $\widetilde{\tau}(T_xM^m)$, as follows:

$$\widetilde{\tau}(M^n) = \sum_{1\le\alpha<\beta\le n} K(e_\alpha \wedge e_\beta) = \frac{1}{2}\sum_{\xi=1}^{m} Ric(e_\xi). \tag{2.11}$$

It is clear that the above inequality is congruent to the following equation which will be frequently used throughout the paper from now on:

$$2\widetilde{\tau}(M^n) = \sum_{1\le\alpha<\beta\le n} K(e_\alpha \wedge e_\beta) = \frac{1}{2}\sum_{\xi=1}^{m} Ric(e_\xi). \tag{2.12}$$

The following consequences are obtained from (2.2) and (2.8) as follows:

$$\tau(N_1^p) = \sum_{r=n+1}^{m} \sum_{1\le i<j\le p} \Big(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\Big) + \widetilde{\tau}(N_1^p). \tag{2.13}$$

Similarly, we have

$$\tau(N_2^q) = \sum_{r=n+1}^{m} \sum_{1\le a<b\le q} \Big(\sigma^r_{aa}\sigma^r_{bb} - (\sigma^r_{ab})^2\Big) + \widetilde{\tau}(N_2^q). \tag{2.14}$$

Next, we will assume that N_1^p and N_2^q are two Riemannian manifolds endowed with the Riemannian metrics g_1 and g_2, respectively. Let f be a smooth function defined on N_1^p. Then the warped product manifold $M^n = N_1^p \times_f N_2^q$ is the manifold $N_1^p \times N_2^q$ given by the Riemannian metric $g = g_1 + f^2 g_2$ [10]. Assuming that the $M^n = N_1^p \times_f N_2^q$ is a warped product manifold, then for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$, we find that

$$\nabla_Z X = \nabla_X Z = (X \ln f)Z. \tag{2.15}$$

From [Eq. (3.3) [17]], we have

$$\sum_{\alpha=1}^{p}\sum_{\beta=1}^{q} K(e_\alpha \wedge e_\beta) = \frac{q\Delta f}{f}. \tag{2.16}$$

Let $\{e_1, \ldots, e_n\}$ be an local orthonormal frame of the vector field M^n. Then, the gradient of the function φ and its squared norm are defined as:

$$\nabla\varphi = \sum_{i=1}^{n} e_i(\varphi)e_i. \tag{2.17}$$

and

$$\|\nabla\varphi\|^2 = \sum_{i=1}^{n}\big((\varphi)e_i\big)^2. \tag{2.18}$$

Remark 2.1. A warped product manifold $M^n = N_1^p \times_f N_2^q$ is said to be *trivial* or simply a Riemannian product manifold if the warping function f is constant.

We have an important lemma for further use in our proof.

Lemma 2.1 ([18]). *Let $f \in \mathcal{F}(N_1^p)$, then the gradient of the lift $f \circ \pi$ of f to $M^n = N_1^p \times_f N_2^q$ is the lift to M^n of the gradient of f on N_1^p, where $\pi : N_1^p \times_f N_2^q \to N_1^p$ is the projection map. This means that $\nabla(f \circ \pi) = \nabla f$.*

3. Proof of theorems

3.1. Proof of Theorem 1.2

First of all, we assume that the warped product submanifold $M^n = N_1^p \times_f N_2^q$ is a N_1^p-minimal warped product submanifold, and we will use a similar technique for the other case. From Gauss equation (2.2)(i), we have

$$n^2\|H\|^2 = 2\tau(M^n) + \|\sigma\|^2 - 2\widetilde{\tau}(M^n). \tag{3.1}$$

Assuming $\{e_1 \ldots e_p, e_{p+1} \ldots e_n\}$ to be a local orthonormal frame fields of M^n such that $\{e_1 \ldots e_p\}$ are tangent to N_1^p and $\{e_{p+1} \ldots e_n\}$ are tangent N_2^q. So, the unit tangent vector $\xi = e_A \in \{e_1 \ldots e_n\}$, can be expanded (3.1) as follows:

$$n^2\|H\|^2 = 2\tau(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\Big\{\big(\sigma_{11}^r + \cdots + \sigma_{nn}^r - \sigma_{AA}^r\big)^2 + \big(\sigma_{AA}^r\big)^2\Big\} - \sum_{r=n+1}^{m}\sum_{1\le\alpha\ne\beta\le n}\sigma_{\alpha\alpha}^r\sigma_{\beta\beta}^r - 2\widetilde{\tau}(M^n).$$

It is equivalent to

$$\begin{aligned} n^2\|H\|^2 =& 2\tau(M^n) + \sum_{r=n+1}^{m}\Big\{\big(\sigma_{11}^r + \cdots + \sigma_{nn}^r\big)^2 + \big(2\sigma_{AA}^r - (\sigma_{11}^r + \cdots + \sigma_{nn}^r)\big)^2\Big\} \\ &+ 2\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n}(\sigma_{\alpha\beta}^r)^2 - 2\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n}\sigma_{\alpha}^r\sigma_{\beta}^r - 2\widetilde{\tau}(M^n). \end{aligned}$$

Because we have considered that M^n is a N_1^p-minimal warped product submanifold, we derive

$$\begin{aligned} n^2\|H\|^2 =& \sum_{r=n+1}^{m}\Big\{\big(\sigma_{p+1p+1}^r + \cdots + \sigma_{nn}^r\big)^2 + \big(2\sigma_{AA}^r - (\sigma_{p+1p+1}^r + \cdots + \sigma_{nn}^r)\big)^2\Big\} \\ &+ 2\tau(M^n) + \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n}(\sigma_{\alpha\beta}^r)^2 - \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n}\sigma_{\alpha\alpha}^r\sigma_{\beta\beta}^r \\ &- 2\widetilde{\tau}(M^n) + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\ne A}}(\sigma_{aA}^r)^2 + \sum_{r=n+1}^{m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\ne A}}(\sigma_{\alpha\beta}^r)^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\ne A}}\sigma_{\alpha\alpha}^r\sigma_{\beta\beta}^r. \end{aligned} \tag{3.2}$$

Eq. (2.8), can be written as:

$$\sum_{r=n+1}^{m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}(\sigma^r_{\alpha\beta})^2-\sum_{r=n+1}^{m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}=\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}\widetilde{K}_{\alpha\beta}-\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}K_{\alpha\beta}. \tag{3.3}$$

For the N_1^p-minimality, we have

$$\sum_{r=n+1}^{m}\left\{\left(\sigma^r_{p+1p+1}+\cdots+\sigma^r_{nn}\right)^2\right\}=n^2\|\mathbb{H}\|^2. \tag{3.4}$$

Substituting the value of Eqs. (3.3) and (3.4) in Eq. (3.2), we derive

$$\begin{aligned}\frac{1}{2}n^2\|\mathbb{H}\|^2 =&2\tau(M^n)+\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma^r_{AA}-(\sigma^r_{n_1+1n_1+1}+\cdots+\sigma^r_{nn})\right)^2+\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n}(\sigma^r_{\alpha\beta})^2\\ &-\sum_{r=n+1}^{m}\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}-2\widetilde{\tau}(M^n)+\sum_{r=n+1}^{m}\sum_{\substack{a=1,\\ a\neq A}}(\sigma^r_{aA})^2+\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}\widetilde{K}_{\alpha\beta}-\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}K_{\alpha\beta}.\end{aligned} \tag{3.5}$$

On the other hand, from (2.4), we define

$$\begin{aligned}\tau(M^n)&=\sum_{1\le\alpha<\beta\le n}K(e_\alpha\wedge e_\beta)\\ &=\sum_{i=1}^{p}\sum_{j=p+1}^{n}K(e_i\wedge e_j)+\sum_{1\le i<k\le p}K(e_i\wedge e_k)+\sum_{p+1\le l<o\le n}K(e_l\wedge e_o).\end{aligned} \tag{3.6}$$

Using (2.4) and (2.16), we derive them, as follows:

$$\tau(M^n)=\frac{q\Delta f}{f}+\tau(N_1^p)+\tau(N_2^q). \tag{3.7}$$

From (3.5), (3.6), (3.7) and using (2.7), we derive

$$\begin{aligned}\frac{1}{2}n^2\|\mathbb{H}\|^2 =&\frac{n_2\Delta f}{f}-2\widetilde{\tau}(M^n)+\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}\widetilde{K}_{\alpha\beta}+\widetilde{\tau}(T_xN_1^p)+\widetilde{\tau}(T_xN_2^q)+\sum_{r=n+1}^{m}\left\{\sum_{1\le\alpha<\beta\le n}(\sigma^r_{\alpha\beta})^2-\sum_{\substack{1\le\alpha<\beta\le n\\ \alpha,\beta\neq A}}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}\right\}\\ &+\sum_{r=n+1}^{m}\sum_{\substack{a=1,\\ a\neq A}}(\sigma^r_{aA})^2+\sum_{r=n+1}^{m}\sum_{1\le i\neq j\le p}\left(\sigma^r_{ii}\sigma^r_{jj}-(\sigma^r_{ij})^2\right)+\sum_{r=n+1}^{m}\sum_{p+1\le s\neq t\le n}\left(\sigma^r_{ss}\sigma^r_{tt}-(\sigma^r_{st})^2\right)\\ &+\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma^r_{AA}-(\sigma^r_{p+1p+1}+\cdots+\sigma^r_{nn})\right)^2.\end{aligned} \tag{3.8}$$

Considering the unit tangent vector e_u, we have two choices: e_u is either tangent to the base manifold N_1^p or to the fiber N_2^q. So, first we will prove for the primary case:

Case-I If e_A is tangent to N_1^p, then we fix a unit tangent vector from $\{e_1\dots e_p\}$ to be e_A and consider $\xi=e_A=e_1$. Next, from (2.10) and (3.8), we get:

$$\begin{aligned}\mathcal{R}ic_M(\xi)\le&\frac{1}{2}n^2\|\mathbb{H}\|^2-\frac{q\Delta f}{f}+2\widetilde{\tau}(M^n)-\sum_{2\le\alpha<\beta\le n}\widetilde{K}_{\alpha\beta}-\widetilde{\tau}(N_1^p)-\widetilde{\tau}(N_2^q)-\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma^r_{11}-(\sigma^r_{p+1p+1}+\cdots+\sigma^r_{nn})\right)^2\\ &-\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le n}(\sigma^r_{\alpha\beta})^2+\sum_{r=n+1}^{m}\left\{\sum_{1\le i<j\le p}(\sigma^r_{ij})^2+\sum_{p+1\le s<t\le n}(\sigma^r_{st})^2\right\}\\ &-\sum_{r=n+1}^{m}\left\{\sum_{1\le i<j\le p}\sigma^r_{ii}\sigma^r_{jj}+\sum_{r=n+1}^{2m}\sum_{p+1\le s\neq t\le n}\sigma^r_{ss}\sigma^r_{tt}\right\}+\sum_{r=n+1}^{m}\sum_{2\le\alpha<\beta\le n}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}.\end{aligned} \tag{3.9}$$

Using the definition of the Riemannian submanifolds, from (2.1) and (2.5), one obtains

$$\begin{aligned}\mathcal{R}ic_M(\xi) \leq &\frac{1}{2}n^2\|\mathbb{H}\|^2 - \frac{q\Delta f}{f} + pq + n - 1 - \sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n}(\sigma^r_{\alpha\beta})^2 - \frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma^r_{11} - (\sigma^r_{p+1p+1} + \cdots + \sigma^r_{nn})\right)^2\\ &+ \sum_{r=n+1}^{m}\left\{\sum_{1\leq i<j\leq n_1}(\sigma^r_{ij})^2 + \sum_{p+1\leq s<t\leq n}(\sigma^r_{st})^2\right\} - \sum_{r=n+1}^{m}\left\{\sum_{1\leq i<j\leq p}\sigma^r_{ii}\sigma^r_{jj} + \sum_{r=n+1}^{m}\sum_{p+1\leq s\neq t\leq n}\sigma^r_{ss}\sigma^r_{tt}\right\}\\ &+ \sum_{r=n+1}^{m}\sum_{2\leq\alpha<\beta\leq n}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}.\end{aligned} \tag{3.10}$$

On the other hand, after we do some computations in the last two terms of (3.10), one obtains:

$$\sum_{r=n+1}^{m}\left\{\sum_{1\leq i<j\leq p}(\sigma^r_{ij})^2 + \sum_{p+1\leq s<t\leq n}(\sigma^r_{st})^2\right\} - \sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n}(\sigma^r_{\alpha\beta})^2 = \sum_{r=n+1}^{m}\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}(\sigma^r_{\alpha\beta})^2. \tag{3.11}$$

Similarly, we have

$$\sum_{r=n+1}^{m}\left\{\sum_{1\leq i<j\leq p}\sigma^r_{ii}\sigma^r_{jj} + \sum_{r=n+1}^{m}\sum_{p+1\leq s\neq t\leq n}\sigma^r_{ss}\sigma^r_{tt} - \sum_{2\leq\alpha<\beta\leq n}\sigma^r_{\alpha}\sigma^r_{\beta}\right\} = \sum_{r=n+1}^{m}\left(\sum_{j=2}^{p}\sigma^r_{11}\sigma^r_{jj} - \sum_{\alpha=2}^{p}\sum_{\beta=p+1}^{n}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}\right) \tag{3.12}$$

Replacing (3.12) in (3.10), we derive

$$\begin{aligned}\mathcal{R}ic_M(\xi) \leq &\frac{1}{2}n^2\|\mathbb{H}\|^2 - \frac{q\Delta f}{f} + pq + p + q - 1 - \frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma^r_{11} - (\sigma^r_{p+1p+1} + \cdots + \sigma^r_{nn})\right)^2\\ &- \sum_{r=n+1}^{m}\left(\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}(\sigma_{\alpha\beta})^2 + \sum_{b=2}^{p}\sigma^r_{11}\sigma^2_{bb} - \sum_{\alpha=2}^{p}\sum_{\beta=p+1}^{n}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta}\right).\end{aligned} \tag{3.13}$$

As for N_1^p-minimal warped product submanifold M^n, we compute the following simplification

$$\begin{aligned}\sum_{r=n+1}^{m}\sum_{\alpha=2}^{p}\sum_{\beta=p+1}^{n}\sigma^r_{\alpha\alpha}\sigma^r_{\beta\beta} &= \sum_{r=n+1}^{m}\sum_{\beta=p+1}^{n}\left\{g\big(\sigma(e_2,e_2),e_r\big) + \cdots + g\big(\sigma(e_p,e_p),e_r\big)\right\}\sigma^r_{\beta\beta}\\ &= \sum_{r=n+1}^{m}\sum_{\beta=p+1}^{n}\left\{g\big(\sigma(e_1,e_1),e_r\big) + \cdots + g\big(\sigma(e_p,e_p),e_r\big) - g\big(\sigma(e_1,e_1),e_r\big)\right\}\sigma^r_{\beta\beta}\\ &= -\sum_{r=n+1}^{m}\sum_{\beta=p+1}^{n}\sigma^r_{11}\sigma^r_{\beta\beta}.\end{aligned} \tag{3.14}$$

Similarly, we have

$$\sum_{r=n+1}^{m}\sum_{b=2}^{p}\sigma^r_{11}\sigma^r_{bb} = -\sum_{r=n+1}^{m}(\sigma_{11})^2. \tag{3.15}$$

On the other hand, we deduce that

$$\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma^r_{11} - (\sigma^r_{p+1p+1} + \cdots + \sigma^r_{nn})\right)^2 + \sum_{r=n+1}^{m}\sum_{\beta=p+1}^{n}\sigma^r_{11}\sigma^r_{\beta\beta} = 2\sum_{r=n+1}^{m}(\sigma_{11})^2 + \frac{1}{2}n^2\|\mathbb{H}\|^2. \tag{3.16}$$

Using (3.14) and (3.15) in Eq. (3.13), after the evaluation of (3.16), we finally get

$$\begin{aligned}\mathcal{R}ic_M(\xi) \leq &\frac{1}{2}n^2\|\mathbb{H}\|^2 - \frac{q\Delta f}{f} + pq + n - 1 - \frac{1}{4}\sum_{r=n+1}^{m}\left(\sigma^r_{p+1p+1} + \cdots + \sigma^r_{nn}\right)\\ &- \sum_{r=n+1}^{m}\left\{(\sigma_{11})^2 - \sum_{\beta=p+1}^{n}\sigma^r_{11}\sigma^r_{\beta\beta} + \frac{1}{4}\left(\sigma^r_{\sigma+1\sigma+1} + \cdots + \sigma^r_{nn}\right)\right\}.\end{aligned} \tag{3.17}$$

The last term of inequality (3.17) is equal to $\frac{n^2}{4}\|\mathbb{H}\|^2$ and using also the N_1^p-minimality, we get:

$$\mathcal{R}ic_M(\xi)+\frac{q\Delta f}{f}\leq\frac{1}{4}n^2\|\mathbb{H}\|^2+pq+p+q-1-\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma_{11}^r-\sum_{\beta=p+1}^{n}\sigma_{\beta\beta}\right)^2, \tag{3.18}$$

which implies the inequality (1.1). For the other case, we have

Case:-2 If e_A is tangent to N_2^q, then we fix the unit tangent vector field from $e_{p+1},\dots e_n$ such that $\xi=e_n$. From (2.10) to (3.9) using a similar approach as in the first case, one obtains:

$$\begin{aligned}\frac{1}{2}n^2\|\mathbb{H}\|^2\geq{}&\mathcal{R}ic_M(\xi)+\frac{q\Delta f}{f}-2\widetilde{\tau}(M^n)+\sum_{1\leq\alpha<\beta\leq n-1}\widetilde{K}_{\alpha\beta}+\widetilde{\tau}(N_1^p)+\widetilde{\tau}(N_2^q)\\&+\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma_{nn}^r-(\sigma_{p+1p+1}^r+\dots+\sigma_{nn}^r)\right)^2+\sum_{r=n+1}^{m}\sum_{\beta=1}^{n-1}\sigma_{nn}^r\sigma_{\beta\beta}^r\\&+\sum_{r=n+1}^{m}\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}(\sigma_{\alpha\beta}^r)^2-\sum_{r=n+1}^{m}\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n-1}\sigma_{\alpha\alpha}^r\sigma_{\beta\beta}^r.\end{aligned} \tag{3.19}$$

After some computations, one obtains:

$$\begin{aligned}&\sum_{r=n+1}^{m}\left\{\frac{1}{2}\left((\sigma_{p+1p+1}^r+\dots+\sigma_{nn}^r)-2\sigma_{nn}^r\right)^2+\sum_{\beta=n+1}^{n-1}\sigma_{nn}^r\sigma_{\beta\beta}\right\}\\&\quad=\sum_{r=n+1}^{m}\left\{\frac{1}{2}\left(\sigma_{p+1p+1}^r+\dots+\sigma_{nn}^r\right)^2+(\sigma_{nn}^r)^2-\sum_{\beta=n+1}^{n-1}\sigma_{nn}^r\sigma_{\beta\beta}\right\}.\end{aligned} \tag{3.20}$$

Since, M^n is N_1^p-minimal, then

$$\sum_{r=n+1}^{m}\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}(\sigma_{\alpha\beta}^r)^2=0. \tag{3.21}$$

Using a similar technique, as in (3.11), also replacing (3.20)–(3.21) in (3.19), we obtain the following inequality

$$\frac{1}{4}n^2\|\mathbb{H}\|^2\geq\mathcal{R}ic_M(\xi)+\frac{q\Delta f}{f}-pq-n+1+\frac{1}{2}\sum_{r=n+1}^{m}\left(2\sigma_{nn}^r-\sum_{\beta=p+1}^{n}\sigma_{\beta\beta}\right)^2,$$

which again implies the inequality (1.1). To derive the inequality (1.1), when warped product submanifold M^n is N_2^q-minimal, we will use a similar method, as we applied in the first case. Hence we conclude that the inequality (1.1) holds for both N_i^j-minimal isometric immersion for $i=1$ *or* 2 and $j=p,q$.

Now we will verify the equality cases in the inequality (1.1). Let us consider the relative null space $\mathcal{N}_x$ of the warped product submanifold M^n in a unit sphere $\mathbb{S}^m$ which is defined in Eq. (2.3). For $A\in\{e_1\dots e_n\}$, a unit tangent vector e_A to M^n at x satisfies the equality sign of the inequality (1.1), if and only if the following condition holds:

$$(i)\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}\sigma_{\alpha\beta}^r=0,\ \ (ii)\ \sum_{\substack{b=1\\b\neq A}}^{n}\sigma_{bA}^r=0,\ \ (iii),\ 2\sigma_{AA}^r=\sum_{\beta=p+1}^{n}\sigma_{\beta\beta}, \tag{3.22}$$

such that $r\in\{e_{n+1}\dots,m\}$. The first condition (i) implies that the M^n is a mixed totally geodesic warped product submanifold. Using the minimality and combining with (ii) and (iii) of (3.22), it can be easily seen that the unit tangent vector $\xi=e_A$ lies in the relative null space $\mathcal{N}_x$ at x. The converse part is straightforward and hence, we complete the proof of (1) from inequality (1.1).

Moreover, for N_1^p-minimal isometric warped product submanifold, the equality condition in (1.1) holds if and only if:

$$(i)\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}\sigma_{\alpha\beta}=0,\ \ (ii)\ \sum_{b=1}^{n}\sum_{\substack{A=1\\b\neq A}}^{n}\sigma_{bA}^r=0,\ \ (iii),\ 2\sigma_{\alpha\alpha}^r=\sum_{\beta=n_1+1}^{n}\sigma_{\beta\beta}, \tag{3.23}$$

where $\alpha\in\{1\dots,p\}$ and $r\in\{n+1\dots,m\}$. As M^n is a N_1^p-minimal, then from the third term of (3.23) we get that $\sigma_{\alpha\alpha}^r=0,\ \alpha\in\{1\dots,p\}$. So, combining these conditions with the second term (ii) of (3.23), we find that M^n is a N_1^p-totally geodesic warped product submanifold in a unit sphere $\mathbb{S}^n$. This proves the statement (a) of (2).

Assuming that M^n is a N_1^p-minimal, then the equality sign holds in (1.1) for all unit tangent vectors to N_2^q at x if and only if the following statements, hold:

$$(i) \quad \sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n} \sigma_{\alpha\beta}^r = 0, \quad (ii) \quad \sum_{b=1}^{p}\sum_{\substack{A=1\\ b\neq A}}^{n} \sigma_{bA}^r = 0, \quad (iii), \quad 2\sigma_{LL}^r = \sum_{\beta=p+1}^{n} \sigma_{\beta\beta}, \tag{3.24}$$

such that $L \in \{p+1\ldots, n\}$ and $r \in \{n+1, \ldots, m\}$. Two cases arise from the third condition (iii) from (3.24), that is

$$h_{LL}^r = 0, \ \forall L \in \{p+1\ldots, n\} \ \& \ r \in \{n+1, \ldots, m\}, \ \ or \ \dim N_2^{n_2} = 2. \tag{3.25}$$

If the first part of (3.25) holds, then in the light of the second condition from (3.24), we get that M^n is an N_2^q-totally geodesic warped product submanifold in a unit sphere $\mathbb{S}^m$. This is the first statement of the part (b) of (2) of the theorem. For the other part, we consider that M^n is not N_2^q-totally geodesic warped product submanifold and $\dim N_2^q = 2$. Then, from (ii) of (3.24), we hypothesize that M^n is N_2^2-totally umbilical warped product submanifold in-unit sphere $\mathbb{S}^m$. Hence the part (b) of (2) is proved completely. Now, we prove statement (3), using (3.23) and (3.24) together, and then we will use part (a) and (b) of (2). Thus, let us consider that $\dim N_2^q \neq 2$. Since, from (a) and (b) of a statement (3) respectively, we get that M^n is N_1^p-totally geodesic warped product submanifold in a unit sphere $\mathbb{S}^m$, this means that M^n is totally geodesic warped product submanifold in $\mathbb{S}^m$. Moreover, for the other case, we assume that the previous does not hold. Then from parts (a) and (b) of statement (2) one obtains that M^n is mixed totally geodesic and N_1^p-totally geodesic warped product submanifold in $\mathbb{S}^m$ with $\dim N_2^q = 2$. As for the last assertion to show that M^n is a totally umbilical warped product submanifold into $\mathbb{S}^n$, it is sufficient to prove that M^n is N_2^q-totally umbilical and is N_1^p-totally geodesic warped product submanifold in $\mathbb{S}^m$, remaining results obtained, directly from (b) and (a), respectively. This gives the complete proof part (3). Using a similar technique as in the above case, we can prove the theorem when M^n is N_2^q-minimal warped product submanifold in the unit sphere $\mathbb{S}^m$. This completes the proof of the theorem.

4. Geometric mechanics and applications

4.1. Harmonic function on compact warped product manifolds

In this subsection, we will consider that M^n is a compact submanifold with an empty boundary $\partial M = \emptyset$. The following famous result regarding such a manifold was proved in [11] as follows:

Lemma 4.1 (*Hopf's Lemma [11]*)**.** *Let M^n be compact, connected Riemannian manifold such that the Laplacian of the positive differentiable function $\varphi \in C^\infty(M^n)$ is non-negative, such that $\Delta\varphi \geq 0(\Delta\varphi \leq 0)$. Then φ is a constant function.*

Applying the above lemma, we prove the following result:

Theorem 4.1. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from an n-dimensional connected and compact warped product submanifold M^n into an m-dimensional unit sphere $\mathbb{S}^m$. Then M^n is a simply Riemannian product manifold if and only if the following inequality is satisfied:*

$$\mathcal{R}ic_M(\xi) = \frac{n^2}{4}\|\mathbb{H}\|^2 + p(q+1) + q + 1. \tag{4.1}$$

Proof. From (1.1) and (2.16), we have

$$\frac{n^2}{4}\|\mathbb{H}\|^2 \geq \mathcal{R}ic_M(\xi) + q\frac{\Delta f}{f} - pq - p - q + 1$$

Assuming that Eq. (4.1) holds, then from the above result, we get:

$$q\frac{\Delta f}{f} \leq 0,$$

which implies that $\Delta f \leq 0$. Using Lemma 4.1 we conclude that the warping function f is constant. From Remark 2.1 one obtains that M^n is a Riemannian product submanifold or a trivial warped product submanifold. The converse part is straightforward from (4.1). This completes the proof of the theorem.

For any positive differentiable function $\varphi \in C^\infty(M^n)$, the Hessian tensor of the function φ is a symmetric 2-covariant tensor field on M^n defined by

$$\Delta\varphi = -trace\mathrm{H}^\varphi = -trace\mathrm{H}^{lnf}. \tag{4.2}$$

Thus, from the above relation and Theorem 1.2, verifying the relations between Ricci curvature, the squared norm of mean curvature and Hessian tensor of the warping function, one obtains the following:

Corollary 4.1. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from an n-dimensional warped product submanifold M^n into a m-dimensional unit sphere $\mathbb{S}^m$. Thus, for each unit vector field $\xi \in T_x N_i$, with $i = 1$ or $i = 2$, we have the following inequality:*

$$\mathcal{R}ic_M(\xi) \leq \frac{n^2}{4}\|\mathbb{H}\|^2 - q\frac{trace\mathrm{H}^f}{f} + p(q+1) + q + 1.$$

4.2. An application to eigenvalue estimate

A lower bound on the Ricci curvature implies various bound on the geometric quantities. Let M^n be a complete non-compact Riemannian manifold and fix an arbitrary point x in M^n. Let M^n be a Riemannian manifold and $\lambda_1(M^n)$ denote the first eigenvalue of the Dirichlet boundary value problem

$$\begin{aligned} \Delta\varphi &= \lambda\varphi \ \ in\ M^n \\ \varphi &= 0 \ \ on\ \partial M^n, \end{aligned} \tag{4.3}$$

where Δ is the Laplacian on M^n and defined as $\Delta\varphi = -div(\nabla\varphi)$. From the monotonicity principle we find that $r < t$ which means that $\lambda_1(M_r^n) > \lambda_1(M_t^n)$. Therefore, the $\lim_{r\longrightarrow\infty}\lambda_1(\mathcal{D}_t)$ exists and the following definition can be given:

$$\lambda_1(M) = \lim_{r\longrightarrow\infty}\lambda_1(\mathcal{D}_r) \tag{4.4}$$

The above limit is independent of choice of the center x. By using the first non-zero eigenvalue of the Laplacian operator, Cheng [14,22] proved the eigenvalue comparison theorem which states that if M is complete and isometric to the standard unit sphere then $Ric(M) \geq 1$ and $d(M) = \pi$. Using the proof of Cheng [21] and also the studies of Palmer [[37], Lemma 1. p53], let us assume that φ is a non-constant warping function. Then the maximum (minimum) principle on the eigenvalue λ_1 yields (see, for instance, [9,14])

$$\lambda_1\int_{M^n}\varphi^2 dV \leq \int_{M^n}\|\nabla\varphi\|^2 dV, \tag{4.5}$$

with equality holding if and only if the condition $\Delta\varphi = \lambda_1\varphi$, holds. We give now the following:

Theorem 4.2. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from an n-dimensional compact warped product submanifold M^n into an m-dimensional unit sphere $\mathbb{S}^m$. Let the warping function $\ln f$ is an eigenfunction of the Laplacian of M^n associated to the first eigenvalue $\lambda_1(M^n)$ of the Dirichlet boundary problem (4.3), then the following inequality holds*

$$\int_{M^n}\mathcal{R}ic_M(\xi)dV \leq \frac{n^2}{4}\int_{M^n}\|\mathbb{H}\|^2 dV + q\lambda_1\int_{M^n}(\ln f)^2 dV + q\Big(p+1+\frac{p-1}{q}\Big)Vol(M^n). \tag{4.6}$$

The equality cases are same in Theorem 1.2.

Proof. As M^n is compact, this means that M^n is bounded. This implies that M^n have lower and upper bounds. Let $\lambda_1 = \lambda_1(M)$ and φ be a solution of (4.3) to the corresponding λ_1. It seems that φ does not change the sign in M^n. Therefore, we can rewrite (1.1) in the following form

$$\mathcal{R}ic_M(\xi) - q\|\nabla(\ln f)\|^2 \leq \frac{n^2}{4}\|\mathbb{H}\|^2 + q\Big(p+1+\frac{p-1}{q} - \Delta(\ln f)\Big), \tag{4.7}$$

where ∇ is the gradient on M^n. Integrating the above equation along the Riemannian volume form dV, we derive

$$\int_{M^n}\mathcal{R}ic_M(\xi)dV - q\int_{M^n}\|\nabla(\ln f)\|^2 dV \leq \frac{n^2}{4}\int_{M^n}\|\mathbb{H}\|^2 dV + q\Big(p+1+\frac{p-1}{q}\Big)Vol(M^n). \tag{4.8}$$

If λ_1 is an eigenvalue of the eigenfunction $\ln f$ such that $\Delta\ln f = \lambda_1\ln f$ with $\varphi = \ln f$ in (4.5), then equality in (4.5) holds,

$$\int_{M^n}\|\nabla\ln f\|^2 dV = \lambda_1\int_{M^n}(\ln f)^2 dV. \tag{4.9}$$

Using (4.9), we get

$$\int_{M^n}\mathcal{R}ic_M(\xi)dV - q\lambda_1\int_{M^n}(\ln f)^2 dV \leq \frac{n^2}{4}\int_{M^n}\|\mathbb{H}\|^2 dV + q\Big(p+1+\frac{p-1}{q}\Big)Vol(M^n). \tag{4.10}$$

This completes the proof of the theorem.

4.3. Relation and classification in Poisson equation

There are many conditions when physical quantities are described by a Poisson's equation such as the gravitational potential in the presence of mass, the electrostatic potential in light of distribution charge and study-state temperature in the presence of sinks as well. Similarly, the Laplacian equation describes stationary process in physics such as: the displacement of a membrane, the gravitational potential in the absence of mass, the steady heat flow equation in the absence of source of heat, the velocity potential for some fluids. Let $\mathcal{D} \subset M^n$ be a bounded, closed set. Then a variation of $\varphi \in \mathcal{D}$ is a one parameter family of functions $\varphi(s,t)$ where $s \in (-\epsilon, \epsilon)$ and $t \in M^n$ such that

$$\varphi(0,t) = \varphi(t) \tag{4.11}$$

$$\varphi(s,x) = \varphi(s), \quad x \in M^n - \{\mathcal{D}\}. \tag{4.12}$$

It is denoted by the following

$$\delta\varphi(x) = \frac{\partial \varphi(s,x)}{\partial s}_{|s=0}, \; i = \bar{1,n}. \tag{4.13}$$

Therefore the integral

$$I = \int_{\mathcal{D}} L d_{vg}, \tag{4.14}$$

is said to be stationary under the above variation if

$$\frac{dI}{ds}_{|s=0} = 0, \tag{4.15}$$

where we denote the volume element $d_{vg} = \sqrt{|g| dx^1, \dots, dx^m}$ and $|g| = |det(g_{ij})|$. The integral (4.14) is stationary under any variation of φ if and only if the following Euler–Lagrange equations are satisfied

$$\sum_{k=1}^{m} \left(\frac{\partial L}{\partial(\varphi^i_{:k})} \right) = \frac{\partial L}{\partial(\varphi^i)}, \, i = \bar{1,n}. \tag{4.16}$$

Let $f, \rho \in \mathcal{F}(M)$ on a Riemannian manifold (M, g) and let us consider the Lagrangian as:

$$L = \frac{\|f\|^2}{2} - \rho f. \tag{4.17}$$

If we use the Euler–Lagrange equation, derived from Eq. (4.14) with the right hand side $\frac{\partial L}{\partial f} = -\rho$, then Eq. (4.17) becomes the Poisson's equation:

$$\Delta f = \rho. \tag{4.18}$$

The following lemma is a useful result to obtain the solution of Poisson's equation.

Lemma 4.2 ([11]). *Let $k \in \mathbb{R}$, then an equation on the sphere $\mathbb{S}^m$ such that $\Delta f = k$ has a solution if and only if $k = 0$. In this case solutions are constant.*

Using the above result, due to Poisson's equation (4.18), we give now, the following:

Theorem 4.3. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from an n-dimensional compact warped product submanifold M^n into an m-dimensional unit sphere $\mathbb{S}^m$. If the warping function f is the solution of Poisson equation, then M^n is a Riemannian product and the following inequality holds.*

$$\mathcal{R}ic_M(\xi) \leq \frac{n^2}{4}\|\mathbb{H}\|^2 + q\left(p + 1 + \frac{p-1}{q}\right). \tag{4.19}$$

Proof. For any $k \in \mathbb{R}$, the equation on the sphere $\mathbb{S}^m$ for warped product submanifold M^n is:

$$\Delta(lnf) = k \tag{4.20}$$

From the hypothesis of Theorem 4.3, we know that f is the solution of Poisson's equation (4.18) and using Lemma 4.2 lead us to the conclusion that f is a solution of (4.20) if and only if $k = 0$. Then, one obtains:

$$\Delta(lnf) = 0. \tag{4.21}$$

On the other hand, Lemma 4.2 also shows that the solutions are constant. This means that f is a constant function, that is, M^n is a trivial warped product or simply Riemannian product manifold and hence, we have

$$\|\nabla \ln f\|^2 = 0. \tag{4.22}$$

Therefore, using (4.21) and (4.22) in (1.1), we get the required result.

As a consequence of the Poisson's function and using Poisson's equation (4.18), we have

Corollary 4.2. *If $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, is a N_1^p-minimal isometric immersion from an n-dimensional compact warped product submanifold M^n into an m-dimensional unit sphere $\mathbb{S}^m$ such that the warping function f satisfies the Poisson equation, then we have*

$$\mathcal{R}ic_M(\xi) + q\left(\frac{\rho}{f}\right) \leq \frac{n^2}{4}\|\mathbb{H}\|^2 + q\left(p + 1 + \frac{p-1}{q}\right). \tag{4.23}$$

where ρ is the Poisson's function defined in the Poisson's equation (4.18).

Proof. Using (4.18) in (1.1), we get the result and this completes the proof of the corollary.

4.4. Dirichlet energy and Lagrangian formalisms on warped product and their classifications

Let φ be positive differentiable function defined on a compact Riemannian manifold M^n such that $\varphi \in \mathcal{F}(M^n)$. Then *Dirichlet energy* of a function φ is defined in $\big[$pp. 44 [11]$\big]$ as:

$$\mathrm{E}(\varphi) = \frac{1}{2}\int_{M^n} \|\nabla\varphi\|^2 dV, \tag{4.24}$$

where dV is the volume element of M^n. Similarly, the *Lagrangian* for a function φ on Riemannian manifold M^n is given in [11], as follows:

$$L_\varphi = \frac{1}{2}\|\nabla\varphi\|^2. \tag{4.25}$$

The Euler–Lagrange equation for the Lagrangian (4.25) is:

$$\Delta\varphi = 0. \tag{4.26}$$

If we consider M^n to be a compact oriented Riemannian manifold without boundary such that $\partial M^n = \emptyset$, then we have a strong result which link the relation between Dirichlet energy, Ricci curvature and the squared mean curvature as follows:

Theorem 4.4. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from a compact oriented warped product submanifold M^n into the unit sphere $\mathbb{S}^m$. Then we have the following inequality*

$$\mathrm{E}(\ln f) \geq \frac{1}{2q}\left\{\int_{M^n} \mathcal{R}ic_M(\xi)dV - \frac{n^2}{4}\int_{M^n} \|\mathbb{H}\|^2 dV - \Big(pq + p + q - 1\Big)Vol(M^n)\right\}, \tag{4.27}$$

where dV is the volume element and $\mathrm{E}(\ln f)$ denotes the Dirichlet energy of the warping function lnf with respect to dV.

Proof. Let φ be a positive function defined on a compact oriented Riemannian manifold without boundary. Yano–Kon proved in [41] the following result $\int_{M^n} \Delta\varphi = 0$. Applying this result to the warping function $\ln f$, we get

$$\int_{M^n} \Delta(ln f) = 0. \tag{4.28}$$

Integrating inequality (1.1) along the volume element dV on a compact oriented warped product submanifold M^n, one obtains:

$$\int_{M^n} \mathcal{R}ic_M(\xi)dV \leq \frac{n^2}{4}\int_{M^n} \|\mathbb{H}\|^2 dV + \int_{M^n} \|\nabla \ln f\|^2 dV + \Big(pq + p + q - 1\Big)\int_{M^n} dV - \int_{M^n} \Delta(ln f)dV.$$

Using (4.28), the above inequality, became:

$$\int_{M^n} \mathcal{R}ic_M(\xi)dV \leq \frac{n^2}{4}\int_{M^n} \|\mathbb{H}\|^2 dV + \int_{M^n} \|\nabla \ln f\|^2 dV + \Big(pq + p + q - 1\Big)\int_{M^n} dV. \tag{4.29}$$

Therefore, in the *Dirichlet energy* formula (4.24) by setting $\varphi = \ln f$ and using also (4.29), we get the required result. This completes the proof of the theorem.

Another important result which can be obtained directly from Euler–Lagrange equation (4.26) in terms of the Lagrangian L_φ is defined in (4.25) as follows:

Corollary 4.3. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from a compact oriented warped product submanifold M^n into the unit sphere $\mathbb{S}^m$ such that the warping function $\ln f$ satisfies the Euler–Lagrange equation. Then, the* *following inequality, holds:*

$$L_{\ln f} \geq \frac{1}{2q}\left\{\mathcal{R}ic_M(\xi) - \frac{n^2}{4}\|\mathbb{H}\|^2 - \Big(pq + p + q - 1\Big)\right\}, \tag{4.30}$$

where $L_{\ln f}$ is the Lagrangian defined in Eq. (4.25).

Proof. Using (4.25) and (4.26) in (1.1), we get the inequality (4.30). This completes the proof of the corollary.

4.5. Relation between Ricci curvature and Hamiltonian of the warping function

In the point $x \in M^n$, the Hamiltonian for a local orthonormal frame can be written (see more detail in [11]) as follows:

$$\mathsf{H}(p, x) = \frac{1}{2}\sum_{i=1}^{n} p(e_i)^2. \tag{4.31}$$

If we replace $p = d\varphi$ in the above equation, (where d is a differentiable operator), then from (2.17), one obtains:

$$\mathsf{H}(d\varphi, x) = \frac{1}{2}\sum_{i=1}^{n} d\varphi(e_i)^2 = \frac{1}{2}\sum_{i=1}^{n} e_i(\varphi)^2 = \frac{1}{2}\|\nabla\varphi\|^2. \tag{4.32}$$

Using the above formula, we get:

Corollary 4.4. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from a compact warped product submanifold M^n into the unit sphere $\mathbb{S}^m$. Then the Ricci-Hamilton inequality can be written as follows:*

$$\mathsf{H}(d\ln f, x) \geq \frac{1}{2q}\left\{\mathcal{R}ic_M(\xi) - \frac{n^2}{4}\|\mathbb{H}\|^2 - \Big(pq + p + q - 1\Big)\right\}, \tag{4.33}$$

where $\mathsf{H}(d\ln f, x)$ is the Hamiltonian of the warping function $\ln f$.

Proof. Using (4.32) into (1.1), we get (4.33).

4.6. Some triviality results on warped products

We can give some interesting applications of Hopf's lemma assuming that M^n is a compact Riemannian manifold with boundary such that $\partial M \neq \emptyset$. We prove the following theorem:

Theorem 4.5. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from a compact connected oriented warped product submanifold M^n into the unit sphere $\mathbb{S}^m$. If the following relation holds:*

$$\mathsf{E}(\ln f) = \frac{1}{2q}\left\{\int_{M^n}\mathcal{R}ic_M(\xi)dV - \frac{n^2}{4}\int_{M^n}\|\mathbb{H}\|^2 dV - \Big(pq + p + q - 1\Big)Vol(M^n)\right\}, \tag{4.34}$$

then M^n is a Riemannian product manifold.

Proof. Integrating in (1.1), we get:

$$\int_{M^n}\mathcal{R}ic_M(\xi)dV \leq \frac{n^2}{4}\int_{M^n}\|\mathbb{H}\|^2 dV + \Big(pq + p + q - 1\Big)\int_{M^n} dV + q\int_{M^n}\|\nabla\ln f\|^2 dV - \int_{M^n}\Delta\ln f dV. \tag{4.35}$$

Using (4.24) in the above equation and if (4.34) is satisfied, then from (4.35) one obtains:

$$-\int_{M^n}\Delta(\ln f)dV \leq 0 \implies \Delta(\ln f) \geq 0. \tag{4.36}$$

Hence, now using Lemma 4.1 we conclude that $\ln f$ must be a constant. This means that $\ln f = a$ for any constant a which implies that $f = e^a$. So, we proved that the warped product submanifold M^n is a usual Riemannian product manifold.

4.7. Some topological obstructions

This subsection is devoted to the work of Obata [36] which is characterized specific Riemannian manifolds by ordinary differential equations. He derived the necessary and sufficient condition for an n-dimensional complete and connected Riemannian manifold (M, g) to be isometric to the n-sphere $\mathbb{S}^n$ if there exists a non-constant smooth function φ on M^n that satisfies the differential equation $H_\varphi = -c\varphi g$, where H_φ stands for the Hessian of φ. Then we obtain the following

Theorem 4.6. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$, be a N_1^p-minimal isometric immersion from a compact oriented warped product submanifold M^n into the unit sphere $\mathbb{S}^m$ with non-negative Ricci curvature and satisfying the following equality*

$$\left\|Hess(\ln f)\right\|^2 = -\frac{3\lambda_1}{p}\left\{\frac{n^2}{4q}\|\mathbb{H}\|^2 + \left(p+1+\frac{p-1}{q}\right)\right\}, \tag{4.37}$$

where $\lambda_1 > 0$ is a positive eigenvalue associated to eigenfunction $\ln f$. Then, the base manifold N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature equal to $\frac{\lambda_1}{p}$.

Proof. Let us define the following equation as

$$\|Hess(\ln f) - t\ln fI\|^2 = \|Hess(\ln f)\|^2 + t^2(\ln f)^2\|I\|^2 - 2t\ln f g(Hess(\ln f), I).$$

But we know that $\|I\|^2 = trace(II^*) = p$ and $g(Hess(\ln f), I^*) = tr(Hess(\ln f)I^*) = trHess(\ln f)$. Then the proceeding equation takes the form

$$\left\|Hess(\ln f) - t\ln fI\right\|^2 = \|Hess(\ln f)\|^2 + pt^2(\ln f)^2 - 2t\ln f\,\Delta\ln f. \tag{4.38}$$

Assuming λ_1 is an eigenvalue of the eigenfunction then $\Delta\ln f = \lambda_1\ln f$. Thus we get

$$\left\|Hess(\ln f) - t\ln fI\right\|^2 = \|Hess(\ln f)\|^2 + \left(pt^2 - 2t\lambda_1\right)(\ln f)^2. \tag{4.39}$$

On the other hand, we obtain

$$\Delta\varphi^2 = 2\varphi\Delta\varphi + \|\nabla\varphi\|^2$$

Then setting $\varphi = \ln f$, we have

$$\Delta(\ln f)^2 = 2\ln f\,\Delta\ln f + \|\nabla\ln f\|^2.$$

or

$$\lambda_1(\ln f)^2 = 2\lambda_1(\ln f)^2 + \|\nabla\ln f\|^2,$$

which implies that

$$(\ln f)^2 = -\frac{1}{\lambda_1}\|\nabla\ln f\|^2. \tag{4.40}$$

It follows from (4.39) and (4.40), we find that

$$\left\|Hess(\ln f) - t\ln fI\right\|^2 = \|Hess(\ln f)\|^2 + \left(2t - \frac{pt^2}{\lambda_1}\right)\|\nabla\ln f\|^2. \tag{4.41}$$

In particular, $t = -\frac{\lambda_1}{p}$ on (4.41) by taking integration, we get

$$\int_{M^n}\left\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\right\|^2 dV = \int_{M^n}\|Hess(\ln f)\|^2 dV - \frac{3\lambda_1}{p}\int_{M^n}\|\nabla\ln f\|^2 dV. \tag{4.42}$$

Again, integrating on (1.1) and involving the Green lemma, we have

$$\int_{M^n} \mathcal{R}ic_M(\xi)dV \leq \frac{n^2}{4}\int_{M^n} \|\mathbb{H}\|^2 dV + q\int_{M^n} \|\nabla(\ln f)\|^2 dV + q\Big(p+1+\frac{p-1}{q}\Big)Vol(M^n). \tag{4.43}$$

From (4.42) and (4.43), we derive

$$\begin{aligned}\frac{1}{q}\int_{M^n} \mathcal{R}ic_M(\xi)dV \leq &\frac{n^2}{4q}\int_{M^n} \|\mathbb{H}\|^2 dV - \frac{p}{3\lambda_1}\int_{M^n} \Big\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\Big\|^2 dV \\ &+ \frac{p}{3\lambda_1}\int_{M^n} \|Hess(\ln f)\|^2 dV + \Big(p+1+\frac{p-1}{q}\Big)Vol(M^n).\end{aligned} \tag{4.44}$$

As we assumed that the Ricci curvature is non-negative then $\mathcal{R}ic(\xi) \geq 0$, thus

$$\begin{aligned}\int_{M^n} \Big\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\Big\|^2 dV \leq &\frac{3n^2\lambda_1}{4pq}\int_{M^n} \|\mathbb{H}\|^2 dV + \int_{M^n} \|Hess(\ln f)\|^2 dV \\ &+ \frac{3\lambda_1}{p}\Big(p+1+\frac{p-1}{q}\Big)Vol(M^n).\end{aligned} \tag{4.45}$$

If the following equality holds from (4.37), then

$$\int_{M^n} \|Hess(\ln f)\|^2 dV = -\frac{3\lambda_1}{p}\left\{\frac{n^2}{4q}\int_{M^n} \|\mathbb{H}\|^2 dV - \Big(p+1+\frac{p-1}{q}\Big)Vol(M^n)\right\}. \tag{4.46}$$

From Eq. (4.45) one obtains:

$$\int_{M^n} \Big\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\Big\|^2 dV \leq 0. \tag{4.47}$$

But it is well known that

$$\int_{M^n} \Big\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\Big\|^2 dV \geq 0. \tag{4.48}$$

Combining Eqs. (4.47) and (4.48), we get

$$\Big\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\Big\|^2 = 0 \implies Hess(\ln f) = -\frac{\lambda_1}{p}\ln fI. \tag{4.49}$$

Since the warping function $\ln f$ of nontrivial warped product manifold M^n is non-constant, Eq (4.49), gives Obata's [36] differential equation with constant $c = \frac{\lambda_1}{p} > 0$ as $\lambda_1 > 0$, and therefore N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature $\frac{\lambda_1}{p}$. This completes the proof of the theorem.

As an application to Theorem 4.6, we give now the following:

Corollary 4.5. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$ be a minimal isometric immersion from a compact oriented warped product submanifold M^n into the unit sphere $\mathbb{S}^m$ with non-negative Ricci curvature and satisfying the equality*

$$\|Hess(\ln f)\|^2 = -\frac{3\lambda_1}{p}\Big(p+1+\frac{p-1}{q}\Big), \tag{4.50}$$

where λ_1 is a positive eigenvalue associated to eigenfunction $\ln f$. Then N_1^p is isometric to the sphere $\mathbb{S}^p(\frac{\lambda_1}{p})$ with constant sectional curvature equal to $\frac{\lambda_1}{p}$.

Another consequence of Theorem 4.6 is the following:

Corollary 4.6. *$\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$ be a minimal isometric immersion from a compact oriented warped product submanifold M^n into the unit sphere $\mathbb{S}^m$ with non-negative Ricci curvature and satisfying the equality (4.50) such that $\lambda_1 = p$. Then N_1^p is isometric to the sphere $\mathbb{S}^p$.*

In [26], Rio, Kupeli and Unal characterized Euclidean sphere using a standard differential equation which is the another version of Obata's differential equation. If a complete Riemannian manifold M^n admits a real valued non-constant function φ such $\Delta\varphi + \lambda_1\phi = 0$ such that $\lambda_1 < 0$, then M^n is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ϕ satisfies the equation that $\frac{d^2\phi}{dt^2} + \lambda_1\phi = 0$. Using this concept, we give now the following result:

Theorem 4.7. *Let $\ell : M^n = N_1^p \times_f N_2^q \longrightarrow \mathbb{S}^m$ be a N_1^p-minimal isometric immersion from a complete warped product submanifold M^n into the unit sphere $\mathbb{S}^m$ with positive Ricci curvature and satisfying the following equality*

$$\|Hess(\ln f)\|^2 = -\frac{3\lambda_1}{p}\left\{\frac{n^2}{4q}\|\mathbb{H}\|^2 + \left(p+1+\frac{p-1}{q}\right)\right\}, \tag{4.51}$$

where $\lambda_1 < 0$ is a negative eigenvalue of the eigenfunction $\ln f$. Then N_1^p is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ϕ satisfies the differential equation

$$\frac{d^2\phi}{dt^2} + \lambda_1\phi = 0. \tag{4.52}$$

Proof. In the hypothesis of the theorem, we assumed that the Ricci curvature is positive and hence, using the Myers's theorem which states that a complete Riemannian manifold with positive Ricci curvature is compact we conclude that M^n is a compact warped product submanifold and with free boundary. Now from (4.44), we have

$$\begin{aligned}\frac{1}{q}\int_{M^n}\mathcal{R}ic_M(\xi)dV \leq{}& \frac{n^2}{4q}\int_{M^n}\|\mathbb{H}\|^2dV - \frac{p}{3\lambda_1}\int_{M^n}\left\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\right\|^2dV\\ &+ \frac{p}{3\lambda_1}\int_{M^n}\|Hess(\ln f)\|^2dV + \left(p+1+\frac{p-1}{q}\right)Vol(M^n)\end{aligned}$$

As we assumed that the Ricci curvature is positive $Ric(\xi) > 0$, then we get:

$$\begin{aligned}\int_{M^n}\left\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\right\|^2dV <{}& \frac{3n^2\lambda_1}{4q}\int_{M^n}\|\mathbb{H}\|^2dV + \int_{M^n}\|Hess(\ln f)\|^2dV\\ &+ \frac{3\lambda_1}{p}\left(p+1+\frac{p-1}{q}\right)Vol(M^n)\end{aligned} \tag{4.53}$$

If Eq. (4.50) is satisfied, then from (4.53), we get

$$\left\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\right\|^2 < 0,$$

which implies that

$$\left\|Hess(\ln f) + \frac{\lambda_1}{p}\ln fI\right\|^2 = 0. \tag{4.54}$$

In this case $\lambda_1 < 0$, we invoke the result from [26], therefore N_1^p is isometric to a warped product of the Euclidean line and a complete Riemannian manifold, where the warping function on $\mathbb{R}$ satisfies the differential equation (4.52). This completes the proof of theorem.

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