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Approximately J^*-homomorphisms: A fixed point approach

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ABSTRACT

The functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to the true solution of (ξ). A functional equation is *superstable* if every solution satisfying the equation approximately is an exact solution of it. Using fixed point methods, we prove the stability and superstability of J^*-homomorphisms between J^*-algebras for the generalized Jensen-type functional equation $f(\frac{x+y}{2})+f(\frac{x-y}{2})=f(x)$.

1. Introduction

By a J^*-algebra we mean a closed subspace A of a C^*-algebra such that $xx^*x \in A$ whenever $x \in A$. Many familiar spaces are J^*-algebras [1], for example: (i) every Cartan factor of type I, i.e., the space of all bounded operators $B(H, K)$ between Hilbert spaces H and K; (ii) every Cartan factor of type IV, i.e., a closed $*$-subspace A of $B(H)$ in which the square of each operator in A is a scalar multiple of the identity operator on H; (iii) every JC^*-algebra; (iv) every ternary algebra of operators [2]. A J^*-homomorphism between J^*-algebras A and B is defined to be a linear mapping $h : A \to B$ such that

$$h(aa^*a) = h(a)h(a)^*h(a)$$

for all $a \in A$.

In particular, every $*$-homomorphism between C^*-algebras is a J^*-homomorphism. In [3], the stability of J^*-homomorphisms between J^*-algebras has been studied (see also [4,5]).

The stability of functional equations was first introduced by Ulam [6] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the

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homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [7] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [8] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Rassias [8] is called the generalized Hyers–Ulam stability.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.4}$$

for all $x \in E$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$, and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

During the last few decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [9–13].

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (See also [14–19].)

In this paper, we will use the fixed point alternative of Cădariu and Radu to prove the stability and superstability of J^*-homomorphisms between J^*-algebras for the generalized Jensen-type functional equation

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x).$$

Throughout this paper assume that A, B are two J^*-algebras.

2. Main results

Before proceeding to the main results, we will state the following theorem.

Theorem 2.1 (*The Fixed Point Alternative [20]*). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$d(T^m x, T^{m+1}x) = \infty$ for all $m \ge 0$,

or other exists a natural number m_0 such that:

- ⋆ *$d(T^m x, T^{m+1}x) < \infty$ for all $m \ge m_0$;*
- ⋆ *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
- ⋆ *y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;*
- ⋆ *$d(y, y^*) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

Lemma 2.2 (*[5]*). *Let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$, where X and Y are linear spaces. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 2.3. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : A^3 \to [0, \infty)$ such that*

$$\left\|\mu f\left(\frac{x+zz^*z+y}{2}\right) + \mu f\left(\frac{x+zz^*z-y}{2}\right) - f(\mu x) - \mu f(z)f(z)^*f(z)\right\| \le \phi(x,y,z) \tag{2.1}$$

for all $\mu \in \mathbb{T} := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that

$$\phi(x,y,z) \le 2L\phi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{2.2}$$

for all $x, y, z \in A$, then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\| \le \frac{L}{1-L}\phi(x,0,0) \tag{2.3}$$

for all $x \in A$.

Proof. It follows from (2.2) that

$$2^{-j}\phi(2^j x, 2^j y, 2^j z) \le L^j \phi(x,y,z)$$

for all $x, y, z \in A$ and all integers j. Hence

$$\lim_{j\to\infty} 2^{-j}\phi(2^jx, 2^jy, 2^jz) = 0 \tag{2.4}$$

for all $x, y, z \in A$. Putting $\mu = 1$ and $y = z = 0$ in (2.2), we obtain

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \phi(x, 0, 0) \tag{2.5}$$

for all $x \in A$. Hence

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \le \frac{1}{2}\phi(2x, 0, 0) \le L\phi(x, 0, 0) \tag{2.6}$$

for all $x \in A$. Consider the set $X := \{g \mid g : A \to B, g(0) = 0\}$ and introduce the generalized metric on X:

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \le C\phi(x, 0, 0) \text{ for all } x \in A\}.$$

It is easy to show that (X, d) is complete. Now we define the linear mapping $J : X \to X$ by

$$J(g)(x) = \frac{1}{2}g(2x)$$

for all $x \in A$. It is easy to show that $d(J(g), J(h)) \le Ld(g, h)$ for all $g, h \in X$ (see [20]). It follows from (2.6) that $d(f, J(f)) \le L$. By Theorem 2.1, $\{J^nf\}$ converges to a unique fixed point h of J in the set $X_1 := \{g \in X : d(f, g) < \infty\}$. So h satisfies $h(2x) = 2h(x)$ and

$$h(x) = \lim_{n\to\infty} \frac{1}{2^n}f(2^nx), \qquad d(f, h) \le \frac{1}{1-L}d(f, J(f))$$

for all $x \in A$. Therefore $d(f, h) \le \frac{L}{1-L}$. This implies the inequality (2.3). Put $z = 0$ in (2.1). It follows from the definition of J and (2.4) that

$$\begin{aligned}\left\|\mu h\left(\frac{x+y}{2}\right) + \mu h\left(\frac{x-y}{2}\right) - h(\mu x)\right\| &= \lim_{n\to\infty} \frac{1}{2^n}\|\mu f(2^{n-1}(x+y)) + \mu f(2^{n-1}(x-y)) - f(2^n\mu x)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\phi(2^nx, 2^ny, 0) = 0\end{aligned}$$

for all $x, y \in A$. So

$$\mu h\left(\frac{x+y}{2}\right) + \mu h\left(\frac{x-y}{2}\right) = h(\mu x) \tag{2.7}$$

for all $x, y \in A$. Putting $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$ in (2.7), we get

$$\mu h(u) + \mu h(v) = h(\mu u + \mu v)$$

for all $u, v \in A$. Since $h(0) = 0$, h is additive and $h(\mu x) = \mu h(x)$ for all $\mu \in \mathbb{T}$ and all $x \in A$. Hence, h is $\mathbb{C}$-linear by Lemma 2.2. Setting $x = y = 0$ and $\mu = 1$ in (2.1), we have

$$\begin{aligned}\|h(zz^*z) - h(z)h(z)^*h(z)\| &= \left\|2h\left(\frac{zz^*z}{2}\right) - h(z)h(z)^*h(z)\right\| \\ &= \lim_{n\to\infty} \frac{1}{8^n}\left\|2f\left(\frac{8^nzz^*z}{2}\right) - f(2^nz)f(2^nz)^*f(2^nz)\right\| \\ &\le \lim_{n\to\infty} \frac{1}{8^n}\phi(0, 0, 2^nz) \le \lim_{n\to\infty} \frac{1}{2^n}\phi(0, 0, 2^nz) = 0\end{aligned}$$

for all $z \in A$. Thus $h : A \to B$ is a J^*-homomorphism satisfying (2.3), as desired. □

We prove the following generalized Hyers–Ulam stability problem for J^*-homomorphisms on J^*-algebras.

Corollary 2.4. *Let $p \in (0, 1)$ and $\delta, \theta \ge 0$ be real numbers. Suppose $f : A \to B$ satisfies $f(0) = 0$ and*

$$\left\|\mu f\left(\frac{x + zz^*z + y}{2}\right) + \mu f\left(\frac{x + zz^*z - y}{2}\right) - f(\mu x) - \mu f(z)f(z)^*f(z)\right\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. Then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\| \le \frac{2^p\delta}{2 - 2^p} + \frac{2^p\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Set $\phi(x, y, z) := \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in A$. Then we get the desired result by $L = 2^{p-1}$ in Theorem 2.3. □

Remark 2.5. Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi : A^3 \to [0, \infty)$ such that

$$\left\| \mu f\left(\frac{x + zz^*z + y}{2}\right) + \mu f\left(\frac{x + zz^*z - y}{2}\right) - f(\mu x) - \mu f(z)f(z)^*f(z) \right\| \le \Phi(x, y, z)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. Let $0 < L < 1$ be a constant such that $2\Phi(x, y, z) \le L\Phi(2x, 2y, 2z)$ for all $x, y, z \in A$. By a method similar to that of the proof of Theorem 2.3, one can show that there exists a unique J^*-homomorphism $h : A \to B$ satisfying

$$\|f(x) - h(x)\| \le \frac{1}{1 - L}\Phi(x, 0, 0)$$

for all $x \in A$.

For the case $\Phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ (where θ is a non-negative real number and $p > 1$), there exists a unique J^*-homomorphism $h : A \to B$ satisfying

$$\|f(x) - h(x)\| \le \frac{2^p\theta}{2^p - 2}\|x\|^p$$

for all $x \in A$.

The case in which $p = 1$ was excluded in Corollary 2.4 and Remark 2.5. Indeed the results are not valid when $p = 1$. Here we use Gajda's example [21] to give a counter-example.

Proposition 2.6. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by*

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{for } |x| \ge 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by the formula

$$f(x) := \sum_{n=0}^{\infty} 2^{-n}\phi(2^n x).$$

Let

$$D_\mu f(x, y, z) := \mu f\left(\frac{x + z\bar{z}z + y}{2}\right) + \mu f\left(\frac{x + z\bar{z}z - y}{2}\right) - f(\mu x) - \mu f(z)\overline{f(z)}f(z)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in \mathbb{C}$. Then f satisfies

$$|D_\mu f(x, y, z)| \le 36(|x| + |y| + |z|) \tag{2.8}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in \mathbb{C}$, and the range of $|f(x) - A(x)|/|x|$ for $x \ne 0$ is unbounded for each additive function $A : \mathbb{C} \to \mathbb{C}$.

Proof. It is clear that f is bounded by 2 on $\mathbb{C}$. If $|x| + |y| + |z| = 0$ or $|x| + |y| + |z| \ge 1$, then

$$\left|D_\mu f(x, y, z)\right| \le 14 \le 14(|x| + |y| + |z|).$$

Now suppose that $0 < |x| + |y| + |z| < 1$. Then there exists an integer $k \ge 0$ such that

$$\frac{1}{2^{k+1}} \le |x| + |y| + |z| < \frac{1}{2^k}. \tag{2.9}$$

Therefore

$$2^m|x + z\bar{z}z \pm y|, 2^m|\mu x|, 2^m|z| < 1$$

for all $m = 0, 1, \dots, k - 1$. From the definition of f and (2.9), we have

$$|f(z)| \le k|z| + \sum_{n=k}^{\infty} 2^{-n}|\phi(2^n z)| \le k|z| + \frac{2}{2^k},$$

$$\begin{aligned} |D_\mu f(x, y, z)| \le k|z|^3 + \frac{6}{2^k} + |f(z)|^3 &\le (k + k^3)|z|^3 + \frac{8}{2^k} + \frac{6k^2 + 12k}{4^k}|z| \\ &\le \frac{k^3 + 6k^2 + 13k}{4^k}|z| + \frac{8}{2^k} \\ &\le 20|z| + 16(|x| + |y| + |z|) \\ &\le 36(|x| + |y| + |z|). \end{aligned}$$

Therefore f satisfies (2.8). Let $A : \mathbb{C} \to \mathbb{C}$ be an additive function such that

$$|f(x) - A(x)| \le \beta |x|$$

for all $x \in \mathbb{C}$. Then there exists a constant $c \in \mathbb{C}$ such that $A(x) = cx$ for all rational numbers x. So we have

$$|f(x)| \le (\beta + |c|)|x| \tag{2.10}$$

for all rational numbers x. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If x is a rational number in $(0, 2^{1-m})$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. So

$$f(x) \ge \sum_{n=0}^{m-1} 2^{-n} \phi(2^n x) = mx > (\beta + |c|)x$$

which contradicts (2.10). □

Now we establish the superstability of J^*-homomorphisms as follows.

Theorem 2.7. *Let $|r| > 1$, and let $f : A \to B$ be a mapping satisfying $f(rx) = rf(x)$ for all $x \in A$. Let $\phi : A^3 \to [0, \infty)$ be a mapping such that*

$$\|\mu f(x + zz^*z + y) + \mu f(x - y) - 2f(\mu x) - \mu f(z)f(z)^* f(z)\| \le \phi(x, y, z) \tag{2.11}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that $\phi(x, y, z) \le |r| L \phi(\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$ for all $x, y, z \in A$, then f is a J^*-homomorphism.

Proof. By using equation $f(rx) = rf(x)$ and (2.11), we have $f(0) = 0$ and

$$\|\mu f(x + y) + \mu f(x - y) - 2f(\mu x)\| \le |r|^{-n} \phi(r^n x, r^n y, 0), \tag{2.12}$$

$$\|f(zz^*z) - f(z)f(z)^* f(z)\| \le |r|^{-3n} \phi(0, 0, r^n z) \tag{2.13}$$

for all $x, y \in A$ and all integers n. It follows from $\phi(x, y, z) \le |r| L \phi(\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$ that

$$\lim_{n\to\infty} |r|^{-n} \phi(r^n x, r^n y, r^n z) = 0$$

for all $x, y, z \in A$. Hence we get from (2.12) and (2.13) that

$$\mu f(x + y) + \mu f(x - y) = 2f(\mu x), \qquad f(zz^*z) = f(z)f(z)^* f(z)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. So f is additive and $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}$ and all $x \in A$. By Lemma 2.2, f is $\mathbb{C}$-linear and we conclude that f is a J^*-homomorphism. □

The following theorem gives a similar result to Theorem 2.7 and we omit its proof.

Theorem 2.8. *Let $0 < |r| < 1$, and let $f : A \to B$ be a mapping satisfying $f(rx) = rf(x)$ for all $x \in A$. Let $\phi : A^3 \to [0, \infty)$ be a mapping satisfying (2.11). If there exists a constant $0 < L < 1$ such that $|r|\phi(x, y, z) \le L\phi(rx, ry, rz)$ for all $x, y, z \in A$, then f is a J^*-homomorphism.*

Corollary 2.9. *Let $0 < |r| \ne 1$, $p \in (0, 1)$ and $\delta, \theta \ge 0$ be real numbers. Suppose that $f : A \to B$ is a mapping satisfying $f(rx) = rf(x)$ for all $x \in A$ and the following inequality:*

$$\|\mu f(x + zz^*z + y) + \mu f(x - y) - 2f(\mu x) - \mu f(z)f(z)^* f(z)\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. Then f is a J^*-homomorphism.

Proof. Set $\phi(x, y, z) := \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ all $x, y, z \in A$. For $|r| > 1$, let $L = |r|^{p-1}$ and for $0 < |r| < 1$, let $L = |r|^{1-p}$. Then we get the desired result by Theorem 2.7 (for $|r| > 1$) and Theorem 2.8 (for $0 < |r| < 1$). □

Corollary 2.10. *Let $0 < |r| \ne 1$, $p > 1$ and $\theta \ge 0$ be real numbers. Suppose that $f : A \to B$ is a mapping satisfying $f(rx) = rf(x)$ for all $x \in A$ and the following inequality:*

$$\|\mu f(x + zz^*z + y) + \mu f(x - y) - 2f(\mu x) - \mu f(z)f(z)^* f(z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x, y, z \in A$. Then f is a J^*-homomorphism.

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