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Evolving states in nonlocal type dynamics of composite systems

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ABSTRACT

We discuss a model of nonlocal dynamics describing non-dissipative interaction of quantum systems. Within this framework, the evolution of the composite system is governed by an operator equation

$$-\mathrm{i}\dot{K} = KH + \hat{H}K + \beta K f(K^*K).$$

Here, H and $\hat{H}$ are time-independent self-adjoint Hamiltonians, $x \mapsto f(x)$ is a real analytic function, and β is a real parameter. We demonstrate that the equation is completely solvable in the sense that a solution $K = K(t)$ may be represented as a composition of three factors, each determined from a decoupled linear problem. Namely, if $K(0) = K_0$ then

$$K(t) = \exp[\mathrm{i}\hat{H}t] \circ K_0 \circ \exp[\mathrm{i}\beta f(K_0^*K_0)t] \circ \exp[\mathrm{i}Ht].$$

1. Introduction

The operator equation central to our discussion – see Eq. (17) – describes the evolution of a composite quantum state via a nonlocal, nonlinear type dynamics. It is a direct generalization of an equation, which has already been introduced in [1]. In that early stage the equation emerged from an attempt to reinterpret certain nonlinear Maxwell equations, [2,3], and to explain their curiously nonclassical phenomenology. Strictly speaking, that prototype equation corresponds to the case $f(x) = x^{-1}$ and $\hat{H} = 0$. Its solutions were described in [4] within the framework of finite-rank operators. Taking a step toward equations with two local Hamiltonians is as natural mathematically as it is necessary from the physical point of view. Mathematically, it accomplishes a symmetrization of the equation with respect to the Hermitian adjoint. The physical motivation emerges from an interpretation of the equation on grounds of the Quantum Information Theory (QIT), which we are lead to by applications. The main application thus far is that proposed in [5], where the equation is used to develop a PDE model of Quantum Hall type systems. In retrospect, the model may be interpreted through nonlocal pairing of distinct quantum systems (one magnetic and one electronic). This has been brought to light in [6,7], where a link with nonlocality, quantum entanglement, and other notions of QIT has been made explicit. An equation based on two independent Hamiltonians has first been discussed in [8], albeit for the finite-dimensional case with $f(x) = x^{-1}$ only. Subsequently, I have given a general description of the stationary states in [9], and a discussion of the spectra, mainly for $f(x) = \log x$, in [10]. Remarkably, in this type of dynamics the spectrum (i.e. the set of the energy levels of the stationary states of the composite system) can have a rich and nontrivial structure, e.g. it is shown in [10] that for a certain choice of the constituents the spectrum is a Cantor set. This inherent complexity of the spectra encourages my belief that the nonlocal dynamics provides quite a versatile modelling tool, adaptable to a variety of tasks one encounters in quantum and nano-system engineering. An additional attractive feature of the nonlocal dynamics is how it blends many different themes, sometimes leading to

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unexpected new perspectives at the important classical objects, e.g. it is demonstrated in [10] that for a certain choice of the constituents, the Riemann zeta function (that takes $s = 1/\beta$ as its independent variable) comes to the fore in the characterization of the ground state of the composite system.

In this article I discuss the time-dependent equation in a fairly general, infinite-dimensional setting. The main result is an explicit formula for the solutions. It takes considerable effort to extend the finite-dimensional version of the results to infinite-dimensions. The approach that seems to work best is based on the notion of strong derivatives, and this is the one taken here. In consequence, the reader will find that the analytic structures employed below blend smoothly with those needed in the formulation of the Hille–Yosida type theorems for the one-parameter semigroups. The results presented here rely upon those classical theorems, and may be viewed as their direct extension toward nonlinear problems. The classical Ref. [11] contains a very elegant exposition of the one-parameter semigroup theory, and has been particularly helpful in the process of setting the main result on a rigorous foundation.

Operator equations are natural and frequently encountered in the quantum theory. Perhaps the most known examples are the von Neumann equation (VNE), [12], and the quantum Markovian master equation (QMME), [13,14]. A number of authors, motivated by different pursuits, has considered nonlinear generalizations of QMME, [15–19], or VNE, [20–22]. All those equations appear to have an essentially different structure than the one considered here, and require different methods in order to describe solutions. Perhaps the general type of equations discussed in [23–25] have more in common with the one considered here: while based on a single local Hamiltonian, they can have the same type of nonlinearity as the one considered here. However, an analysis of the solutions is not undertaken therein. The reader may wish to consult the Appendix in [9] for further comments about nonlinear operator equations encountered in the physics and mathematics literature, and their relation to the equation considered here. In my view, one of the essential distinct features of Eq. (17) – both mathematically and physically – is an insertion of nonlinearity in conformity with the Hamiltonian framework and with the standard kinematics of composite quantum systems. A good question to be addressed in future is how to extend the method developed here to discuss a model enhanced with a Jaynes–Cummings, [26], or similar type interaction term in addition to the nonlinear term (see [27] for a brief discussion of the compatibility, but also differences, between the two types of dynamics). It is also very intriguing to ask if these methods could be extended and applied to dissipative-type dynamics. Such an extension would by no means be trivial, see e.g. Remark 5 at the end of the article.

2. Local quantum dynamics

We will briefly review the linear quantum dynamics in operator formalism; the reader interested in a thorough exposition of this material may consult [11]. Let $\mathbf{H}$ be a separable Hilbert space. The state of a quantum system is a unit length vector $\psi \in \mathbf{H}$. Time evolution of the state may be described via a one-parameter family of isometry transformations $V(t) : \mathbf{H} \longrightarrow \mathbf{H}$, $t \in (-\infty, \infty)$. Moreover, the path $V(t)$ is described infinitesimally via equation

$$-\mathrm{i}\dot{V}(t) = V(t)H \quad \text{with an initial condition : } V(0) = I. \tag{1}$$

The operator H is the Hamiltonian of the quantum system. It is a self-adjoint linear operator, densely defined in $\mathbf{H}$. We denote its domain by $\mathbf{D}(H)$. $\mathbf{D}(H) \subset \mathbf{H}$ is a dense linear subspace of $\mathbf{H}$, and for all $\psi \in \mathbf{D}(H)$, $H\psi \in \mathbf{H}$. The derivatives $\dot{V}(t)$ and Eq. (1) are understood in the strong sense, i.e.

$$-\mathrm{i}\dot{V}(t)\psi = V(t)H\psi, \quad \psi \in \mathbf{D}(H),$$

where $\dot{V}(t)\psi := \lim_{h\to 0} \frac{1}{h}(V(t+h) - V(t))\psi$. As is well known, [11], Ch. IX, the initial value problem has a unique solution $V = V(t) : \mathbf{H} \longrightarrow \mathbf{H}$ with $V(0) = I$. Moreover, $V(t)$ satisfies the group property

$$V(t+s) = V(t)V(s), \quad \text{for arbitrary } t, s.$$

Since $V(t)\,V(-t) = U(t-t) = U(0) = I$, we have $V(-t) = V(t)^{-1}$. In addition, $V(t)$ is an isometry, i.e. $V(t)^* = V(t)^{-1} = V(-t)$. Finally, V commutes with H, i.e.

$$V(t)H\psi = HV(t)\psi \quad \text{for } \psi \in \mathbf{D}(H). \tag{2}$$

It is worthwhile mentioning that $V(t)$ is constructed via a (strong) limit formula

$$V(t)\psi = \lim_{n\to\infty} \left(1 - \mathrm{i}\frac{t}{n}H\right)^{-n} \psi, \quad \psi \in \mathbf{H}.$$

It is common to use the notation $V(t) = \exp[\mathrm{i}Ht]$. Since in our case H is a self-adjoint operator, $V(t)$ may also be defined via the functional calculus, [28], i.e.

$$V(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}P_\lambda,$$

where $\mathrm{d}P_\lambda$ is the projection valued measure induced by H $\left(\text{and } H = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}P_\lambda\right)$. Let us also take note of the fact that $\dot{V}(t)\psi$ exists if and only if $\psi \in \mathbf{D}(H)$, see [28, VIII. 4 (Stone's theorem)]. This fact too may be connected with the functional calculus. Indeed, $\dot{V}$ may be represented in the form

$$\dot{V}(t) = \int_{-\infty}^{\infty} \mathrm{i}\lambda\, \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}P_\lambda.$$

Thus, the operator is well-defined on the set

$$\mathbf{D} = \left\{ \psi \in \mathbf{H} : \int_{-\infty}^{\infty} |\mathrm{i}\lambda\, \mathrm{e}^{\mathrm{i}\lambda t}|^2\, \mathrm{d}\langle\psi|P_\lambda|\psi\rangle < \infty \right\} = \left\{ \psi \in \mathbf{H} : \int_{-\infty}^{\infty} \lambda^2\, \mathrm{d}\langle\psi|P_\lambda|\psi\rangle < \infty \right\} = \mathbf{D}(H).$$

In order to fix the notation, we introduce another quantum system described in the same way. It is endowed with another separable Hilbert space $\widehat{\mathbf{H}}$, state vectors denoted φ, and the evolution determined by a local Hamiltonian $\hat{H}$ via the equation

$$-\mathrm{i}\dot{U} = \hat{H}U = U\hat{H}, \quad U : \widehat{\mathbf{H}} \to \widehat{\mathbf{H}}, \qquad U(0) = I. \tag{3}$$

Again, the time derivative of $U(t)$ is understood in the strong sense, and Eq. (3) is understood in the strong sense. Naturally, $U(t)$ has all the same properties as $V(t)$.

Observe also that the strong limit is interchangeable with the operation of adjoint. In particular

$$(\dot{U}(t))^* = (U(t)^*)^{\cdot}. \tag{4}$$

(This identity is also a direct consequence of the functional calculus.) Note that the domain of this operator (as defined by either side of the identity) is $\mathbf{D}(\hat{H})$. It follows that

$$\mathrm{i}\dot{U}^* = U^*\hat{H} = \hat{H}U^*, \quad \text{similarly } \mathrm{i}\dot{V}^* = V^*H = HV^*. \tag{5}$$

We will also use the product rule for strong derivatives. Consider a composition $A(t)B(t)$, where $A = A(t)$ and $B = B(t)$ are each a one-parameter family of bounded operators. An elementary argument shows that if an argument vector ψ is such that both $\dot{B}\psi$ and $\dot{A}B\psi$ exist, then $(AB)^{\cdot}\psi = \dot{A}B\psi + A\dot{B}\psi$.

3. Nonlocal dynamics of a composite system

Let $\mathbf{H}$ and $\widehat{\mathbf{H}}$ be two separable Hilbert spaces, attributed to two quantum systems. Depending on the systems either Hilbert space may be infinite dimensional or finite-dimensional. We wish to consider a composite system consisting of the two subsystems. According to the standard rules of quantum mechanics, such system is associated with the tensor product space $\widehat{\mathbf{H}}\otimes\mathbf{H}$, [29]. This structure is the foundation for many *kinematic* concepts of quantum mechanics, e.g. nonlocality, violation of Bell inequalities, or quantum teleportation, [29], as well as some models of dynamics, such as the Jaynes–Cummings model for cavity QED, [26,30]. In what follows we discuss a specific type of dynamics of a composite system, previously explored in [7,8].

We find it convenient to work with $\mathbf{H}^*$ (the dual space) rather than $\mathbf{H}$ itself—naturally all properties of the subsystem can be inscribed in one space as well as the other, and the distinction is only formal. However, the algebraic structure of the composite system space $\widehat{\mathbf{H}} \otimes \mathbf{H}^*$ is better adapted to our purposes, as it enables an operator interpretation of the vectors. Indeed, consider a Hilbert–Schmidt (*H–S* for short) operator of the form

$$K = \sum K_{mn}|\varphi_m\rangle\langle\psi_n| : \mathbf{H} \to \widehat{\mathbf{H}} \tag{6}$$

where each of the two collections $\{\psi_n \in \mathbf{H}\}$, $\{\varphi_m \in \widehat{\mathbf{H}}\}$ furnishes an orthonormal basis. Recall that the set of *H–S* operators $HS(\mathbf{H}, \widehat{\mathbf{H}}) = \{K : \mathbf{H} \to \widehat{\mathbf{H}} : \mathrm{Tr}\,(KK^*) < \infty\}$ with the Hermitian product

$$\langle K|L\rangle = \mathrm{Tr}\left(KL^*\right) \tag{7}$$

is a separable Hilbert space. Indeed, there is an obvious canonical isomorphism

$$HS(\mathbf{H}, \widehat{\mathbf{H}}) \cong \widehat{\mathbf{H}} \otimes \mathbf{H}^*.$$

This isomorphism enables one to establish a '1–1' identification of *H–S* operators with the (pure) states of a composite quantum system, e.g. K is identified with $|\Psi_c\rangle = \sum K_{mn}|\varphi_m\rangle \otimes \langle\psi_n| \in \widehat{\mathbf{H}} \otimes \mathbf{H}^*$. The composite system at hand is composed of subsystems $\widehat{\mathbf{H}}$ and $\mathbf{H}^*$. Let us define

$$\rho := K^*K : \mathbf{H} \to \mathbf{H}, \qquad \hat{\rho} := KK^* : \widehat{\mathbf{H}} \to \widehat{\mathbf{H}}. \tag{8}$$

Now, if K is a *H–S* operator, then ρ and $\hat{\rho}$ are trace class operators. In addition, assuming $\langle K|K\rangle = 1$ one automatically has $\mathrm{Tr}\,\rho = 1 = \mathrm{Tr}\,\hat{\rho}$. Such normalization conditions allow one to interpret ρ and $\hat{\rho}$ as density operators, characterizing the mixed states of the subsystems. Observe that

$$\begin{aligned} \rho^T &= \mathrm{Tr}_{\widehat{\mathbf{H}}}\,(|\Psi_c\rangle\langle\Psi_c|) : \mathbf{H}^* \to \mathbf{H}^*, \\ \hat{\rho} &= \mathrm{Tr}_{\mathbf{H}}\,(|\Psi_c\rangle\langle\Psi_c|) : \widehat{\mathbf{H}} \to \widehat{\mathbf{H}}. \end{aligned} \tag{9}$$

In other words, our definition of $\hat{\rho}$ fully coincides with the standard definition, prevalent in quantum mechanics, while our ρ is the transpose of the standard one.

Recall that *H–S* operators are compact. Since compact operators form a two-sided ideal, [31,32], trace class operators are *a fortiori* compact. In particular, ρ has a discrete spectrum. It consists of either a finite set of non-negative eigenvalues, or an infinite set of positive eigenvalues and its accumulation point, 0, say

$$1 \geq \rho_1 \geq \rho_2 \geq \cdots 0.$$

In the latter case 0 may but need not be an eigenvalue. Identities (8) directly imply that the sequence of nonzero eigenvalues of ρ (listed with multiplicities) is identical with that of $\hat{\rho}$. Therefore the spectra of both density operators are identical. However, in the infinite-dimensional case it is *a priori* possible that 0 is an eigenvalue of ρ but not $\hat{\rho}$, or vice versa. (To illustrate this with an example, one can take $\widehat{\mathbf{H}} = \mathbf{H}$, and let $K := \sum_{n=1}^{\infty} \frac{1}{n} |\varphi_n\rangle\langle\varphi_{n+1}|$. In this case ρ has a nontrivial kernel, while $\hat{\rho}$ is *one-to-one*, but not *onto*.)

We now turn attention to the dynamics of the composite system $\widehat{\mathbf{H}} \otimes \mathbf{H}^*$. When the subsystems $\widehat{\mathbf{H}}$ and $\mathbf{H}^*$ are isolated, their evolution is governed by the rules of local quantum dynamics outlined in Section 2. At this stage, however, we wish to allow the systems to interact. We assume a particular model of interaction, which is based on the energy functional

$$E(\rho, \hat{\rho}) = \mathrm{Tr}[H\rho] + \mathrm{Tr}[\hat{\rho}\hat{H}] + \beta \mathrm{Tr} F(\rho). \tag{10}$$

Here, F is an analytic function, such that $F(x)$ is real for real argument $x \in [0, \infty)$, and β is a real parameter. (In some applications, it may be appropriate to consider F that is defined only for $x \neq 0$, such as $F = \log$. The discussion then requires an *a priori* assumption that ρ be nonsingular, or a modification of the meaning of $\log \rho$. In order to keep technical complexity in check we will adhere to the simpler scenario.) Note that

$$\mathrm{Tr}\, F(\rho) = \sum_{k=1}^{\infty} F(\rho_k) = \mathrm{Tr}\, F(\hat{\rho}).$$

Note that the first two terms of (10) represent the expectation value of energy stored in the two subsystems, and so the third term is the energy of interaction between the subsystems. The actual value of E depends on the composite system state K, albeit indirectly, through the partial traces. It seems helpful to emphasize at this point that the spatial proximity or remoteness of the two subsystems is of no consequence to the composite system state and, *a fortiori*, has no bearing on the value of E. The model of interaction *is not* geometric. Hence, it is best described as nonlocal.

Note also, that for a special choice of $F(x) = -x \ln x$, the third term represents the von Neumann entropy of either subsystem, [12]. There are essential reasons to consider a general function F in its place. First, doing so does not make the theory any more complex—on the contrary, it helps to reveal its underlying structure. Second, other choices of F are of interest when one is modelling quantum systems, e.g. $F = \log$ is featured in [1,4,5]. At the same time, the interpretation of $\mathrm{Tr}\, F(\rho)$ as entropy puts emphasis on the formal analogy between our suggestion to calculate the energy of a composite system via $E(\rho)$ and the statistical treatment of quantum ensembles, [33]. We emphasize, however, that here ρ describes a specific single subsystem in a mixed state, as opposed to a statistical ensemble. Also, in contrast to the techniques employed in quantum statistical mechanics, we will use the functional $E(\rho)$ to derive the dynamics.

We wish to describe the evolution of the composite system state. (Recall that a state vector is identified with an operator denoted K.) Substituting decomposition (8) into (10) yields

$$\Xi(K) = \frac{1}{2} E(K^*K) = \frac{1}{2} \mathrm{Tr}[K^*KH] + \mathrm{Tr}[K^*\hat{H}K] + \beta \mathrm{Tr}\, F(K^*K). \tag{11}$$

With the *H–S* product this can be expressed in the form

$$2\Xi(K) = \langle KH|K\rangle + \langle \hat{H}K|K\rangle + \beta \mathrm{Tr}\, F(K^*K). \tag{12}$$

Ξ assumes finite values on the set

$$\Gamma_{H,\hat{H},F} := \{K \in HS(\mathbf{H}, \widehat{\mathbf{H}}) : |\langle KH|K\rangle| < \infty, |\langle \hat{H}K|K\rangle| < \infty, \mathrm{Tr}\, F(K^*K) < \infty\}. \tag{13}$$

One may view $\Gamma_{H,\hat{H},F}$ as an object analogous to a Sobolev space, e.g. we have:

Proposition 3.1. *Let $F(x) = \sum_{n=0}^{\infty} F_n x^n$ be an entire function such that $\sum_{n=0}^{\infty} |F_n|\, |x|^n < \infty$ for all x. Then $\Gamma_{H,\hat{H},F}$ is a dense linear subspace of $HS(\mathbf{H}, \widehat{\mathbf{H}})$.*

Proof. We first show that $\Gamma_{H,\hat{H},F}$ is a linear space. Let $K \neq 0$, so that $\mathrm{Tr}\,(K^*K) = c$ with positive c. Since $\mathrm{Tr}\left(\frac{1}{c}K^*K\right) = 1$, all eigenvalues of $\frac{1}{c}K^*K$ are nonnegative and bounded above by 1. Therefore $\left(\frac{1}{c}K^*K\right)^n \leq \frac{1}{c}K^*K$, and $\mathrm{Tr}\,(K^*K)^n \leq c^n$. This implies

$$|\mathrm{Tr}\, F(K^*K)| \leq \mathrm{Tr}|\sum_{n=0}^{\infty} F_n (K^*K)^n| \leq \mathrm{Tr} \sum_{n=0}^{\infty} |F_n| (K^*K)^n = \sum_{n=0}^{\infty} |F_n| \mathrm{Tr}(K^*K)^n \leq \sum_{n=0}^{\infty} |F_n| c^n < \infty.$$

Therefore, $\operatorname{Tr} F(K^*K)$ is finite for all H–S operators. Now, since $\langle KH|K\rangle = \langle KH^{\frac{1}{2}}|KH^{\frac{1}{2}}\rangle$, condition $|\langle KH|K\rangle| < \infty$ is equivalent to $KH^{\frac{1}{2}} \in HS(\mathbf{H}, \widehat{\mathbf{H}})$. The latter condition is manifestly linear, and defines a linear subspace in $HS(\mathbf{H}, \widehat{\mathbf{H}})$. The analogous condition $\hat{H}^{\frac{1}{2}}K \in HS(\mathbf{H}, \widehat{\mathbf{H}})$ is also linear. This shows that $\Gamma_{H,\hat{H},F}$ is a linear subspace of H–S.

In order to see that $\Gamma_{H,\hat{H},F}$ is dense in H–S, we observe the following: Let $K : \mathbf{H} \to \widehat{\mathbf{H}}$ be of the form

$$K = \sum K_{mn}|\varphi_m\rangle\langle\psi_n|, \quad \psi_n \in \mathbf{D}(H), \varphi_m \in \mathbf{D}(\hat{H}) \text{ (finite sum)}. \tag{14}$$

Obviously, $K \in \Gamma_{H,\hat{H},F}$. At the same time, operators of this form are a dense subset of H–S. Indeed, since the set of all finite rank operators is dense in H–S, see e.g. [32], it suffices to check that finite rank operators can be approximated by operators of type (14). This is seen as follows. Given

$$K = \sum K_{mn}|u_m\rangle\langle v_n|, \quad u_m \in \widehat{\mathbf{H}},\ v_n \in \mathbf{H} \text{ (finite sum)},$$

pick approximating sequences $\mathbf{D}(\hat{H}) \ni \varphi_n^l \to u_n$, and $\mathbf{D}(H) \ni \psi_m^l \to v_m$, and let

$$K^{(l)} := \sum K_{mn}|\varphi_m^l\rangle\langle\psi_n^l|.$$

It is a straightforward exercise to verify that $\|K^{(l)} - K\| \to 0$ (convergence in H–S norm). This completes the proof. □

The real-imaginary decomposition of the scalar product, $\langle K|L\rangle = (K, L) + \mathrm{i}\omega(K, L)$ introduces a symplectic form ω. It is useful to observe the identity

$$2\omega(L, -\mathrm{i}K) = \langle K|L\rangle + \langle L|K\rangle. \tag{15}$$

We will show that the gradient of functional Ξ may be expressed as

$$D\Xi(K)[\Phi] = \omega(\Phi, -\mathrm{i}KH - \mathrm{i}\hat{H}K - \mathrm{i}Kf(K^*K)), \quad \text{where } f = F'. \tag{16}$$

The calculation of the gradient of the two sesquilinear terms is straightforward. The nonlinear part is more subtle and needs to be discussed in more detail. We wish to calculate $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \operatorname{Tr} F[(K+\varepsilon\Phi)^*(K+\varepsilon\Phi)]$. Let us denote $\rho_\varepsilon = (K+\varepsilon\Phi)^*(K+\varepsilon\Phi)$, and introduce the resolvent

$$R(\varepsilon, z) := (\rho_\varepsilon - z)^{-1}, \quad \text{and} \quad R(z) := R(0, z).$$

The Dunford–Taylor formula, [11], reads

$$F[(K + \varepsilon\Phi)^*(K + \varepsilon\Phi)] = -\frac{1}{2\pi\,\mathrm{i}}\int_\gamma F(z)R(\varepsilon, z)\mathrm{d}z, \quad (\varepsilon \text{ sufficiently small}).$$

The integral converges in the sense of operator norm, *a fortiori* in the strong sense. Here γ is a closed smooth curve in the resolvent set of ρ_0, which encircles the spectrum of ρ_0. Note that if $\varepsilon \to 0$, then $\|\rho_\varepsilon - \rho_0\| \to 0$ (operator norm). Therefore γ lies entirely in the resolvent set and retains the spectrum of ρ_ε in its interior, for $\varepsilon > 0$ sufficiently small, see [11, Ch. IV, Theorems 3.11 and 2.23]. Next, it follows from the second Neumann series, [11], that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} F[(K + \varepsilon\Phi)^*(K + \varepsilon\Phi)] = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma F(z)R(z)\left[\Phi^*K + K^*\Phi\right]R(z)\mathrm{d}z.$$

Since the integral converges in the strong sense we can interchange integration with trace. In this way we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \operatorname{Tr} F[(K + \varepsilon\Phi)^*(K + \varepsilon\Phi)] &= \operatorname{Tr}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} F[(K + \varepsilon\Phi)^*(K + \varepsilon\Phi)] \\ &= \frac{1}{2\pi\,\mathrm{i}}\int_\gamma F(z)\operatorname{Tr}\left[R(z)\left[\Phi^*K + K^*\Phi\right]R(z)\right]\mathrm{d}z \\ &= \frac{1}{2\pi\,\mathrm{i}}\int_\gamma F(z)\operatorname{Tr}\left[R(z)^2\left[\Phi^*K + K^*\Phi\right]\right]\mathrm{d}z.\end{aligned}$$

Subsequently, we note that $\frac{\partial}{\partial z}R(z) = R(z)^2$ (e.g. via the first Neumann series). Therefore, integrating the last expression by parts we obtain

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma F(z)R(z)^2\,\mathrm{d}z = -\frac{1}{2\pi\,\mathrm{i}}\int_\gamma f(z)R(z)\mathrm{d}z = f(K^*K) \quad (f = F').$$

Therefore

$$\langle Kf(K^*K)|\Phi\rangle = \operatorname{Tr}\left[f(K^*K)\Phi^*K\right] = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma F(z)\operatorname{Tr}\left[R(z)^2\Phi^*K\right]\mathrm{d}z,$$

and also

$$\langle \Phi | Kf(K^*K)\rangle = \mathrm{Tr}\left[f(K^*K)K^*\Phi\right] = \frac{1}{2\pi \mathrm{i}}\int_\gamma F(z)\mathrm{Tr}\left[R(z)^2K^*\Phi\right]\mathrm{d}z.$$

Collecting the results, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{Tr}\, F[(K+\varepsilon\Phi)^*(K+\varepsilon\Phi)] = \langle Kf(K^*K)|\Phi\rangle + \langle \Phi|Kf(K^*K)\rangle = 2\omega(\Phi, -\mathrm{i}f(K^*K)K),$$

where we have applied (15). In view of (16), the Hamiltonian flow of Ξ has the form

$$-\mathrm{i}\dot{K} = KH + \hat{H}K + \beta Kf(K^*K). \tag{17}$$

An analysis of this evolution equation is the main focus of this article.

In the closing of this section, we make the following useful observation.

Lemma 3.1. *Let K be a bounded operator and let f be analytic in an open set containing the spectrum of K^*K. Then $Kf(K^*K) = f(KK^*)K$.*

Proof. Observe that $K(K^*K - z) = (KK^* - z)K$. Therefore, for z from the resolvent set we have

$$K(K^*K - z)^{-1} = (KK^* - z)^{-1}K.$$

Let γ be a curve in the resolvent set, encircling the spectrum of K^*K. The Dunford–Taylor formula implies:

$$Kf(K^*K) = -\frac{1}{2\pi\mathrm{i}}K\int_\gamma f(z)(K^*K - z)^{-1}\,\mathrm{d}z = -\frac{1}{2\pi\mathrm{i}}\int_\gamma f(z)(KK^* - z)^{-1}\,\mathrm{d}zK = f(KK^*)K.$$

Note that the spectrum of K^*K coincides with the spectrum of KK^*, with the possible exception of a single point 0, so that the same curve γ can be used in both integrals. □

4. A special class of solutions

Here, we assume that H (resp. $\hat{H}$) have a purely discrete spectrum, consisting of eigenvalues h_i ($\hat{h}_i$), say, $h_0 \le h_1 \le \cdots$ (resp. $\hat{h}_0 \le \hat{h}_1 \le \cdots$). In this case, it is not hard to find a matrix representation of the operators satisfying Eq. (17). Indeed, let us assume

$$H = \sum_{n=0}^{\infty} h_n|\psi_n\rangle\langle\psi_n| \tag{18}$$

and

$$\hat{H} = \sum_{n=0}^{\infty} \hat{h}_n|\varphi_n\rangle\langle\varphi_n| \tag{19}$$

where $(\psi_n)_{n=0}^\infty$ (respectively, $(\varphi_n)_{n=0}^\infty$) is a complete list of the eigenfunctions of H (resp. $\hat{H}$). The crucial step is to introduce an Ansatz

$$K(t) = \sum_{n=0}^{\infty} k_n|\varphi_n\rangle\langle\psi_n|. \tag{20}$$

Substituting (18)–(20) into Eq. (17) we obtain

$$\mathrm{i}\dot{k}_n = k_n\left(h_n + \hat{h}_n + \beta f(|k_n|^2)\right), \quad n = 0, 1, 2, \ldots.$$

Let the polar representation of the complex coefficients be

$$k_n = r_n\mathrm{e}^{\mathrm{i}\theta_n}$$

We readily obtain $\dot{r}_n = 0$, so that $r_n(t) = r_n(0) = r_n$, and $\dot{\theta}_n = h_n + \hat{h}_n + \beta f(r_n^2)$. Summarizing, the solution is represented in the form

$$K(t) = \sum_{n=0}^{\infty} k_n(0)\,\mathrm{e}^{\mathrm{i}t\left[h_n + \hat{h}_n + \beta f(r_n^2)\right]}|\varphi_n\rangle\langle\psi_n|. \tag{21}$$

Let us observe that in this case the mixed states of the subsystems are frozen in time, i.e. $\rho(t) = \rho(0)$, $\hat{\rho}(t) = \hat{\rho}(0)$. In general, the mixed states may evolve in time, but their eigenvalue collections will be time-independent—see Corollary 5.1, and the remark that follows it. It is also interesting to make the following observations (we will see later, cf. Corollary 5.2, that they remain true for general solutions as well):

Proposition 4.1. *Let $K(t)$ be given by (21). We have*

1. *If $K(0)$ is a bounded operator, then $K(t)$ is bounded for all t.*
2. *If $K(0)$ is compact, then $K(t)$ is compact for all t.*
3. *If $K(0)$ is a Hilbert–Schmidt operator, then $K(t)$ is Hilbert–Schmidt for all t.*
4. *If $K(0)$ is a finite-rank operator, then $K(t)$ is finite-rank for all t.*

Proof. The class of $K(t)$ is decided by the properties of the sequence $(|k_n(t)|)_{n=0}^{\infty}$, which is time-independent, i.e. $|k_n(t)| = |k_n(0)| =: |k_n|$ for all t. The operator norm $\|K(t)\| = \sup\{|k_n(t)| : n = 0, 1, 2, \ldots\}$ is the same for all t. We conclude that if $K(0)$ is bounded, then $K(t)$ is bounded for all t. For the operators $K(t)$ to be compact it is necessary and sufficient that the sequence $(|k_n|)$ is either finite (finite-rank operator), or infinite with the only accumulation point being 0. The operators $K(t)$ will be H–S if $\sum_n |k_n|^2 < \infty$. □

5. General solutions

In this section we assume for simplicity that f is continuous on $[0, \infty)$. This excludes functions that have a pole at 0 such as $f = \log$. However, we need this assumption to ensure that when K is a bounded operator, $f(K^*K)$ is also bounded. Also, we emphasize that we do not introduce any restrictions on the spectra of H or $\hat{H}$. As we will see, general solutions of Eq. (17) have a triunitary structure, i.e. they can be expressed via an appropriate product of the initial condition and three distinct operator exponentials. We begin with a formal definition of solutions.

Definition 5.1. An nontrivial (i.e. $\neq 0$) operator $K = K(t) : \mathbf{H} \to \widehat{\mathbf{H}}$, $t \in (-\infty, \infty)$ is a solution of Eq. (17) if it satisfies the following three conditions:

1. $K(t)$ is a bounded operator for all t.
2. All $K(t)$ map the domain of H into the domain of $\hat{H}$, i.e.

$$K(t)[\mathbf{D}(H)] \subset \mathbf{D}(\hat{H}), \quad \text{arbitrary } t. \tag{22}$$

3. Eq. (17) holds in the strong sense, i.e.

$$-\mathrm{i}\dot{K}\psi = KH\psi + \hat{H}K\psi + \beta Kf(K^*K)\psi, \quad \psi \in \mathbf{D}(H). \tag{23}$$

Here, $f(K^*K) = f(K(t)^*K(t))$ is interpreted via the spectral theorem, and $\dot{K}(t) = \frac{\mathrm{d}}{\mathrm{d}t}K(t)$ denotes the strong derivative.

The following observation is crucial in describing the solutions of (17).

Lemma 5.1. *Assume that $K(t) : \mathbf{H} \to \widehat{\mathbf{H}}$ is a solution of (17), and define*

$$T(t) := U^*(t) \circ K(t) \circ V(t)^*,$$

where V and U satisfy (1) and (3) (respectively). The operator $T(t) : \mathbf{H} \to \widehat{\mathbf{H}}$ is bounded, and satisfies

$$-\mathrm{i}\dot{T}\psi = \beta Tf(T^*T)\psi, \quad \psi \in \mathbf{H}. \tag{24}$$

Proof. Since $K(t)$ are bounded by definition, and $U(t)$ and $V(t)$ are unitary, $T(t)$ are also bounded. Observe that for a $\psi \in \mathbf{D}(H)$ one has

$$-\mathrm{i}\dot{T}\psi = -\mathrm{i}\dot{U}^*KV^*\psi - \mathrm{i}U^*\dot{K}V^*\psi - \mathrm{i}U^*K\dot{V}^*\psi.$$

We will inspect the right-hand side term by term. First, applying (5), we have

$$-\mathrm{i}U^*K\dot{V}^*\psi = -U^*KHV^*\psi, \quad \psi \in \mathbf{D}(H).$$

Second, (5) implies $V^*\psi \in \mathbf{D}(H)$, and the *a priori* assumption (22) ensures $KV^*\psi \in \mathbf{D}(\hat{H})$. Therefore, applying (5), again, we have

$$-\mathrm{i}\dot{U}^*KV^*\psi = -U^*\hat{H}KV^*\psi, \quad \psi \in \mathbf{D}(H).$$

Furthermore, substituting $\psi = V^*\psi$ in (23), we obtain

$$-\mathrm{i}U^*\dot{K}V^*\psi = U^*KHV^*\psi + U^*\hat{H}KV^*\psi + \beta U^*Kf(K^*K)V^*\psi.$$

The last four equations put together imply

$$-\mathrm{i}\dot{T}\psi = \beta U^*Kf(K^*K)V^*\psi.$$

Observe that since $T^*T = VK^*KV^*$, we have $f(T^*T) = Vf(K^*K)V^*$. This shows that Eq. (24) is satisfied for all $\psi \in \mathbf{D}(H)$. However, since $T(t)$ are continuous, the composition operators $T(t)f(T(t)^*T(t))$ are also continuous. Therefore, the equation extends from the dense subset $\mathbf{D}(H)$ to the entire space $\mathbf{H}$. □

Note that Eq. (24) involves only *bounded* operators. Next, we will characterize its solutions.

Lemma 5.2. *If $T = T(t)$ satisfies Eq. (24) and $T(0) = T_0$, then*

$$T(t) = T_0 \exp[\mathrm{i}\beta f(T_0^* T_0)t]. \tag{25}$$

Proof. Since the strong derivative and the adjoint operations commute (24) implies

$$\mathrm{i}\dot{T}^*\phi = \beta f(T^*T)T^*\phi, \quad \phi \in \widehat{\mathbf{H}}.$$

Using this, we calculate

$$-\mathrm{i}(T^*T)^{\cdot}\psi = -\mathrm{i}\dot{T}^*T\psi - \mathrm{i}T^*\dot{T}\psi = \beta\left[-f(T^*T)T^*T + T^*Tf(T^*T)\right]\psi = 0$$

for an arbitrary $\psi \in \mathbf{H}$. Therefore $T(t)^*T(t) = T_0^*T_0$, and T satisfies

$$-\mathrm{i}\dot{T} = \beta T f(T_0^* T_0). \tag{26}$$

Recall that $f = f(x)$ is real for $x \in [0, 1]$ by assumption, which ensures selfadjointness of the bounded operator $f(T_0^*T_0)$. It follows that $T(t)$ is as in (25). □

The following theorem is the main result, characterizing solutions of (17).

Theorem 5.1. *All solutions of Eq. (17) have the triunitary structure characterized in statement* I . *Statement* II *gives sufficient conditions on the initial condition $K(0) = K_0$ that ensure the existence of a solution.*

(I) *If $K(t)$ is a solution of Eq. (17) with initial condition $K(0) = K_0$, then*

$$K(t) = U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V(t), \tag{27}$$

where V and U satisfy (1) and (3) (respectively). In particular a solution is uniquely determined by its initial value $K(0)$.

(II) *Assume that a bounded operator $K_0 : \mathbf{H} \to \widehat{\mathbf{H}}$ satisfies*

(i) $K_0[\mathbf{D}(H)] \subset \mathbf{D}(\hat{H})$.

(ii) $\exp[\mathrm{i}\beta f(K_0^*K_0)t][\mathbf{D}(H)] \subset \mathbf{D}(H)$.

Then, operator $K(t)$ determined by K_0 via (27) is a solution of (17).

Proof. (I) It follows from Lemmas 5.1 and 5.2 that if $K(t)$ is a solution of (17) with $K(0) = K_0$, then $K(t) = U(t)T(t)V(t)$, where $T(t)$ satisfies (25) with $T_0 = T(0) = K_0$. Therefore, $K(t)$ has the form (27).
(II) Let $K(t)$ be defined by (27) with K_0 such that all three conditions in II are satisfied. Since $K(t)$ is automatically bounded, we only need to verify that it satisfies (23). Let $\psi \in \mathbf{D}(H)$. We calculate

$$\begin{aligned} -\mathrm{i}\dot{K}\psi \;=\; & -\mathrm{i}\dot{U}(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V(t)\psi + \beta U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]f(K_0^*K_0)V(t)\psi \\ & - \mathrm{i}U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]\dot{V}(t)\psi. \end{aligned} \tag{28}$$

We will examine the right-hand side term by term. First, it follows from (1) directly that the last term on the right-hand side may be represented in the form

$$-\mathrm{i}U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]\dot{V}(t)\psi = U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V(t)H\psi = K(t)H\psi.$$

Next, we inspect the middle term. Observe that $K(t)^*K(t) = V(t)^*K_0^*K_0V(t)$ and subsequently

$$V(t)^* f(K_0^*K_0)V(t) = f(K(t)^*K(t)).$$

Therefore, inserting $V(t)V(t)^* = I$, we obtain

$$\begin{aligned} U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]f(K_0^*K_0)V(t)\psi \;&=\; U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V(t)V(t)^* f(K_0^*K_0)V(t)\psi \\ &= K(t)f(K(t)^*K(t))\psi. \end{aligned}$$

Finally, observe that $V\psi \in \mathbf{D}(H)$ by virtue of (2). Hence, condition (ii) ensures that $\exp[\mathrm{i}\beta f(K_0^*K_0)t]V\psi \in \mathbf{D}(H)$. Condition (i), in turn, ensures that $K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V\psi \in \mathbf{D}(\hat{H})$. Thus, we can substitute $\psi = K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V\psi$ in (3), which leads to

$$\begin{aligned} -\mathrm{i}\dot{U}(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V(t)\psi \;&=\; \hat{H}U(t)K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]V(t)\psi \\ &= \hat{H}K(t)\psi. \end{aligned}$$

Collecting the above expressions for the right-hand side terms in (28), we see that $K(t)$ satisfies (23). This completes the proof of the theorem. □

As an immediate corollary, we obtain:

Corollary 5.1. *Let $K(t)$ be a solution of Eq. (17). The evolution of the corresponding mixed state $\rho = K^*K$, which represents the first subsystem, is governed by the local H-dynamics, i.e.*

$$\rho(t) = K(t)^*K(t) = V(t)^*K_0^*K_0V(t), \quad \text{or infinitesimally: } \mathrm{i}\dot{\rho}(t) = [H, \rho] \quad \text{(strong)}. \tag{29}$$

Similarly, the mixed state of the second subsystem, $\hat{\rho} = KK^*$, evolves according to

$$\hat{\rho} = U(t)\hat{\rho}_0U(t)^*, \qquad -\mathrm{i}\dot{\hat{\rho}}(t) = [H, \hat{\rho}] \quad \text{(strong)}. \tag{30}$$

Proof. Observe that $\exp[\mathrm{i}\beta f(K_0^*K_0)t]$ commutes with $K_0^*K_0$. With this in mind, both (29) and (30) follow directly from (27). The infinitesimal versions are a consequence of (1) and (3). □

Remark 1. The above corollary shows that the mixed states of subsystems evolve according to the von Neumann equation. Therefore the subsystem dynamics is local and linear. The nonlinearity inherent in (17) cannot be detected via an experiment conducted on a subsystem. It also means that interaction of the subsystems is non-dissipative. Indeed, dissipative Markovian interaction is described by the master equation in Lindblad form, [13,12,29], which in addition to the commutator requires a few extra terms.

Remark 2. It is interesting to observe that even if $K(0) = K_0$ is normal, i.e. the two Hilbert spaces are identified $\mathbf{H} = \widehat{\mathbf{H}}$, and $K_0^*K_0 = K_0K_0^*$, $K = K(t)$ does not need to be normal for $t \neq 0$. Indeed, the operators are normal if and only if $V(t) = U(t)^*$.

Corollary 5.2. *If $K_0^*K_0$ is a trace class operator, then $K(t)^*K(t)$ is a trace class operator for all t, and*

$$\mathrm{Tr}(K^*K) = \mathrm{Tr}(K_0^*K_0). \tag{31}$$

Moreover, if $F(K_0K_0^*)$ is trace class, then

$$\mathrm{Tr}\, F(K^*K)^{\cdot} = 0. \tag{32}$$

Similarly, if $K_0HK_0^*$ is trace class, then

$$\mathrm{Tr}(KHK^*)^{\cdot} = 0. \tag{33}$$

Also, if $K_0^*\hat{H}K_0$ is trace class, then

$$\mathrm{Tr}(K^*\hat{H}K)^{\cdot} = 0. \tag{34}$$

A fortiori, *if all compositions listed above are trace class, then*

$$\Xi(K)^{\cdot} = 0 \quad \text{(conservation of energy)}. \tag{35}$$

Proof. By Corollary 5.1 $K^*K = V(t)^*K_0^*K_0V(t)$, which implies $F(K^*K) = V(t)^*F(K_0^*K_0)V(t)$. Identities (31) and (32) are a direct consequence. Next, applying Lemma 3.1 with the function $\exp[\mathrm{i}\beta f(x)t]$ instead of $f(x)$, we obtain

$$K_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t] = \exp[\mathrm{i}\beta f(K_0K_0^*)t]K_0.$$

Next, substitute (27) and, using the above identity and (2), obtain

$$\begin{aligned} KHK^* &= UK_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]VHV^* \exp[-\mathrm{i}\beta f(K_0^*K_0)t]K_0^*U^* \\ &= UK_0 \exp[\mathrm{i}\beta f(K_0^*K_0)t]HVV^* \exp[-\mathrm{i}\beta f(K_0^*K_0)t]K_0^*U^* \\ &= U \exp[\mathrm{i}\beta f(K_0K_0^*)t]K_0HK_0^* \exp[-\mathrm{i}\beta f(K_0K_0^*)t]U^* \\ &= \left(U \exp[\mathrm{i}\beta f(K_0K_0^*)t]\right) K_0HK_0^* \left(U \exp[\mathrm{i}\beta f(K_0K_0^*)t]\right)^*. \end{aligned}$$

This shows that KHK^* and $K_0HK_0^*$ are unitarily equivalent, and proves (33). Identity (34) follows from an analogous argument. In view of the definition (11), identity (35) is a direct consequence of all the previous ones, and the well known cyclicity of the trace $\mathrm{Tr}(KHK^*) = \mathrm{Tr}(K^*KH)$. □

Corollary 5.3. *If K_0 is a compact operator, then $K(t)$ is compact for all t.*

Proof. Since the set of compact operators is an ideal, the claim follows directly from (27). □

Corollary 5.4. *If $K = K(t)$ satisfies (17), and $K(0) = K_0$ is Fredholm, then $K(t)$ is Fredholm for all t. Moreover,*

$$\mathrm{index}\, K(t) = \mathrm{index}\, K_0.$$

Proof. In view of (27) for a fixed $tK(t)$ is a composition of K_0 with unitary operators. Therefore, the two operators are either simultaneously Fredholm, or simultaneously fail to be Fredholm. They belong in the same index class, because a composition of operators is a sum of the indices of those operators, and the index of a unitary operator is zero, [32]. □

Remark 3 (*Polar Representation of Solutions*)**.** As is well-known, [31], any bounded operator, say, $K : \mathbf{H} \to \widehat{\mathbf{H}}$ admits a polar decomposition in the form

$$K = WR,$$

where the self-adjoint non-negative operator $R = \sqrt{K^*K} : \mathbf{H} \to \mathbf{H}$ is defined via the spectral decomposition theorem, and W is a partial isometry, i.e. an isometry of $\operatorname{Ker} W = (\operatorname{Im} R)^{\perp}$ onto $\operatorname{Im} W \subseteq \widehat{\mathbf{H}}$. In particular,

$$W^*W = E, \qquad WW^* = F, \tag{36}$$

where E is the orthogonal projection onto $(\operatorname{Ker} W)^{\perp}$ in $\mathbf{H}$, and F is the orthogonal projection onto $\operatorname{Im} W = \operatorname{Im} K$ in $\widehat{\mathbf{H}}$.

While formula (27) provides a complete description of solutions to Eq. (17), it is interesting to display the solution in the polar decomposition form $K(t) = W(t)R(t)$. First, let $K_0 = W_0R$. Second, observe that $\rho = K^*K = R^2$ and by Corollary 5.1, $R^2 = \rho(t) = V(t)^*\rho_0V(t)$. It follows that

$$R(t) = V(t)^*R_0V(t). \tag{37}$$

On the other hand (27) yields

$$\begin{aligned} K &= U(t)W_0R_0 \exp[\mathrm{i}\beta f(R_0^2)t]V(t) \\ &= U(t)W_0 \exp[\mathrm{i}\beta f(R_0^2)t]R_0V(t) \\ &= U(t)W_0 \exp[\mathrm{i}\beta f(R_0^2)t]V(t)V(t)^*R_0V(t). \end{aligned}$$

Therefore,

$$W(t) = U(t)W_0 \exp[\mathrm{i}\beta f(R_0^2)t]V(t). \tag{38}$$

It is interesting to observe that if $H = 0$ (while $\hat{H}$ is nontrivial), then $V = I$ is time-independent, and so is R, i.e. only the partial isometry $W(t)$ evolves in time. Solutions of this type were investigated already in [4], albeit in a less general setting in which K was a priori assumed to be a finite-rank operator.

Remark 4 (*Application to Integro-Differential Equations*)**.** Let K be defined via integral kernel function $k \in L^2([0,1] \times [0,1])$, i.e. for $u \in L^2[0,1]$ set

$$Ku(x) = \int k(x,y)u(y)\mathrm{d}y.$$

This is a basic model for Hilbert–Schmidt (H–S) operators. It is easy to see that $K : L^2[0,1] \to L^2[0,1]$ is a bounded operator, whose operator norm equals the L^2-norm of the kernel k [32]. An operator K^* is also a H–S operator with kernel $k^*(x,y) = \overline{k(y,x)}$. For the sake of this example, we assume that H and $\hat{H}$ are also H–S with respective kernels h and $\hat{h}$. Let for simplicity $f(x) = x$, and consider equation

$$-\mathrm{i}\dot{K} = KH + \hat{H}K + \beta KK^*K. \tag{39}$$

Equivalently, the equation may be written as an integro-differential equation for the kernel k, i.e.

$$-\mathrm{i}\dot{k}(x,y) = \int k(x,z)h(z,y)\mathrm{d}z + \int \hat{h}(x,z)k(z,y)\mathrm{d}z + \beta \iint k(x,t)\overline{k(z,t)}k(z,y)\mathrm{d}z\,\mathrm{d}t. \tag{40}$$

Theorem 5.1 may be applied to see that the solution has the triunitary structure (27). In particular, the Theorem indicates a nontrivial conclusion that the solution will remain a H–S operator for all times. Formula (27) may also be applied to the numerical investigation of the discretized problem.

Remark 5. It is natural to ask if the main result will still hold if β is allowed to assume complex (or purely imaginary) values. The answer is negative. Indeed, if β is complex, then the claim in Lemma 5.2 is no longer true. In consequence, formula (27) cannot be expected to hold in that case.

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