



Elastic curves with constant curvature at rest in the hyperbolic plane

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ABSTRACT

We extend the classical variational model for elastic curves that are circular at rest to the hyperbolic plane $\mathbb{H}^2(-1)$. For simplicity, we call them λ -elastica. We show that there are three types of critical curves according to their symmetries: rotational, translational and horocyclical type curves. By explicitly solving the Euler–Lagrange equation and giving a closedness criterion in each case, we can show that there exists a 2-parameter family of closed rotational λ -elastica and that there exists an “eight”-shaped example of closed translational λ -elastica in $\mathbb{H}^2(-1)$. However, we prove that there are no examples of closed λ -elastica of horocyclical type. The second variation formula is applied to study the stability of the constant curvature solution multiple covers. As an application, we combine these results with a Lorentzian version of the Hopf map to construct examples of closed elastic membranes in the anti de Sitter 3-space. A numerical approach is used to gain insight into the space of closed λ -elastica. One plausible consequence of this numerics is that the “eight”-shaped critical curve mentioned before appears to be the only closed translational λ -elastica and that it is also a candidate for a local minimum of elastic energy.

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1. Introduction

Imagine that we bend a thin stiff rod, with circular cross-section in the Euclidean space, which is in the shape of a straight line when no forces act on it. Then the bent rod can be described as a regular curve $\gamma : I \rightarrow \mathbb{R}^3$, whose elastic energy is given by $\int_{\gamma} \kappa^2$, where κ is the curvature of γ . More generally, if the stiff rod has an arbitrary shape of curvature κ_0 in its undeformed state, then the bending energy can be written as

$$\mathcal{F}(\gamma) = \int_{\gamma} (\kappa - \kappa_0)^2. \quad (1)$$

Equilibrium positions of the stiff bent rod can be studied by computing the minimizers (or, more generally, the critical points) of (1) under the boundary conditions determined by the concrete problem that we wish to analyze [1]. If γ is a plane curve and $\kappa_0 = 0$, the above problem corresponds to the classical model for elastic curves proposed by D. Bernoulli around 1740. The case $\kappa_0 = \text{constant} \neq 0$ would correspond to the situation in which the curve is a circle at rest (alternatively, it could be interpreted as a global measure of how far our curve is from being a circle). This geometric approach can be used to model stiff polymers, vortices in fluids, superconductors, membranes and mechanical properties of DNA molecules (for more details, see [2,3] and the references therein).

Of course, assuming that κ stands for the geodesic curvature, the variational problem induced by (1) can be considered in any Riemannian manifold $\mathbb{M}^n$. From now on, we restrict ourselves to the case in which κ_0 is a constant $-\lambda \in \mathbb{R}$

$$\mathcal{L}_{\lambda}(\gamma) = \int_{\gamma} (\kappa + \lambda)^2, \quad (2)$$

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Table 1Values of d for which periodic solutions of (9) exist if $0 < \lambda < 1$.

Roots of Q_d	I $0 < \lambda < \eta < 1$	II $\eta = \lambda$	III $\eta < \lambda < 1$	
$0 < \alpha_3 < \alpha_4$	$-\delta_2 < d \leq -\delta_1$	$-\delta_2 < d < -\delta_1$	$-\delta_2 < d < -\xi$	1
$0 = \alpha_3 < \alpha_4$	–	$d = -\xi = -\delta_1$	$d = -\xi$	2
$\alpha_3 < 0 < \alpha_4$	–	–	$-\xi < d \leq -\delta_1$	3
$\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$	$-\delta_1 < d < -\xi$	–	–	4
$\alpha_1 < \alpha_2 < 0 = \alpha_3 < \alpha_4$	$d = -\xi$	–	–	5
$\alpha_1 < \alpha_2 < \alpha_3 < 0 < \alpha_4$	$-\xi < d < 0$	$-\delta_1 = -\xi < d < 0$	$-\delta_1 < d < 0$	6
$\alpha_1 < 0 < \alpha_4$	$0 < d$	$0 < d$	$0 < d$	7

Table 2Values of d for which periodic solutions of (9) exist if $1 < \lambda$.

Roots of Q_d	IV $\lambda = 1$	V $1 < \lambda < 2$	VI $\lambda = 2$	VII $\lambda > 2$	
$0 < \alpha_3 < \alpha_4$	$-\delta_2 < d < -\xi$	$-\delta_2 < d < -\xi$	$-\delta_2 < d < 0$	$-\delta_2 < d \leq 0$	1
$0 = \alpha_3 < \alpha_4$	$d = -\xi$	$d = -\xi$	$d = 0$	–	2
$\alpha_3 < 0 < \alpha_4$	$-\xi < d < 0$	$-\xi < d \leq 0$	–	–	3
$\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$	–	–	–	$0 < d < -\xi$	4
$\alpha_1 < \alpha_2 < 0 = \alpha_3 < \alpha_4$	–	–	–	$-\xi = d$	5
$\alpha_1 < \alpha_2 < \alpha_3 < 0 < \alpha_4$	–	$0 < d < -\delta_1$	$0 < d < -\delta_1$	$-\xi < d < -\delta_1$	6
$\alpha_1 < 0 < \alpha_4$	$0 < d$	$-\delta_1 < d$	$-\delta_1 < d$	$-\delta_1 < d$	7

and $\mathbb{M}^n$ is a real space form of constant curvature c , $\mathbb{M}^n(c)$. Moreover, we shall focus our attention on closed critical curves due to their special geometric significance. Then, it can be proved that closed critical curves of (2) in real space forms must lie fully in either a 2-dimensional or a 3-dimensional totally geodesic space of $\mathbb{M}^n(c)$ [4,5]. Thus, we may assume from the beginning that γ lies in either $\mathbb{M}^2(c)$ or $\mathbb{M}^3(c)$. If $\lambda = 0$, critical points of (2) correspond to the classical Bernoulli's elastica and will be called *free elastica* following [5]. For simplicity, critical points of (2) when $\lambda \neq 0$ will be referred to as λ -*elastica* in this paper.

Assume first that $\lambda = 0$. Classical Bernoulli's plane elastica possible shapes were discovered by Euler [6]. There are no closed plane free elastica, but if one admits a constraint on the length, then in addition to non-closed examples also circles and Bernoulli's eight-curves appear. Classical plane elastica have also been analyzed under different boundary conditions by Linner [7]. If $\mathbb{M}^2(c)$ is either the 2-sphere $\mathbb{S}^2(1)$ or the hyperbolic plane $\mathbb{H}^2(-1)$, closed free elastica have been studied and classified by Langer and Singer in [5]. There are no closed free elastic curves in $\mathbb{R}^3$ (again, a penalty on the length allows the existence of closed elastica for any torus knot type (m, n) with $m > 2n$ [8,9]) and closed free elastica in $\mathbb{S}^3(1)$ have been classified in [10]. The classification of closed free elastica in $\mathbb{H}^3(-1)$ is still open.

Now, assume that $\lambda \neq 0$. Clearly, since the total curvature $\int \kappa$ is constant on a regular homotopy class of plane curves, no examples of plane λ -elastica other than Euler's elastica appear. On the other hand, there are no non-trivial closed λ -elastica in $\mathbb{R}^3$ with constant curvature [11], but a complete analysis of non-constant curvature λ -elastica in $\mathbb{R}^3$ is still missing. As for the other 2-dimensional space forms, closed λ -elastica in $\mathbb{S}^2(1)$ have been analyzed in [12]. Hence, what remains to do is to study closed λ -elastica in $\mathbb{H}^2(-1)$. This will be our task in this work. We point out that this functional is affected by the orientation of the curve: if $\gamma(s)$ is a critical point of $\mathcal{L}_\lambda$, then $\gamma(-s)$ is a critical point of $\mathcal{L}_{-\lambda}$. Hence, we may assume $\lambda \geq 0$.

In Section 2 we compute the first variation formula and see that restriction to the space of closed curves forces its boundary term to vanish. Then, we can obtain a first integral of the Euler–Lagrange (EL) equation. It is a first order non-linear ODE expressed in terms of the curvature κ of the critical curve γ , whose periodic solutions must be found. Periodic solutions are explicitly determined and classified in terms of two parameters: λ itself and a constant of integration d . It is a complex classification which is summarized in Tables 1 and 2. In Section 3 we use the symmetry of the EL equation along with Langer and Singer's techniques [5], to establish a closedness criterion for critical curves whose curvatures are given by the periodic solutions. It turns out that the sign of d determines the symmetries of the critical curves. Accordingly, we may have three different kinds of critical curves: curves of rotational type, of horocyclical type, or of translational type (see Section 3 for more details). This gives raise to different closedness criteria which we use to prove that there exists a 2-parameter family of closed critical points of rotational type (Proposition 4) and that there exists at least one closed critical point of translational type (Proposition 7). We also prove that there are no closed critical points of horocyclical type (Proposition 5).

On the other hand, an stability analysis would require to obtain the second variation formula of $\mathcal{L}_\lambda(\gamma)$. It has been computed in [4,13], for example, but it is very difficult to handle in general. We will be able to partially sort out this problem by considering critical points of constant curvature. In Section 4 we determined the multiple covers of the closed constant curvature critical curves that are stable. Very little is known about the stability of closed critical curves with non-constant curvature. Numerical considerations made in Section 6 suggest that the “eight”-figure critical point found in Proposition 7 is a local minimum for this energy. Some additional numerical insight into the space of closed solutions is also made in Section 6. This analysis reveals plausible properties which we have been able to prove in some cases, while they remain open in others.

In a different, yet related, context, it was around 1810 that Sophie Germain set a model for the *elastic energy* of a surface S in her study of the elastic plates. By using a generalization of Germain's energy, Canham and Helfrich [14,15], introduced a mathematical model for the study of biological membranes. The *shape energy* is defined by

$$\mathbf{E}(S) = \int_S (b(H - c_0)^2 + cG) \cdot dA + a \int_S dA + d \int_\Omega dV, \quad (3)$$

where, $a, b, c, d \in \mathbb{R}$, c_0 is a constant (the *spontaneous curvature* of S), dA is the area element of S , dV is the volume element of the domain bounded by S and H and G are the mean and Gaussian curvature of the surface, respectively. Critical points of (3) serve as models for *elastic surfaces or membranes* in $\mathbb{R}^3$ (which are called *vesicles* if S is closed). Many authors have studied the existence of vesicles with different topological configuration in $\mathbb{R}^3$, especially by using numerical methods, and confront them with experimental observations (see, for instance, [16–18]).

Of course, it makes full sense to study the variational problem associated to (3) in any Riemannian manifold and it is worth to observe that *minimal surfaces, surfaces of constant mean curvature and Willmore surfaces* [19], are particular examples of membranes for suitable choices of the parameters in (3). Actually, it can be used also to provide very attractive energy actions in string and superstring theories and M -theory (see, [20–22] and the references given there). Many examples of vesicles in $\mathbb{S}^3(1)$ have been constructed using as an essential tool the Hopf map [23,24]. In order to show that the third homotopy group of $\mathbb{S}^2$ is not trivial, H. Hopf introduced in 1931 the so called Hopf map from the three sphere onto the two sphere [25]. This map has been successfully apply to study many different important problems both in Mathematics as well as in Physics (see [26] for a recent survey on the subject). There is a version of the Hopf map in the Lorentzian background. This map (which we also called Hopf map) goes from the three anti de Sitter space onto the hyperbolic plane and can be also viewed as a circle principal bundle (see Section 5 for more details).

If we try to analyze the EL equation induced by (3), we run into a very difficult differential equations which is hopeless to solve. Fortunately, as an application of our results, in Section 5 we are able to construct examples of closed elastic surfaces in the anti de Sitter space $\text{AdS}_3(-1)$ by means of the Hopf map, the Palais' symmetric criticality principle [27], and the closed λ -elastic curves in $\mathbb{H}^2(-1)$ previously obtained.

Finally, some long technical results are banished to the Appendix. Various tedious computations, computer programming and numerical algorithms are not included.

2. Solutions to the Euler–Lagrange equations

We denote by ∇ , $R(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ the Levi–Civita connection, the Riemannian curvature tensor and the standard inner product of $\mathbb{H}^2(-1)$, respectively. Let $\gamma : \mathbb{I} = [0, 1] \rightarrow \mathbb{H}^2(-1)$ be an immersed C^∞ curve. As usual, $\gamma'(t)$ will be the tangent vector field to $\gamma(t)$, and $T(t)$, $N(t)$ the unit tangent and unit normal vector fields respectively. Also, the speed, oriented curvature and arclength parameter of $\gamma(t)$ will be represented by $v = \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}}$, $\kappa(t)$ and s , respectively.

For a fixed real number λ , we consider the functional

$$\mathcal{L}_\lambda(\gamma) := \int_\gamma (\kappa + \lambda)^2 ds = \int_0^1 (\kappa + \lambda)^2 v dt, \quad (4)$$

acting on the following space of curves

$$\Omega = \{\gamma : \mathbb{I} \rightarrow \mathbb{H}^2(-1); \gamma'(t) \neq 0, \gamma(0) = p, \gamma(1) = q, \gamma'(0) = e, \gamma'(1) = \tilde{e}\}, \quad (5)$$

where $p, q \in \mathbb{H}^2(-1)$ are fixed points, and $e \in T_p\mathbb{H}^2(-1)$, $\tilde{e} \in T_q\mathbb{H}^2(-1)$ are fixed tangent vectors. As we have said before, critical points of $\mathcal{L}_\lambda$ will be called λ -*elastic curves*. Our approach to the study of (4) is inspired by Langer and Singer [5].

For a given $\gamma \in \Omega$, we consider a C^∞ variation of γ by curves in Ω , that is a C^∞ function $\Gamma : (-\epsilon, \epsilon) \times [0, 1] \rightarrow \mathbb{H}^2(-1)$ such that $\Gamma(0, t) = \gamma(t)$ and $\Gamma(u, t) = \gamma_u(t) \in \Omega$. Denote by $W = \frac{\partial \Gamma}{\partial u}(0, t)$ the variation field of Γ . Then using standard arguments which involve integration by parts, the Frenet equations of γ and a technical result similar to Lemma 1.1 of [5], one can obtain the first variation formula of this functional (when $\lambda = 0$, this formula was given in [5])

$$\frac{d\mathcal{L}_\lambda}{du}(0) = \int_0^L \langle W, \mathcal{E} \rangle ds + \mathcal{B}(W, \gamma) \Big|_0^L, \quad (6)$$

where L is the length of γ and the Euler–Lagrange and boundary operators are given respectively by

$$\mathcal{E} = \nabla_T \mathcal{J} + R(\mathcal{K}, T)T; \quad \mathcal{B}(W, \gamma) = \langle \nabla_T W, \mathcal{K} \rangle - \langle W, \mathcal{J} \rangle, \quad (7)$$

with $\mathcal{K}$ and $\mathcal{J}$ being vector fields along γ defined by

$$\mathcal{K} = 2(\kappa - \lambda)N; \quad \mathcal{J} = (\kappa^2 - \lambda^2)T + 2\kappa_s N. \quad (8)$$

The boundary operator vanishes identically under our boundary conditions. Moreover, since we are mainly interested in closed critical points, we restrict ourselves to $\tilde{\Omega}$, the space of closed curves in $\mathbb{H}^2(-1)$. Thus we have that $\gamma \in \tilde{\Omega}$ is a critical

point of $\mathcal{L}_\lambda$, if and only if, $\mathcal{E} = 0$. Hence, by using the Frenet equations we see that $\gamma \in \tilde{\mathcal{Q}}$ is a critical point of $\mathcal{L}_\lambda$, if and only if, its curvature $\kappa(s)$ satisfies the following Euler–Lagrange (EL) equation

$$2\kappa_{ss} + (\kappa + \lambda)(\kappa^2 - \lambda\kappa - 2) = 0, \quad (9)$$

where the subscript s denotes derivative with respect to the arclength parameter. Constant solutions of (9) are given by $\kappa = -\lambda$ or $\kappa = (\lambda \pm \sqrt{\lambda^2 + 8})/2$. Curves in $\mathbb{H}^2(-1)$ with constant curvature verifying $0 \leq \|\kappa\| \leq 1$ are geodesics ($\kappa = 0$), or equidistant curves ($0 < \|\kappa\| < 1$), or horocycles ($\kappa = 1$), and they never close up. The only curves of constant curvature which can close up in $\mathbb{H}^2(-1)$ are those with $\|\kappa\| > 1$ (hyperbolic circles). Therefore, the hyperbolic circles of geodesic curvature $\kappa = -\lambda$, if $\lambda > 1$; of curvature $\kappa = (\lambda - \sqrt{\lambda^2 + 8})/2$, if $0 < \lambda < 1$; and of curvature $\kappa = (\lambda + \sqrt{\lambda^2 + 8})/2$, if $\lambda > 0$, will be closed critical curves of $\mathcal{L}_\lambda$.

Assume now that $\kappa(s)$ is a non-constant solution of (9). Since our goal are the closed critical curves of (4), we must investigate first the periodic solutions of (9). It is easy to see that

$$4\kappa_s^2 + (\kappa^2 - \lambda^2)^2 - 4(\kappa + \lambda)^2 = d, \quad (10)$$

with $d \in \mathbb{R}$, is a first integral of (9). Let us use the following notation: for any $d \in \mathbb{R}$, $Q_d(x) = d - (x^2 - \lambda^2)^2 + 4(x + \lambda)^2$; $Q_o(x) = Q_d(x)$ with $d = 0$; $\eta_0 = -\lambda$, $\eta_1 = (\lambda - \sqrt{\lambda^2 + 8})/2$, $\eta_2 = (\lambda + \sqrt{\lambda^2 + 8})/2$; $Q_o(\eta_1) = \delta_1$, and $Q_o(\eta_2) = \delta_2$ (observe that $Q_o(\eta_0) = 0$).

Then, writing the Euler–Lagrange equation (9) as an autonomous system

$$\begin{aligned} z' &= f(z), \\ z &= (\kappa, \kappa_s), \\ f : D = \mathbb{R}^2 &\longrightarrow \mathbb{R}^2, \end{aligned} \quad (11)$$

we see that $a_0 = (-\lambda, 0)$, $a_1 = (\eta_1, 0)$ and $a_2 = (\eta_2, 0)$ are equilibrium points of (11). Also, the equation of the orbits in the phase space can be obtained from (10)

$$4y^2 + (x^2 - \lambda^2)^2 - 4(x + \lambda)^2 = d. \quad (12)$$

Thus periodic solutions of (9) would correspond to closed orbits in D . Because of the integral (10), all solutions of (11) are bounded. Therefore, solutions are of two types: equilibrium points, and 1-dimensional submanifolds. The 1-dimensional submanifolds are, in turn, of two subtypes: embedded circles (periodic solutions) and embedded copies of the real line. By examining (12) one sees that limit points of the latter must be equilibrium points (x, y) where $Q_o''(x) > 0$ (saddle point). Conversely, an equilibrium point (x, y) for which $Q_o''(x) > 0$ must be a limit point of a non-periodic solution. Finally, centers of (11) correspond to equilibrium points for which $Q_o''(x) < 0$ and centers are surrounded by closed orbits (periodic solutions).

One can see that if $0 < \lambda < 1$, then $Q_o''(\eta_i) < 0$, $i \in \{1, 2\}$, and $Q_o(x)$ reaches local maxima δ_i at η_i , $i \in \{1, 2\}$. It reaches also a local minimum 0 at η_0 . Hence, a_1 and a_2 are centers and a_0 is a saddle point in this case. If $\lambda > 1$, then $Q_o''(\eta_1) > 0$ and $Q_o''(\eta_2) < 0$, $i \in \{0, 2\}$, thus $Q_o(x)$ reaches local maxima 0 and δ_2 at η_0 and η_2 , respectively, and a local minimum δ_1 at η_1 . Now a_0 and a_2 are centers and a_1 is a saddle point. Finally, if $\lambda = 1$, then $Q_o(x)$ has a local maximum δ_2 at η_2 and a saddle point at $\eta_1 = \eta_0 = -1$.

Closed orbits of these cases can be studied by direct analysis of their phase planes. Thus, for a given λ , the phase plane of (11) can be drawn once we know the roots of $Q_d(x)$, which obviously depend on d . After a long but straightforward computation, one can check that all possible values of λ and d for which we can find non-constant periodic solutions of (9) are summarized in Tables 1 and 2, where we are denoting $\xi(\lambda) = 4\lambda^2 - \lambda^4$ and $\eta = \sqrt{3\sqrt{3} - 5}$.

All the closed orbits of (12) are obtained for the values of $d \in \mathbb{R}$ given in these tables and they correspond to the periodic solutions of (9). Moreover, one can derive more specific information from them. For instance, we look at the first column of Table 1. We say that a curve $\gamma(s)$ is *convex* if $\kappa(s)$ does not change its sign and it is called *strictly convex* if $\kappa(s) \neq 0$, $\forall s$. Otherwise, we say that $\gamma(s)$ is *non-convex*. Then

Case I. $0 < \lambda < \eta < 1$. According to Table 1, we have the following possibilities:

Case I.1. $d \in (-\delta_2, -\delta_1]$.

For any $d \in (-\delta_2, -\delta_1]$, the polynomial $Q_d(x)$ has two simple positive roots $0 < \alpha_3^d < \alpha_4^d$ and the corresponding periodic solution $\kappa_d(s)$ varies between its minimum $\kappa_d(s_1) = \alpha_1^d$ and its maximum $\kappa_d(s_2) = \alpha_2^d$. Also $\kappa_d(s)$ is symmetric with respect to $s = s_2$ and the critical curve γ_d associated to κ_d is strictly convex.

Cases I.2. and I.3. These cases are not possible.

Case I.4. $d \in (-\delta_1, -\xi)$.

If $d \in (-\delta_1, -\xi)$, then $Q_d(x)$ has four simple roots satisfying $\alpha_1^d < \alpha_2^d < -\lambda < 0 < \alpha_3^d < \alpha_4^d$ and there are two periodic solutions $\hat{\kappa}_d(s)$ and $\tilde{\kappa}_d(s)$ whose corresponding critical curves are denoted by $\hat{\gamma}_d(s)$, $\tilde{\gamma}_d(s)$. Now $\hat{\kappa}_d(s)$ varies between its minimum $\hat{\kappa}_d(s_1) = \alpha_1^d$ and its maximum $\hat{\kappa}_d(s_2) = \alpha_2^d$ and is symmetric with respect to $s = s_2$. Similarly, $\tilde{\kappa}_d(s)$ varies between its minimum $\tilde{\kappa}_d(s_3) = \alpha_3^d$ and its maximum $\tilde{\kappa}_d(s_4) = \alpha_4^d$. Also $\tilde{\kappa}_d(s)$ is symmetric with respect to $s = s_4$. Both critical curves $\hat{\gamma}_d$ and $\tilde{\gamma}_d$ are strictly convex.

Case I.5. $d = -\xi$.

This is similar to the above case, but now $\tilde{\kappa}_d(s)$ varies between its minimum $\tilde{\kappa}_d(s_3) = 0$ and its maximum $\tilde{\kappa}_d(s_4) = \alpha_4^d$. Thus, $\hat{\gamma}_d$ is strictly convex and $\tilde{\gamma}_d$ is convex but not strictly convex.

Case I.6. $d \in (-\xi, 0)$.

In this case $Q_d(x)$ has four simple roots satisfying $\alpha_1^d < \alpha_2^d < -\lambda < \alpha_3^d < 0 < \alpha_4^d$ and there are two periodic solutions $\hat{\kappa}_d(s)$ and $\tilde{\kappa}_d(s)$ with corresponding critical curves $\hat{\gamma}_d(s)$, $\tilde{\gamma}_d(s)$. Again, $\hat{\kappa}_d(s)$ varies between its minimum $\hat{\kappa}_d(s_1) = \alpha_1^d$ and its maximum $\hat{\kappa}_d(s_2) = \alpha_2^d$. Also $\hat{\kappa}_d(s)$ is symmetric with respect to $s = s_2$ and the critical curve $\hat{\gamma}_d$ associated to $\hat{\kappa}_d$ is convex. Analogously, $\tilde{\kappa}_d(s)$ varies between its minimum $\tilde{\kappa}_d(s_3) = \alpha_3^d$ and its maximum $\tilde{\kappa}_d(s_4) = \alpha_4^d$, it is symmetric with respect to $s = s_4$ and the associated critical curve $\tilde{\gamma}_d$ is non-convex.

Case I.7. $d > 0$

If $d > 0$, then $Q_d(x)$ has two simple roots with different sign $\alpha_1^d < 0 < \alpha_4^d$. The discussion is similar to the above cases, but this time the curvature changes its sign and the corresponding curve is non-convex

The remaining cases II to VII can be discussed analogously.

Actually, we can solve (10) explicitly with the aid of formulas I.3.145 and I.3.147 of [28] after some long and tedious computations that we omit here. Assume first that we are working with those values of λ and d of the above tables which give rise to four roots $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ of $Q_d(x)$. Then, we obtain two solutions of (10) which we denote simply by $\kappa(s)$ and $\tilde{\kappa}(s)$ regardless of the values of λ and d . The first one is given by

$$\kappa(s) = \frac{\alpha_2(\alpha_4 - \alpha_1) - \alpha_4(\alpha_2 - \alpha_1)cn^2(rs, M)}{(\alpha_4 - \alpha_1) - (\alpha_2 - \alpha_1)cn^2(rs, M)}, \quad (13)$$

where $r = \sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}/4$, $M = \sqrt{\frac{(\alpha_4 - \alpha_3)(\alpha_2 - \alpha_1)}{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}}$ and $cn(rs, M)$ is the Jacobi Elliptic cosine. It oscillates between $\alpha_1 = \kappa(0)$ and $\kappa(K(M)/r) = \alpha_2$, $K(M)$ being the complete elliptic integral of the first kind. The second solution $\tilde{\kappa}(s)$ is obtained by interchanging $1 \rightarrow 3, 2 \rightarrow 4$ in the right term of (13) and it oscillates between $\alpha_3 = \tilde{\kappa}(0)$ and $\tilde{\kappa}(K(M)/r) = \alpha_4$.

Analogously, the solution in those cases in which the values of the parameters λ and d produce two roots $\alpha_3 < \alpha_4$ can be written as

$$\kappa(s) = \frac{(q\alpha_4 + p\alpha_3) - (q\alpha_4 - p\alpha_3)cn(rs, M)}{(p + q) + (p - q)cn(rs, M)}, \quad (14)$$

with

$$p^2 = (\alpha_4 + \alpha_3)^2 + 2\alpha_4^2 - 2\lambda^2 + 4; \quad M = \frac{1}{2}\sqrt{\frac{(\alpha_4 - \alpha_3)^2 - (p - q)^2}{pq}}, \quad (15)$$

$$q^2 = (\alpha_4 + \alpha_3)^2 + 2\alpha_3^2 - 2\lambda^2 + 4; \quad r = \frac{\sqrt{pq}}{2}. \quad (16)$$

Now $\kappa(s)$ oscillates between $\alpha_3 = \kappa(0)$ and $\kappa(2K(M)/r) = \alpha_4$.

3. Closed critical curves

So far, we have determined in (13) and (14) all possible periodic solutions to (10). They are the candidates for the curvature of our critical curves $\gamma(s)$. Given a periodic function defined by (13) or (14), there exists a unique (up to isometries) curve in the hyperbolic plane whose curvature is given by such a function, but this curve is not closed necessarily. So we need to establish conditions under which a curve $\gamma(s)$ whose curvature is given by a periodic solution of (9) closes up.

Following the notation in the beginning of Section 2, a vector field X along a curve γ is called a *Killing field along γ* if $X(v) = X(\kappa) = 0$, where $v = \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}}$, [5]. Killing vector fields along γ are the solutions, X , of the following linear system (Lemma 1.1 of [5])

$$X(v) = \langle \nabla_T X, T \rangle = 0, \quad (17)$$

$$X(\kappa) = \langle \nabla_T^2 X - X, N \rangle = 0. \quad (18)$$

Obviously, the restriction to γ of any Killing field of $\mathbb{H}^2(-1)$ gives a Killing vector field along γ . The converse of this also holds, [5]. Consequently, we have : $W \in T_\gamma \Omega$ is a Killing vector field along γ if and only if it extends to a Killing field (that will be also denoted by W) on $\mathbb{H}^2(-1)$.

Remark 1. Assume that X is a Killing field of $\mathbb{H}^2(-1)$. Then, all integral curves of X are curves of constant geodesic curvature. If X admits a geodesic α as an integral curve, and $c(t)$ is another integral curve of the same Killing field X through a point p , then $d(c(t), \alpha) = d(p, \alpha)$, for all t and, therefore, $c(t)$ is an equidistant curve with respect to α . Conversely, if a Killing

field X admits an equidistant integral curve to some geodesic α , then α is also an integral curve of X . Hence, the one parameter group of isometries determined by X corresponds to the group of translations along the geodesic α . Similarly, if X admits a hyperbolic circle as integral curve, then all other integral curves of X must be circles with the same center and the associated one parameter group in this case consists of rotations around the common center which is a zero of the Killing field. Conversely, if a Killing field has a zero, its integral curves are circles with a common center. Finally, if X admits a horocycle as integral curve, then the rest of its integral curves must be horocycles with the same point at infinity as well, and the one parameter group associated to X consists of parallel displacements.

Hence, the following definitions make sense, [5]: a Killing field X on the $\mathbb{H}^2(-1)$ is called *translational* if it has an integral geodesic. If it has a zero, then X is called *rotational*, while *horocyclical* means that X admits an integral curve being a horocycle. Analogously, a Killing field on a curve γ is said to be translational (respectively, rotational or horocyclical) if its extension to $\mathbb{H}^2(-1)$ is translational Killing field (respectively, rotational or horocyclical).

For any $\lambda > 0$ and $d \in \mathbb{R}$ satisfying the conditions of Tables 1 and 2, one gets one or two periodic solutions of (9) κ_d^λ and $\tilde{\kappa}_d^\lambda$. In order to ease the notation, any periodic solution is denoted simply by $\kappa(s)$ and we shall use $\gamma(s)$ for the corresponding curve in $\mathbb{H}^2(-1)$ (unique up to isometries) determined by $\kappa(s)$. Let us define a vector field $\mathcal{J}$ on $\gamma(s)$ by (8). Then, we have

Proposition 2. Assume that $\lambda > 0$ and let us consider the energy $\mathcal{L}_\lambda$ acting on Ω . For any $d \in \mathbb{R}$ satisfying the conditions of Tables 1 and 2, let $\kappa(s)$ be any periodic solution to (9) corresponding to d , and let $\gamma(s)$ denote the critical curve of $\mathcal{L}_\lambda$ whose curvature is given by $\kappa(s)$. Then, the vector field on $\gamma(s)$ given by $\mathcal{J} = (\kappa^2 - \lambda^2)T + 2\kappa_s N$ is: (i) a rotational Killing field along $\gamma(s)$, if $d < 0$; (ii) a horocyclical Killing field along $\gamma(s)$, if $d = 0$; and (iii) a translational Killing field along $\gamma(s)$, if $d > 0$.

Proof. Since $\gamma(s)$ is a solution of the Euler–Lagrange equation for $\mathcal{L}_\lambda$, its curvature function satisfies (9) and so $\nabla_T \mathcal{J} = 2(\kappa + \lambda)N$ and $\nabla_T^2 \mathcal{J} = -2\kappa(\kappa + \lambda)T + 2\kappa_s N$. Consequently, by using (9), (17) and (18), we see that $\mathcal{J}$ is a Killing vector field along γ . As we have mentioned before, it can be extended to a Killing field on the whole hyperbolic plane, which we also denote by $\mathcal{J}$.

Now, assume that $p_o = \gamma(s_o)$ and that s_o is a local maximum of $\kappa(s)$. Then $\kappa_s(s_o) = 0$ and $\mathcal{J}$ is tangent to $\gamma(s)$ at p_o . If the curvature at s_o satisfies $\kappa(s_o)^2 = \lambda^2$, then (10) gives that $d \leq 0$. Moreover, $d = 0$ can only happen if $\kappa(s_o) = -\lambda$. But in such a case, by analyzing $Q_d(x)$ and the phase plane of (11), one would have that either $\lambda \leq 1$, in which case the non-constant corresponding solutions of (9) would be non-periodic, contradicting our assumption on the periodicity of $\kappa(s)$, or $\lambda > 1$, and the non-constant corresponding periodic solutions of (9) would satisfy $-\lambda < \lambda - 2 \leq \kappa(s) \leq \lambda + 2$, which contradicts $\kappa(s_o) = -\lambda$. Thus, the case $d = 0$ is not possible at a local maximum of $\kappa(s)$, s_o , where $\kappa(s_o)^2 = \lambda^2$, and we must have $d < 0$. But then, $\mathcal{J}(s_o) = 0$, and having the Killing field $\mathcal{J}$ a zero we conclude that it must be of rotational type as pointed out in Remark 1.

Now, assume that s_o is a local maximum of $\kappa(s)$ where $\kappa(s_o)^2 \neq \lambda^2$. Let β be the integral curve of $\mathcal{J}$ passing through $\gamma(s_o) = p_o$ at time t_o . According to Remark 1, β has constant curvature, say κ_β . Denote by $\bar{T}$ the unit tangent vector to β . Using (8) we see that $d\|\mathcal{J}\|^2/ds = 8\kappa_s\kappa_{ss} + 4(\kappa^2 - \lambda^2)\kappa\kappa_s$, which means that $d\|\mathcal{J}\|/ds = d\|\mathcal{J}\|^{-1}/ds = 0$ at s_o . Combining this and the Euler–Lagrange equation (9) we get

$$(\nabla_{\bar{T}} \bar{T})(p_o) = \left(\frac{D}{dt} \frac{\mathcal{J}}{\|\mathcal{J}\|} \right)(t_o) = \frac{1}{\|\mathcal{J}\|} \left(\frac{D}{ds} \mathcal{J} \right)(s_o) = \frac{2}{(\kappa(s_o) - \lambda)} N(s_o), \quad (19)$$

and, therefore, the curvature of β is $\kappa_\beta = 2/(\kappa(s_o) - \lambda)$. Moreover, for any $\lambda > 0$ we have that: (i) if $d < 0$, then the roots of $Q_d(x)$ lie in the interval $(\lambda - 2, \lambda + 2)$, which implies in turn $\lambda - 2 < \kappa(s) < \lambda + 2$. Consequently, $\|\kappa_\beta\| > 1$; (ii) if $d = 0$, the roots of $Q_d(x)$ are $-\lambda$, $\lambda - 2$ and $\lambda + 2$. In this case, while $-\lambda$ corresponds to a constant solution $\kappa(s)$, the non-constant periodic solutions satisfy $\lambda - 2 \leq \kappa(s) \leq \lambda + 2$, which means that $\|\kappa_\beta\| = 1$; (iii) Finally, if $d > 0$, the roots of $Q_d(x)$ are outside of the interval $[\lambda - 2, \lambda + 2]$ and then, extremals of $\kappa(s)$ are outside of $[\lambda - 2, \lambda + 2]$ as well. Thus, now we have $\|\kappa_\beta\| > 1$. Yet again, in all three cases the conclusion is drawn from Remark 1. $\square$

According to the three different cases of the above statement, we distinguish the following possibilities

3.1. Rotational case: $d < 0$

We take coordinates $x(u, \theta) = (\sinh u \cos \theta, \sinh u \sin \theta, \cosh u)$ in the hyperboloid's upper sheet model of $\mathbb{H}^2(-1)$, with metric coefficients $g_{11} = 1$, $g_{12} = 0$, $g_{22} = \sinh^2 u$. Let $\gamma(s) = x(u(s), \theta(s))$ be a critical point of the energy $\mathcal{L}_\lambda$ (under arbitrary boundary conditions) and choose $\mathcal{J}$ so that $b\mathcal{J} = x_\theta$. Denote by $T(s)$ the unit tangent to $\gamma(s)$ and take the unit normal $N(s)$, so that $\{T(s), N(s)\}$ is positively oriented. Imagine that $\kappa(s)$, the curvature of $\gamma(s)$, reaches a local maximum at s_o and let β be the integral curve of $\mathcal{J}$ through $\gamma(s_o)$. From (19) we have $\kappa_\beta = 2/(\kappa(s_o) - \lambda)$. On the other hand, it can be seen that $\|x_\theta\| = 1/(\kappa_\beta^2 - 1)$. Therefore, we can combine these two last equations with $b\mathcal{J} = x_\theta$ and (10) to conclude that $b^2 d = -1$. Moreover, using the definition of J given in Proposition 2 and since $\|T(s)\| = 1$, we get

$$(u')^2 + (\theta')^2 \sinh^2 u = 1, \quad (20)$$

$$\theta' \sinh^2 u = \langle x_\theta, T \rangle = \langle b\mathcal{J}, T \rangle = b(\kappa^2 - \lambda^2), \quad (21)$$

$$u' \sinh u = \langle x_\theta, N \rangle = \langle b\mathcal{J}, N \rangle = 2b\kappa_s. \quad (22)$$

From (22) we have $\cosh u = 2b(\kappa + \lambda) + \mu$, but using again (10), $b\mathcal{J} = x_\theta$ and $bd^2 = -1$ one gets $\mu = 0$, hence

$$\cosh u = 2b(\kappa + \lambda), \quad (23)$$

which combined with (21) gives

$$\theta'(s) = \sqrt{-d} \frac{\kappa^2 - \lambda^2}{4(\kappa + \lambda)^2 + d}. \quad (24)$$

Finally, the parametrization of γ in this coordinate system of $\mathbb{H}^2(-1)$ will be

$$\gamma(s) = \sqrt{\frac{4(\kappa + \lambda)^2 + d}{-d}} \left(\cos \left(\sqrt{-d} \int \frac{\kappa^2 - \lambda^2}{4(\kappa + \lambda)^2 + d} \right), \sin \left(\sqrt{-d} \int \frac{\kappa^2 - \lambda^2}{4(\kappa + \lambda)^2 + d} \right), \frac{2(\kappa + \lambda)}{\sqrt{4(\kappa + \lambda)^2 + d}} \right). \quad (25)$$

Now, for any d satisfying $-\delta_2 < d < 0$, let κ be the corresponding periodic solution of (9) with period h_d . Let γ_d denote the curve in $\mathbb{H}^2(-1)$ whose curvature is given by κ and define the *angular change* of γ_d in one period of the curvature by

$$\Lambda_d := \sqrt{-d} \int_0^{h_d} \frac{(\kappa^2 - \lambda^2)}{d + 4(\kappa + \lambda)^2} ds. \quad (26)$$

Hence, from (25) and (26) we have

Proposition 3. Fix a real value $\lambda > 0$. Then for any d satisfying $-\delta_2 < d < 0$, the curve γ_d is a closed λ -elastic curve, if and only if, its angular change in one period of the curvature Λ_d is a rational multiple of π .

Thus, in order to prove the existence of closed critical curves we must analyze the range of variation of Λ_d as d moves in $(-\delta_2, 0)$. For a given λ , the polynomial $Q_d(\kappa)$ with $d \in (-\delta_2, 0)$ may have either two real roots $\alpha_1 \leq \alpha_4$ or four real roots $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$. In the latter case, we only will consider the solution $\kappa(s)$ which moves between α_3 and α_4 and define $\Lambda^\lambda : (-\delta_2, 0) \rightarrow \mathbb{R}$ by $\Lambda^\lambda(d) = \Lambda_d$, where Λ_d is given in (26). Then using (10), $\Lambda^\lambda(d)$ turns out to be

$$\Lambda^\lambda(d) = \int_{\alpha_3}^{\alpha_4} \sqrt{-d} \frac{(\kappa^2 - \lambda^2)}{(d/4 + (\kappa + \lambda)^2) \sqrt{Q_d(\kappa)}} d\kappa. \quad (27)$$

We see that $\Lambda^\lambda(d)$ moves continuously as d moves in $(-\delta_2, 0)$ with two exceptions: those values in which d gives rise to multiple roots of $Q_d(\kappa)$; and when $d = -16\lambda^2$. Multiple roots of $Q_d(\kappa)$ affect the integrability of (27) but we have seen that these values of d do not give rise to periodic solutions of (10). Anyway, $Q_d(\kappa)$ does not have multiple roots when $\lambda > 0$ and $d \in (-\delta_2, 0)$. On the other hand, if $d = -16\lambda^2$ then we have that $\alpha_3 = \lambda$ is a root of $Q_d(\kappa)$. Hence $\kappa(K(M)/r) = \alpha_3 = \lambda$ and $\kappa_s(K(M)/r) = 0$. Thus $\mathcal{J}$ given by (8) vanishes at $s_3 = K(M)/r$, so that when the arclength parameter takes the value $s = s_3$, the corresponding curve $\gamma(s)$ will pass through the hyperboloid model's lowest point. Therefore $d = -16\lambda^2$ causes a gap of length 2π in $\Lambda^\lambda(d)$.

As before, for a given λ we denote $\eta_1 = (\lambda - \sqrt{\lambda^2 + 8})/2$, $\eta_2 = (\lambda + \sqrt{\lambda^2 + 8})/2$ and $\psi = -16\lambda^2$. We also define $\Lambda_1 = \Lambda^\lambda(\psi)$. Then, we will prove in Appendix that

Remember that if $d \in (-\delta_1, 0)$ we have two periodic solutions of (10) and that, in this case, we are only considering the one satisfying $\alpha_3 \leq \kappa \leq \alpha_4$ in the definition of $\Lambda^\lambda(d)$. For these values of d , the angular change corresponding to the solution verifying $\alpha_1 \leq \kappa \leq \alpha_2$ satisfies $\lim_{d \rightarrow -\delta_1^+} \Lambda^\lambda(d) = -2\pi\eta_1/\sqrt{\eta_1^2 + 2}$. Hence, we have

Proposition 4. For any $\lambda > 0$, $d \in (-\delta_2, 0)$, the angular change $\Lambda^\lambda(d)$ moves continuously in $(-\delta_2, \psi) \cup (\psi, 0)$ and, therefore, there exist infinitely many closed critical curves of $\mathcal{L}_\lambda$ in the hyperbolic plane of rotational type. More concretely, for any $m, n \in \mathbb{Z}$ satisfying

1. $\frac{2\pi m}{n} \in (-\pi, \Lambda_1 - \pi) \cup (\pi, \frac{-2\pi\eta_1}{\sqrt{\eta_1^2 + 2}}) \cup (\Lambda_1 + \pi, \frac{2\pi\eta_2}{\sqrt{\eta_2^2 + 2}})$, if $0 < \lambda < 1$; or
2. $\frac{2\pi m}{n} \in (\Lambda_1 - \pi, 0) \cup (\Lambda_1 + \pi, \frac{2\pi\eta_2}{\sqrt{\eta_2^2 + 2}})$, if $1 < \lambda$,

there exists a closed critical point $\gamma_{m,n}$ of $\mathcal{L}_\lambda$ in the hyperbolic plane of rotational type, which closes up after n periods of its curvature and m trips around the center of the disk model for the hyperbolic plane.

The shape of rotational type closed critical points can be deduced from the above considerations. For instance, fix $0 < \lambda < 1$ and choose $d \in (-\delta_1, 0)$. For this value of d , consider $\kappa(s)$ the periodic solution of (10) satisfying $\alpha_1 < \kappa(s) < \alpha_2$. Assume that condition 1 of Proposition 4 is satisfied and therefore that $\kappa(s)$ gives rise to a closed critical curve $\gamma(s)$. Take a point s_0 such that $\kappa_s(s_0) = 0$. From (8) and (10) we have $(d\|\mathcal{J}\|^2/ds)(s_0) = 0$ and $(d^2\|\mathcal{J}\|^2/ds^2)(s_0) = 8(\kappa + \lambda)\kappa_{ss}(s_0)$, but from (9) we have $\kappa_{ss}(s_0) = -8(\kappa + \lambda)(\kappa - (\lambda - \sqrt{\lambda^2 + 8})/2)(\kappa - (\lambda + \sqrt{\lambda^2 + 8})/2)(s_0)$. Since $\kappa_s(s_0) = 0$, we must have either $\kappa(s_0) = \alpha_1$ or $\kappa(s_0) = \alpha_2$. Imagine that $\kappa(s_0) = \alpha_1$, that is, $\kappa(s)$ reaches a minimum at s_0 , then $\gamma(s)$ is tangent to the circle Σ_1 of curvature $2/(\alpha_1 - \lambda)$ as we said before. Moreover, since $\alpha_1 < (\lambda - \sqrt{\lambda^2 + 8})/2$ in this case we have $\kappa_{ss}(s_0) > 0$ and $(d^2\|\mathcal{J}\|^2/ds^2)(s_0) < 0$, i.e. $\gamma(s)$ reaches the farthest points from the hyperboloid's bottom at the minima (maxima, in absolute value) of its curvature. Doing a similar discussion if $\kappa_s(s_0) = 0$ and $\kappa(s_0) = \alpha_2$, we see that $\gamma(s)$ oscillates between

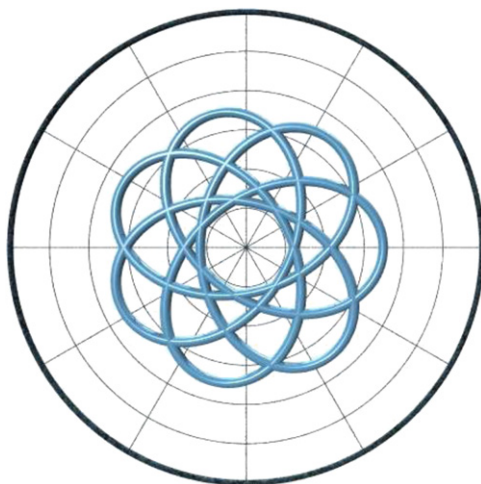


Fig. 1. Rotational type convex critical point for $\lambda = 0.2$.

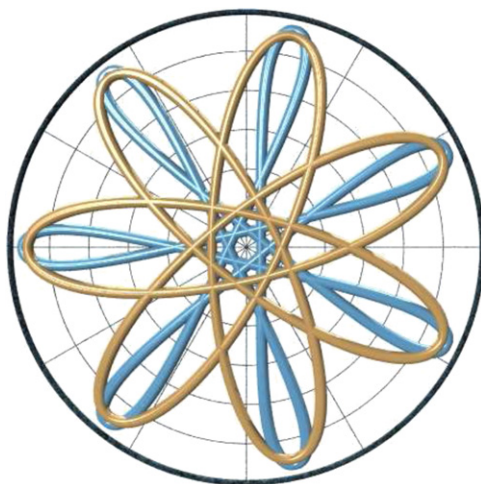


Fig. 2. Two rotational type solutions for the same value of d when $\lambda = 0.2$.

two circles Σ_1 and Σ_2 of curvatures $2/(\alpha_1 - \lambda)$ and $2/(\alpha_2 - \lambda)$ respectively, touching them tangentially at the points in which the curvature of $\gamma(s)$ reaches its minima and maxima respectively. Analogous arguments can be made for the other admissible values of λ , d (see Figs. 1–6).

3.2. Horocyclical case: $d = 0$

This time we proceed to selecting coordinates

$$x(u, w) = \left(\frac{2ue^{2w}}{u^2e^{2w} + 4}, \frac{4e^w}{u^2e^{2w} + 4} \right),$$

in the upper semiplane model of $\mathbb{H}^2(-1)$, with metric coefficients given by $g_{11} = e^{2w}/4$, $g_{12} = 0$, $g_{22} = 1$. Let $\gamma(s) = x(u(s), w(s))$ be a critical point of the energy $\mathcal{L}_\lambda$ (under arbitrary boundary conditions) and choose $\mathcal{J}$ so that $b\mathcal{J} = x_w$. Imagine that $\kappa(s)$ reaches a local maximum at s_o and let $\beta = x(u, w_o)$ be the integral horocycle of $\mathcal{J}$ through $\gamma(s_o)$. Then, combining $\kappa(s_o) = \lambda + 2$, g_{11} and the definition of $\mathcal{J}$, one has $b = \pm \frac{e^{2w_o}}{8(\lambda+1)}$.

Now, consider $T(s)$ the unit tangent to $\gamma(s)$ and take the unit normal $N(s)$, so that $\{T(s), N(s)\}$ is positively oriented. Then $N(s) = -v'(s)\frac{2}{e^w}x_u + u'(s)\frac{e^w}{2}x_w$ and

$$\begin{aligned} u' \frac{e^{2w}}{4} &= \langle x_u, T \rangle = \langle b\mathcal{J}, T \rangle = b(\kappa^2 - \lambda^2), \\ -w' \frac{e^w}{2} &= \langle x_u, N \rangle = \langle b\mathcal{J}, N \rangle = 2b\kappa_s. \end{aligned}$$

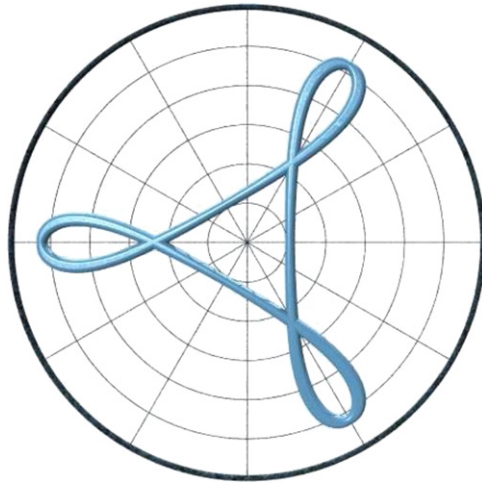


Fig. 3. Rotational type non-convex critical point for $\lambda = 0.6$.

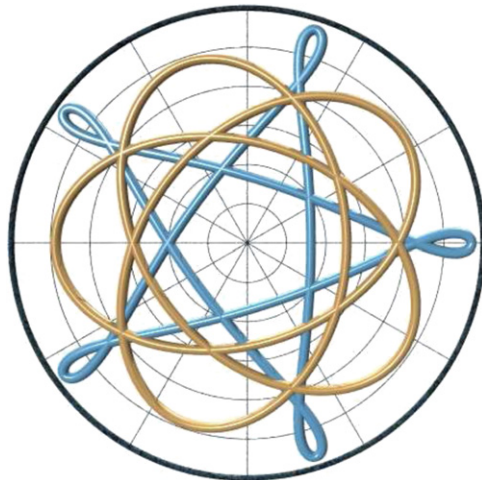


Fig. 4. Two rotational type solutions for the same value of d when $\lambda = 0.6$.

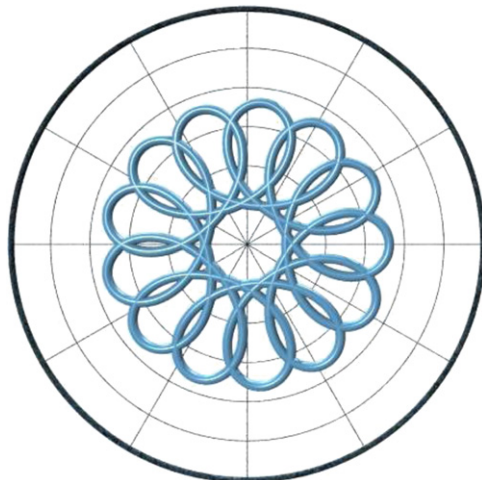


Fig. 5. Rotational type convex solution for $\lambda = 1.5$.

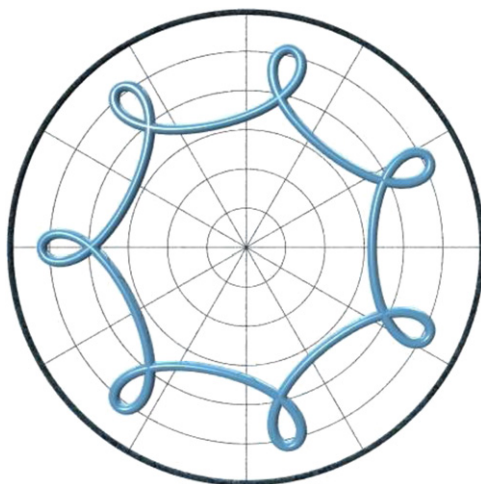


Fig. 6. Rotational type non-convex solution for $\lambda = 1.5$.

Table 3

Behavior of the angular change for $\lambda > 1$.

	$\lim \Lambda^\lambda(d)$
$d \rightarrow -\delta_2^+$	$\frac{2\pi\eta_2}{\sqrt{\eta_2^2+2}}$
$d \rightarrow \psi^\pm$	$\Lambda_1 \mp \pi$
$d \rightarrow 0^\pm$	0
$d \rightarrow -\delta_1$	$-\infty$
$d \gg 0$	$O(\frac{1}{\sqrt[3]{d}}) > 0$

From the second equation we get $\frac{e^w}{2} = -2b(\kappa + \lambda) + \eta$. We may assume $b < 0$, then the above equation and $b = -\frac{e^{2w_0}}{8(\lambda+1)}$ imply $\eta = 0$. Hence $u'(s) = \frac{\kappa - \lambda}{4b(\kappa + \lambda)}$.

Therefore, if κ is a periodic solution of (9) with period h_d for $d = 0$, then the corresponding critical point $\gamma(s)$ closes up, if and only if,

$$\Lambda_d := \int_0^{h_d} \frac{\kappa - \lambda}{4b(\kappa + \lambda)} ds, \quad (28)$$

vanishes. According to Table 3 one can expect to find periodic solutions $\kappa(s)$ of (10) only if $1 < \lambda$. In this case, $\kappa(s)$ moves between its minimum $\lambda - 2$ and its maximum $\lambda + 2$. But $Q_d(\kappa) = \frac{1}{4}(\kappa + \lambda)^2(4 - (\lambda - \kappa)^2)$ and using (10) and formula 2.269 of [28] we have

$$\begin{aligned} \Lambda_d(\lambda) &= 2 \int_{\lambda-2}^{\lambda+2} \frac{du}{d\kappa} d\kappa \\ &= -\frac{1}{b} \int_{2\lambda-2}^{2\lambda+2} \frac{(\kappa - 2\lambda)d\kappa}{\kappa^2 \sqrt{4 - (\kappa - 2\lambda)^2}} = \frac{1}{b(\lambda^2 - 1)} \int_{2\lambda-2}^{2\lambda+2} \frac{d\kappa}{\kappa \sqrt{4 - (\kappa - 2\lambda)^2}} \\ &= \frac{1}{b(\lambda^2 - 1)} \left[\frac{1}{\sqrt{4(1 - \lambda^2)}} \arcsin \left(\frac{8(1 - \lambda^2) + 4\lambda\kappa}{\kappa \sqrt{16}} \right) \right]_{2\lambda-2}^{2\lambda+2} \\ &= (\lambda^2 - 1)^{-\frac{3}{2}} \frac{\pi}{2b}. \end{aligned}$$

Therefore

Proposition 5. *There exist no closed critical curves of $\mathcal{L}_\lambda$ in the hyperbolic plane of horocyclical type for any $\lambda > 0$.*

Fig. 7 shows a non-closed critical point of horocyclical type.

3.3. Translational case: $d > 0$

We take coordinates $x(u, \theta) = (e^u \cos \theta, e^u \sin \theta)$ in the upper semiplane model of $\mathbb{H}^2(-1)$. The metric coefficients now are $g_{11} = g_{22} = 1/\sin^2 \theta$, $g_{12} = 0$. Let $\gamma(s) = x(u(s), \theta(s))$ be a critical point of the energy $\mathcal{L}_\lambda$ (under arbitrary boundary

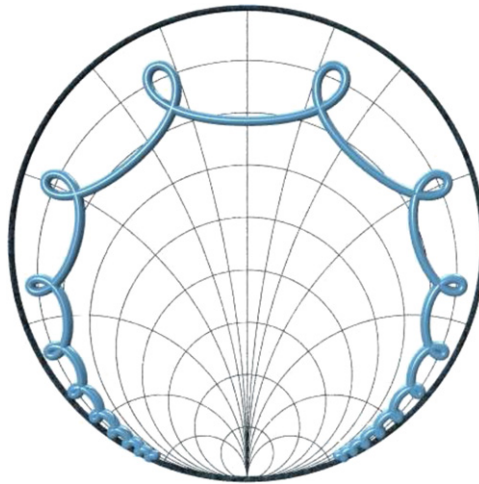


Fig. 7. Non-closed horocyclical type solution, $d = 0$.

conditions) and choose $\mathcal{J}$ so that $b\mathcal{J} = x_u$. Denote by $T(s)$ the unit tangent to $\gamma(s)$ and take the unit normal $N(s)$, so that $\{T(s), N(s)\}$ is positively oriented. Imagine that $\kappa(s)$ reaches a local maximum at s_0 and let β be the integral curve of $\mathcal{J}$ through $\gamma(s_0)$. From (19) we have $\kappa_\beta = 2/(\kappa(s_0) - \lambda)$. On the other hand, it can be seen that $\|x_u\| = 1/(1 - \kappa_\beta^2)$. Therefore, we can combine these two last equations with $b\mathcal{J} = x_u$ and (10) to conclude that $b^2d = 1$. Moreover, since γ is arclength parameterized $\|T(s)\| = 1$, we get

$$(u')^2 + (\theta')^2 = \sin^2 \theta, \quad (29)$$

$$u' \frac{1}{\sin^2 \theta} = \langle x_u, T \rangle = \langle b\mathcal{J}, T \rangle = b(\kappa^2 - \lambda^2), \quad (30)$$

$$-\theta' \frac{1}{\sin^2 \theta} = \langle x_u, N \rangle = \langle b\mathcal{J}, N \rangle = 2b\kappa_s. \quad (31)$$

From (31) we have $\cosh u = 2b(\kappa + \lambda) + \mu$, but using again (10) $b\mathcal{J} = x_\theta$ and $bd^2 = 1$ one gets $\mu = 0$, hence

$$\cot \theta = 2b(\kappa + \lambda), \quad (32)$$

which combined with (30) gives

$$u'(s) = \sqrt{d} \frac{\kappa^2 - \lambda^2}{4(\kappa + \lambda)^2 + d}. \quad (33)$$

Hence, in this parametrization a critical point $\gamma(s)$ can be written as

$$\gamma(s) = e^{\sqrt{d} \int \frac{\kappa^2 - \lambda^2}{4(\kappa + \lambda)^2 + d}} \left(\frac{2(\kappa + \lambda)}{\sqrt{4(\kappa + \lambda)^2 + d}}, \sqrt{\frac{d}{4(\kappa + \lambda)^2 + d}} \right). \quad (34)$$

Thus, if κ is a periodic solution of (9) with period h_d for $d > 0$, and we denote by $\gamma(s)$ the corresponding critical curve, we have

Proposition 6. Fix a real value $\lambda > 0$. Then for any d satisfying $0 < d$, the curve γ is a closed λ -elastic curve, if and only if, its angular change in one period of the curvature Λ_d is 0, i.e.

$$\Lambda_d := \sqrt{-d} \int_0^{h_d} \frac{(\kappa^2 - \lambda^2)}{d + 4(\kappa + \lambda)^2} = 0. \quad (35)$$

Therefore, using Table 3, we have

Proposition 7. For any $\lambda > 1$ the angular change $\Lambda^\lambda(d)$ moves continuously in $(0, -\delta_1) \cup (-\delta_1, \infty)$ and crosses the x -axis in the second interval. Therefore, there exist a closed critical curve of $\mathcal{L}_\lambda$ in the hyperbolic plane of translational type.

Fig. 8 shows a “eight”-shaped curve which is a closed critical point of translational type.

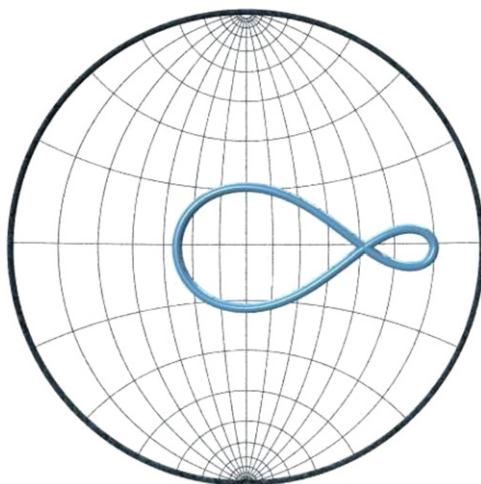


Fig. 8. Translational type closed solution, $d > 0$.

4. Second variation formula

Given a closed critical point γ of $\mathcal{L}_\lambda$, we say that γ is stable if $\frac{d^2 \mathcal{L}_\lambda}{du^2}(\gamma) = \frac{d^2 \mathcal{L}_\lambda}{du^2}(0) \geq 0$. Denote its curvature by κ and take a normal variation of γ with variation normal field $W = \phi N$. Then, from (6) one can compute the second variation formula at γ

$$\begin{aligned} \frac{d^2 \mathcal{L}_\lambda}{du^2}(0) &= \int_\gamma \langle W, \nabla_W \mathcal{E} \rangle ds \\ &= \int_\gamma \{(3\kappa^2 - \lambda^2 - 2)\phi W(\kappa) + 2\phi W(\kappa_{ss})\} ds, \end{aligned}$$

where $W(\kappa) = \phi_{ss} + \phi(\kappa^2 - 1)$ and $W(\kappa_{ss}) = \phi_{ssss} + \phi_{ss}(\kappa^2 - 1) + 5\kappa\kappa_s\phi_s + (3\kappa_s^2 + 4\kappa\kappa_{ss})\phi$.

Now, using the Euler–Lagrange equation and integrating by parts, we get

$$\frac{d^2 \mathcal{L}_\lambda}{du^2}(0) = \int_\gamma (2\phi_{ss}^2 + B\phi\phi_s + A\phi^2) ds, \quad (36)$$

where $A = 6\kappa_s^2 - \kappa^4 + 3(\lambda^2 + 1)\kappa^2 + 8\lambda\kappa + 2 + \lambda^2$ and $B = 5\kappa^2 - \lambda^2 - 4$.

Case 1. $\lambda > 1$.

In this case we denote by ϵ_o^m and ϵ_2^m the m -multiple covers of the curves with constant curvature $\eta_o = -\lambda$ and $\eta_2 = (\lambda + \sqrt{\lambda^2 + 8})/2$, respectively. They are circles in $\mathbb{H}^2(-1)$ which are critical points for $\mathcal{L}_\lambda$. By substitution of $\kappa = \eta_o$ in (36) we obtain

$$\frac{d^2 \mathcal{L}_\lambda}{du^2}(0) = 2 \int_\gamma (\phi_{ss} + (\lambda^2 - 1)\phi)^2 ds \geq 0, \quad (37)$$

therefore, as one would have been expected from the beginning, we have

Proposition 8. Given any $\lambda > 1$, ϵ_o^m is stable for every m .

Now, evaluating (36) at $\gamma = \epsilon_2^m$, we obtain

$$\frac{d^2 \mathcal{L}_\lambda}{du^2}(0) = \int_\gamma (2\phi_{ss}^2 + B\phi\phi_s + A\phi^2) ds, \quad (38)$$

where $A = 4 + 17\lambda^2/2 + \lambda^4 + (7\lambda\sqrt{\lambda^2 + 8})/2 + \lambda^3\sqrt{\lambda^2 + 8}$ and $B = 6 + 3\lambda^2/2 + (5\lambda\sqrt{\lambda^2 + 8})/2$. On the other hand, ϕ must have a Fourier series of the form $a_o/2 + \sum_{n=1}^{\infty} a_n \cos(ns/mp) + b_n \sin(ns/mp)$ where p is determined in terms of λ by $p = \sinh \vartheta$ and $\coth \vartheta = \eta_2$. Therefore, from (38) one obtains

$$\frac{d^2 \mathcal{L}_\lambda}{du^2}(0) = \pi mp \left\{ \frac{Aa_o^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \left(\frac{n^2}{m^2 p^2} - \mu^2 \right) \left(\frac{n^2}{m^2 p^2} - \delta^2 \right) \right\}, \quad (39)$$

where $1 < \mu^2 < \delta^2$ are the roots of $2x^2 - Bx + A$. So, for a given m , the stability problem for η_2^m reduces to searching for a natural number n such that $\mu^2 < n^2/(m^2 p^2) < \delta^2$. On the other hand one may check that $\mu^2 p^2 = 1$ and that A and B are non-negative, therefore

Proposition 9. Assume that $\lambda > 1$ and let ε_2^m be the m -cover of the circle with constant curvature η_2 . Then, ε_2^m is stable, if and only if, $\|m\| < 1/(\delta p - 1)$, where $\delta^2 p^2 = (8 + \lambda^2 + 3\lambda\sqrt{\lambda^2 + 8})/(4 + 2\lambda^2 + 2\lambda\sqrt{\lambda^2 + 8})$.

Moreover, since $1/(\delta p - 1)$ increases monotonically from $2/(\sqrt{6} - 2)$ as λ moves from 1 to ∞ , we get

Corollary 10. ε_2^1 is always stable for any $\lambda > 1$.

Case 2. $0 < \lambda < 1$.

In this case we denote by ε_1^m and ε_2^m the m -multiple covers of the curves with constant curvature $\eta_1 = (\lambda - \sqrt{\lambda^2 + 8})/2$ and $\eta_2 = (\lambda + \sqrt{\lambda^2 + 8})/2$, respectively. They are circles in $\mathbb{H}^2(-1)$ which are critical points for $\mathcal{L}_\lambda$. The proof of the following proposition is similar to that of Proposition 8 and its corollary (in this case, $1/(\delta p - 1)$ decreases monotonically from $1/(\sqrt{2} - 1)$ and $2/(\sqrt{6} - 2)$ as λ moves from 0 to 1)

Proposition 11. Assume that $0 < \lambda \leq 1$ and let ε_2^m be the m -cover of the circle with constant curvature η_2 . Then, ε_2^m is stable, if and only if, $< 1/(\delta p - 1)$, where now $\delta^2 p^2 = \frac{8 + \lambda^2 - 3\lambda\sqrt{\lambda^2 + 8}}{4 + 2\lambda^2 - 2\lambda\sqrt{\lambda^2 + 8}}$. In particular, ε_2^1 is always stable.

Finally, evaluating (36) at $\gamma = \varepsilon_1^m$, we obtain again formulas (38) and (39), where now $A = 4 + 17\lambda^2/2 + \lambda^4 - 7\lambda\sqrt{\lambda^2 + 8}/2 - \lambda^3\sqrt{\lambda^2 + 8}$ and $B = 6 + 3\lambda^2/2 - (5\lambda\sqrt{\lambda^2 + 8})/2$. Also, one may check that $\mu^2 p^2 = 1$ and that A and B are non-negative, therefore by similar computations

Proposition 12. Assume that $0 < \lambda \leq 1$ and let ε_1^m be the m -cover of the circle with constant curvature η_1 . Then,

1. If $0 \leq \lambda < 1/\sqrt{10}$ the only stable m -covers are ε_1^1 ;
2. If $1/\sqrt{10} \leq \lambda \leq 1$, then in addition to ε_1^1 the only stable m -cover is ε_1^2 .

5. Hopf vesicles in the anti de Sitter 3-space

The Hopf map from the anti de Sitter space to the hyperbolic can be described as follows (for more details, see, for instance, [20]). Let us consider the complex plane $\mathbb{C}^2$ endowed with the semi-Riemannian metric $\langle z, w \rangle = \text{Real}(z_1 \bar{w}_1 - z_2 \bar{w}_2)$. The hyperquadric $\langle z, z \rangle = -r^2$ can be naturally identified with the anti de Sitter space $\text{AdS}_3(-1/r^2)$ of curvature $-1/r^2$. The restriction to $\text{AdS}_3(-1/r^2)$ of the map $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$\pi(z_1, z_2) = \frac{1}{2r}(\bar{z}_1 z_1 - \bar{z}_2 z_2, 2z_2 \bar{z}_1), \quad (40)$$

sends the anti de Sitter space to $\mathbb{H}^2(-4/r^2)$, the hyperbolic plane of curvature $(-4/r^2)$. This restriction

$$\pi : \text{AdS}_3(-\frac{1}{r^2}) \rightarrow \mathbb{H}^2(-\frac{4}{r^2}), \quad (41)$$

is known also as the Hopf map and it is semi-Riemannian submersion.

Now, take any Frenet curve γ in $\mathbb{H}^2(-4/r^2)$. Then, the complete lift of γ via π , denoted by $\mathbf{T}_\gamma = \pi^{-1}(\gamma)$, is a Lorentzian flat surface which is called the Hopf tube shaped on γ . If γ is closed, then $\mathbf{T}_\gamma = \pi^{-1}(\gamma)$ is a Lorentzian flat torus. These tori are embedded when the associated curves have no self-intersections. Moreover if γ is an immersed closed curve in $\mathbb{H}^2(-4/r^2)$ of length L and enclosing an area A , then its Hopf torus $\mathbf{T}_\gamma = \pi^{-1}(\gamma)$ is isometric to the Lorentzian flat torus $\mathbb{L}^2/\Gamma$, where Γ is the lattice in $\mathbb{L}^2$ generated by $(2\pi r, 0)$ and $(2A, L)$, [20].

The Hopf map can also be regarded as principal fiber bundle with structure group $\mathbb{S}^1$. This group acts by isometries on $\text{AdS}_3(-1/r^2)$, $e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$, so that the orbit space is $\mathbb{H}^2(-4/r^2)$.

On the other hand, it is not difficult to check that the $\mathbb{S}^1$ -invariant surfaces of $\text{AdS}_3(-1/r^2)$ are precisely the Lorentzian Hopf tubes $\mathbf{T}_\gamma$ which we have just defined.

We want to construct examples of elastic surfaces (that is critical points of the elastic energy (3)) in $\text{AdS}_3(-1/r^2)$. Without loss of generality we assume from now on that $r = 1$ and, for simplicity, we eliminate the constraints on the area and enclosed volume on the energy functional. In other words, we take $a = 0 = d$ in (3). Thus, we consider the elastic energy

$$\mathcal{E}(S) = \int_S (H - \mu)^2 dA + \eta \int_S G dA, \quad (42)$$

acting on the space of immersions of a surface in $\text{AdS}_3(-1)$. We obtain,

Proposition 13. A Hopf torus $\mathbf{T}_\gamma$ is an elastic surface in $\text{AdS}_3(-1)$, that is a critical point of (42), if and only if, γ is a closed critical point of $\int_\gamma (\kappa - 2\mu)^2$ in the hyperbolic plane $\mathbb{H}^2(-4)$.

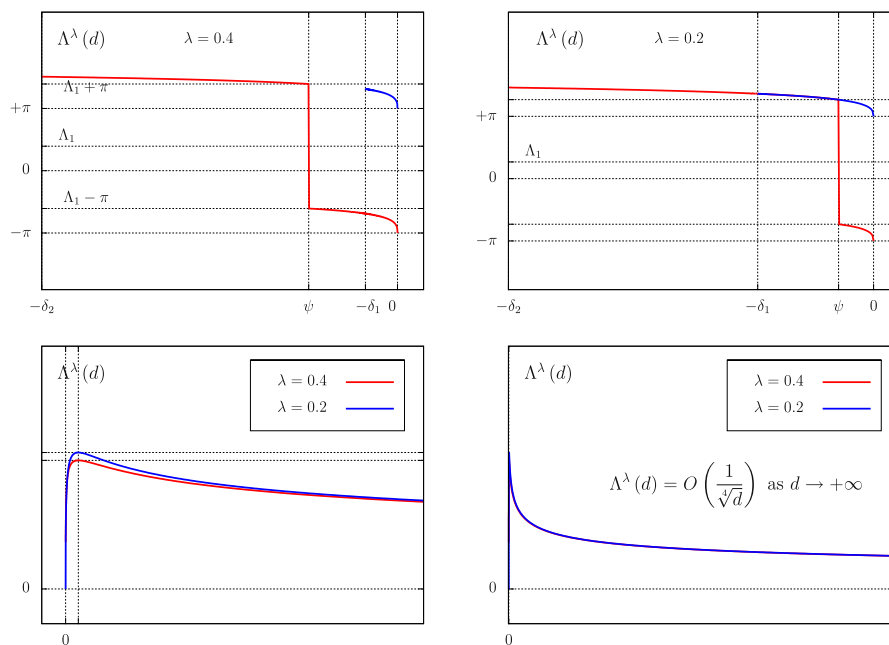


Fig. 9. Angular change for $\lambda < 1$. We show the graphics corresponding to $\lambda = 0.2$ and $\lambda = 0.4$. As we see they practically overlap for $d \gg 0$.

Proof. Let us denote by $\mathcal{M} = \{f : T \rightarrow \mathbf{AdS}_3(-1); f \text{ is an immersion}\}$ the manifold of immersions of a torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ into $\mathbf{AdS}_3(-1)$. Obviously, the elastic energy (42) is invariant under the action of $\mathbb{S}^1$ on $\mathbf{AdS}_3(-1)$. We consider the energy functional $\mathcal{E}$ acting on $\mathcal{M}$ and denote by $\mathcal{M}_{\mathbb{S}^1}$ the subspace of $\mathcal{M}$ formed by the $\mathbb{S}^1$ -invariant immersions. As we have pointed out previously, $\mathcal{M}_{\mathbb{S}^1}$ is nothing but the family of Hopf tori. Let us denote by $\mathcal{K}$ and by $\mathcal{K}_{\mathbb{S}^1}$ the sets of critical points of $\mathcal{E} : \mathcal{M} \rightarrow \mathbb{R}$ and $\mathcal{E} : \mathcal{M}_{\mathbb{S}^1} \rightarrow \mathbb{R}$, respectively. Then, we can apply the Palais' symmetric criticality principle [27] obtaining $\mathcal{K} \cap \mathcal{M}_{\mathbb{S}^1} = \mathcal{K}_{\mathbb{S}^1}$. Hence, in order to find the critical Hopf tori for $\mathcal{E}$ it is enough to consider the critical points of $\mathcal{E} : \mathcal{M}_{\mathbb{S}^1} \rightarrow \mathbb{R}$.

Now, the Hopf fibration $\pi : \mathbf{AdS}_3(-1) \rightarrow \mathbb{H}^2(-4)$ is a Riemannian submersion with geodesic fibers. Then given $\gamma : [0, L] \rightarrow \mathbb{H}^2(-4)$ a unit parameterized closed curve and its corresponding Hopf torus on T_γ , one can use the O'Neill formulas to see that the mean curvature of T_γ , denoted by $H(t, s)$, and the geodesic curvature of γ , $\kappa(s)$, are related by $H(t, s) = \kappa(s)/2$, where t is the fiber parameter. Thus, by substitution in (42) and using the Gauss Bonnet theorem, we obtain

$$\mathcal{E}(T_\gamma) = \int_{T_\gamma} (H - \mu)^2 dA = \int \int \left(\frac{\kappa(s)}{2} - \mu \right)^2 ds \cdot dt = \frac{\pi}{2} \int_0^L (\kappa(s) - 2\mu)^2 ds, \quad (43)$$

which proves the statement. $\square$

Combining Propositions 4 and 13, one obtains plenty of examples of Hopf vesicles in $\mathbf{AdS}_3(-1)$. For instance,

Corollary 14. For any pair of integer numbers $m, n \in \mathbb{Z}$ satisfying condition (i) (respectively, condition (ii)) of Proposition 4 if $-1/2 < \mu < 0$ (respectively, if $\mu < -1/2$), there exists a Hopf torus in $\mathbf{AdS}_3(-1)$ which is a closed critical point of the elastic energy (42).

Observe that if $\mu > -2$ then the absolute minimum of (42) is zero and it is reached at the Hopf torus with constant mean curvature μ shaped on the circle of $\mathbb{H}^2(-4)$ with constant curvature 2μ . Moreover, by using (13), (14), (25) and (40) one can give explicit parameterizations of these critical points (up to one quadrature) and, therefore, obtain explicit solutions to the EL equation associated to the elastic energy (42) in $\mathbf{AdS}_3(-1)$.

6. Numerical approach

For a given $\lambda > 0$, the angular change $\Lambda^\lambda(d)$ behavior as d approaches to the ends of its intervals of continuity will be computed in Appendix. This has been used in Section 3 to show the existence of closed critical points. In order to gain some insight into the space of closed critical points, in this section we develop some numerical experiments concerning the angular change $\Lambda^\lambda(d)$ and the energy $\mathcal{E}^\lambda(d)$ of a closed critical curve. They give us both qualitative and quantitative information on the behavior of this space of curves.

First, we have checked the monotonicity of $\Lambda^\lambda(d)$ numerically which is shown here in Figs. 9 and 10 for several significant cases. As for the critical points of rotational type, for any $\lambda > 0$, pictures corresponding to $d < 0$ of Figs. 9 and 10

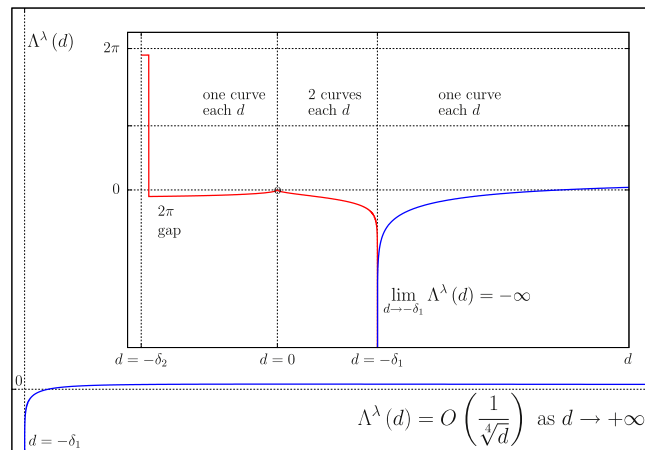


Fig. 10. Angular change for $\lambda > 1$. The picture corresponds to $\lambda = 4$. The inset shows the graphics for small values of d .

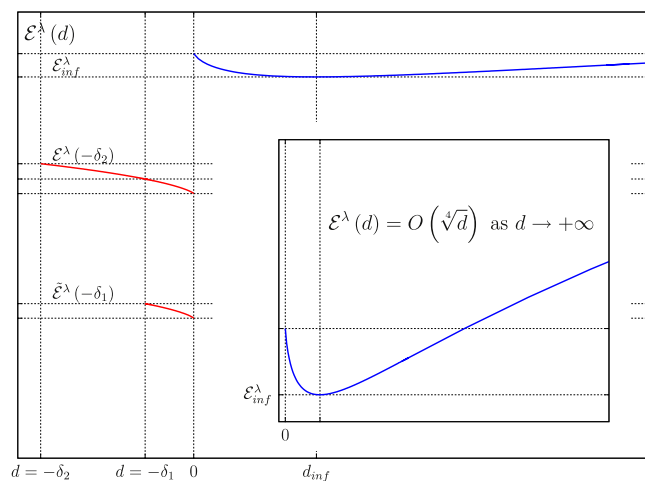


Fig. 11. Variation of the energy in one period of the curvature for $\lambda < 1$. The inset points to the “eight”-shaped figure as a local minimizer.

suggest that, for any pair $m, n \in \mathbb{Z}$ with $\gcd(m, n) = 1$ and satisfying the conditions of Proposition 4, the associated closed critical point $\gamma_{m,n}$ is unique.

In contrast, pictures corresponding to $d > 0$ of Figs. 9 and 10 insinuate that there is no uniqueness of closed critical points of translational type, for a given pair $m, n \in \mathbb{Z}$ with $\gcd(m, n) = 1$. That is, in this case we might have geometrically different closed critical curves (i.e., with different curvature functions) corresponding to the same pair (m, n) . However, we have not found any closed critical curve of translational type when $\lambda < 1$, and, if $\lambda > 1$, we have found only one closed critical point of translational type, $d > 0$ (see Proposition 7). It corresponds to the “eight”-shaped Fig. 8. The second row of Fig. 9 and the right side of Fig. 10 indicate that there are no more closed critical points of translational type.

On the other hand, evaluation of the second variation formula for the non-constant curvature critical points is extremely difficult. In order to get some extra information about the candidates for local minima, one may try to analyze the energy's behavior in one period of the curvature. The energy of a closed critical point in one period of its curvature is denoted by $\mathcal{E}^\lambda(d)$. When two solutions appear for the same value of λ and d , the energy of both curves is represented by $\mathcal{E}^\lambda(d)$ and $\tilde{\mathcal{E}}^\lambda(d)$, respectively.

As it was mentioned before, although one can obtain explicit formulas for the energy in one period of the curvature in terms of elliptic functions and nice relations between $\mathcal{E}^\lambda(d)$ and $\tilde{\mathcal{E}}^\lambda(d)$ (see the Appendix), they are so complicated in general that any qualitative property we could draw would require a huge amount of computations. Rather than doing so, we have used such formulas to make a numerical study. In Figs. 11 and 12, the behavior of $\mathcal{E}^\lambda(d)$ and $\tilde{\mathcal{E}}^\lambda(d)$ is shown for different values of λ .

If $\lambda < 1$, then Fig. 11 seems to contradict our Propositions 11 and 12. In fact, hyperbolic circles with curvatures η_1 and η_2 are reached as d approaches to $-\delta_1$ and $-\delta_2$ respectively, and they look like local maxima for the energy instead of local minima. However, we should be aware that the picture corresponds to $\mathcal{E}^\lambda(d)$ and $\tilde{\mathcal{E}}^\lambda(d)$ which are the energies in one period of the curvature and that, in the “proximity” of the constant curvature closed critical points (that is for values of d close to

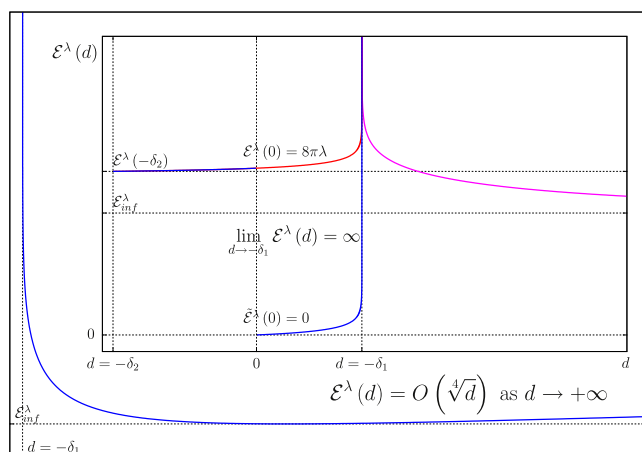


Fig. 12. Variation of the energy in one period of the curvature for $\lambda > 1$. Again, it suggests that the “eight”-shaped figure as a local minimizer.

$-\delta_1$ and $-\delta_2$) other closed critical points need more than one period of their curvature to be able to close up (see Figs. 9 and 10). It also appears that there is a local minimum for certain $d > 0$ (Fig. 11). But we have not found closed critical points of translational type when $\lambda < 1$. So we have not detected closed local minima other than the circles with curvatures η_1 and η_2 in this case (Propositions 11 and 12).

If $\lambda > 1$, Fig. 12 suggests that, in addition to the rotational type local minima with constant curvature that we have found in Propositions 8 and 9, there exists another non-constant curvature local minimum corresponding to a certain $d_0 > 0$ (translational type). Numerical computations indicate that this local minimum corresponds to the “eight”-shaped critical curve (see, Fig. 8). It is the only translational type closed critical point that we have detected numerically.

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Appendix

A.1. Angular change

Case 1. $\lambda < 1$. We have that $-\delta_2 < -\delta_1 < 0$ in this case.

Case 1.1. If $d \in (-\delta_1, 0)$, then we obtain two solutions of (10) for each value of d . The first solution moves between $\alpha_1 \leq \kappa(s) \leq \alpha_2$ while the second one varies in $\alpha_3 \leq \kappa(s) \leq \alpha_4$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, are the four roots of $Q_d(x)$. Since $d < 0$, the angular change is

$$\Lambda^\lambda(d) = \sqrt{-d} \int_0^\rho \frac{(\kappa^2 - \lambda^2)}{d + 4(\kappa + \lambda)^2},$$

which will be denoted by Λ^a and Λ^b for the first and second solution respectively. Then, an algebraic manipulation of the integrand combined with formulas 3.147 and 3.151 of [28] give

$$\begin{aligned} \Lambda^a &= \sqrt{-d} \int_{\alpha_1}^{\alpha_2} \left\{ 1 - \frac{\lambda + \frac{\sqrt{-d}}{4}}{\kappa + \lambda + \frac{\sqrt{-d}}{2}} - \frac{\lambda - \frac{\sqrt{-d}}{4}}{\kappa + \lambda - \frac{\sqrt{-d}}{2}} \right\} \frac{d\kappa}{\sqrt{Q_d(\kappa)}} \\ &= \frac{2\sqrt{-d}}{\sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}} \left\{ \left[1 - \frac{\lambda + \frac{\sqrt{-d}}{4}}{\alpha_3 + \lambda + \frac{\sqrt{-d}}{2}} - \frac{\lambda - \frac{\sqrt{-d}}{4}}{\alpha_3 + \lambda - \frac{\sqrt{-d}}{2}} \right] K(M) \right. \\ &\quad - \frac{\left(\lambda + \frac{\sqrt{-d}}{4} \right) (\alpha_3 - \alpha_2)}{\left(\alpha_3 + \frac{\sqrt{-d}}{2} + \lambda \right) \left(\alpha_2 + \frac{\sqrt{-d}}{2} + \lambda \right)} \Pi(v_1^a, M) \\ &\quad \left. - \frac{\left(\lambda - \frac{\sqrt{-d}}{4} \right) (\alpha_3 - \alpha_2)}{\left(\alpha_3 - \frac{\sqrt{-d}}{2} + \lambda \right) \left(\alpha_2 - \frac{\sqrt{-d}}{2} + \lambda \right)} \Pi(v_2^a, M) \right\}, \end{aligned} \quad (44)$$

and

$$\begin{aligned} \Lambda^b &= \sqrt{-d} \int_{\alpha_3}^{\alpha_4} \left\{ 1 - \frac{\lambda + \frac{\sqrt{-d}}{4}}{\kappa + \lambda + \frac{\sqrt{-d}}{2}} - \frac{\lambda - \frac{\sqrt{-d}}{4}}{\kappa + \lambda - \frac{\sqrt{-d}}{2}} \right\} \frac{d\kappa}{\sqrt{Q_d(\kappa)}} \\ &= \frac{2\sqrt{-d}}{\sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}} \left\{ \left[1 - \frac{\lambda + \frac{\sqrt{-d}}{4}}{\alpha_2 + \lambda + \frac{\sqrt{-d}}{2}} - \frac{\lambda - \frac{\sqrt{-d}}{4}}{\alpha_2 + \lambda - \frac{\sqrt{-d}}{2}} \right] K(M) \right. \\ &\quad + \frac{\left(\lambda + \frac{\sqrt{-d}}{4} \right) (\alpha_3 - \alpha_2)}{\left(\alpha_3 + \frac{\sqrt{-d}}{2} + \lambda \right) \left(\alpha_2 + \frac{\sqrt{-d}}{2} + \lambda \right)} \Pi(v_1^b, M) \\ &\quad \left. + \frac{\left(\lambda - \frac{\sqrt{-d}}{4} \right) (\alpha_3 - \alpha_2)}{\left(\alpha_3 - \frac{\sqrt{-d}}{2} + \lambda \right) \left(\alpha_2 - \frac{\sqrt{-d}}{2} + \lambda \right)} \Pi(v_2^b, M) \right\}, \end{aligned} \quad (45)$$

where $M^2 = \frac{(\alpha_4 - \alpha_3)(\alpha_2 - \alpha_1)}{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}$, $K(M)$ is the complete elliptic integral of first kind of modulus M , $\Pi(v, M)$ the complete elliptic integral of third kind of modulus M , and

$$v_1^a = \frac{(\alpha_2 - \alpha_1) \left(\alpha_3 + \lambda + \frac{\sqrt{-d}}{2} \right)}{(\alpha_3 - \alpha_1) \left(\alpha_2 + \lambda + \frac{\sqrt{-d}}{2} \right)}, \quad v_2^a = \frac{(\alpha_2 - \alpha_1) \left(\alpha_3 + \lambda - \frac{\sqrt{-d}}{2} \right)}{(\alpha_3 - \alpha_1) \left(\alpha_2 + \lambda - \frac{\sqrt{-d}}{2} \right)}, \quad (46)$$

$$v_1^b = \frac{(\alpha_4 - \alpha_3) \left(\alpha_2 + \lambda + \frac{\sqrt{-d}}{2} \right)}{(\alpha_4 - \alpha_2) \left(\alpha_3 + \lambda + \frac{\sqrt{-d}}{2} \right)}, \quad v_2^b = \frac{(\alpha_4 - \alpha_3) \left(\alpha_2 + \lambda - \frac{\sqrt{-d}}{2} \right)}{(\alpha_4 - \alpha_2) \left(\alpha_3 + \lambda - \frac{\sqrt{-d}}{2} \right)}. \quad (47)$$

Since $\lambda < 1$ we have $\alpha_2 < -\lambda < \alpha_3$, hence $v_1^{a,b}$ and $v_2^{a,b}$ are negative. Then, we arrive to the following two options for the relationship between Π and the Heuman Lambda function Λ_o :

$$\Pi(v, M) = \frac{1}{1-v} K(M) + \frac{\pi v [\Lambda_o(\beta, M) - 1]}{2\sqrt{v(1-v)(v-M^2)}}, \quad (48)$$

$$\Pi(v, M) = \frac{M^2}{M^2 - v} K(M) - \frac{\pi v \Lambda_o(\psi, M)}{2\sqrt{v(1-v)(v-M^2)}}, \quad (49)$$

where $\beta = \arcsin(1/\sqrt{1-v})$ and $\psi = \arcsin(\sqrt{v/(v-M^2)})$.

Observe that, if $\lambda \geq 0$ and $d < 0$,

$$\sqrt{-Q_d \left(-\lambda - \frac{\sqrt{-d}}{2} \right)} = \sqrt{-d} \left(\lambda + \frac{\sqrt{-d}}{4} \right), \quad (50)$$

and

$$\sqrt{-Q_d \left(-\lambda + \frac{\sqrt{-d}}{2} \right)} = \begin{cases} -\sqrt{-d} \left(\lambda - \frac{\sqrt{-d}}{4} \right), & d < -16\lambda^2, \\ \sqrt{-d} \left(\lambda - \frac{\sqrt{-d}}{4} \right), & d > -16\lambda^2. \end{cases} \quad (51)$$

Hence, using (46)–(48), (50) and (51), we have

$$\begin{aligned} \Lambda^a &= \frac{2\sqrt{-d}}{\sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}} \frac{(\alpha_1^2 - \lambda^2)}{((\alpha_1 + \lambda)^2 + \frac{d}{4})} K(M) - \Lambda_o(\beta_1^a, M) \pi \\ &\quad + \begin{cases} \Lambda_o(\beta_2^a, M) \pi, & d < -16\lambda^2, \\ 2\pi - \Lambda_o(\beta_2^a, M) \pi, & d > -16\lambda^2. \end{cases} \end{aligned} \quad (52)$$

Analogously, using (46), (47) and (49)–(51), we have

$$\begin{aligned} \Lambda^b = & \frac{2\sqrt{-d}}{\sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}} \frac{(\alpha_1^2 - \lambda^2)}{((\alpha_1 + \lambda)^2 + \frac{d}{4})} K(M) - \Lambda_o(\psi_1^b, M)\pi \\ & + \begin{cases} \Lambda_o(\psi_2^b, M)\pi, & d < -16\lambda^2, \\ -\Lambda_o(\psi_2^b, M)\pi, & d > -16\lambda^2. \end{cases} \end{aligned} \quad (53)$$

Since $v_1^a v_1^b = M^2 = v_2^a v_2^b$, we obtain $\psi_2^b = \beta_2^a$. Analogously, we have $\psi_1^b = \beta_1^a$. Therefore, comparing (52) and (53), we finally obtain

$$\Lambda^a - \Lambda^b = \begin{cases} 0, & d < -16\lambda^2, \\ 2\pi, & d > -16\lambda^2. \end{cases} \quad (54)$$

As before, we use the following notation $\eta_1 = (\lambda - \sqrt{\lambda^2 + 8})/2$ and $\eta_2 = (\lambda + \sqrt{\lambda^2 + 8})/2$. Now, if $d \rightarrow -\delta_1$, then $M \rightarrow 0$; $v_i^a \rightarrow 0$; $\beta_i^a \rightarrow \pi/2$; $\alpha_1 \rightarrow \eta_1$; $\alpha_2 \rightarrow \eta_1$; $\alpha_3 \rightarrow -\eta_1 + \frac{2}{\eta_1}\sqrt{2 - \eta_1^2}$ and $\alpha_4 \rightarrow -\eta_1 - \frac{2}{\eta_1}\sqrt{2 - \eta_1^2}$. Therefore using (10), one gets

$$\lim_{d \rightarrow -\delta_1} \Lambda^a = -\frac{2\pi\eta_1}{\sqrt{(\eta_1^2 + 2)}}, \quad (55)$$

which combined with (54) gives

$$\lim_{d \rightarrow -\delta_1} \Lambda^b = -\frac{2\pi\eta_1}{\sqrt{\eta_1^2 + 2}} - \begin{cases} 0, & d < -16\lambda^2 \\ 2\pi, & d > -16\lambda^2. \end{cases} \quad (56)$$

If $d \rightarrow -\delta_2$, then Q_d has only two roots and we do not have an explicit expression for the angular change. Nevertheless, one can see that the period of the curvatures in this area is $\rho = 4K(M)/r$, where $r = \sqrt{pq}/2$. As $d \rightarrow -\delta_2$, then $M \rightarrow 0$ ($\eta_2 > 0$) and $r \rightarrow \frac{1}{\eta_2}\sqrt{(\eta_2^2 - 1)(\eta_2^2 + 2)}$. Hence, looking at (10) again and noticing that the limit curve would correspond to the curve with constant curvature η_2 and period $\rho = \frac{2\pi\eta_2}{\sqrt{(\eta_2^2 - 1)(\eta_2^2 + 2)}}$, one can see that

$$\lim_{d \rightarrow -\delta_2} \Lambda^\lambda(d) = \frac{2\pi\eta_2}{\sqrt{(\eta_2^2 + 2)}}. \quad (57)$$

The analysis when $d \rightarrow \pm\psi$ is very much like the corresponding case in [12].

Now, assume that $d \rightarrow 0^-$. Then $(\alpha_2 + \lambda) \rightarrow 0$ and $(\alpha_3 + \lambda) \rightarrow 0$. Denote $(\alpha_3 + \lambda)$ by ϵ . From the definition of $Q_d(\kappa)$, we obtain

$$d = -4(1 - \lambda^2)\epsilon^2 - 4\lambda\epsilon^3 + \epsilon^4 \approx -4(1 - \lambda^2)\epsilon^2. \quad (58)$$

By using the derivative of $Q_d(\kappa)$, one can see that $-\epsilon - o(\epsilon^2) < (\alpha_2 + \lambda) < -\epsilon$. In other words $(\alpha_2 + \lambda) \approx -\epsilon$, which means that

$$\lim_{d \rightarrow 0^-} \frac{(\alpha_2 + \lambda)}{(\alpha_3 + \lambda)} = -1. \quad (59)$$

Hence, $(\alpha_3 - \alpha_2) \approx 2\epsilon$. On the other hand, $M^2 = \frac{(\alpha_4 - \alpha_3)(\alpha_2 - \alpha_1)}{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)} \rightarrow 1$ as $d \rightarrow 0^-$, therefore,

$$1 - M^2 = \frac{(\alpha_4 - \alpha_1)}{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)} (\alpha_3 - \alpha_2) = o(\epsilon). \quad (60)$$

Combining (60) with the well known fact that, when $M \rightarrow 1$, $K(M)$ behaves asymptotically as $\frac{1}{2} \ln(16/(1 - M^2))$, we obtain $K(M) = O(\ln(1/\epsilon))$. Using this along with (58) and (60) we have that

$$\lim_{d \rightarrow 0^-} \frac{2\sqrt{-d}}{\sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}} \frac{(\alpha_1^2 - \lambda^2)}{((\alpha_1 + \lambda)^2 + \frac{d}{4})} K(M) = 0, \quad (61)$$

which takes care of the first part of formula (52). As for the remaining parts, we observe that $M \rightarrow 1$ as $d \rightarrow 0^-$ and that $\Lambda_o(x, 1) = 2x/\pi$. Thus,

$$\lim_{d \rightarrow 0^-} \Lambda_o(\beta_1^a, M) = \frac{2}{\pi} \lim_{d \rightarrow 0^-} \beta_1^a, \quad \lim_{d \rightarrow 0^-} \Lambda_o(\beta_2^a, M) = \frac{2}{\pi} \lim_{d \rightarrow 0^-} \beta_2^a. \quad (62)$$

Combining (46) with formulas (58) to (60) one sees that $v_1^a \approx \frac{(1 + \sqrt{1 - \lambda^2})\epsilon}{(-1 + \sqrt{1 - \lambda^2})\epsilon}$ and $v_2^a \approx \frac{(1 - \sqrt{1 - \lambda^2})\epsilon}{(-1 - \sqrt{1 - \lambda^2})\epsilon}$, which combined in turn with the definition of β , $\beta_i^a = \arcsin(1/\sqrt{1 - v_i^a})$, $i \in \{1, 2\}$, implies that $\sin^2(\lim_{d \rightarrow 0^-} \beta_1^a) + \sin^2(\lim_{d \rightarrow 0^-} \beta_2^a)$

$= (1 - \sqrt{1 - \lambda^2})/2 + (1 + \sqrt{1 - \lambda^2})/2 = 1$. Therefore,

$$\lim_{d \rightarrow 0^-} \{\beta_1^a + \beta_2^a\} = \frac{\pi}{2}. \quad (63)$$

Finally, (52) and (61)–(63) give

$$\lim_{d \rightarrow 0^-} \Lambda^a(d) = 2\pi - \lim_{d \rightarrow 0^-} \{\Lambda_0(\beta_1^a, M)\pi + \Lambda_0(\beta_2^a, M)\pi\} \quad (64)$$

$$= 2\pi - 2 \lim_{d \rightarrow 0^-} \{\beta_1^a + \beta_2^a\} = \pi, \quad (65)$$

and, as a consequence of formula (54), we also have

$$\lim_{d \rightarrow 0^-} \Lambda^b(d) = -\pi. \quad (66)$$

Case 1.2. If $d > 0$, we observe first from (65) and (66) that

$$\lim_{d \rightarrow 0^+} \Lambda^{\lambda}(d) = 0. \quad (67)$$

On the other hand, for d large enough the polynomial

$$Q_d(\kappa) = (\kappa - \alpha_1)(\alpha_4 - \kappa)((\kappa - m)^2 + n^2),$$

has only two roots, denoted here by α_1 and α_4 , and where $d = -\alpha_1\alpha_4(m^2 + n^2)$, $m = -(\alpha_1 + \alpha_4)/2 < 0$ and $n^2 = 3m^2 - \alpha_1\alpha_4 - 4 - 2\lambda^2$.

We decompose the angular change into three parts,

$$\Lambda^{\lambda}(d) = 2 \left\{ \int_{\alpha_1}^{-\lambda} \varphi(\kappa) d\kappa + \int_{-\lambda}^{\lambda} \varphi(\kappa) d\kappa + \int_{\lambda}^{\alpha_4} \varphi(\kappa) d\kappa \right\}, \quad (68)$$

denoted respectively by $\Lambda_{(\alpha_1, -\lambda)}$, $\Lambda_{(-\lambda, \lambda)}$ and $\Lambda_{(\lambda, \alpha_4)}$, where

$$\varphi(\kappa) = \sqrt{d} \frac{\kappa^2 - \lambda^2}{(d + 4(\kappa + \lambda)^2) \sqrt{Q_d(\kappa)}}. \quad (69)$$

The function $\varphi(\kappa)$ is negative only if $\kappa \in (-\lambda, \lambda)$. Therefore, $\Lambda_{(\alpha_1, -\lambda)}$ and $\Lambda_{(\lambda, \alpha_4)}$ are always positive while $\Lambda_{(-\lambda, \lambda)}$ is always negative. We wish to prove that they have the following asymptotic behavior with respect to d : $\Lambda_{\alpha_1, -\lambda} = O(1/\sqrt[4]{d})$, $\Lambda_{(-\lambda, \lambda)} = O(1/d)$ and $\Lambda_{\lambda, \alpha_4} = O(1/\sqrt[4]{d})$.

1. Assume first that $\kappa \in (-\lambda, \lambda)$, then $\varphi(\kappa)$ is bounded below by

$$|\varphi(\kappa)| \geq \frac{\sqrt{d}(\lambda^2 - \kappa^2)}{(d + 16\lambda^2) \sqrt{Q_d(\lambda)}}. \quad (70)$$

On the other hand, since $\lambda < 1$ one has

$$|\varphi(\kappa)| \leq \frac{\lambda^2 - \kappa^2}{\sqrt{dQ_d(-\lambda)}}. \quad (71)$$

Therefore, from (70) the angular change in this case satisfies

$$|\Lambda_{(-\lambda, \lambda)}| \geq \frac{8\sqrt{d}\lambda^3}{3\sqrt{(d + 16\lambda^2)^3}} = O\left(\frac{1}{d}\right). \quad (72)$$

Moreover, we have from (71) that $|\Lambda_{(-\lambda, \lambda)}| \leq \frac{8\lambda^3}{3d} = O\left(\frac{1}{d}\right)$.

2. Assume now that $\kappa \in (\alpha_1, -\lambda)$. Similar considerations can be made if $\kappa \in (\lambda, \alpha_4)$. Observe now that

$$|\varphi(\kappa)| \leq \left(\frac{\alpha_1^2 - \lambda^2}{d} \right) \frac{\sqrt{d}}{\sqrt{(\kappa - \alpha_1)(\alpha_4 + \lambda)((-\lambda - m)^2 + n^2)}}. \quad (73)$$

Since $\lambda < 1$ we obtain

$$|\varphi(\kappa)| \geq \frac{\sqrt{d}(\kappa^2 - \lambda^2)}{(d + 4(\alpha_1 + \lambda)^2) \sqrt{Q_d(\eta_1)}}. \quad (74)$$

Moreover, from $\int_{\alpha_1}^{-\lambda} \frac{d\kappa}{\sqrt{(\kappa - \alpha_1)}} = 2\sqrt{-\lambda - \alpha_1}$, we find using (73) the following upper bound

$$|\Lambda_{(\alpha_1, -\lambda)}| \leq \left(\frac{4(\alpha_1^2 - \lambda^2) \sqrt{-\lambda - \alpha_1}}{\sqrt{d(\alpha_4 + \lambda)((-\lambda - m)^2 + n^2)}} \right). \quad (75)$$

Table 4
Behavior of the angular change for $0 < \lambda < 1$.

	$\lim \Lambda^\lambda(d)$
$d \rightarrow -\delta_2^+$	$\frac{2\pi\eta_2}{\sqrt{\eta_2^2+2}}$
$d \rightarrow \psi^\pm$	$\Lambda_1 \mp \pi$
$d \rightarrow 0^-$	$-\pi$
$d \rightarrow 0^+$	0
$d \gg 0$	$O(\frac{1}{\sqrt[4]{d}}) > 0$

Analogously, from (74) we obtain for $\lambda < 1$ ($\delta_1 > 0$, here)

$$|\Lambda_{(\alpha_1, -\lambda)}| \geq \frac{2(\lambda^2(3\alpha_1 + 2\lambda) - \alpha_1^3)}{3(d + 4(\alpha_1 + \lambda)^2)} \sqrt{\frac{d}{d + \delta_1}}.$$

Notice that $m \rightarrow 0$, $\alpha_4/\alpha_1 \rightarrow -1$ and $n^2/\alpha_1^2 \rightarrow 1$ as $d \rightarrow +\infty$. Hence, one may conclude that $\Lambda_{(\alpha_1, -\lambda)} = O(1/\sqrt[4]{d})$.

Finally, since

$$\begin{aligned} \lim_{d \rightarrow +\infty} \sqrt[4]{d} \Lambda^\lambda(d) &= \lim_{d \rightarrow +\infty} \sqrt[4]{d} (\Lambda_{(\alpha_1, -\lambda)} + \Lambda_{(-\lambda, \lambda)} + \Lambda_{(\lambda, \alpha_4)}) \\ &= \lim_{d \rightarrow +\infty} \sqrt[4]{d} (\Lambda_{(\alpha_1, -\lambda)} + \Lambda_{(\lambda, \alpha_4)}) = C > 0, \end{aligned}$$

we conclude that the angular change is positive as $d \rightarrow \infty$. This finishes the proof of Table 4.

Case 2.

Case 2.1. If $\lambda > 1$ and $d \in (-\delta_2, 0) \cup (0, -\delta_1 + 1)$, the analysis is very similar to the one made in the spherical case, [12].

Case 2.1. If $\lambda > 1$ and $d \gg 0$. In this case the polynomial Q_d has also two roots and the analysis is totally parallel to that of case 1.2.

1. When $\kappa \in (-\lambda, \lambda)$ formulas (70) and (72) still hold, while since $\lambda > 1$ one gets

$$|\varphi(\kappa)| \leq \frac{\lambda^2 - \kappa^2}{\sqrt{dQ_d(\eta_1)}}, \quad (76)$$

from which we see that $|\Lambda_{(-\lambda, \lambda)}| \leq \frac{8\lambda^3}{3\sqrt{d(d+\delta_1)}} = O(\frac{1}{d})$ (notice that $\delta_1 < 0$ in this case).

2. If $\kappa \in (\alpha_1, -\lambda)$ (or $\kappa \in (\lambda, \alpha_4)$), the analysis is also similar to case 1.2. Formulas (73) and (75) remain valid, but now since $\lambda > 1$ we have

$$|\varphi(\kappa)| \geq \frac{\sqrt{d}(\kappa^2 - \lambda^2)}{(d + 4(\alpha_1 + \lambda)^2)\sqrt{Q_d(-\lambda)}}, \quad (77)$$

from where one has

$$|\Lambda_{(\alpha_1, -\lambda)}| \geq \frac{2(\lambda^2(3\alpha_1 + 2\lambda) - \alpha_1^3)}{3(d + 4(\alpha_1 + \lambda)^2)}. \quad (78)$$

Again $m \rightarrow 0$, $\alpha_4/\alpha_1 \rightarrow -1$ and $n^2/\alpha_1^2 \rightarrow 1$ as $d \rightarrow +\infty$, what implies that $\Lambda_{(\alpha_1, -\lambda)} = O(1/\sqrt[4]{d})$.

Proceeding as in the previous case, one can obtain Table 3.

A.2. Computations of the energy

For any value of λ and d given in Tables 1 and 2, we denote by $\mathcal{E}^\lambda(d)$ the energy of the corresponding critical point in one period of its curvature. We want to analyze in this section how $\mathcal{E}^\lambda(d)$ behaves. We use the following notation

$$\mathcal{E}_{\delta_i}^\lambda = -\frac{8(-1)^i(\eta_i^2 - 1)^2\pi}{\eta_i\sqrt{(\eta_i^2 + 2)(\eta_i^2 - 1)}}, \quad i \in \{1, 2\}; \quad \mathcal{E}_0^\lambda = 8(1 - \lambda^2)^{1/2} + 8\lambda \left[\arcsin(\lambda) - \frac{\pi}{2} \right].$$

Assume for the moment that $0 < \lambda < 1$. We compute first $\lim_{d \rightarrow -\delta_2} \mathcal{E}^\lambda(d)$. In this case the polynomial Q_d has two roots and it can be written in the following way

$$Q_d(\kappa) = (\alpha_3 - \kappa)(\kappa - \alpha_4)((\kappa - m)^2 + n^2),$$

where

$$m = -\frac{(\alpha_3 + \alpha_4)}{2}, \quad n^2 = 3m^2 - \alpha_3\alpha_4 - 4 - 2\lambda^2. \quad (79)$$

Table 5
Behavior of the energy for $\lambda < 1$.

	$d \rightarrow -\delta_2$	$d \rightarrow -\delta_1$	$d \rightarrow 0^-$	$d \rightarrow 0^+$	$d \gg 0$
$\lim \mathcal{E}^\lambda(d)$	$\mathcal{E}_{\delta_2}^\lambda$	$\mathcal{E}_{\delta_1}^\lambda + 8\lambda\pi$	$\mathcal{E}_0^\lambda + 8\lambda\pi$	$2\mathcal{E}_0^\lambda + 8\lambda\pi$	$O(\sqrt[4]{d})$
$\lim \tilde{\mathcal{E}}^\lambda(d)$		$\mathcal{E}_{\delta_1}^\lambda$	$\mathcal{E}_0^\lambda$		

Table 6
Behavior of the energy for $\lambda > 1$.

	$d \rightarrow -\delta_2$	$d \rightarrow 0^-$	$d \rightarrow \delta_2^\pm$	$d \gg 0$
$\lim \mathcal{E}^\lambda(d)$	$\mathcal{E}_{\delta_2}$	$8\pi\lambda$	∞	$O(\sqrt[4]{d})$
$\lim \tilde{\mathcal{E}}^\lambda(d)$		0	∞	

If $d \rightarrow -\delta_2$, the curvatures of this family approach to the constant η_2 and the different parameters defined in (15), (16) and (79) approach to

$$\alpha_3 = \alpha_4 = \eta_2, \quad M = 0, \quad n^2 = -2\lambda\eta_1 = 4\frac{\lambda}{\eta_2} = 4 - \frac{8}{\eta_2^2},$$

$$m = -\eta_2, \quad \rho = \frac{2\pi}{r}, \quad r = \frac{1}{\eta_2} \sqrt{(\eta_2^2 + 2)(\eta_2^2 - 1)}.$$

Therefore $\lim_{d \rightarrow -\delta_2} \mathcal{E}^\lambda(d) = \int_0^\rho (\eta_2 + \lambda)^2 = \mathcal{E}_{\delta_2}^\lambda$.

Now we compute $\lim_{d \rightarrow -\delta_1^+} \mathcal{E}^\lambda(d)$. If $d \rightarrow -\delta_1^+$, we are working in the zone where two critical curves appear for each value of d , so that we have two families of solutions. As $d \rightarrow -\delta_1^+$ the first family, corresponding to the solutions whose curvature functions vary between the roots α_1 and α_2 of the polynomial Q_d , approaches to the curve with constant curvature $\kappa = \eta_1$, which has the following values for the parameters given in (15) and (16)

$$M = 0, \quad \alpha_1 = \alpha_2 = \eta_1, \quad \alpha_3 = -\eta_1 + \frac{2}{\eta_1} \sqrt{2 - \eta_1^2},$$

$$\alpha_4 = -\eta_1 - \frac{2}{\eta_1} \sqrt{2 - \eta_1^2}, \quad r = -\frac{1}{2\eta_1} \sqrt{(\eta_1^2 + 2)(\eta_1^2 - 1)}.$$

Then, denoting by $\tilde{\mathcal{E}}^\lambda(d)$ the energy of a critical point of this family in one period of its curvature, we have $\lim_{d \rightarrow -\delta_1^+} \tilde{\mathcal{E}}^\lambda(d) = \mathcal{E}_{\delta_1}^\lambda$.

The second family corresponds to the solutions whose curvature functions vary between the roots α_3 and α_4 of the polynomial Q_d . Their curvatures are given by (13) which in this case satisfy

$$\lim_{d \rightarrow -\delta_1^+} \kappa(s) = \alpha_2 + \frac{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_2)}{(\alpha_3 - \alpha_2) + (\alpha_4 - \alpha_3) \cos^2(rs)}.$$

Now, taking into account formulas 2.563 of [28] and using the following notation

$$E_1(x) = -\sqrt{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_2)} \arctan \left(\sqrt{\frac{(\alpha_4 - \alpha_2)}{(\alpha_3 - \alpha_2)}} \cot(x) \right),$$

$$E_2(x) = -\frac{1}{2} \frac{(\alpha_4 + \alpha_3 - 2\alpha_2)}{\sqrt{(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)}} \arctan \left(\sqrt{\frac{(\alpha_4 - \alpha_2)}{(\alpha_3 - \alpha_2)}} \cot(x) \right) - \frac{1}{2} \frac{(\alpha_4 - \alpha_3) \sin x \cos x}{(\alpha_3 - \alpha_2) + (\alpha_4 - \alpha_3) \cos^2 x},$$

and

$$E(x) = \frac{1}{r} \{ (\alpha_2 + \lambda)^2 x + 2(\alpha_2 + \lambda)E_1(x) + (\alpha_4 - \alpha_2)(\alpha_3 - \alpha_2)E_2(x) \},$$

we obtain that

$$\lim_{d \rightarrow -\delta_1^+} \mathcal{E}^\lambda(d) = E(\pi) - E(0) = \mathcal{E}_{\delta_1}^\lambda + 8\lambda\pi.$$

Similar arguments can be used in the remaining cases and after long (and usually much more complicated) computations one gets Table 5 (see also Fig. 11).

Dealing with the case $\lambda > 1$ in an analogous manner, one can prove also that Table 6 holds (see also Fig. 12).

References

- [1] L.D. Landau, E.M. Lifshitz, *Theory of Elasticity*, vol. 7 Course of Theoretical Physics, vol. 7, MA: Butterworth Heinemann, Boston, 1986.
- [2] R.E. Goldstein, S.A. Langer, Nonlinear dynamics of stiff polymers, *Phys. Rev. Lett.* 75 (1995) 1094–1097.
- [3] R. Kamien, The geometry of soft materials: a primer, *Rev. Mod. Phys.* 74 (2002) 953–971.
- [4] J. Arroyo, Presión calibrada total. Estudio Variacional y aplicaciones al problema de Chen-Willmore, Ph. D. Thesis. Univ. del País Vasco, Leioa, Spain, January 2001.
- [5] J. Langer, D.A. Singer, The total squared curvature of closed curves, *J. Diff. Geom.* 20 (1984) 1–22.
- [6] L. Euler, Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti Bousquet, Lausannae et Genevae E 65A. O.O. Ser. I vol. 24, 1744.
- [7] A. Linner, Curve straightening and the Palais–Smale condition, *Trans. A.M.S.* 350 (1998) 3743–3765.
- [8] J. Langer, D.A. Singer, Knotted elastic curves in $\mathbb{R}^3$, *J. London Math. Soc.* 30 (1984) 512–520.
- [9] D.A. Singer, Lectures on elastic curves and rods, *Curvature and Variational Modeling in Physics and Biophysics*, in: O.J. Garay, E. García-Río, R. Vázquez-Lorenzo (Eds.), AIP Conference Proceedings 1002, Melville, New York, 2008, pp. 3–32.
- [10] J. Arroyo, O.J. Garay, J.J. Mencía, Extremals of curvature energy actions on spherical closed curves, *J. Geom. Phys.* 51 (2004) 101–125.
- [11] R. Capovilla, C. Chrysomalakos, J. Guven, Hamiltonian for curves, *J. Phys. A: Math. Gen.* 35 (2002) 6571–6587.
- [12] J. Arroyo, O.J. Garay, J.J. Mencía, Elastic circles in $\mathbb{S}^2(1)$, *J. Physics A: Math. Gen.* 39 (2006) 2307–2324.
- [13] J. Arroyo, O.J. Garay, J.J. Mencía, Closed generalized elastic curves in $\mathbb{S}^2(1)$, *J. Geom. and Phy.* 48 (2003) 339–353.
- [14] P.B. Canham, The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell, *J. Theor. Biol.* 26 (1970) 61–81.
- [15] W. Helfrich, Elastic properties of lipid bilayers: theory and possible experiments, *Z. Naturforsch* 28c (1973) 693–703.
- [16] X. Michalet, D. Bensimon, Vesicles of toroidal topology: observed morphology and shape transformations, *J. Phys. II France* 5 (1995) 263–287.
- [17] J.C.C. Nitsche, Boundary value problems for variational integrals involving surface curvatures, *Q. Appl. Math.* 60 (1993) 363–387.
- [18] U. Seifert, K. Berndl, R. Lipowsky, Shape transformations of vesicles: phase diagram for spontaneous curvature and bilayer coupling model, *Phys. Rev. A* 44 (1991) 1182.
- [19] T.J. Willmore, *Total Curvature in Riemannian Geometry*, E Horwood Limited, Chichester, 1982.
- [20] M. Barros, A. Ferrández, P. Lucas, Conformal tension in string theories and M-theory, *Nuclear Phys. b* 584 (2000) 719–748.
- [21] M. Barros, Simple geometric models with applications to physics, *Curvature and Variational Modeling in Physics and Biophysics*, in: O.J. Garay, E. García-Río, R. Vázquez-Lorenzo (Eds.), AIP Conference Proceedings 1002, Melville, New York, 2008, pp. 71–113.
- [22] A.L. Kholodenko, V.V. Nesterenko, Classical dynamics of the rigid string from the Willmore funcional, *J. Geom. Phys.* 16 (1995) 15–26.
- [23] J. Arroyo, O.J. Garay, Hopf vesicles in $\mathbb{S}^3(1)$, *Global Diff. Geom. The mathematical Legacy of A. Gray. Contemporary Math.* 288 (2001) 258–262.
- [24] U. Pinkall, Hopf tori in $\mathbb{S}^3$, *Inventiones Math.* 81 (1985) 379–386.
- [25] H. Hopf, ber die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, *Math. Ann.* 104 (1931) 637–665.
- [26] H.K. Urbantke, The Hopf fibration seven times in physics, *J. Geom. and Phy.* 46 (2003) 125–150.
- [27] R.S. Palais, Critical point theory and the minimax principle, *Global Analysis, Proc. Sympos. Pure Math.* 15 (1970) 185–212.
- [28] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1980.