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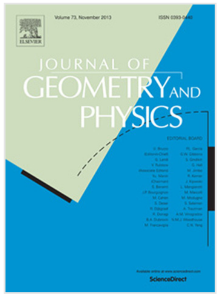

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A bicategory of reduced orbifolds from the point of view of differential geometry

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Abstract

We describe a bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ of reduced orbifolds in the framework of classical differential geometry (i.e. without any explicit reference to the notions of Lie groupoids or differentiable stacks, but only using orbifold atlases, local lifts and changes of charts). In order to construct such a bicategory, we firstly define a 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ whose objects are reduced orbifold atlases (on any paracompact, second countable, Hausdorff topological space). The definition of morphisms is obtained as a slight modification of a definition by A. Pohl, while the definitions of 2-morphisms and compositions of them are new in this setup. Using the bicalculus of fractions described by D. Pronk, we are able to construct the bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ from the 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. We prove that $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ is equivalent to the bicategory of reduced orbifolds described in terms of proper, effective, étale Lie groupoids by D. Pronk and I. Moerdijk and to the well-known 2-category of reduced orbifolds constructed from a suitable class of differentiable Deligne-Mumford stacks.

Keywords: Reduced orbifolds, Lie groupoids, differentiable stacks, 2-categories, bicategories
2010 MSC: 14D20, 14H51, 14H60, 14F45

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Introduction

A well-known issue in mathematics is that of modeling geometric objects where points have non-trivial groups of automorphisms. In topology and differential geometry the standard approach to these objects (when each point has a finite group of automorphisms) is through orbifolds. This concept was formalized for the first time by Ikiro Satake in 1956 in [Sa] with some different hypotheses than the current ones, although the informal idea dates back at least to Henri Poincaré (for example, see [Poi]). Currently there are at least 3 main approaches to orbifolds:

(1) via orbifold atlases and "good maps" between them, as described in [CR],

(2) via the 2-category of proper, étale (Lie) groupoids, "localized" with respect to weak equivalences (see for example [Pr], [M] and [MM]),

(3) via a family of C^∞-Deligne-Mumford stacks (see for example [J1] and [J2]).

On the one hand, the approach in (1) gives rise to a 1-category. On the other hand, the approach in (2) gives rise to a bicategory (i.e. almost a 2-category, where compositions of 1-morphisms is associative only up to canonical 2-morphisms) and the approach in (3) gives rise to a 2-category. It was proved in [Pr] that (2) and (3) are equivalent as bicategories. Since (2) and (3) are compatible approaches, then one might argue that:

(i) there should also exist a non-trivial structure of 2-category or bicategory, having as objects orbifold atlases or equivalence classes of them (i.e. orbifold structures);

(ii) the structure of (i) should be compatible with the approaches of (2) and (3), and it should replace the approach of (1) (since (1) gives rise only to a 1-category instead of a 2-category or bicategory).

In the present paper we will manage to prove both (i) and (ii) for the family of all *reduced* orbifolds, i.e. orbifolds that are locally modeled on open connected sets of some $\mathbb{R}^n$, modulo finite groups acting smoothly and *effectively* on them. In order to do that, we proceed as follows.

- We describe a 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ whose objects are reduced orbifold atlases on any paracompact, second countable, Hausdorff topological space. The definition of morphisms is obtained as a slight modification of an analogous definition given by Anke Pohl in [Po], while the notion of 2-morphisms (and compositions of them) is new in this setup (see Definitions 1.9). Such notions are useful for differential geometers mainly because they don't require any previous knowledge of Lie groupoids and/or differentiable stacks. $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ is a 2-category, but it is still not the structure that we want to get in (i); indeed in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ different orbifold atlases representing the same orbifold structure in general are not related by an isomorphism or by an internal equivalence.
- We recall briefly the definition of the 2-category $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$, whose objects are proper, effective, étale differentiable groupoids, and we describe in Theorem 3.15 a 2-functor $\mathcal{F}^{\,\mathrm{red}} : (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \to (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$.

- In [Pr] Dorette Pronk proved that the class $\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}$ of all weak equivalences in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ (also known as essential equivalences) admits a right bicalculus of fractions. Roughly speaking, this amounts to saying that there are a bicategory $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$ and a pseudofunctor

$$\mathcal{U}_{\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}} : (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}) \longrightarrow (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$$

  that sends each weak equivalence to an internal equivalence, and that is universal with respect to this property. The bicategory obtained in this way is the bicategory that we mentioned in (2) above, if we restrict to the case of reduced orbifolds.

- In $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ we select a class $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$ of morphisms (that we call "refinements" of reduced orbifold atlases, see Definition 5.1), and we prove that such a class admits a right bicalculus of fractions. Therefore, we are able to construct a bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ and a pseudofunctor

$$\mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}} : (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \longrightarrow (\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb}) := (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})\left[\mathbf{W}^{-1}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}\right]$$

  that sends each refinement to an internal equivalence, and that is universal with respect to this property (see Proposition 6.1). Objects in this new bicategory are again reduced orbifold atlases; a morphism from an atlas $\mathcal{X}$ to an atlas $\mathcal{Y}$ is any triple consisting of a reduced orbifold atlas $\mathcal{X}'$, a refinement $\mathcal{X}' \to \mathcal{X}$ and a morphism $\mathcal{X}' \to \mathcal{Y}$. In other terms, a morphism from $\mathcal{X}$ to $\mathcal{Y}$ is given firstly by replacing $\mathcal{X}$ with a "refined" atlas $\mathcal{X}'$ (keeping track of the refinement), then by considering a morphism from $\mathcal{X}'$ to $\mathcal{Y}$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. We refer to Description 6.3 for the notion of 2-morphisms in this bicategory.

- Lastly, using the results about bicategories of fractions that we proved in our previous papers [T3] and [T4], we are able to prove that:

  Theorem A (Proposition 7.5 and Theorem 8.3). *There is pseudofunctor $\mathcal{G}^{\,\mathrm{red}}$ (explicitly constructed), making the next diagram commute; assuming the axiom of choice, $\mathcal{G}^{\,\mathrm{red}}$ is an equivalence of bicategories.*

$$\begin{tikzcd}
(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \arrow[r, "\mathcal{F}^{\,\mathrm{red}}"] \arrow[d, "\mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}}"'] \arrow[dr, phantom, "\curvearrowright"] & (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}) \arrow[d, "\mathcal{U}_{\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}}"] \\
(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb}) \arrow[r, dashed, "\mathcal{G}^{\,\mathrm{red}}"'] & (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right].
\end{tikzcd} \tag{0.1}$$

  Since (2) and (3) are equivalent approaches by [Pr], this implies at once that:

  Theorem B (Theorem 8.4). $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ *is equivalent to the 2-category $(\mathbf{Orb}^{\mathbf{eff}})$ of effective orbifolds described as a full 2-subcategory of the 2-category of C^∞-Deligne-Mumford stacks.*

In all this paper we will not use explicitly the language of stacks; however, it is important to remark that:

- in the language of (differentiable) stacks, the notion of objects is complicated and does not provide a simple geometric intuition, since it is based on the the notions of pseudofunctor (or category fibered in groupoids), Grothendieck topology and descent conditions. However, having managed to understand stacks, 1-morphisms and 2-morphisms are almost straightforward to define and the resulting structure is that of a 2-category;

- in the language used in the present paper, objects are very easy to describe since they are simply reduced orbifold atlases; as we mentioned above, morphism are also easy to describe. However, compositions of 1-morphisms and the notion of 2-morphisms require more care (see Description 6.3); moreover the resulting structure will be that of a bicategory, hence compositions will be associative only up to canonical 2-morphisms.

One important problem remains still open: we have described a bicategory that solves problems (i) and (ii) by *restricting* to reduced orbifolds. Is it possible to give an analogous description of a bicategory ($\mathcal{O}\mathbf{rb}$) also in the more general case of (possibly) non-reduced orbifolds? Since the bicategories of (2) and (3) are also defined (and equivalent) in this more general setup, in principle this should be possible, but it would require much more work.

1. Reduced orbifold atlases

Let us review some basic definitions about reduced orbifolds.

Definition 1.1. [MP, § 1] Let X be a paracompact, second countable, Hausdorff topological space and let $X' \subseteq X$ be open and non-empty. Then a *reduced orbifold chart* (also known as *reduced uniformizing system*) *of dimension* n for X' is the datum of a *connected* open subset $\widetilde{X}$ of $\mathbb{R}^n$, a *finite* group G of smooth automorphisms of $\widetilde{X}$ and a continuous, surjective and G-invariant map $\pi : \widetilde{X} \twoheadrightarrow X'$, which induces an homeomorphism between $\widetilde{X}/G$ and X' (where we give to $\widetilde{X}/G$ the quotient topology)

Remark 1.2. We will always assume that G *acts effectively*; the orbifolds that have this property are usually called *reduced* or *effective*. Some of the current literature on orbifolds assumes that $\widetilde{X}$ is only a connected smooth manifold instead of an open connected subset of some $\mathbb{R}^n$. This makes a difference for the definition of charts, but the arising notion of orbifold is not affected by that.

The following definition is a special case of [Po, § 2.1].

Definition 1.3. Let us fix any pair of reduced charts $(\widetilde{X}_1, G_1, \pi_1)$ and $(\widetilde{X}_2, G_2, \pi_2)$ for subsets X_1, X_2 of X. Then a *change of charts* from $(\widetilde{X}_1, G_1, \pi_1)$ to $(\widetilde{X}_2, G_2, \pi_2)$ is any diffeomorphism $\lambda : \widetilde{Y}_1 \xrightarrow{\sim} \widetilde{Y}_2$ such that:

- $\widetilde{Y}_1$ is any connected component of $\pi_1^{-1}(Y)$ for some open non-empty subset Y of X_1 (since the action of G_1 on $\widetilde{X}_1$ permutes such connected components, then $\pi_1(\widetilde{Y}_1) = Y$);
- $\widetilde{Y}_2$ is an open subset of $\widetilde{X}_2$;
- $\pi_2 \circ \lambda = \pi_1|_{\widetilde{Y}_1}$.

Using [MP, Lemma A.2] and the fact that λ is a diffeomorphism, it turns out that Y is contained also in X_2 and that $\widetilde{Y}_2$ is a connected component of $\pi_2^{-1}(Y)$. So the inverse of any change of charts is again a change of charts. If λ is any change of charts, we denote by $\operatorname{dom}\lambda$ its domain and by $\operatorname{cod}\lambda$ its codomain.

If $\widetilde{x} \in \operatorname{dom}\lambda$, then $\operatorname{germ}_{\widetilde{x}}\lambda$ denotes the germ of λ at $\widetilde{x}$. An *embedding* is any change of charts λ as above, such that $\operatorname{dom}\lambda = \widetilde{X}_1$. Two charts as before are called *compatible* if for each pair $\widetilde{x}_1 \in \widetilde{X}_1$, $\widetilde{x}_2 \in \widetilde{X}_2$ with $\pi_1(\widetilde{x}_1) = \pi_2(\widetilde{x}_2)$, there exists a change of charts λ from $(\widetilde{X}_1, G_1, \pi_1)$ to $(\widetilde{X}_2, G_2, \pi_2)$, with $\widetilde{x}_1 \in \operatorname{dom}\lambda$. Up to composing λ with an element of G_2, this is equivalent to the existence of a change of charts λ such that $\widetilde{x}_1 \in \operatorname{dom}\lambda$ and $\lambda(\widetilde{x}_1) = \widetilde{x}_2$.

Using [MM, Lemma 2.10], the following definition is equivalent to [MP, § 1].

Definition 1.4. Let X be a paracompact, second countable, Hausdorff topological space; a *reduced orbifold atlas of dimension* n on X is any family $\mathcal{X} = \{(\widetilde{X}_i, G_i, \pi_i)\}_{i\in I}$ of reduced orbifolds charts of dimension n, such that $\{X_i := \pi_i(\widetilde{X}_i)\}_{i\in I}$ is an open cover of X, and any two charts of $\mathcal{X}$ are compatible. Given any pair $(i, i') \in I \times I$, we denote by $\mathcal{C}h(\mathcal{X}, i, i')$ the set of all changes of charts λ from $(\widetilde{X}_i, G_i, \pi_i)$ to $(\widetilde{X}_{i'}, G_{i'}, \pi_{i'})$ and we set $\mathcal{C}h(\mathcal{X}) := \coprod_{(i,i')\in I\times I} \mathcal{C}h(\mathcal{X}, i, i')$. If $\mathcal{X}'$ is another reduced orbifold atlas for X, we say that it is *equivalent* to $\mathcal{X}$ if and only if any chart of $\mathcal{X}$ is compatible with any chart of $\mathcal{X}'$ (equivalently, if the union of $\mathcal{X}$ and $\mathcal{X}'$ is still an atlas). A *reduced orbifold* of dimension n is any pair $(X, [\mathcal{X}])$ consisting of a paracompact, second countable, Hausdorff topological space X and a class $[\mathcal{X}]$ of equivalent atlases on X.

Definition 1.5. Let us fix any pair of reduced orbifold atlases $\mathcal{X} = \{(\widetilde{X}_i, G_i, \pi_i)\}_{i\in I}$ and $\mathcal{Y} = \{(\widetilde{Y}_j, H_j, \chi_j)\}_{j\in J}$ for X and Y respectively. Then a *representative of a morphism* from $\mathcal{X}$ to $\mathcal{Y}$ is any tuple $\hat{f} := (f, \overline{f}, \{\widetilde{f}_i\}_{i\in I}, P_f, \nu_f)$ that satisfies the following conditions:

(M1) $f : X \to Y$ is any continuous map;

(M2) $\overline{f} : I \to J$ is any set map, such that $f(\pi_i(\widetilde{X}_i)) \subseteq \chi_{\overline{f}(i)}(\widetilde{Y}_{\overline{f}(i)})$ for each $i \in I$;

(M3) for each $i \in I$, $\widetilde{f}_i$ is a C^∞-map $\widetilde{X}_i \to \widetilde{Y}_{\overline{f}(i)}$ that is a *local lift of* f with respect to the orbifold charts $(\widetilde{X}_i, G_i, \pi_i) \in \mathcal{X}$ and $(\widetilde{Y}_{\overline{f}(i)}, H_{\overline{f}(i)}, \chi_{\overline{f}(i)}) \in \mathcal{Y}$, i.e. $\chi_{\overline{f}(i)} \circ \widetilde{f}_i = f \circ \pi_i$;

(M4) P_f is any subset of $\mathcal{C}h(\mathcal{X})$ with the property that for any $\lambda \in \mathcal{C}h(\mathcal{X})$ and for any $\widetilde{x} \in \operatorname{dom}\lambda$, there exists $\hat{\lambda} \in P_f$, such that $\widetilde{x} \in \operatorname{dom}\hat{\lambda}$ and $\operatorname{germ}_{\widetilde{x}}\lambda = \operatorname{germ}_{\widetilde{x}}\hat{\lambda}$;

(M5) $\nu_f : P_f \to \mathcal{C}h(\mathcal{Y})$ is any set map that assigns to each $\lambda \in P_f(i, i')$ a change of charts $\nu_f(\lambda) \in \mathcal{C}h(\mathcal{Y}, \overline{f}(i), \overline{f}(i'))$, such that:

  (a) $\operatorname{dom}\nu_f(\lambda)$ is an open set containing $\widetilde{f}_i(\operatorname{dom}\lambda)$,

  (b) $\operatorname{cod}\nu_f(\lambda)$ is an open set containing $\widetilde{f}_{i'}(\operatorname{cod}\lambda)$,

  (c) $\widetilde{f}_{i'} \circ \lambda = \nu_f(\lambda) \circ \widetilde{f}_i|_{\operatorname{dom}\lambda}$,

  (d) for all $i \in I$, for all $\lambda, \lambda' \in P_f(i, -)$ and for all $\widetilde{x}_i \in \operatorname{dom}\lambda \cap \operatorname{dom}\lambda'$ with $\operatorname{germ}_{\widetilde{x}_i}\lambda = \operatorname{germ}_{\widetilde{x}_i}\lambda'$, we have

$$\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda) = \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda'),$$

  (e) for all $(i, i', i'') \in I^3$, for all $\lambda_1 \in P_f(i, i')$, for all $\lambda_2 \in P_f(i', i'')$ and for all $\widetilde{x}_i \in \lambda_1^{-1}(\operatorname{cod}\lambda_1 \cap \operatorname{dom}\lambda_2)$, we have

$$\operatorname{germ}_{\widetilde{f}_{i'}(\lambda_1(\widetilde{x}_i))}\nu_f(\lambda_2) \cdot \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda_1) = \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda_3),$$

  where λ_3 is any element of $P_f(i, i'')$ such that $\operatorname{germ}_{\widetilde{x}_i}\lambda_3 = \operatorname{germ}_{\widetilde{x}_i}\lambda_2 \circ \lambda_1$ (it exists by (M4)),

  (f) for all $i \in I$, for all $\lambda \in P_f(i, i)$ and for all $\widetilde{x}_i \in \operatorname{dom}\lambda$ such that $\operatorname{germ}_{\widetilde{x}_i}\lambda = \operatorname{germ}_{\widetilde{x}_i}\operatorname{id}_{\widetilde{X}_i}$, we have

$$\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda) = \operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\operatorname{id}_{\widetilde{Y}_{\overline{f}(i)}}.$$

Given another representative $\hat{f} := (f', \overline{f}', \{\widetilde{f}'_i\}_{i\in I}, P_{f'}, \nu_{f'})$ of a morphism from $\mathcal{X}$ to $\mathcal{Y}$, we say that $\hat{f}$ is *equivalent* to $\hat{f}'$ if and only if $f = f'$, $\overline{f} = \overline{f}'$, $\widetilde{f}_i = \widetilde{f}'_i$ for all $i \in I$, and

$$\mathrm{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\,\nu_f(\lambda) = \mathrm{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\,\nu_{f'}(\lambda') \tag{1.1}$$

for all $i \in I$, for all $\lambda \in P_f(i,-)$, $\lambda' \in P_{f'}(i,-)$ and for all $\widetilde{x}_i \in \mathrm{dom}\,\lambda \cap \mathrm{dom}\,\lambda'$ with $\mathrm{germ}_{\widetilde{x}_i}\,\lambda = \mathrm{germ}_{\widetilde{x}_i}\,\lambda'$. This defines an equivalence relation (it is reflexive by (M5d)). The equivalence class of $\hat{f}$ will be denoted by

$$[\hat{f}] = \left(f, \overline{f}, \left\{\widetilde{f}_i\right\}_{i\in I}, [P_f, \nu_f]\right) : \mathcal{X} \longrightarrow \mathcal{Y} \tag{1.2}$$

and it will be called a *morphism of reduced orbifold atlases from* $\mathcal{X}$ *to* $\mathcal{Y}$ *over the continuous map* $f : X \to Y$.

Remark 1.6. In the notations of [Po], condition (M4) is the condition that P_f *generates* the pseudogroup $\mathcal{C}h(\mathcal{X})$ inside the larger pseudogroup $\Psi(\mathcal{X})$ defined and used in [Po]; such a pseudogroup is obtained by taking into account all changes of charts of $\mathcal{X}$ with a more general definition than the one used in the present paper. In [Po] there are other two technical conditions (axioms of "quasi-pseudogroup"), but they are implied by (M4) in our case, so we can omit them. Under this remark, our next definition of morphism of orbifold atlases $\mathcal{X} \to \mathcal{Y}$ is equivalent to the definition of "orbifold map with domain atlas $\mathcal{X}$ and range atlas $\mathcal{Y}$" stated in [Po, Definitions 4.4 and 4.10] (even if the representatives for the maps described in [Po] are different from the representatives used in the present paper).

Construction 1.7. Let us fix orbifold atlases $\mathcal{X}$, $\mathcal{Y}$ (as above) and $\mathcal{Z} = \{(\widetilde{Z}_l, K_l, \eta_l)\}_{l\in L}$, and any two morphisms

$$[\hat{f}] = \left(f, \overline{f}, \left\{\widetilde{f}_i\right\}_{i\in I}, [P_f, \nu_f]\right) : \mathcal{X} \longrightarrow \mathcal{Y}, \qquad [\hat{g}] = \left(g, \overline{g}, \left\{\widetilde{g}_j\right\}_{j\in J}, [P_g, \nu_g]\right) : \mathcal{Y} \longrightarrow \mathcal{Z}. \tag{1.3}$$

Then we define a composition

$$[\hat{g}] \circ [\hat{f}] := \left(g \circ f, \overline{g} \circ \overline{f}, \left\{\widetilde{g}_{\overline{f}(i)} \circ \widetilde{f}_i\right\}_{i\in I}, [P_{g\circ f}, \nu_{g\circ f}]\right) : \mathcal{X} \longrightarrow \mathcal{Z}. \tag{1.4}$$

Here we construct the class $[P_{g\circ f}, \nu_{g\circ f}]$ as follows. Firstly, we fix any representative (P_g, ν_g) for $[P_g, \nu_g]$; since $P_g \subseteq \mathcal{C}h(\mathcal{Y})$ satisfies condition (M4), following the lines of [Po, Construction 5.9] (with the only differences due to Remark 1.6) we can construct a subset $P_{g\circ f} \subseteq \mathcal{C}h(\mathcal{X})$, satisfying property (M4), and a set map $\nu_f^{\mathrm{ind}} : P_{g\circ f} \to P_g \subseteq \mathcal{C}h(\mathcal{Y})$, such that $(P_{g\circ f}, \nu_f^{\mathrm{ind}})$ is a representative for $[P_f, \nu_f]$. Then we simply define $\nu_{g\circ f} := \nu_g \circ \nu_f^{\mathrm{ind}}$. The construction of the pair $(P_{g\circ f}, \nu_f^{\mathrm{ind}})$ depends on the choices, but using axioms (M) it is easy to prove that (1.4) does not depend on such choices. Moreover, we have:

Lemma 1.8. *The composition* (1.4) *is associative.*

The proof is obvious for what concerns the composition of maps of the form $f, \overline{f}$ and $\widetilde{f}_i$; the proof of the associativity on the pairs of the form $[P_f, \nu_f]$ is straightforward, so we omit it. As we said in the introduction, our first aim is to construct a 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ of reduced orbifold atlases. Roughly speaking, a 2-category is the datum of objects, morphisms and "morphism between morphisms" (known as 2-*morphisms*), together with identities and compositions of morphisms and 2-morphisms (for more details we refer e.g. to [Lei]). First of all, we give the following:

Definition 1.9. Let us fix any pair of reduced orbifold atlases $\mathcal{X} = \{(\widetilde{X}_i, G_i, \pi_i)\}_{i\in I}$, $\mathcal{Y} := \{(\widetilde{Y}_j, H_j, \chi_j)\}_{j\in J}$ over X and Y respectively. Moreover, let us fix two morphisms $[\hat{f}^m] := (f, \overline{f}^m, \{\widetilde{f}^m_i\}_{i\in I}, [P_{f^m}, \nu_{f^m}]) : \mathcal{X} \to \mathcal{Y}$ for $m = 1, 2$, over *the same* continuous function $f : X \to Y$. Then a *representative of a 2-morphism* from $[\hat{f}^1]$ to $[\hat{f}^2]$ is any set of data $\delta := \{(\widetilde{X}^a_i, \delta^a_i)\}_{i\in I,\, a\in A(i)}$, such that:

(2Ma) for all $i \in I$ the set $\{\widetilde{X}^a_i\}_{a\in A(i)}$ is an open covering of $\widetilde{X}_i$;

(2Mb) for all $i \in I$ and for all $a \in A(i)$, δ_i^a is a change of charts in $\mathcal{Y}$, such that

$$\widetilde{f}_i^1\left(\widetilde{X}_i^a\right) \subseteq \operatorname{dom}\delta_i^a \subseteq \widetilde{Y}_{\overline{f}^1(i)}, \quad \widetilde{f}_i^2\left(\widetilde{X}_i^a\right) \subseteq \operatorname{cod}\delta_i^a \subseteq \widetilde{Y}_{\overline{f}^2(i)};$$

(2Mc) for all $i \in I$ and for all $a \in A(i)$, we have $\widetilde{f}_i^2|_{\widetilde{X}_i^a} = \delta_i^a \circ \widetilde{f}_i^1|_{\widetilde{X}_i^a}$;

(2Md) for all $i \in I$, for all $a, a' \in A(i)$ and for all $\widetilde{x}_i \in \widetilde{X}_i^a \cap \widetilde{X}_i^{a'}$, we have $\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\delta_i^a = \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\delta_i^{a'}$;

(2Me) for all $(i,i') \in I \times I$, for all $(a,a') \in A(i) \times A(i')$, for all $\lambda \in \mathcal{C}h(\mathcal{X}, i, i')$ and for all $\widetilde{x}_i \in \operatorname{dom}\lambda \cap \widetilde{X}_i^a$ such that $\lambda(\widetilde{x}_i) \in \widetilde{X}_{i'}^{a'}$, there exist data

$$(P_{f^m}, \nu_{f^m}) \in [P_{f^m}, \nu_{f^m}], \quad \lambda^m \in P_{f^m}(i,i') \quad \text{for } m = 1,2, \tag{1.5}$$

such that

$$\widetilde{x}_i \in \operatorname{dom}\lambda^m, \quad \operatorname{germ}_{\widetilde{x}_i}\lambda^m = \operatorname{germ}_{\widetilde{x}_i}\lambda \quad \text{for } m = 1,2 \tag{1.6}$$

and

$$\operatorname{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)}\nu_{f^2}(\lambda^2) \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\delta_i^a = \operatorname{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))}\delta_{i'}^{a'} \cdot \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\nu_{f^1}(\lambda^1). \tag{1.7}$$

Given another representative of a 2-morphisms from $[\hat{f}^1]$ to $[\hat{f}^2]$ as follows

$$\overline{\delta} := \left\{\left(\widetilde{X}_i^{\overline{a}}, \overline{\delta}_i^{\overline{a}}\right)\right\}_{i \in I, \overline{a} \in \overline{A}(i)},$$

we say that δ is *equivalent* to $\overline{\delta}$ if and only if for all $i \in I$, for all pairs $(a, \overline{a}) \in A(i) \times \overline{A}(i)$ and for all $\widetilde{x}_i \in \widetilde{X}_i^a \cap \widetilde{X}_i^{\overline{a}}$ (if non-empty), we have

$$\operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\delta_i^a = \operatorname{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)}\overline{\delta}_i^{\overline{a}}.$$

This gives rise to an equivalence relation (it is reflexive by (2Md)). We denote by $[\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ the class of any δ as before and we say that $[\delta]$ is a 2-*morphism* from $[\hat{f}^1]$ to $[\hat{f}^2]$. We denote by $[\delta]^{-1}$ the class of the collection $\{(\widetilde{X}_i^a, (\delta_i^a)^{-1})\}_{i \in I, a \in A(i)}$.

Remark 1.10. Let us suppose that there exist data as in (1.5) that satisfy conditions (1.6) and (1.7). Let $(P'_{f^m}, \nu'_{f^m}, \lambda'^m)$ for $m = 1,2$ be another set of data as in (1.5) that satisfies condition (1.6). Then by (M5d) we conclude that $\operatorname{germ}_{\widetilde{f}_i^m(\widetilde{x}_i)}\nu_{f^m}(\lambda^m) = \operatorname{germ}_{\widetilde{f}_i^m(\widetilde{x}_i)}\nu'_{f^m}(\lambda'^m)$ for $m = 1,2$, so (1.7) is verified also by the new set of data. Therefore, (2Me) is equivalent to:

(2Me)$'$ for all $(i,i') \in I \times I$, for all $(a,a') \in A(i) \times A(i')$, for all $\lambda \in \mathcal{C}h(\mathcal{X}, i, i')$, for all $\widetilde{x}_i \in \operatorname{dom}\lambda \cap \widetilde{X}_i^a$ such that $\lambda(\widetilde{x}_i) \in \widetilde{X}_{i'}^{a'}$ and *for all the data* (1.5) that satisfy (1.6), we have that (1.7) holds.

2. The 2-category ($\mathcal{R}$ed $\mathcal{A}$tl)

We assume here that the reader is familiar with the notions of 2-category, bicategory, pseudofunctor and 2-functor; we refer to [Lei, § 1.5] for a general overview on this subject.

Construction 2.1. Let us fix two reduced orbifold atlases $\mathcal{X} = \{(\widetilde{X}_i, G_i, \pi_i)\}_{i \in I}$, $\mathcal{Y} = \{(\widetilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ for X and Y respectively, any continuous map $f : X \to Y$ and any triple of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ over f:

$$[\hat{f}^m] := \left(f, \overline{f}^m, \left\{\widetilde{f}_i^m\right\}_{i \in I}, [P_{f^m}, \nu_{f^m}]\right) \quad \text{for} \quad m = 1,2,3.$$

In addition, let us fix any 2-morphism $[\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ and any 2-morphism $[\sigma] : [\hat{f}^2] \Rightarrow [\hat{f}^3]$. We want to define a *vertical composition* $[\sigma] \odot [\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^3]$; in order to do that, let us fix any representative

$$\sigma = \left\{ \left(\widetilde{X}_i^b, \sigma_i^b \right) \right\}_{i \in I,\, b \in B(i)}$$

for $[\sigma]$. Then it is not difficult to construct a (in general non-unique) representative

$$\delta = \left\{ \left(\widetilde{X}_i^a, \delta_i^a \right) \right\}_{i \in I,\, a \in A(i)}$$

for $[\delta]$, such that for each $i \in I$ and for each $(a, b) \in A(i) \times B(i)$, the set map δ_i^a restricted to the set $\widetilde{Y}_i^{a,b} := (\delta_i^a)^{-1}(\operatorname{cod} \delta_i^a \cap \operatorname{dom} \sigma_i^b)$ (if non-empty) is again a change of charts of $\mathcal{Y}$, so that also $\theta_i^{a,b} := \sigma_i^b \circ \delta_i^a|_{\widetilde{Y}_i^{a,b}}$ is a change of charts of $\mathcal{Y}$. Then for each $i \in I$ and for each $(a, b) \in A(i) \times B(i)$ we set $\widetilde{X}_i^{a,b} := \widetilde{X}_i^a \cap \widetilde{X}_i^b$; if $\widetilde{X}_i^{a,b}$ is non-empty, then by (2Mb) $\widetilde{Y}_i^{a,b}$ is also non-empty. Then we define:

$$\theta := \left\{ \left(\widetilde{X}_i^{a,b}, \theta_i^{a,b} \right) \right\}_{i \in I,\, (a,b) \in A(i) \times B(i) \text{ s.t. } \widetilde{X}_i^{a,b} \neq \varnothing}.$$

A straightforward proof shows that:

Lemma 2.2. *The collection θ defined above is a representative of a 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^3]$. Moreover, the class $[\theta]$ does not depend on the choices of representatives δ for $[\delta]$ and σ for $[\sigma]$.*

Therefore, it makes sense to give the following definition.

Definition 2.3. Given any pair $[\delta]$, $[\sigma]$ as before, we define their *vertical composition* as:

$$[\sigma] \odot [\delta] := [\theta] : [\hat{f}^1] \Longrightarrow [\hat{f}^3].$$

Construction 2.4. Let us fix any triple of reduced orbifold atlases $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ for X, Y and Z respectively (with the same notations used above), and any set of morphisms

$$[\hat{f}^m] := \left(f, \overline{f}^m, \left\{ \widetilde{f}_i^m \right\}_{i \in I}, [P_{f^m}, \nu_{f^m}] \right) : \mathcal{X} \longrightarrow \mathcal{Y} \quad \text{for} \quad m = 1, 2,$$

$$[\hat{g}^m] := \left(g, \overline{g}^m, \left\{ \widetilde{g}_j^m \right\}_{j \in J}, \left[P_{g^m}, \nu_{g^m} \right] \right) : \mathcal{Y} \longrightarrow \mathcal{Z} \quad \text{for} \quad m = 1, 2.$$

Moreover, let us choose any 2-morphism $[\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ and any 2-morphism $[\xi] : [\hat{g}^1] \Rightarrow [\hat{g}^2]$. Our aim is to define an *horizontal* composition $[\xi] * [\delta] : [\hat{g}^1] \circ [\hat{f}^1] \Rightarrow [\hat{g}^2] \circ [\hat{f}^2]$. In order to do that, we fix any representative (P_{g^1}, ν_{g^1}) for $[P_{g^1}, \nu_{g^1}]$. Then it is easy to construct a representative $\delta := \{(\widetilde{X}_i^a, \delta_i^a)\}_{i \in I,\, a \in A(i)}$ for $[\delta]$, such that for each $i \in I$ and for each $a \in A(i)$, the change of charts δ_i^a belongs to P_{g^1}. We choose also any representative $\xi := \{(\widetilde{Y}_j^c, \xi_j^c)\}_{j \in J, c \in C(j)}$ for $[\xi]$. Let us fix any $i \in I, a \in A(i), c \in C(\overline{f}^2(i))$ and let us consider the (possibly empty) sets

$$\widetilde{Z}_i^{a,c} := \left(\nu_g^1(\delta_i^a) \right)^{-1} \left(\operatorname{cod} \nu_g^1(\delta_i^a) \cap \operatorname{dom} \xi^c_{\overline{f}^2(i)} \right), \qquad \widetilde{X}_i^{a,c} := \widetilde{X}_i^a \cap \left(\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1 \right)^{-1} \left(\widetilde{Z}_i^{a,c} \right).$$

Let us fix any point $\overline{x}_i \in \widetilde{X}_i^{a,c}$ and let us set $\overline{z}_i := \widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}_i^1(\overline{x}_i) \in \widetilde{Z}_i^{a,c}$. Then we can easily construct a (non-unique) open connected subset $\widetilde{Z}_i^{a,c,\overline{x}_i} \subseteq \widetilde{Z}_i^{a,c}$, such that:

- $\overline{z}_i \in \widetilde{Z}_i^{a,c,\overline{x}_i}$;
- for all $h \in H_{\overline{g}^1 \circ \overline{f}^1(i)}$ such that $h(\overline{z}_i) = \overline{z}_i$, we have $h(\widetilde{Z}_i^{a,c,\overline{x}_i}) = \widetilde{Z}_i^{a,c,\overline{x}_i}$;
- for all $h \in H_{\overline{g}^1 \circ \overline{f}^1(i)}$ such that $h(\overline{z}_i) \neq \overline{z}_i$, we have $h(\widetilde{Z}_i^{a,c,\overline{x}_i}) \cap \widetilde{Z}_i^{a,c,\overline{x}_i} = \varnothing$

(in this way, $\nu^1_g(\delta^a_i)$ is a change of charts of $\mathcal{Z}$ if restricted to $\widetilde{Z}^{a,c,\overline{x}_i}_i$). Then we set

$$\gamma := \left\{\left(\widetilde{X}^a_i \cap \left(\widetilde{g}^1_{\overline{f}^1(i)} \circ \widetilde{f}^1_i\right)^{-1}(\widetilde{Z}^{a,c,\overline{x}_i}_i),\, \xi^c_{\overline{f}^2(i)} \circ \nu^1_g(\delta^a_i)\Big|_{\widetilde{Z}^{a,c,\overline{x}_i}_i}\right)\right\}_{i\in I,\,(a,c)\in A(i)\times C(\overline{f}^2(i)),\,\overline{x}_i\in\widetilde{X}^{a,c}_i}.$$

A direct check proves that:

Lemma 2.5. *The collection γ defined above is a representative of a 2-morphism from $[\hat{g}^1]\circ[\hat{f}^1]$ to $[\hat{g}^2]\circ[\hat{f}^2]$. Moreover, the class $[\gamma]$ does not depend on the representatives (P_{g^1},ν_{g^1}), δ and ξ chosen for $[P^1_g,\nu^1_g]$, $[\delta]$ and $[\xi]$ respectively, nor on the choices of the sets $\widetilde{Z}^{a,c,\overline{x}_i}_i$ made above.*

So it makes sense to give the following definition.

Definition 2.6. Given any pair $[\delta]$, $[\xi]$ as before, we define their *horizontal composition* as:

$$[\xi]*[\delta] := [\gamma] : [\hat{g}^1]\circ[\hat{f}^1] \Longrightarrow [\hat{g}^2]\circ[\hat{f}^2].$$

Then we have:

Proposition 2.7. *The definitions of* reduced *orbifold atlases, morphisms, 2-morphisms, and compositions $\circ$, $\odot$ and $*$ give rise to a 2-category (that we denote by $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$), where every 2-morphism is invertible.*

Proof. In order to construct a 2-category, we define some data as follows.

(1) The class of objects is the set of all the reduced orbifold atlases $\mathcal{X}$ for any paracompact, second countable, Hausdorff topological space X.

(2) If $\mathcal{X}$ and $\mathcal{Y}$ are reduced atlases for X and Y respectively, we define a small category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})(\mathcal{X},\mathcal{Y})$ as follows: the space of objects is the set of all morphisms $[\hat{f}]:\mathcal{X}\to\mathcal{Y}$ over any continuous map $f:X\to Y$; for any pair of morphisms $[\hat{f}]$ and $[\hat{g}]$ over f and g respectively, using Definition 1.9 we set:

$$\Big((\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})(\mathcal{X},\mathcal{Y})\Big)([\hat{f}],[\hat{g}]) := \begin{cases} \text{all 2-morphisms } [\delta]:[\hat{f}]\Rightarrow[\hat{g}] & \text{if } f=g,\\ \varnothing & \text{otherwise.}\end{cases} \tag{2.1}$$

The composition in any such category is the vertical composition $\odot$, that is clearly associative; the identity over any object $[\hat{f}]$ is given by the class $i_{[\hat{f}]}$ of the collection $\{(\widetilde{X}_i,\mathrm{id}_{\widetilde{Y}_{\overline{f}(i)}})\}_{i\in I}$. Then the inverse of any $[\delta]$ as above is the class $[\delta]^{-1}$ described in Definition 1.9.

(3) For every reduced orbifold atlas $\mathcal{X}=\{(\widetilde{X}_i,G_i,\pi_i)\}_{i\in I}$ on a topological space X, we define the *identity* of $\mathcal{X}$ as the morphism

$$\mathrm{id}_{\mathcal{X}} := \left(\mathrm{id}_X,\mathrm{id}_I,\left\{\mathrm{id}_{\widetilde{X}_i}\right\}_{i\in I},\left[\mathcal{C}h(\mathcal{X}),\mathrm{id}_{\mathcal{C}h(\mathcal{X})}\right]\right):\mathcal{X}\longrightarrow\mathcal{X}.$$

(4) For every triple of reduced atlases $\mathcal{X},\mathcal{Y},\mathcal{Z}$, we define a functor "composition"

$$(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})(\mathcal{X},\mathcal{Y})\times(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})(\mathcal{Y},\mathcal{Z})\longrightarrow(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})(\mathcal{X},\mathcal{Z})$$

as $\circ$ on any pair of morphisms and as $*$ on any pair of 2-morphisms. We want to prove that this gives rise to a functor. It is easy to see that identities are preserved, so one needs only to prove the *interchange law* (see [Bo, Proposition 1.3.5]). In this setup, this amounts to proving that for any diagram as follows

$$\begin{tikzcd}[column sep=huge]
\mathcal{X} \arrow[r, bend left=60, "{[\hat{f}^1]}"] \arrow[r, "{[\hat{f}^2]}" description] \arrow[r, bend right=60, "{[\hat{f}^3]}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow{[\delta]}"] \arrow[r, phantom, shift right=4ex, "\Downarrow{[\sigma]}"] & \mathcal{Y} \arrow[r, bend left=60, "{[\hat{g}^1]}"] \arrow[r, "{[\hat{g}^2]}" description] \arrow[r, bend right=60, "{[\hat{g}^3]}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow{[\xi]}"] \arrow[r, phantom, shift right=4ex, "\Downarrow{[\eta]}"] & \mathcal{Z},
\end{tikzcd}$$

we have $([\eta]\odot[\xi])*([\sigma]\odot[\delta]) = ([\eta]*[\sigma])\odot([\xi]*[\delta])$. This is long but completely straightforward, so we omit it.

$\square$

3. From reduced orbifold atlases to proper, effective, étale groupoids

The aim of this section is to define a 2-functor $\mathcal{F}^{\mathrm{red}}$ from $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ to the 2-category of proper, effective, étale Lie groupoids. We recall briefly the necessary definitions and notations.

Definition 3.1. [Ler, Definition 2.11] A *Lie groupoid* is the datum of two smooth (Hausdorff, paracompact) manifolds $\mathscr{X}_0, \mathscr{X}_1$ and five smooth maps:

- two submersions $s,t:\mathscr{X}_1 \rightrightarrows \mathscr{X}_0$, called *source* and *target* of the Lie groupoid;
- $m:\mathscr{X}_1\,{}_t\!\times_s \mathscr{X}_1 \to \mathscr{X}_1$, called *multiplication*;
- $i:\mathscr{X}_1\to\mathscr{X}_1$, known as the *inverse* map of the Lie groupoid;
- $e:\mathscr{X}_0\to\mathscr{X}_1$, called *identity*;

which satisfy the following axioms:

(LG1) $s\circ e = 1_{\mathscr{X}_0} = t\circ e$;

(LG2) if pr_1 and pr_2 are the two projections $\mathscr{X}_1\,{}_t\!\times_s \mathscr{X}_1 \to \mathscr{X}_1$, then $s\circ m = s\circ \mathrm{pr}_1$ and $t\circ m = t\circ \mathrm{pr}_2$;

(LG3) the two morphisms $m\circ(\mathrm{id}_{\mathscr{X}_1}\times m)$ and $m\circ(m\times \mathrm{id}_{\mathscr{X}_1})$ from $\mathscr{X}_1\,{}_t\!\times_s \mathscr{X}_1\,{}_t\!\times_s \mathscr{X}_1$ to $\mathscr{X}_1$ are equal;

(LG4) the two morphisms $m\circ(e\circ s, \mathrm{id}_{\mathscr{X}_1})$ and $m\circ(\mathrm{id}_{\mathscr{X}_1}, e\circ t)$ from $\mathscr{X}_1$ to $\mathscr{X}_1$ are both equal to the $\mathrm{id}_{\mathscr{X}_1}$;

(LG5) $i\circ i = \mathrm{id}_{\mathscr{X}_1}$, $s\circ i = t$, $m\circ(\mathrm{id}_{\mathscr{X}_1}, i) = e\circ s$ and $m\circ(i, \mathrm{id}_{\mathscr{X}_1}) = e\circ t$.

For simplicity, we will denote any Lie groupoid as before by $\mathscr{X}$. In the following pages, even if we will deal with several Lie groupoids, we will denote by s the source morphism of any such object, and analogously for the morphisms t, m, e and i. This will not create any problem, since it will be always clear from the context what is the Lie groupoid we are working with. We denote by denote by $\mathscr{X}_0/\mathscr{X}_1$ the topological quotient of $\mathscr{X}_0$ obtained by identifying any two points x_0 and x_0' if and only if there is $x_1\in\mathscr{X}_1$, such that $s(x_1)=x_0$ and $t(x_1)=x_0'$; we denote by $\mathrm{pr}_{\mathscr{X}}:\mathscr{X}_0 \twoheadrightarrow \mathscr{X}_0/\mathscr{X}_1$ the quotient map.

Definition 3.2. [M, § 2.1] Given two Lie groupoids $\mathscr{X}$ and $\mathscr{Y}$, a *morphism* between them is any pair $\psi=(\psi_0,\psi_1)$, where $\psi_0:\mathscr{X}_0\to\mathscr{Y}_0$ and $\psi_1:\mathscr{X}_1\to\mathscr{Y}_1$ are smooth maps, which together commute with all structure morphisms of the two Lie groupoids. In other words, we require that $s\circ\psi_1=\psi_0\circ s$, $t\circ\psi_1=\psi_0\circ t$, $\psi_0\circ e = e\circ\psi_0$, $\psi_1\circ m = m\circ(\psi_1\times\psi_1)$ and $\psi_1\circ i = i\circ\psi_1$.

Definition 3.3. [PS, Definition 2.3] Let us fix two morphisms of Lie groupoids $\psi^m:\mathscr{X}\to\mathscr{Y}$ for $m=1,2$. Then a *natural transformation* $\alpha:\psi^1\Rightarrow\psi^2$ is the datum of any smooth map $\alpha:\mathscr{X}_0\to\mathscr{Y}_1$, such that the following conditions hold:

(NT1) $s\circ\alpha=\psi_0^1$ and $t\circ\alpha=\psi_0^2$;

(NT2) $m\circ(\alpha\circ s,\psi_1^2) = m\circ(\psi_1^1,\alpha\circ t)$.

There are well-known notions of identities, compositions of morphisms, vertical and horizontal compositions of natural transformations (constructed as the corresponding notions in the 2-category of small categories). These data give rise to the 2-category of Lie groupoids, usually denoted by $(\mathcal{L}\mathbf{ie}\,\mathcal{G}\mathbf{pd})$ (see [PS, § 2.1]).

Definition 3.4. [M, § 1.2 and § 1.5] A Lie groupoid $\mathscr{X}$ is called *proper* if the map $(s,t):\mathscr{X}_1\rightarrow\mathscr{X}_0\times\mathscr{X}_0$ is proper; it is called *étale* if the map s (equivalently, t) is étale (i.e. a local diffeomorphism). Since each étale map is a submersion, in general we will simply write "étale groupoid" instead of "étale Lie groupoid".

Remark 3.5. Let $\mathscr{X}$ be a proper étale groupoid and let us fix any pair of points $x_0,x_0'\in\mathscr{X}_0$. Since both s and t are étale, for every point x_1 in $\mathscr{X}_1$ such that $s(x_1)=x_0$ and $t(x_1)=x_0'$, we can find a sufficiently small open neighborhood W_{x_1} of x_1 where both s and t are invertible. Then the set map

$$t\circ(s|_{W_{x_1}})^{-1}:\,s(W_{x_1})\longrightarrow t(W_{x_1}).$$

is a diffeomorphism from an open neighborhood of x_0 to an open neighborhood of x_0', and it commutes with the projection $\mathrm{pr}_{\mathscr{X}}$. So for each pair of points x_0,x_0' as above we can define a set map

$$\begin{gathered}\kappa_{\mathscr{X}}(x_0,x_0',-):\{x_1\in\mathscr{X}_1,\ \text{such that}\, s(x_1)=x_0\ \text{and}\ t(x_1)=x_0'\}\longrightarrow\\ \longrightarrow\{\mathrm{germ}_{x_0}f,\quad\forall\ \text{diffeomorphisms}\ f\ \text{around}\ x_0,\ \text{such that}\ f(x_0)=x_0'\ \text{and}\ \mathrm{pr}_{\mathscr{X}}\circ f=\mathrm{pr}_{\mathscr{X}}\}\end{gathered}\tag{3.1}$$

by setting:

$$\kappa_{\mathscr{X}}(x_0,x_0',x_1):=\mathrm{germ}_{x_0}\left(t\circ(s|_{W_{x_1}})^{-1}\right)=\mathrm{germ}_{x_1}t\cdot\left(\mathrm{germ}_{x_1}s\right)^{-1}.$$

We claim that $\kappa_{\mathscr{X}}(x_0,x_0',-)$ *is surjective.* For that, we have to consider two cases separately; if $\mathrm{pr}_{\mathscr{X}}(x_0)\neq\mathrm{pr}_{\mathscr{X}}(x_0')$, then both the first and the second set in (3.1) are empty, so $\kappa_{\mathscr{X}}(x_0,x_0',-)$ is a bijection. If $\mathrm{pr}_{\mathscr{X}}(x_0)=\mathrm{pr}_{\mathscr{X}}(x_0')$, this means that there is a (in general non-unique) point $x_1\in\mathscr{X}_1$, such that $s(x_1)=x_0$ and $t(x_1)=x_0'$. Let us fix any $\mathrm{germ}_{x_0}f$ as in the second line of (3.1) (for some diffeomorphism f such that $f(x_0)=x_0'$ and $\mathrm{pr}_{\mathscr{X}}\circ f=\mathrm{pr}_{\mathscr{X}}$). Then the function

$$g:=s\circ(t|_{W_{x_1}})^{-1}\circ f|_{f^{-1}(t(W_{x_1}))}\tag{3.2}$$

is a diffeomorphism around x_0, it fixes x_0 and it commutes with $\mathrm{pr}_{\mathscr{X}}$. As a simple consequence of [N, Theorem 2.3], there is a (in general non-unique) point $\widetilde{x}_1$ in $\mathscr{X}_1$, such that $s(\widetilde{x}_1)=x_0=t(\widetilde{x}_1)$ and $\kappa_{\mathscr{X}}(x_0,x_0,\widetilde{x}_1)=\mathrm{germ}_{x_0}g$. Using (3.2), this implies that

$$\mathrm{germ}_{x_0}f=\kappa_{\mathscr{X}}(x_0,x_0',x_1)\cdot\kappa_{\mathscr{X}}(x_0,x_0,\widetilde{x}_1)=\kappa_{\mathscr{X}}(x_0,x_0',m(\widetilde{x}_1,x_1)),$$

so we have proved that $\kappa_{\mathscr{X}}(x_0,x_0',-)$ is surjective.

Definition 3.6. [M, Example 1.5] Let us fix any proper, étale groupoid $\mathscr{X}$. We say that $\mathscr{X}$ is *effective* (or *reduced*) if $\kappa_{\mathscr{X}}(x_0,x_0,-)$ is injective for every $x_0\in\mathscr{X}_0$.

Using this definition and Remark 3.5, it is not difficult to prove the following:

Lemma 3.7. *Let us fix any proper, effective, étale groupoid $\mathscr{X}$. Then the set map $\kappa_{\mathscr{X}}(x_0,x_0',-)$ is a bijection for every pair of points x_0,x_0' in $\mathscr{X}_0$.*

Definition 3.8. We define the 2-categories $(\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$, $(\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$ and $(\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$ as the full 2-subcategories of $(\mathcal{L}\mathbf{ie}\,\mathcal{G}\mathbf{pd})$ obtained by restricting to étale groupoids, respectively to proper, étale Lie groupoids, respectively to proper, effective, étale groupoids (morphisms and 2-morphisms are simply restricted according to that).

Construction 3.9. (adapted from [Pr2, § 4.4] and from [Po, Construction 2.4]) Let us fix any reduced orbifold atlas $\mathcal{X}=\{(\widetilde{X}_i,G_i,\pi_i)\}_{i\in I}$ of dimension n. Then we define $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})$ as the following Lie groupoid.

- $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_0:=\coprod_{i\in I}\widetilde{X}_i$, with manifold structure given by the fact that each $\widetilde{X}_i$ is an open subset of $\mathbb{R}^n$.
- As a set, we define

  $$\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_1:=\left\{\mathrm{germ}_{\widetilde{x}_i}\lambda,\quad\forall\,i\in I,\ \forall\,\lambda\in\mathcal{C}h(\mathcal{X},i,-),\ \forall\,\widetilde{x}_i\in\mathrm{dom}\,\lambda\right\};$$

  its topological and differentiable structure are given by the germ topology and by the germ differentiable structure. In particular for each $i\in I$ and each $\lambda\in\mathcal{C}h(\mathcal{X},i,-)$, the subset $\left\{\mathrm{germ}_{\widetilde{x}_i}\lambda,\,\forall\,\widetilde{x}_i\in\mathrm{dom}\,\lambda\right\}$ is open and diffeomorphic to $\mathrm{dom}\,\lambda\subseteq\widetilde{X}_i\subseteq\mathbb{R}^n$ (see [Po, Construction 2.4] for more details).

- The structure maps are defined as follows:

$$s(\operatorname{germ}_{\widetilde{x}_i}\lambda):=\widetilde{x}_i,\quad t(\operatorname{germ}_{\widetilde{x}_i}\lambda):=\lambda(\widetilde{x}_i),\quad m(\operatorname{germ}_{\widetilde{x}_i}\lambda,\operatorname{germ}_{\lambda(\widetilde{x}_i)}\lambda'):=\operatorname{germ}_{\lambda(\widetilde{x}_i)}\lambda'\cdot\operatorname{germ}_{\widetilde{x}_i}\lambda,$$
$$i(\operatorname{germ}_{\widetilde{x}_i}\lambda):=\operatorname{germ}_{\lambda(\widetilde{x}_i)}\lambda^{-1},\quad e(\widetilde{x}_i):=\operatorname{germ}_{\widetilde{x}_i}\operatorname{id}_{\widetilde{X}_i}. \tag{3.3}$$

A direct check proves that s and t are both étale, that m,e,i are smooth and that axioms (LG1) – (LG5) are satisfied, so $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})$ is an étale groupoid.

It is straightforward to see that:

Lemma 3.10. *For every reduced orbifold atlas $\mathcal{X}$, the étale groupoid $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})$ is proper and effective, i.e. it belongs to $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$.*

Until now we have associated to each object of $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ an object of $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$; we want to do the same for morphisms and 2-morphisms.

Construction 3.11. (adapted from [Po, Proposition 4.7]) Let us fix any pair of reduced orbifold atlases $\mathcal{X}:=\{(\widetilde{X}_i,G_i,\pi_i)\}_{i\in I}$ and $\mathcal{Y}:=\{(\widetilde{Y}_j,H_j,\chi_j)\}_{j\in J}$ for X and Y respectively and any morphism $[\hat{f}]:\mathcal{X}\to\mathcal{Y}$, with representative $\hat{f}:=(f,\overline{f},\{\widetilde{f}_i\}_{i\in I},P_f,\nu_f)$. As we did above for $\mathcal{X}$, the groupoid $\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})$ is defined by

$$\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_0:=\coprod_{j\in J}\widetilde{Y}_j,\quad \mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_1:=\Big\{\operatorname{germ}_{\widetilde{y}_j}\omega,\quad \forall\,j\in J,\ \forall\,\omega\in\mathcal{C}h(\mathcal{Y},j,-),\ \forall\,\widetilde{y}_j\in\operatorname{dom}\omega\Big\}.$$

Then we define a set map $\mathcal{F}^{\,\mathrm{red}}([\hat{f}])_0:\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_0\to\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_0$ as

$$\mathcal{F}^{\,\mathrm{red}}([\hat{f}])_0\Big|_{\widetilde{X}_i}:=\widetilde{f}_i:\widetilde{X}_i\longrightarrow\widetilde{Y}_{\overline{f}(i)}\subseteq\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_0$$

for all $i\in I$. Now let x_1 be any point in $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_1$ and let $\widetilde{x}_i:=s(x_1)\in\widetilde{X}_i$ for some $i\in I$. Since P_f satisfies condition (M4), then there is a (non-unique) $\lambda\in P_f(i,-)$ such that $x_1=\operatorname{germ}_{\widetilde{x}_i}\lambda$. We set $\mathcal{F}^{\,\mathrm{red}}([\hat{f}])_1(x_1):=\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda)\in\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_1$. By (M5d) $\mathcal{F}^{\,\mathrm{red}}([\hat{f}])_1$ is well-defined; by (M5e) and (M5f) the pair $\mathcal{F}^{\,\mathrm{red}}([\hat{f}]):=(\mathcal{F}^{\,\mathrm{red}}([\hat{f}])_0,\mathcal{F}^{\,\mathrm{red}}([\hat{f}])_1)$ is a morphism of Lie groupoids $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})\to\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})$, and it does not depend on $\hat{f}$, nor on the choice of λ as above, but only on $[\hat{f}]$.

Construction 3.12. Now let us fix any pair of atlases $\mathcal{X}$ and $\mathcal{Y}$ as in Construction 3.11, and any pair of morphisms $[\hat{f}^1],[\hat{f}^2]:\mathcal{X}\to\mathcal{Y}$ over a continuous map $f:X\to Y$, with representatives $\hat{f}^m:=(f,\overline{f}^{\,m},\{\widetilde{f}^m_i\}_{i\in I},P_{f^m},\nu_{f^m})$ for $m=1,2$. Moreover, let us fix any 2-morphism $[\delta]:[\hat{f}^1]\Rightarrow[\hat{f}^2]$ and any representative $\delta:=\{(\widetilde{X}^a_i,\delta^a_i)\}_{i\in I,\,a\in A(i)}$ for it. Then we define a set map

$$\mathcal{F}^{\,\mathrm{red}}([\delta]):\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_0=\coprod_{i\in I}\widetilde{X}_i\to\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_1,\quad \mathcal{F}^{\,\mathrm{red}}([\delta])(\widetilde{x}_i):=\operatorname{germ}_{\widetilde{f}^1_i(\widetilde{x}_i)}\delta^a_i\quad\forall\,i\in I,\,\forall\,a\in A,\,\forall\,\widetilde{x}_i\in\widetilde{X}^a_i.$$

$\mathcal{F}^{\,\mathrm{red}}([\delta])$ is well-defined by property (2Md) for δ; using the germ topology on $\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_1$ and (M3) it is a smooth map. Using (3.3), (2Mc) and Remark 1.10, we get that $\mathcal{F}^{\,\mathrm{red}}([\delta])$ is a natural transformation $\mathcal{F}^{\,\mathrm{red}}([\hat{f}^1])\Rightarrow\mathcal{F}^{\,\mathrm{red}}([\hat{f}^2])$. Using the last part of Definition 1.9, it depends only on $[\delta]$ and not on the representative δ chosen above

A direct check proves that:

Lemma 3.13. *For every pair of morphisms $[\hat{f}]:\mathcal{X}\to\mathcal{Y}$ and $[\hat{g}]:\mathcal{Y}\to\mathcal{Z}$, we have $\mathcal{F}^{\,\mathrm{red}}([\hat{g}]\circ[\hat{f}])=\mathcal{F}^{\,\mathrm{red}}([\hat{g}])\circ\mathcal{F}^{\,\mathrm{red}}([\hat{f}])$. For every reduced orbifold atlas $\mathcal{X}$ we have $\mathcal{F}^{\,\mathrm{red}}(\operatorname{id}_{\mathcal{X}})=\operatorname{id}_{\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})}$; for every morphism $[\hat{f}]$ between reduced orbifold atlases we have $\mathcal{F}^{\,\mathrm{red}}(i_{[\hat{f}]})=i_{\mathcal{F}^{\,\mathrm{red}}([\hat{f}])}$.*

Moreover, using the definition of vertical and horizontal compositions in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ (as induced by restriction from the bigger 2-category $(\mathcal{L}\mathbf{ie}\,\mathcal{G}\mathbf{pd})$ of Lie groupoids), it is easy to see that:

Lemma 3.14. *For each diagram in* $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ *as follows*

$$\begin{tikzcd}[column sep=huge]
\mathcal{X} \arrow[r, bend left=50, "{[\hat{f}^1]}"{name=A}] \arrow[r, "{[\hat{f}^2]}"{name=B}] \arrow[r, bend right=50, "{[\hat{f}^3]}"'{name=C}] & \mathcal{Y} \arrow[r, bend left, "{[\hat{g}^1]}"{name=D}] \arrow[r, bend right, "{[\hat{g}^2]}"'{name=E}] & \mathcal{Z},
\arrow[from=A, to=B, phantom, "\Downarrow{[\delta]}"]
\arrow[from=B, to=C, phantom, "\Downarrow{[\sigma]}"]
\arrow[from=D, to=E, phantom, "\Downarrow{[\xi]}"]
\end{tikzcd}$$

$\mathcal{F}^{\,\mathrm{red}}([\sigma]\odot[\delta]) = \mathcal{F}^{\,\mathrm{red}}([\sigma])\odot\mathcal{F}^{\,\mathrm{red}}([\delta])$ *and* $\mathcal{F}^{\,\mathrm{red}}([\xi]*[\delta]) = \mathcal{F}^{\,\mathrm{red}}([\xi])*\mathcal{F}^{\,\mathrm{red}}([\delta])$.

Lemmas 3.10, 3.13 and 3.14 prove that:

Theorem 3.15. $\mathcal{F}^{\,\mathrm{red}}$ *is a* 2*-functor from* $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ *to* $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$.

We state some properties of $\mathcal{F}^{\,\mathrm{red}}$ that we are going to use soon.

Lemma 3.16. (adapted from [Po, Proposition 4.9]) *Let us fix any pair of reduced orbifold atlases* $\mathcal{X},\mathcal{Y}$ *and any morphism* $\psi:\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})\to\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})$*. Then there is a unique morphism* $[\hat{f}]:\mathcal{X}\to\mathcal{Y}$ *in* $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$*, such that* $\mathcal{F}^{\,\mathrm{red}}([\hat{f}])=\psi$.

Proof. Let us $\mathcal{X}$ be the collection $\{(\widetilde{X}_i,G_i,\pi_i)\}_{i\in I}$ over X and let $\mathcal{Y}$ be the collection $\{(\widetilde{Y}_j,H_j,\chi_j)\}_{j\in J}$ over Y. Since each $\widetilde{X}_i$ is connected by definition of orbifold atlas, then the morphism $\psi_0:\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_0\to\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_0$ induces a set map $\overline{f}:I\to J$ such that $\psi_0(\widetilde{X}_i)\subseteq\widetilde{Y}_{\overline{f}(i)}$ for every $i\in I$. For each $i\in I$ we set $\widetilde{f}_i:=\psi_0|_{\widetilde{X}_i}:\widetilde{X}_i\to\widetilde{Y}_{\overline{f}(i)}$. Moreover, we define a set map $f:X\to Y$ by

$$f(\pi_i(\widetilde{x}_i)):=\chi_{\overline{f}(i)}\circ\widetilde{f}_i(\widetilde{x}_i)\quad\forall\, i\in I,\ \forall\,\widetilde{x}_i\in\widetilde{X}_i. \tag{3.4}$$

Using Construction 3.11, f is well-defined. Moreover, it is continuous since on each open set of X of the form $X_i=\pi_i(\widetilde{X}_i)$ f coincides with the continuous map $\chi_{\overline{f}(i)}\circ\widetilde{f}_i$. Following the proof of [Po, Proposition 4.9], one can construct a subset $P_f\subseteq\mathcal{C}h(\mathcal{X})$, satisfying condition (M4), and a set map $\nu_f:P_f\to\mathcal{C}h(\mathcal{Y})$, such that for each $\lambda\in P_f$ and for each $\widetilde{x}_i\in\operatorname{dom}\lambda$ we have:

$$\operatorname{germ}_{\widetilde{f}_i(\widetilde{x}_i)}\nu_f(\lambda)=\psi_1(\operatorname{germ}_{\widetilde{x}_i}\lambda). \tag{3.5}$$

Using Definition 3.2, it is easy to see that the collection $\hat{f}:=(f,\overline{f},\{\widetilde{f}_i\}_{i\in I},P_f,\nu_f)$ is a representative of a morphism from $\mathcal{X}$ to $\mathcal{Y}$. The collection $\hat{f}$ depends on some choices, but the class $[\hat{f}]$ depends only on ψ. A direct check using (3.4) and (3.5) proves that $\mathcal{F}^{\,\mathrm{red}}([\hat{f}])=\psi$, and that $[\hat{f}]$ is the unique morphism with such a property. □

Lemma 3.17. *Let us fix any pair of reduced orbifold atlases* $\mathcal{X}$ *and* $\mathcal{Y}$*, any pair of morphisms* $[\hat{f}^m]:\mathcal{X}\to\mathcal{Y}$ *for* $m=1,2$*, and any natural transformation* $\alpha:\mathcal{F}^{\,\mathrm{red}}([\hat{f}^1])\Rightarrow\mathcal{F}^{\,\mathrm{red}}([\hat{f}^2])$*. Then there exists a unique* 2*-morphism* $[\delta]:[\hat{f}^1]\Rightarrow[\hat{f}^2]$ *such that* $\mathcal{F}^{\,\mathrm{red}}([\delta])=\alpha$.

Proof. For each $m=1,2$, let us fix a representative $\hat{f}^m:=(f,\overline{f},\{\widetilde{f}^m_i\}_{i\in I},P_{f^m},\nu_{f^m})$ for $[\hat{f}^m]$. By Definition 3.3, α is a smooth map from $\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_0$ to $\mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})_1$, such that $s\circ\alpha=\mathcal{F}^{\,\mathrm{red}}([\hat{f}^1])_0$; so for each $i\in I$ and for each $\overline{x}_i\in\widetilde{X}_i\subseteq\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})_0$, we can choose a change of charts $\delta^{\overline{x}_i}$ of $\mathcal{Y}$, such that

$$\alpha(\overline{x}_i)=\operatorname{germ}_{\mathcal{F}^{\,\mathrm{red}}([\hat{f}^1])_0(\overline{x}_i)}\delta^{\overline{x}_i}=\operatorname{germ}_{\widetilde{f}^1_i(\overline{x}_i)}\delta^{\overline{x}_i}. \tag{3.6}$$

Using Construction 3.9 (for $\mathcal{Y}$ instead of $\mathcal{X}$), the set

$$S(\delta^{\overline{x}_i}) := \left\{ \mathrm{germ}_{\widetilde{y}_{\overline{f}^1(i)}} \delta^{\overline{x}_i}, \quad \forall\, \widetilde{y}_{\overline{f}^1(i)} \in \mathrm{dom}\, \delta^{\overline{x}_i} \right\} \subseteq \mathcal{F}^{\mathrm{red}}(\mathcal{Y})_1 \tag{3.7}$$

is open in $\mathcal{F}^{\mathrm{red}}(\mathcal{Y})_1$, hence $\alpha^{-1}(S(\delta^{\overline{x}_i}))$ is open in $\mathcal{F}^{\mathrm{red}}(\mathcal{X})_0$. Therefore the set $\widetilde{X}_i^{\overline{x}_i} := \alpha^{-1}(S(\delta^{\overline{x}_i})) \cap \widetilde{X}_i$ is an open set in $\widetilde{X}_i$. We want to prove that the collection $\delta := \{(\widetilde{X}_i^{\overline{x}_i}, \delta^{\overline{x}_i})\}_{i \in I, \overline{x}_i \in \widetilde{X}_i}$ is a representative of a 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^2]$. So let us fix any $i \in I$, any $\overline{x}_i \in \widetilde{X}_i$ and any $\widetilde{x}_i \in \widetilde{X}_i^{\overline{x}_i}$; the point $\alpha(\widetilde{x}_i)$ belongs to (3.7), hence it is of the form $\mathrm{germ}_{\widetilde{y}_{\overline{f}^1(i)}} \delta^{\overline{x}_i}$ for a unique point $\widetilde{y}_{\overline{f}^1(i)}$. Using condition (NT1) for α (see Definition 3.3), we have: $\widetilde{y}_{\overline{f}^1(i)} = s(\mathrm{germ}_{\widetilde{y}_{\overline{f}^1(i)}} \delta^{\overline{x}_i}) = s(\alpha(\widetilde{x}_i)) = \mathcal{F}^{\mathrm{red}}([\hat{f}^1])_0(\widetilde{x}_i) = \widetilde{f}_i^1(\widetilde{x}_i)$. Therefore

$$\alpha(\widetilde{x}_i) = \mathrm{germ}_{\widetilde{y}_{\overline{f}^1(i)}} \delta^{\overline{x}_i} = \mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta^{\overline{x}_i} \tag{3.8}$$

(in other terms, (3.6) holds not only for the point $\overline{x}_i$, but also for any $\widetilde{x}_i$ in $\widetilde{X}_i^{\overline{x}_i}$). Again by (NT1) we have $t \circ \alpha = \mathcal{F}^{\mathrm{red}}([\hat{f}^2])_0$, so for each $\widetilde{x}_i \in \widetilde{X}_i^{\overline{x}_i}$ we have

$$\widetilde{f}_i^2(\widetilde{x}_i) = \mathcal{F}^{\mathrm{red}}([\hat{f}^2]))_0(\widetilde{x}_i) = t \circ \alpha(\widetilde{x}_i) \overset{(3.8)}{=} \left(\mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta^{\overline{x}_i} \right) \overset{(3.3)}{=} \delta^{\overline{x}_i} \circ \widetilde{f}_i^1(\widetilde{x}_i),$$

so in particular

$$\widetilde{f}_i^1(\widetilde{X}_i^{\overline{x}_i}) \subseteq \mathrm{dom}\, \delta^{\overline{x}_i} \quad \text{and} \quad \widetilde{f}_i^2(\widetilde{X}_i^{\overline{x}_i}) \subseteq \mathrm{cod}\, \delta^{\overline{x}_i};$$

therefore properties (2Ma), (2Mb) and (2Mc) (see Definition 1.9) are verified for δ. If $\overline{x}_i$ and $\hat{x}_i$ are both in $\widetilde{X}_i$ and $\widetilde{x}_i \in \widetilde{X}_i^{\overline{x}_i} \cap \widetilde{X}_i^{\hat{x}_i}$, then by (3.8) we have

$$\mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta^{\overline{x}_i} = \alpha(\widetilde{x}_i) = \mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta^{\hat{x}_i},$$

so (2Md) holds. Now let us fix any $(i, i') \in I \times I$, any $(\overline{x}_i, \hat{x}_{i'}) \in \widetilde{X}_i \times \widetilde{X}_{i'}$, any $\lambda \in \mathcal{C}h(\mathcal{X}, i, i')$ and any $\widetilde{x}_i \in \mathrm{dom}\, \lambda \cap \widetilde{X}_i^{\overline{x}_i}$ such that $\lambda(\widetilde{x}_i) \in \widetilde{X}_{i'}^{\hat{x}_{i'}}$. Since both P_{f^1} and P_{f^2} satisfy condition (M4), then for each $m = 1, 2$ there exists $\lambda^m \in P_{f^m}(i, i')$ such that $\widetilde{x}_i \in \mathrm{dom}\, \lambda^m$ and $\mathrm{germ}_{\widetilde{x}_i} \lambda^m = \mathrm{germ}_{\widetilde{x}_i} \lambda$. We recall (see Construction 3.11) that

$$\mathcal{F}^{\mathrm{red}}([\hat{f}^m])_1(\mathrm{germ}_{\widetilde{x}_i} \lambda) = \mathrm{germ}_{\widetilde{f}_i^m(\widetilde{x}_i)} \nu_{f^m}(\lambda^m) \quad \text{for } m = 1, 2.$$

Therefore:

$$\begin{aligned}
&\mathrm{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_{f^2}(\lambda^2) \cdot \mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta^{\overline{x}_i} = m \left(\mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \delta^{\overline{x}_i}, \mathrm{germ}_{\widetilde{f}_i^2(\widetilde{x}_i)} \nu_{f^2}(\lambda^2) \right) \overset{(3.8)}{=} \\
&\overset{(3.8)}{=} \left(m \circ \left(\alpha \circ s, \mathcal{F}^{\mathrm{red}}([\hat{f}^2])_1 \right) \right) (\mathrm{germ}_{\widetilde{x}_i} \lambda^2) \overset{(\mathrm{NT2})}{=} \left(m \circ \left(\mathcal{F}^{\mathrm{red}}([\hat{f}^1])_1, \alpha \circ t \right) \right) (\mathrm{germ}_{\widetilde{x}_i} \lambda^1) = \\
&= m \left(\mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \nu_{f^1}(\lambda^1), \mathrm{germ}_{\widetilde{f}_{i'}^1(\lambda^1(\widetilde{x}_i))} \delta^{\hat{x}_{i'}} \right) = \mathrm{germ}_{\widetilde{f}_{i'}^1(\lambda(\widetilde{x}_i))} \delta^{\hat{x}_{i'}} \cdot \mathrm{germ}_{\widetilde{f}_i^1(\widetilde{x}_i)} \nu_{f^1}(\lambda^1).
\end{aligned}$$

So also property (2Me) holds. Therefore δ is a representative of a 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^2]$. Different choices of changes of charts of the form $\delta^{\overline{x}_i}$ give rise to different δ's, but their equivalence class $[\delta]$ is the same. By (3.8) we get that $\mathcal{F}^{\mathrm{red}}([\delta]) = \alpha$, and that $[\delta]$ as above is the only 2-morphism with such a property. $\square$

So we have proved that for every pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$ the functor

$$\mathcal{F}^{\mathrm{red}}(\mathcal{X}, \mathcal{Y}) : (\mathcal{R}\mathbf{ed}\, \mathcal{A}\mathbf{tl})(\mathcal{X}, \mathcal{Y}) \longrightarrow (\mathcal{PE\acute{E}}\, \mathcal{G}\mathbf{pd})(\mathcal{F}^{\mathrm{red}}(\mathcal{X}), \mathcal{F}^{\mathrm{red}}(\mathcal{Y}))$$

is a bijection on objects and morphisms (i.e. on 1-morphisms and 2-morphisms of $(\mathcal{R}\mathbf{ed}\, \mathcal{A}\mathbf{tl})$ and of $(\mathcal{PE\acute{E}}\, \mathcal{G}\mathbf{pd})$).

4. Weak equivalences between étale groupoids

As we mentioned in the Introduction, the bicategory of reduced orbifold atlases described in terms of proper, effective and étale groupoids is not $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$, rather a bicategory obtained from $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ by selecting a suitable class of morphisms (weak equivalences, see below) and by "turning" them into internal equivalences. This is a special case of the following setup:

Definition 4.1. ([Pr, § 2.1], restricting to 2-categories for simplicity of exposition) Let us fix any 2-category $\mathscr{C}$ and any class $\mathbf{W}$ of morphisms in $\mathscr{C}$; then the pair $(\mathscr{C},\mathbf{W})$ *admits a right bicalculus of fractions* if and only if the following conditions hold:

(BF1) for every object A of $\mathscr{C}$, the 1-identity id_A belongs to $\mathbf{W}$;

(BF2) $\mathbf{W}$ is closed under compositions;

(BF3) for every morphism $\mathrm{w}: A \rightarrow B$ in $\mathbf{W}$ and for every morphism $f: C \rightarrow B$, there are an object D, a morphism $\mathrm{w}': D \rightarrow C$ in $\mathbf{W}$, a morphism $f': D \rightarrow A$ and an invertible 2-morphism $\alpha: f\circ \mathrm{w}' \Rightarrow \mathrm{w}\circ f'$;

(BF4) (a) given any morphism $\mathrm{w}: B \rightarrow A$ in $\mathbf{W}$, any pair of morphisms $f^1, f^2: C \rightarrow B$ and any $\alpha: \mathrm{w}\circ f^1 \Rightarrow \mathrm{w}\circ f^2$, there are an object D, a morphism $\mathrm{v}: D \rightarrow C$ in $\mathbf{W}$ and a 2-morphism $\beta: f^1\circ \mathrm{v} \Rightarrow f^2 \circ \mathrm{v}$, such that $\alpha * i_{\mathrm{v}} = i_{\mathrm{w}} * \beta$;

(b) if α in (a) is invertible, then so is β;

(c) if $(D', \mathrm{v}': D' \rightarrow C, \beta': f^1\circ \mathrm{v}' \Rightarrow f^2\circ \mathrm{v}')$ is another triple with the same properties of (D, v, β) in (a), then there are an object E, a pair of morphisms $\mathrm{u}: E \rightarrow D$, $\mathrm{u}': E \rightarrow D'$ and an invertible 2-morphism $\zeta: \mathrm{v}\circ \mathrm{u} \Rightarrow \mathrm{v}'\circ \mathrm{u}'$, such that $\mathrm{v}\circ\mathrm{u}$ belongs to $\mathbf{W}$ and $(\beta' * i_{\mathrm{u}'})\odot(i_{f^1} * \zeta) = (i_{f^2} * \zeta)\odot(\beta * i_{\mathrm{u}})$;

(BF5) if $\mathrm{w}: A \rightarrow B$ is a morphism in $\mathbf{W}$, $\mathrm{v}: A \rightarrow B$ is any morphism and if there is an invertible 2-morphism $\alpha: \mathrm{v} \Rightarrow \mathrm{w}$, then also v belongs to $\mathbf{W}$.

Theorem 4.2. *[Pr, Theorem 21] Given any pair* $(\mathscr{C},\mathbf{W})$ *satisfying axioms* (BF)*, there are a bicategory* $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ *(called* (right) bicategory of fractions*) and a pseudofunctor* $\mathcal{U}_{\mathbf{W}}: \mathscr{C} \rightarrow \mathscr{C}\left[\mathbf{W}^{-1}\right]$ *that sends each element of* $\mathbf{W}$ *to an internal equivalence, and that is universal with respect to such a property.*

In [Pr] the theorem above is stated with (BF1) replaced by a slightly stronger hypothesis, but actually all the proofs in [Pr] use only the weaker axiom (BF1), so we can state the theorem of [Pr] as we did above. We refer to [Pr, § 2.2, 2.3, 2.4] and to our paper [T2] for more details on the construction of bicategories of fractions. As we mentioned above, we are interested in the case when the class $\mathbf{W}$ is the class of all the weak equivalences in the bicategory $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$. We recall (see [M, § 2.4]) that a morphism $\psi: \mathscr{X} \rightarrow \mathscr{Y}$ between Lie groupoids is a *weak equivalence* (also known as *essential equivalence*) if and only if the following conditions hold:

(ME1) the smooth map $t\circ\pi^1: \mathscr{Y}_1\,{}_s\!\times_{\psi_0}\,\mathscr{X}_0 \rightarrow \mathscr{Y}_0$ is a surjective submersion (here π^1 is the projection $\mathscr{Y}_1\,{}_s\!\times_{\psi_0}\,\mathscr{X}_0 \rightarrow \mathscr{Y}_1$ and the fiber product is a manifold since s is a submersion);

(ME2) the following diagram is cartesian (it is commutative by Definition 3.2):

$$\begin{tikzcd}
\mathscr{X}_1 \arrow[r, "\psi_1"] \arrow[d, "(s,t)"'] & \mathscr{Y}_1 \arrow[d, "(s,t)"] \\
\mathscr{X}_0 \times \mathscr{X}_0 \arrow[r, "(\psi_0\times\psi_0)"] & \mathscr{Y}_0 \times \mathscr{Y}_0.
\end{tikzcd}$$

Any two Lie groupoids $\mathscr{X}$ and $\mathscr{Y}$ are said to be *Morita equivalent* if and only if there are a Lie groupoid $\mathscr{Z}$ and two weak equivalences as follows:

$$\mathscr{X} \xleftarrow{\psi^1} \mathscr{Z} \xrightarrow{\psi^2} \mathscr{Y}.$$

This is actually an equivalence relation, see for example [MM, Chapter 5]. We denote by $\mathbf{W}_{\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$ the set of all weak equivalences in $(\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$. Analogously, we denote by $\mathbf{W}_{\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$, respectively by $\mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$, the set of all weak equivalences between proper and étale groupoids, respectively between proper, effective and étale groupoids. Then we have the following standard result, that is a direct application of [Pr, § 4.1], [MM, Example 5.21(2) and Proposition 5.26], and of the explicit construction of bicategories of fractions in [Pr].

Proposition 4.3. *The pairs* $((\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}), \mathbf{W}_{\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}})$, $((\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}), \mathbf{W}_{\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}})$ *and* $((\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}), \mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}})$ *admit all a right bicalculus of fractions; moreover, there is a commutative diagram as follows, where each horizontal map is an embedding of full 2-subcategories or full bi-subcategories, and each vertical pseudofunctor sends each weak equivalence to an internal equivalence (and is universal with respect to this property):*

$$\begin{tikzcd}
(\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}) \arrow[r, hook] \arrow[d, "\mathcal{U}_{\mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}}"] \arrow[dr, phantom, "\curvearrowright"] & (\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}) \arrow[r, hook] \arrow[d, "\mathcal{U}_{\mathbf{W}_{\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}}"] \arrow[dr, phantom, "\curvearrowright"] & (\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}) \arrow[d, "\mathcal{U}_{\mathbf{W}_{\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}}"] \\
(\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}\right] \arrow[r, hook] & (\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}\right] \arrow[r, hook] & (\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}\right].
\end{tikzcd}$$

The bicategory $(\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{P}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}\right]$ is usually called the *bicategory of orbifolds* (from the point of view of Lie groupoids); its bi-subcategory $(\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}\right]$ is usually called the *bicategory of effective (or reduced) orbifolds.* We refer to Description 6.4 below for an explicit description of this last bicategory.

5. Refinements and weak equivalences in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$

In this section we introduce the notions of refinements and weak equivalences in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. Using the 2-functor $\mathscr{F}^{\,\mathrm{red}}$, the definition of weak equivalences between reduced orbifold atlases will match with the notion of weak equivalences between proper, effective, étale groupoids (see Lemmas 5.5 and 5.6 below).

Definition 5.1. Let us fix any pair of reduced orbifold atlases $\mathcal{X} = \{(\widetilde{X}_i, G_i, \pi_i)\}_{i\in I}$ on X and $\mathcal{Y}$ on Y and any morphism in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ as follows:

$$[\hat{\mathrm{w}}] := \Big(\mathrm{w}, \overline{\mathrm{w}}, \{\widetilde{\mathrm{w}}_i\}_{i\in I}, [P_{\mathrm{w}}, \nu_{\mathrm{w}}]\Big) : \mathcal{X} \longrightarrow \mathcal{Y}. \tag{5.1}$$

Then we say that $[\hat{\mathrm{w}}]$ is a *refinement* if and only if the following two conditions hold:

(REF1) $X = Y$ and the continuous map $\mathrm{w} : X \to X$ is equal to id_X;

(REF2) for each $i \in I$ the smooth map $\widetilde{\mathrm{w}}_i$ is an open embedding; assuming (REF1), this implies that $[\mathcal{X}] = [\mathcal{Y}]$.

We denote by $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$ the class of all the refinements in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. We say that any $[\hat{\mathrm{w}}]$ as in (5.1) is a *weak equivalence of reduced orbifold atlases* if and only if it satisfies the following conditions:

(WE1) the continuous map $\mathrm{w} : X \to Y$ is an homeomorphism;

(WE2) the atlas $\mathrm{w}_*(\mathcal{X}) := \{(\widetilde{X}_i, G_i, \mathrm{w} \circ \pi_i)\}_{i \in I}$ on Y is equivalent to $\mathcal{Y}$; equivalently, for each $i \in I$ the chart $(\widetilde{X}_i, G_i, \mathrm{w} \circ \pi_i)$ on Y is compatible with every chart of $\mathcal{Y}$.

In particular, *each refinement is a weak equivalence.*

Lemma 5.2. *Let us fix any pair of reduced orbifold atlases $\mathcal{X}$ and $\mathcal{Y}$, and any weak equivalence $[\hat{\mathrm{w}}]$ as in Definition 5.1. Then for each $i \in I$ the smooth map $\widetilde{\mathrm{w}}_i : \widetilde{X}_i \to \widetilde{Y}_{\overline{\mathrm{w}}(i)}$ is étale (i.e. a local diffeomorphism).*

Proof. Let us fix any $i \in I$ and any $\widetilde{x}_i \in \widetilde{X}_i$. By definition of morphism in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, we have

$$\mathrm{w} \circ \pi_i = \chi_{\overline{\mathrm{w}}(i)} \circ \widetilde{\mathrm{w}}_i, \tag{5.2}$$

so $\mathrm{w} \circ \pi_i(\widetilde{x}_i)$ belongs to $\chi_{\overline{\mathrm{w}}(i)}(\widetilde{Y}_{\overline{\mathrm{w}}(i)})$. By (WE2), the chart $(\widetilde{X}_i, G_i, \mathrm{w} \circ \pi_i)$ is compatible with the atlas $\mathcal{Y}$, so in particular it is compatible with $(\widetilde{Y}_{\overline{\mathrm{w}}(i)}, H_{\overline{\mathrm{w}}(i)}, \chi_{\overline{\mathrm{w}}(i)})$. Therefore there exists a change of charts λ from $(\widetilde{X}_i, G_i, \mathrm{w} \circ \pi_i)$ to $(\widetilde{Y}_{\overline{\mathrm{w}}(i)}, H_{\overline{\mathrm{w}}(i)}, \chi_{\overline{\mathrm{w}}(i)})$, such that $\widetilde{x}_i \in \operatorname{dom} \lambda$. By Definition 1.3, we have

$$\chi_{\overline{\mathrm{w}}(i)} \circ \lambda = \mathrm{w} \circ \pi_i|_{\operatorname{dom} \lambda}. \tag{5.3}$$

If we denote by $\overline{\lambda}$ the map $\widetilde{\mathrm{w}}_i \circ \lambda^{-1} : \operatorname{cod} \lambda \longrightarrow \widetilde{Y}_{\overline{\mathrm{w}}(i)}$, then we have:

$$\chi_{\overline{\mathrm{w}}(i)} \circ \overline{\lambda}(\widetilde{y}) = \chi_{\overline{\mathrm{w}}(i)} \circ \widetilde{\mathrm{w}}_i \circ \lambda^{-1}(\widetilde{y}) \overset{(5.2)}{=} \mathrm{w} \circ \pi_i \circ \lambda^{-1}(\widetilde{y}) \overset{(5.3)}{=} \chi_{\overline{\mathrm{w}}(i)}(\widetilde{y}), \qquad \forall\, \widetilde{y} \in \operatorname{dom} \overline{\lambda} = \operatorname{cod} \lambda \subseteq \widetilde{Y}_{\overline{\mathrm{w}}(i)}.$$

Since $\operatorname{dom} \overline{\lambda}$ is connected, then the previous identity together with [MM, Lemma 2.11] proves that there is a unique $h \in H_{\overline{\mathrm{w}}(i)}$ such that $\overline{\lambda} = h|_{\operatorname{cod} \lambda}$. Therefore, $\widetilde{\mathrm{w}}_i|_{\operatorname{dom} \lambda} = \overline{\lambda} \circ \lambda = h \circ \lambda$, so we have proved that for each $i \in I$ the map $\widetilde{\mathrm{w}}_i$ coincides locally with a diffeomorphism. □

Using axiom (M5d), it is easy to prove the following result:

Lemma 5.3. *Let us fix the following data:*

(*a*) *a pair of reduced orbifold atlases $\mathcal{X} := \{(\widetilde{X}_i, G_i, \pi_i)\}_{i \in I}$ over X and $\mathcal{Y} := \{(\widetilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ over Y;*

(*b*) *an homeomorphism $\mathrm{w} : X \xrightarrow{\sim} Y$;*

(*c*) *a set map $\overline{\mathrm{w}} : I \to J$;*

(*d*) *for each $i \in I$, an étale map $\widetilde{\mathrm{w}}_i : \widetilde{X}_i \to \widetilde{Y}_{\overline{\mathrm{w}}(i)}$, such that $\chi_{\overline{\mathrm{w}}(i)} \circ \widetilde{\mathrm{w}}_i = \mathrm{w} \circ \pi_i$.*

Then there is a unique *class $[P_{\mathrm{w}}, \nu_{\mathrm{w}}]$, such that the collection of data $[\hat{\mathrm{w}}] := (\mathrm{w}, \overline{\mathrm{w}}, \{\widetilde{\mathrm{w}}_i\}_{i \in I}, [P_{\mathrm{w}}, \nu_{\mathrm{w}}])$ is a a weak equivalence from $\mathcal{X}$ to $\mathcal{Y}$.*

Combining this with Lemma 5.2, this means that *each weak equivalence $[\hat{\mathrm{w}}]$ is completely determined by an underlying homeomorphism* w *and by a collection of étale local liftings for* w. In particular, each refinement is completely determined by a collection of open embeddings from each chart of $\mathcal{X}$ to some charts of $\mathcal{Y}$, commuting with the projections. Using Lemmas 5.2 and 5.3 it is easy to prove that:

Lemma 5.4. *Let us fix the following data:*

- *finitely many* equivalent *reduced orbifold atlases $\mathcal{X}^1, \cdots, \mathcal{X}^r$ over the same topological space X;*
- *a reduced orbifold atlas $\mathcal{X}'$ over another topological space X';*
- *an homeomorphism $\mathrm{w} : X' \xrightarrow{\sim} X$;*
- *for each $m = 1, \cdots, r$ a weak equivalence $[\hat{\mathrm{w}}^m] : \mathcal{X}' \to \mathcal{X}^m$ defined over* w.

Then there are a reduced orbifold atlas $\mathcal{X}$ over X and a weak equivalence $[\hat{\mathrm{v}}] : \mathcal{X} \to \mathcal{X}'$, such that:

(i) $[\hat{\mathrm{v}}]$ *is defined over* $\mathrm{w}^{-1} : X \xrightarrow{\sim} X'$;

(ii) *each local lift of* $[\hat{\mathrm{v}}]$ *is an open embedding;*

(iii) *for each* $m = 1, \cdots, r$, $[\hat{\mathrm{w}}^m] \circ [\hat{\mathrm{v}}]$ *is a* refinement.

In particular, if $X' = X$ *and* $\mathrm{w} = \mathrm{id}_X$, *then also* $[\hat{\mathrm{v}}]$ *is a refinement.*

The following lemmas are similar to [Po, Propositions 5.3 and 6.2]; the significant difference is given by the fact that we consider all the weak equivalences rather than restricting only to the "lifts of the identity" of [Po].

Lemma 5.5. *If* $[\hat{\mathrm{w}}] : \mathcal{X} \to \mathcal{Y}$ *is a weak equivalence of reduced orbifold atlases, then* $\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}])$ *is a weak equivalence of proper, effective, étale groupoids.*

Lemma 5.6. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be reduced orbifold atlases, let* $\psi : \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}) \to \mathcal{F}^{\,\mathrm{red}}(\mathcal{Y})$ *be a weak equivalence and let* $[\hat{\mathrm{w}}] : \mathcal{X} \to \mathcal{Y}$ *be the unique morphism in* $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ *such that* $\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}]) = \psi$ *(see* Lemma 3.16*). Then* $[\hat{\mathrm{w}}]$ *is a weak equivalence of reduced orbifold atlases.*

So given any two reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$, the bijection of Lemma 3.16

$$\{\text{morphisms } [\hat{f}] : \mathcal{X} \to \mathcal{Y} \text{ in } (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})\} \longrightarrow \{\text{morphisms } \phi : \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}) \to \mathcal{F}^{\,\mathrm{red}}(\mathcal{Y}) \text{ in } (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\}$$

induces a bijection between weak equivalences in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ and weak equivalences in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$.

Lemma 5.7. *Let us fix any proper, effective, étale groupoid* $\mathscr{X}$. *Then there are a reduced orbifold atlas* $\mathcal{X}$ *and a weak equivalence* $\psi : \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}) \to \mathscr{X}$.

Proof. The reduced orbifold atlas $\mathcal{X}$ is obtained as in the last part of the proof of Theorem 4.1 in [MP]. In [T1, Lemmas 4.7, 4.8 and 4.9] we proved that there is a weak equivalence as required. The proofs in [T1] were done in the category of complex manifolds, but they can be easily adapted to the case of smooth manifolds. $\square$

6. The bicategories $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ and $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$

In this section we will prove that the pair $((\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}), \mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$ admits a right bicalculus of fractions and we will give a simple description of the associated bicategory of fractions, that we denote by $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$. We will also give briefly a description of the bicategory of fractions $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$; in the next section we will prove that such two bicategories are equivalent.

Proposition 6.1. *The pair* $((\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}), \mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$ *admits a right bicalculus of fractions, so there are a bicategory* $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb}) := (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})\left[\mathbf{W}^{-1}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}\right]$ *and a pseudofunctor*

$$\mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}} : (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \longrightarrow (\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$$

that sends every refinement of reduced orbifold atlases (i.e. every element of $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$*) to an internal equivalence, and that is universal with respect to such a property.*

Proof. Condition (BF1) is obviously satisfied and (BF2) is an easy consequence of the definition of compositions (see Construction 1.7). Let us consider (BF3), so let us fix any triple of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, any refinement $[\hat{\mathrm{w}}] : \mathcal{X} \to \mathcal{Y}$ and any morphism $[\hat{f}] : \mathcal{Z} \to \mathcal{Y}$. Using Lemmas 5.5 and 3.10 we have that $\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}])$ is a weak equivalence between proper, effective and étale groupoids; moreover by Proposition 4.3

the set $\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}$ satisfies (BF3). Therefore there are a proper, effective, étale Lie groupoid $\mathscr{U}$, a weak equivalence ψ', a morphism ϕ' and a natural transformation α as follows:

$$\begin{tikzcd} & \mathscr{U} \arrow[dl, "\psi'"'] \arrow[dr, "\phi'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \\ \mathcal{F}^{\,\mathrm{red}}(\mathcal{Z}) \arrow[r, "\mathcal{F}^{\,\mathrm{red}}([\hat{f}])"'] & \mathcal{F}^{\,\mathrm{red}}(\mathcal{Y}) & \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}). \arrow[l, "\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}])"] \end{tikzcd}$$

By Lemma 5.7 there are a reduced orbifold atlas $\mathcal{U}$ and a weak equivalence $\psi'' : \mathcal{F}^{\,\mathrm{red}}(\mathcal{U}) \to \mathscr{U}$. Using Lemmas 3.16 and 3.17, there are a pair of morphisms $[\hat{\mathrm{v}}]$, $[\hat{g}]$ and a 2-morphism $[\theta]$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ as follows

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "{[\hat{\mathrm{v}}]}"'] \arrow[dr, "{[\hat{g}]}"] \arrow[d, phantom, "\overset{[\theta]}{\Rightarrow}"] & \\ \mathcal{Z} \arrow[r, "{[\hat{f}]}"'] & \mathcal{Y} & \mathcal{X}, \arrow[l, "{[\hat{\mathrm{w}}]}"] \end{tikzcd}$$

such that the 2-functor $\mathcal{F}^{\,\mathrm{red}}$ maps such a diagram to

$$\begin{tikzcd} & \mathcal{F}^{\,\mathrm{red}}(\mathcal{U}) \arrow[dl, "\psi'\circ\psi''"'] \arrow[dr, "\phi'\circ\psi''"] \arrow[d, phantom, "\overset{\alpha * i_{\psi''}}{\Rightarrow}"] & \\ \mathcal{F}^{\,\mathrm{red}}(\mathcal{Z}) \arrow[r, "\mathcal{F}^{\,\mathrm{red}}([\hat{f}])"'] & \mathcal{F}^{\,\mathrm{red}}(\mathcal{Y}) & \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}). \arrow[l, "\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}])"] \end{tikzcd}$$

Now $\psi' \circ \psi''$ is a weak equivalence (by condition (BF2) for the class $\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}$). So by Lemma 5.6 we have that $[\hat{\mathrm{v}}]$ is a weak equivalence of reduced orbifold atlases. By Lemma 5.4 there is a reduced orbifold atlas $\mathcal{V}$ and a weak equivalence $[\hat{\mathrm{u}}] : \mathcal{V} \to \mathcal{U}$, such that $[\hat{\mathrm{v}}] \circ [\hat{\mathrm{u}}]$ is a refinement. Then we set

$$[\hat{\mathrm{w}}'] := [\hat{\mathrm{v}}] \circ [\hat{\mathrm{u}}], \qquad [\hat{f}'] := [\hat{g}] \circ [\hat{\mathrm{u}}], \qquad [\delta] := [\theta] * i_{[\hat{\mathrm{u}}]}.$$

Since each 2-morphism in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ is invertible, the data $(\mathcal{V}, [\hat{\mathrm{w}}'], [\hat{f}'], [\delta])$ prove that (BF3) holds for $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$. The proof that (BF4) holds follows the same ideas described for (BF3).

Lastly, let us prove condition (BF5), so let us fix any pair of reduced orbifold atlases $\mathcal{X} := \{(\widetilde{X}_i, G_i, \pi_i)\}_{i \in I}$ over X and $\mathcal{Y} := \{(\widetilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ over Y, any pair of morphisms

$$[\hat{\mathrm{w}}^m] := \Big(\mathrm{w}^m, \overline{\mathrm{w}}^m, \{\widetilde{\mathrm{w}}_i^m\}_{i \in I}, [P_{\mathrm{w}^m}, \nu_{\mathrm{w}^m}] \Big) : \ \mathcal{X} \longrightarrow \mathcal{Y}, \quad m = 1, 2$$

and any 2-morphism $[\delta] : [\hat{\mathrm{w}}^1] \Rightarrow [\hat{\mathrm{w}}^2]$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, with representative $\delta := \{(\widetilde{X}_i^a, \delta_i^a)\}_{i \in I, a \in A(i)}$. Moreover, let us suppose that $[\hat{\mathrm{w}}^2]$ is a refinement. This implies that $X = Y$, $\mathrm{w}^2 = \mathrm{id}_X$ and that every smooth map $\widetilde{\mathrm{w}}_i^2$ is an open embedding. Using Definition 1.9, we get that $\mathrm{w}^1 = \mathrm{w}^2 = \mathrm{id}_X$, so condition (REF1) holds for $[\hat{\mathrm{w}}^1]$. Now let us prove also (REF2), so let us fix any $i \in I$, any $a \in A(i)$ and any point $\widetilde{x}_i$ in the open set $\widetilde{X}_i^a$. By (2Mc) we have that $\widetilde{\mathrm{w}}_i^1(\widetilde{x}_i) = (\delta_i^a)^{-1} \circ \widetilde{\mathrm{w}}_i^2(\widetilde{x}_i)$, so $\widetilde{\mathrm{w}}_i^1$ locally coincides with an open embedding, hence $\widetilde{\mathrm{w}}_i^1$ is an étale map. Again by (2Mc) we get

$$\widetilde{x}_i = (\widetilde{\mathrm{w}}_i^2)^{-1} \circ \delta_i^a \circ \widetilde{\mathrm{w}}_i^1(\widetilde{x}_i). \tag{6.1}$$

If we fix any other index $a' \in A(i)$ and any other point $\widetilde{x}'_i \in \widetilde{X}_i^{a'}$, then we have also

$$\widetilde{x}'_i = (\widetilde{\mathrm{w}}_i^2)^{-1} \circ \delta_i^{a'} \circ \widetilde{\mathrm{w}}_i^1(\widetilde{x}'_i). \tag{6.2}$$

Now if $\widetilde{\mathrm{w}}_i^1(\widetilde{x}_i) = \widetilde{\mathrm{w}}_i^1(\widetilde{x}'_i)$, then by condition (2Md) (see Definition 1.9) we have $\delta_i^a \circ \widetilde{\mathrm{w}}_i^1(\widetilde{x}_i) = \delta_i^{a'} \circ \widetilde{\mathrm{w}}_i^1(\widetilde{x}'_i)$. Hence using (6.1) and (6.2) we conclude that $\widetilde{x}_i = \widetilde{x}'_i$, i.e. the map $\widetilde{\mathrm{w}}_i^1$ is injective. Since we have already proved that it is locally an open embedding, we conclude that it is globally an open embedding, hence condition (REF2) holds for $[\hat{\mathrm{w}}^1]$. Therefore, (BF5) is verified for $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$. $\square$

Lemma 6.2. *Given any pair of morphisms $[\hat{f}]:\mathcal{X}\to\mathcal{Y}$ and $[\hat{g}]:\mathcal{Y}\to\mathcal{Z}$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, we have:*

- *(i) if $[\hat{f}]$ and $[\hat{g}]$ belong to $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$, then so does $[\hat{g}]\circ[\hat{f}]$;*
- *(ii) if $[\hat{g}]$ and $[\hat{g}]\circ[\hat{f}]$ belong to $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$, then so does $[\hat{f}]$.*

Proof. (i) is simply (BF2) for the class $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$ (see Proposition 6.1). For (ii), let us suppose that $[\hat{f}]$ and $[\hat{g}]$ are as in (1.3). Since $[\hat{g}]$ is a refinement, then $Y=Z$, $g=\mathrm{id}_Z$ and $\widetilde{g}_j$ is an open embedding for each $j\in J$. Since $[\hat{g}]\circ[\hat{f}]$ is a refinement, then $X=Z$, $g\circ f=\mathrm{id}_Z$ and $\widetilde{g}_{\overline{f}(i)}\circ\widetilde{f}_i$ is an open embedding for each $i\in I$. From this we get that $f=\mathrm{id}_X$ and that $\widetilde{f}_i$ is an open embedding for each $i\in I$, i.e. $[\hat{f}]$ is a refinement. $\square$

For the explicit description of $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ below, we use the original construction of bicategories of fractions in [Pr, § 2.2, 2.3 and 2.4] and our previous paper [T2], where we provided simple constructions of associators and compositions of 2-morphisms in any bicategory of fractions, as well as a definition of 2-morphisms that is equivalent to the one in [Pr], but much shorter. In order to construct explicitly a bicategory of fractions, one has to make some choices as in the following description. By [Pr, Theorem 21], *different choices will give* equivalent *bicategories of fractions where objects,* 1*-morphisms and* 2*-morphisms are the same, but compositions of* 1*-morphisms and* 2*-morphisms are* (*possibly*) *different.*

Description 6.3. In order to describe $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$, for any pair of morphisms

$$\mathcal{X}'\xrightarrow{[\hat{f}]}\mathcal{Y}\xleftarrow{[\hat{\mathrm{v}}]}\mathcal{Y}' \tag{6.3}$$

with $[\hat{\mathrm{v}}]$ refinement, using the axiom of choice we *choose* any reduced orbifold atlas $\mathcal{X}''$, any pair of morphisms $[\hat{\mathrm{v}}']$ and $[\hat{f}']$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ with $[\hat{\mathrm{v}}']$ refinement, and any 2-morphism $[\delta]$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ as follows:

$$\begin{tikzcd} & \mathcal{X}'' \arrow[dl, "{[\hat{\mathrm{v}}']}"'] \arrow[dr, "{[\hat{f}']}"] \arrow[d, phantom, "{[\delta]}\,\Rightarrow"] & \\ \mathcal{X}' \arrow[r, "{[\hat{f}]}"'] & \mathcal{Y} & \mathcal{Y}'. \arrow[l, "{[\hat{\mathrm{v}}]}"] \end{tikzcd} \tag{6.4}$$

Such a choice is always possible by (BF3) (see Proposition 6.1) but in general it is not unique. By [Pr, § 2.2] we only have to force such a choice in the following special cases:

(a) whenever (6.3) is such that $\mathcal{Y}=\mathcal{X}'$ and $[\hat{f}]=\mathrm{id}_{\mathcal{Y}}$, then we have to choose $\mathcal{X}'':=\mathcal{Y}'$, $[\hat{f}']:=\mathrm{id}_{\mathcal{Y}'}$, $[\hat{\mathrm{v}}']:=[\hat{\mathrm{v}}]$ and $[\delta]:=i_{[\hat{\mathrm{v}}]}$;

(b) whenever (6.3) is such that $\mathcal{Y}=\mathcal{Y}'$ and $[\hat{\mathrm{v}}]=\mathrm{id}_{\mathcal{Y}}$, then we have to choose $\mathcal{X}'':=\mathcal{X}'$, $[\hat{f}']:=[\hat{f}]$, $[\hat{\mathrm{v}}']:=\mathrm{id}_{\mathcal{X}'}$ and $[\delta]:=i_{[\hat{f}]}$.

Moreover, for simplicity of computations we impose also the following two condition (compatible with the previous ones):

(c) whenever (6.3) is such that $\mathcal{X}'=\mathcal{Y}'$ and $[\hat{f}]$ coincides with the refinement $[\hat{\mathrm{v}}]$, then we choose $\mathcal{X}'':=\mathcal{X}'$, $[\hat{f}']:=\mathrm{id}_{\mathcal{X}'}$, $[\hat{\mathrm{v}}']:=\mathrm{id}_{\mathcal{X}'}$ and $[\delta]:=i_{[\hat{\mathrm{v}}]}$;

(d) let us fix data as in (6.3) with associated data (6.4). Let X' be the underlying topological space of $\mathcal{X}'$ and Y the underlying topological space of both $\mathcal{Y}$ and $\mathcal{Y}'$; moreover let us fix another pair of topological spaces $\widetilde{X}'$ and $\widetilde{Y}$, and a pair of homeomorphisms $\mathrm{p} : X' \xrightarrow{\sim} \widetilde{X}'$ and $\mathrm{q} : Y \xrightarrow{\sim} \widetilde{Y}$. Then we denote by $\mathrm{p}_*(\mathcal{X}')$ the atlas induced on $\widetilde{X}'$ (as we did in Definition 5.1), and analogously for the atlases $\mathrm{q}_*(\mathcal{Y})$ and $\mathrm{q}_*(\mathcal{Y}')$, both defined on $\widetilde{Y}$; moreover, we denote by $\mathrm{q} \circ [\hat{f}] \circ \mathrm{p}^{-1}$ the induced morphism in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ from $\mathrm{p}_*(\mathcal{X}')$ to $\mathrm{q}_*(\mathcal{Y})$ and by $\mathrm{q} \circ [\hat{\mathrm{v}}] \circ \mathrm{q}^{-1}$ the induced refinement from $\mathrm{q}_*(\mathcal{Y}')$ to $\mathrm{q}_*(\mathcal{Y})$. Then we impose that the data associated to

$$\mathrm{p}_*(\mathcal{X}') \xrightarrow{\mathrm{q} \circ [\hat{f}] \circ \mathrm{p}^{-1}} \mathrm{q}_*(\mathcal{Y}) \xleftarrow{\mathrm{q} \circ [\hat{\mathrm{v}}] \circ \mathrm{q}^{-1}} \mathrm{q}_*(\mathcal{Y}')$$

is given by

$$\begin{tikzcd}
 & \mathcal{X}'' \arrow[dl, "\mathrm{p} \circ [\hat{\mathrm{v}}']"'] \arrow[dr, "\mathrm{q} \circ [\hat{f}']"] \arrow[d, phantom, "i_{\mathrm{q}} * [\delta] \Rightarrow"] & \\
\mathrm{p}_*(\mathcal{X}') \arrow[r, "\mathrm{q} \circ [\hat{f}] \circ \mathrm{p}^{-1}"'] & \mathrm{q}_*(\mathcal{Y}) & \mathrm{q}_*(\mathcal{Y}'). \arrow[l, "\mathrm{q} \circ [\hat{\mathrm{v}}] \circ \mathrm{q}^{-1}"]
\end{tikzcd}$$

From now on, $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$ is *any fixed collection of choices* (6.4) *for any data* (6.3), satisfying conditions (a) – (d). Whenever we fix a pair of morphisms

$$\mathcal{X}^1 \xrightarrow{[\hat{\mathrm{w}}^1]} \mathcal{X} \xleftarrow{[\hat{\mathrm{w}}^2]} \mathcal{X}^2 \tag{6.5}$$

with *both* $[\hat{\mathrm{w}}^1]$ and $[\hat{\mathrm{w}}^2]$ refinements, we denote by

$$\begin{tikzcd}
 & \mathcal{X}^3 \arrow[dl, "{[\hat{\mathrm{u}}^1]}"'] \arrow[dr, "{[\hat{\mathrm{u}}^2]}"] \arrow[d, phantom, "{[\mu]} \Rightarrow"] & \\
\mathcal{X}^1 \arrow[r, "{[\hat{\mathrm{w}}^1]}"'] & \mathcal{X} & \mathcal{X}^2 \arrow[l, "{[\hat{\mathrm{w}}^2]}"]
\end{tikzcd} \tag{6.6}$$

the corresponding choice in $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$, with $[\hat{\mathrm{u}}^1]$ refinement. Having fixed all such data and notations, the bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ can be described as follows. The **objects** of $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ are exactly the objects of $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, i.e. all the reduced orbifold atlases according to Definition 1.4. Given any pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$, a **morphism in** $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ from $\mathcal{X}$ to $\mathcal{Y}$ is given by any triple $(\mathcal{X}', [\hat{\mathrm{w}}], [\hat{f}])$ where $\mathcal{X}'$ is any reduced orbifold atlas, $[\hat{\mathrm{w}}]$ is any *refinement* and $[\hat{f}]$ is any morphism in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, as follows:

$$\mathcal{X} \xleftarrow{[\hat{\mathrm{w}}]} \mathcal{X}' \xrightarrow{[\hat{f}]} \mathcal{Y}$$

Given any pair of reduced atlases $\mathcal{X}, \mathcal{Y}$ and any pair of morphisms $(\mathcal{X}^m, [\hat{\mathrm{w}}^m], [\hat{f}^m]) : \mathcal{X} \to \mathcal{Y}$ for $m = 1, 2$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$, a **2-morphism** from the first morphism to the second one is an equivalence class of data $(\mathcal{X}^4, [\,\hat{\mathrm{t}}\,], [\delta])$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, such that $[\,\hat{\mathrm{t}}\,] : \mathcal{X}^4 \to \mathcal{X}^3$ is a refinement and $[\delta] : [\hat{f}^1] \circ [\hat{\mathrm{u}}^1] \circ [\,\hat{\mathrm{t}}\,] \Rightarrow [\hat{f}^1] \circ [\hat{\mathrm{u}}^2] \circ [\,\hat{\mathrm{t}}\,]$ (here and below the data $\mathcal{X}^3$, $[\hat{\mathrm{u}}^1]$, $[\hat{\mathrm{u}}^2]$ and $[\mu]$ are determined by $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$ as in (6.6)). You can consider the data $(\mathcal{X}^4, [\,\hat{\mathrm{t}}\,], [\delta])$ as a way of filling the following diagram, already partially filled by morphisms $(\mathcal{X}^m, [\hat{\mathrm{w}}^m], [\hat{f}^m])$ for $m = 1, 2$ and by the choice (6.6) fixed in $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$:

$$\begin{tikzcd}
& \mathcal{X}^1 \arrow[ddl, "{[\hat{\mathrm{w}}^1]}"'] \arrow[ddr, "{[\hat{f}^1]}"] & \\
& \mathcal{X}^3 \arrow[u, "{[\hat{\mathrm{u}}^1]}"'] & \\
\mathcal{X} \arrow[r, phantom, "\Downarrow\, {[\mu]} * i_{[\hat{\mathrm{t}}]}"] & \mathcal{X}^4 \arrow[u, "{[\hat{\mathrm{t}}]}"'] \arrow[d, "{[\hat{\mathrm{t}}]}"] \arrow[r, phantom, "\Downarrow\, {[\delta]}"] & \mathcal{Y}, \\
& \mathcal{X}^3 \arrow[d, "{[\hat{\mathrm{u}}^2]}"] & \\
& \mathcal{X}^2 \arrow[uul, "{[\hat{\mathrm{w}}^2]}"] \arrow[uur, "{[\hat{f}^2]}"'] &
\end{tikzcd} \tag{6.7}$$

Any other set of data $(\mathcal{X}'^4, [\hat{\mathrm{t}}'], [\delta'])$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ (such that $[\hat{\mathrm{t}}'] : \mathcal{X}'^4 \to \mathcal{X}^3$ is a refinement and $[\delta'] : [\hat{f}^1] \circ [\hat{\mathrm{u}}^1] \circ [\hat{\mathrm{t}}'] \Rightarrow [\hat{f}^1] \circ [\hat{\mathrm{u}}^2] \circ [\hat{\mathrm{t}}']$) represents the same 2-morphism as $(\mathcal{X}^4, [\hat{\mathrm{t}}], [\delta])$ if and only if there are a reduced orbifold atlas $\mathcal{X}^5$, a refinement $[\hat{\mathrm{z}}] : \mathcal{X}^5 \to \mathcal{X}^4$, a morphism $[\hat{\mathrm{z}}'] : \mathcal{X}^5 \to \mathcal{X}'^4$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, and a 2-morphism $[\sigma] : [\hat{\mathrm{t}}'] \circ [\hat{\mathrm{z}}'] \Rightarrow [\hat{\mathrm{t}}] \circ [\hat{\mathrm{z}}]$, such that the compositions of the following two diagrams are equal:

$$\begin{tikzcd}
& \mathcal{X}'^4 \arrow[r, "{[\hat{\mathrm{t}}']}"] \arrow[d, phantom, "\Downarrow\, {[\sigma]}"] & \mathcal{X}^3 \arrow[r, "{[\hat{\mathrm{u}}^1]}"] & \mathcal{X}^1 \arrow[dr, "{[\hat{f}^1]}"] & \\
\mathcal{X}^5 \arrow[ur, "{[\hat{\mathrm{z}}']}"] \arrow[r, "{[\hat{\mathrm{z}}]}"] \arrow[dr, "{[\hat{\mathrm{z}}']}"'] & \mathcal{X}^4 \arrow[ur, "{[\hat{\mathrm{t}}]}"'] \arrow[dr, "{[\hat{\mathrm{t}}]}"] \arrow[d, phantom, "\Downarrow\, {[\sigma]^{-1}}"] \arrow[rr, phantom, "\Downarrow\, {[\varphi]}"] & & & \mathcal{Y}, \\
& \mathcal{X}'^4 \arrow[r, "{[\hat{\mathrm{t}}']}"'] & \mathcal{X}^3 \arrow[r, "{[\hat{\mathrm{u}}^2]}"'] & \mathcal{X}^2 \arrow[ur, "{[\hat{f}^2]}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \mathcal{X}^3 \arrow[r, "{[\hat{\mathrm{v}}^1]}"] & \mathcal{X}^1 \arrow[dr, "{[\hat{f}^1]}"] & \\
\mathcal{X}^5 \arrow[r, "{[\hat{\mathrm{z}}']}"] & \mathcal{X}^4 \arrow[ur, "{[\hat{\mathrm{t}}']}"] \arrow[dr, "{[\hat{\mathrm{t}}']}"'] \arrow[rrr, phantom, "\Downarrow\, {[\varphi']}"] & & & \mathcal{Y}. \\
& & \mathcal{X}^3 \arrow[r, "{[\hat{\mathrm{v}}^2]}"'] & \mathcal{X}^2 \arrow[ur, "{[\hat{f}^2]}"'] &
\end{tikzcd}$$

We denote by $[\mathcal{X}^4, [\hat{\mathrm{t}}], [\delta]]$ the class of any data as above, and we call it a 2-morphism in $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ from $(\mathcal{X}^1, [\hat{\mathrm{w}}^1], [\hat{f}^1])$ to $(\mathcal{X}^2, [\hat{\mathrm{w}}^2], [\hat{f}^2])$. Given any pair of morphisms in $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ as follows:

$$\mathcal{X} \xleftarrow{[\hat{\mathrm{w}}]} \mathcal{X}' \xrightarrow{[\hat{f}]} \mathcal{Y} \qquad\qquad \mathcal{Y} \xleftarrow{[\hat{\mathrm{v}}]} \mathcal{Y}' \xrightarrow{[\hat{g}]} \mathcal{Z}$$

(with both $[\hat{\mathrm{w}}]$ and $[\hat{\mathrm{v}}]$ refinements), we use the fixed choice (6.4) and we set

$$\Big(\mathcal{Y}', [\hat{\mathrm{v}}], [\hat{g}]\Big) \circ \Big(\mathcal{X}', [\hat{\mathrm{w}}], [\hat{f}]\Big) := \Big(\mathcal{X}'', [\hat{\mathrm{w}}] \circ [\hat{\mathrm{v}}'], [\hat{g}] \circ [\hat{f}']\Big) : \mathcal{X} \longrightarrow \mathcal{Z}. \tag{6.8}$$

In this way in general the composition of morphisms in $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ is associative only up to canonical 2-isomorphisms, so $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ is a bicategory but not a 2-category. Also the construction of **vertical and horizontal compositions for 2-morphisms** is induced by the choices $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$. Since we don't need it in this paper, we refer either to the construction originally described in [Pr, § 2.3], or to the simplified version given in our previous paper [T2]). Using the the same references you can also easily describe the induced universal pseudofunctor $\mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}} : (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \to (\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ mentioned in Proposition 6.1.

Description 6.4. Since we want to induce an equivalence from $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ to $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$, we need also to describe this last bicategory. For this, for every pair of morphisms in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$

$$\mathscr{X}' \xrightarrow{\quad\phi\quad} \mathscr{Y} \xleftarrow{\quad\xi\quad} \mathscr{Y}' \tag{6.9}$$

with ξ weak equivalence, we need to *choose* a set of data $(\mathscr{X}'', \phi', \xi', \theta)$ in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ as below

$$\begin{tikzcd}
& \mathscr{X}'' \arrow[dl, "\xi'"'] \arrow[dr, "\phi'"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] & \\
\mathscr{X}' \arrow[r, "\phi"'] & \mathscr{Y} & \mathscr{Y}', \arrow[l, "\xi"]
\end{tikzcd} \tag{6.10}$$

such that ξ' is a weak equivalence. On such choices we impose the analogous of conditions (a), (b) and (c) listed in Description 6.3; moreover, we fix also the following condition:

(d)$'$ let us suppose that the data of (6.9) are given by

$$\mathscr{F}^{\,\mathrm{red}}(\mathcal{X}') \xrightarrow{\ \phi\ } \mathscr{F}^{\,\mathrm{red}}(\mathcal{Y}) \xleftarrow{\mathscr{F}^{\,\mathrm{red}}([\hat{\mathrm{v}}])} \mathscr{F}^{\,\mathrm{red}}(\mathcal{Y}') \tag{6.11}$$

for some triple of atlases $(\mathcal{X}', \mathcal{Y}, \mathcal{Y}')$ and some refinement $[\hat{\mathrm{v}}] : \mathcal{Y}' \to \mathcal{Y}$. In this case, by Lemma 3.16 there is a unique morphism $[\hat{f}] : \mathcal{X}' \to \mathcal{Y}$ such that $\phi = \mathscr{F}^{\,\mathrm{red}}([\hat{f}])$. Then we impose that the data (6.10) associated to (6.9) are the image via $\mathscr{F}^{\,\mathrm{red}}$ of the data (6.4) *chosen* in $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$ for (6.3).

Note that the condition above is well-defined:

- first of all, since $[\hat{\mathrm{v}}']$ in (6.4) is a refinement, hence a weak equivalence, then by Lemma 5.5 $\mathscr{F}^{\,\mathrm{red}}([\hat{\mathrm{v}}'])$ is a weak equivalence, as required;
- if there is more than one triple $(\mathcal{X}', \mathcal{Y}, \mathcal{Y}')$ as above, then the associated data are the same. This is an easy consequence of the construction of $\mathscr{F}^{\,\mathrm{red}}$, together with condition (d) in Description 6.3.

In particular, whenever we fix any pair of morphisms

$$\mathscr{X}^1 \xrightarrow{\ \psi^1\ } \mathscr{X} \xleftarrow{\ \psi^2\ } \mathscr{X}^2$$

with *both* ψ^1 and ψ^2 weak equivalences, we denote by

$$\begin{tikzcd} & \mathscr{X}^3 \arrow[dl, "\xi^1"'] \arrow[dr, "\xi^2"] \arrow[d, phantom, "\overset{\mu}{\Rightarrow}"] & \\ \mathscr{X}^1 \arrow[r, "\psi^1"'] & \mathscr{X} & \mathscr{X}^2 \arrow[l, "\psi^2"] \end{tikzcd}$$

the corresponding choice, with ξ^1 weak equivalence. We denote by $\mathrm{C}(\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}})$ any set of choices satisfying (a), (b), (c) and (d)$'$. Then the objects of $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$ are all the proper, effective, étale Lie groupoids; a morphism from $\mathscr{X}$ to $\mathscr{Y}$, is given by any set of data as follows, with ψ weak equivalence:

$$\mathscr{X} \xleftarrow{\ \psi\ } \mathscr{X}' \xrightarrow{\ \phi\ } \mathscr{Y}\,.$$

Given morphisms $(\mathscr{X}^m, \psi^m, \phi^m) : \mathscr{X} \to \mathscr{Y}$ in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$ for $m = 1, 2$, a 2-morphism from the first one to the second one is represented by a triple $(\mathscr{X}^4, \tau, \rho)$, such that $\tau : \mathscr{X}^4 \to \mathscr{X}^3$ is a weak equivalence and ρ is a natural transformation from $\phi^1 \circ \xi^1 \circ \tau$ to $\phi^2 \circ \xi^2 \circ \tau$. The equivalence relation on such triples is analogous to the one mentioned in Description 6.3, and any 2-morphism in this setup is denoted by $[\mathscr{X}^4, \tau, \rho] : (\mathscr{X}^1, \psi^1, \phi^1) \Rightarrow (\mathscr{X}^2, \psi^2, \phi^2)$. Compositions are defined analogously to $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$.

7. The pseudofunctor $\mathcal{G}^{\,\mathrm{red}}$

Now we are almost ready to describe a pseudofunctor $\mathcal{G}^{\,\mathrm{red}}$ as in (0.1). For that, we will need the following:

Definition 7.1. [T3, Definition 2.1] Let us fix any bicategory $\mathscr{C}$ and any class $\mathbf{W}$ of morphisms in it (not necessarily satisfying conditions (BF)). Then we define a class $\mathbf{W}_{\mathrm{sat}}$ as the class of all morphisms $f : B \to A$ in $\mathscr{C}$, such that there are a pair of objects C, D and a pair of morphisms $g : C \to B$, $h : D \to C$, such that both $f \circ g$ and $g \circ h$ belong to $\mathbf{W}$. We call $\mathbf{W}_{\mathrm{sat}}$ the (right) *saturation* of $\mathbf{W}$; we say that $\mathbf{W}$ is (right) *saturated* if it coincides with its saturation.

We recall (see [T3, Lemma 3.9 and Proposition 3.11]) that if $(\mathscr{C}, \mathbf{W})$ satisfies conditions (BF), then so does $(\mathscr{C}, \mathbf{W}_{\mathrm{sat}})$, and the bicategories $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ and $\mathscr{C}\left[\mathbf{W}_{\mathrm{sat}}^{-1}\right]$ are equivalent.

Definition 7.2. Let us fix any pair of 2-categories $\mathscr{A}, \mathscr{B}$ and any pair of classes $\mathbf{W}_{\mathscr{A}}$ and $\mathbf{W}_{\mathscr{B}}$ of morphisms in $\mathscr{A}$ and $\mathscr{B}$ respectively, such that both $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ and $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ satisfy conditions (BF), so that there are bicategories of fractions $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ and $\mathscr{B}[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}]$. In order to construct explicitly such bicategories of fractions, we need to fix a set of choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ for the pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ (as we did in Description 6.3) and a set of choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$. Given any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, we say that the set of choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ and $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ are *$\mathcal{F}$-compatible* if and only if the following two conditions hold:

(1) $\mathcal{F}(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$;

(2) given any set of data in $\mathscr{A}$ as follows

$$\begin{tikzcd} A^1_{\mathscr{A}} \arrow[r, "\mathrm{w}^1_{\mathscr{A}}"] & A_{\mathscr{A}} & A^2_{\mathscr{A}} \arrow[l, "\mathrm{w}^2_{\mathscr{A}}"'] \end{tikzcd} \tag{7.1}$$

with both $\mathrm{w}^1_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, if its associated choice in $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ is given by

$$\begin{tikzcd} & A^3_{\mathscr{A}} \arrow[dl, "\mathrm{u}^1_{\mathscr{A}}"'] \arrow[dr, "\mathrm{u}^2_{\mathscr{A}}"] \arrow[d, phantom, "\delta_{\mathscr{A}} \atop \Rightarrow"] & \\ A^1_{\mathscr{A}} \arrow[r, "\mathrm{w}^1_{\mathscr{A}}"'] & A_{\mathscr{A}} & A^2_{\mathscr{A}}, \arrow[l, "\mathrm{w}^2_{\mathscr{A}}"] \end{tikzcd} \tag{7.2}$$

then the choice for

$$\mathcal{F}_0(A^2_{\mathscr{A}}) \xrightarrow{\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})} \mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})} \mathcal{F}_0(A^1_{\mathscr{A}})$$

in $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ is given by the image of (7.2) via $\mathcal{F}$.

Using [T3, Theorem 0.3 and Remark 3.2] (in the special case when $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and $\mathcal{F}$ is a 2-functor) together with [T2, Appendix], we have:

Theorem 7.3. *Let us fix two pairs $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ and $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$, both satisfying conditions* (BF)*, any set of choices $C(\mathbf{W}_{\mathscr{A}})$ for the construction of $\mathscr{A}[\mathbf{W}_{\mathscr{A}}^{-1}]$, and analogously for $C(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$; moreover let us fix any 2-functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, such that the choices $C(\mathbf{W}_{\mathscr{A}})$ and $C(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ are $\mathcal{F}$-compatible. Then there is a pseudofunctor*

$$\mathcal{G} : \mathscr{A}\Big[\mathbf{W}_{\mathscr{A}}^{-1}\Big] \longrightarrow \mathscr{B}\Big[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\Big]$$

such that:

- $\mathcal{U}_{\mathbf{W}_{\mathscr{B},\mathrm{sat}}} \circ \mathcal{F} = \mathcal{G} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$;
- *for each object $A_{\mathscr{A}}$, we have $\mathcal{G}(A_{\mathscr{A}}) = \mathcal{F}(A_{\mathscr{A}})$;*
- *for each morphism $(A'_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}}, f_{\mathscr{A}})$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$, we have $\mathcal{G}(A'_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}}, f_{\mathscr{A}}) = (\mathcal{F}(A'_{\mathscr{A}}), \mathcal{F}(\mathrm{w}_{\mathscr{A}}), \mathcal{F}(f_{\mathscr{A}}))$;*
- *for each 2-morphism $[A^4_{\mathscr{A}}, \mathrm{t}_{\mathscr{A}}, \delta_{\mathscr{A}}] : (A^1_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}}, f^1_{\mathscr{A}}) \Rightarrow (A^2_{\mathscr{A}}, \mathrm{w}^2_{\mathscr{A}}, f^2_{\mathscr{A}})$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$, we have $\mathcal{G}([A^4_{\mathscr{A}}, \mathrm{t}_{\mathscr{A}}, \delta_{\mathscr{A}}]) = [\mathcal{F}(A^4_{\mathscr{A}}), \mathcal{F}(\mathrm{t}_{\mathscr{A}}), \mathcal{F}(\delta_{\mathscr{A}})]$ (here we are assuming that the choice for the pair (7.1) is given by (7.2), hence $\mathrm{t}_{\mathscr{A}} : A^4_{\mathscr{A}} \to A^3_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$ and $\delta_{\mathscr{A}}$ is a 2-morphism from $f^1_{\mathscr{A}} \circ \mathrm{u}^1_{\mathscr{A}} \circ \mathrm{t}_{\mathscr{A}}$ to $f^2_{\mathscr{A}} \circ \mathrm{u}^2_{\mathscr{A}} \circ \mathrm{t}_{\mathscr{A}}$).*

If we want to apply Theorem 7.3 to $\mathcal{F}^{\,\mathrm{red}}$, then we need to compute $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, i.e. the right saturation of the class $(\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$. For this, we can use the following result, taken from [T3, Corollary 4.2(b) and (c) and Proposition 2.11(ii)] (the second part of the statement can be found also in [PS, Lemma 8.1]).

Lemma 7.4. *The class $\mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$ is right saturated, i.e. $\mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd},\mathrm{sat}} = \mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$. Moreover, given any pair of morphisms $\phi : \mathscr{X} \to \mathscr{Y}$ and $\psi : \mathscr{Y} \to \mathscr{Z}$ in $(\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$, if any two maps among ϕ, ψ and $\psi \circ \phi$ are weak equivalences, then so is the third one.*

Proposition 7.5. *There is a pseudofunctor $\mathcal{G}^{\,\mathrm{red}} : (\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb}) \to (\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}\right]$, such that:*

(1) for each reduced orbifold atlas $\mathcal{X}$, $\mathcal{G}^{\,\mathrm{red}}(\mathcal{X}) = \mathcal{F}^{\,\mathrm{red}}(\mathcal{X})$;

(2) for each morphism $(\mathcal{X}', [\hat{\mathrm{w}}], [\hat{f}]) : \mathcal{X} \to \mathcal{Y}$, we have $\mathcal{G}^{\,\mathrm{red}}(\mathcal{X}', [\hat{\mathrm{w}}], [\hat{f}]) = (\mathcal{F}^{\,\mathrm{red}}(\mathcal{X}), \mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}]), \mathcal{F}^{\,\mathrm{red}}([\hat{f}]))$;

(3) for each 2-morphism $[\mathcal{X}^4, [\,\hat{\mathrm{t}}\,], [\delta]] : ([\mathcal{X}^1, [\hat{\mathrm{w}}^1], [\hat{f}^1]) \Rightarrow (\mathcal{X}^2, [\hat{\mathrm{w}}^2], [\hat{f}^2])$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$, the 2-morphism $\mathcal{G}^{\,\mathrm{red}}([\mathcal{X}^4, [\,\hat{\mathrm{t}}\,], [\delta]])$ coincides with the class $[\mathcal{F}^{\,\mathrm{red}}(\mathcal{X}^4), \mathcal{F}^{\,\mathrm{red}}(\,[\,\hat{\mathrm{t}}\,]), \mathcal{F}^{\,\mathrm{red}}([\delta])]$.

Moreover, we have $\mathcal{U}_{\mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}} \circ \mathcal{F}^{\,\mathrm{red}} = \mathcal{G}^{\,\mathrm{red}} \circ \mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}}$.

Proof. Let us apply Theorem 7.3 with $\mathscr{A} := (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, $\mathbf{W}_{\mathscr{A}} := \mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$ (i.e. all the refinements of reduced orbifold atlases), $\mathscr{B} := (\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd})$, $\mathbf{W}_{\mathscr{B}} := \mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$ (i.e. all the weak equivalences of proper, effective, étale groupoids) and $\mathcal{F} := \mathcal{F}^{\,\mathrm{red}}$. By Lemma 7.4 we have $\mathbf{W}_{\mathscr{B},\mathrm{sat}} = \mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$. Moreover, $\mathcal{F}^{\,\mathrm{red}}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}) \subseteq \mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}}$ as a consequence of Lemma 5.5. The choices $\mathrm{C}(\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}})$ and $\mathrm{C}(\mathbf{W}_{\mathcal{P}\mathcal{E}\acute{\mathcal{E}}\,\mathcal{G}\mathbf{pd}})$ that we fixed above are $\mathcal{F}^{\,\mathrm{red}}$-compatible (as a consequence of conditions (a), (b), (c), (d) and (d)′), so we conclude by Theorem 7.3. □

8. $\mathcal{G}^{\,\mathrm{red}}$ is an equivalence of bicategories

The main aim of this paper is to prove that the pseudofunctor $\mathcal{G}^{\,\mathrm{red}}$ constructed above is an equivalence of bicategories. For this, we need [T4, Theorem 0.2], that we state below only in the special framework where $\mathscr{A}$ and $\mathscr{B}$ are 2-categories, $\mathcal{F}$ is a 2-functor (also known as strict pseudofunctor), $\mathcal{U}_{\mathbf{W}_{\mathscr{B}}} \circ \mathcal{F} = \mathcal{G} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$, and the equivalence κ appearing in [T4, Theorem 0.2] is the 2-identity of $\mathcal{U}_{\mathbf{W}_{\mathscr{B}}} \circ \mathcal{F}$.

Theorem 8.1. *[T4, Theorem 0.2] Let us fix any pair of 2-categories $\mathscr{A}$, $\mathscr{B}$ and any pair of classes of morphisms $\mathbf{W}_{\mathscr{A}}$, $\mathbf{W}_{\mathscr{B}}$, such that both $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ and $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ satisfy conditions* (BF). *Moreover, let us fix any 2-functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, such that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$. In addition, let us suppose that there is a pseudofunctor $\mathcal{G} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B}}\right]$ such that $\mathcal{U}_{\mathbf{W}_{\mathscr{B}}} \circ \mathcal{F} = \mathcal{G} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$, and let us assume the axiom of choice. Then $\mathcal{G}$ is an equivalence of bicategories if and only if $\mathcal{F}$ satisfies the following 5 conditions.*

(A1) For any object $A_{\mathscr{B}}$, there are a pair of objects $A_{\mathscr{A}}$ and $A'_{\mathscr{B}}$ and a pair of morphisms $\mathrm{w}^1_{\mathscr{B}} : A'_{\mathscr{B}} \to \mathcal{F}(A_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{w}^2_{\mathscr{B}} : A'_{\mathscr{B}} \to A_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$.

(A2) Let us fix any triple of objects $A^1_{\mathscr{A}}, A^2_{\mathscr{A}}, A^3_{\mathscr{B}}$, any morphism $\mathrm{w}^1_{\mathscr{B}} : A^3_{\mathscr{B}} \to \mathcal{F}(A^1_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and any morphism $\mathrm{w}^2_{\mathscr{B}} : A^3_{\mathscr{B}} \to \mathcal{F}(A^2_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Then there are an object $A^4_{\mathscr{A}}$, a morphism $\mathrm{w}^1_{\mathscr{A}} : A^4_{\mathscr{A}} \to A^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, a morphism $\mathrm{w}^2_{\mathscr{A}} : A^4_{\mathscr{A}} \to A^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$, and a set of data $(A^5_{\mathscr{B}}, \mathrm{z}^1_{\mathscr{B}}, \mathrm{z}^2_{\mathscr{B}}, \gamma^1_{\mathscr{B}}, \gamma^2_{\mathscr{B}})$ as follows

$$\begin{tikzcd}
 & A^3_{\mathscr{B}} \arrow[ld, "\mathrm{w}^1_{\mathscr{B}}"'] \arrow[rd, "\mathrm{w}^2_{\mathscr{B}}"] & \\
\mathcal{F}(A^1_{\mathscr{A}}) \arrow[r, phantom, "\Downarrow \gamma^1_{\mathscr{B}}"] & A^5_{\mathscr{B}} \arrow[u, "\mathrm{z}^1_{\mathscr{B}}"'] \arrow[d, "\mathrm{z}^2_{\mathscr{B}}"] \arrow[r, phantom, "\Downarrow \gamma^2_{\mathscr{B}}"] & \mathcal{F}(A^2_{\mathscr{A}}), \\
 & \mathcal{F}(A^4_{\mathscr{A}}) \arrow[lu, "\mathcal{F}(\mathrm{w}^1_{\mathscr{A}})"] \arrow[ru, "\mathcal{F}(\mathrm{w}^2_{\mathscr{A}})"'] &
\end{tikzcd}$$

such that $\mathrm{z}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$ and both $\gamma^1_{\mathscr{B}}$ and $\gamma^2_{\mathscr{B}}$ are invertible.

(*A3*) *Let us fix any pair of objects* $B_{\mathscr{A}}, A_{\mathscr{B}}$ *and any morphism* $f_{\mathscr{B}} : A_{\mathscr{B}} \to \mathcal{F}(B_{\mathscr{A}})$*. Then there are an object* $A_{\mathscr{A}}$*, a morphism* $f_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$*, an object* $A'_{\mathscr{B}}$*, a morphism* $\mathrm{v}^1_{\mathscr{B}} : A'_{\mathscr{B}} \to A_{\mathscr{B}}$ *in* $\mathbf{W}_{\mathscr{B}}$*, a morphism* $\mathrm{v}^2_{\mathscr{B}} : A'_{\mathscr{B}} \to \mathcal{F}(A_{\mathscr{A}})$ *in* $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ *and an invertible 2-morphism* $\alpha_{\mathscr{B}} : f_{\mathscr{B}} \circ \mathrm{v}^1_{\mathscr{B}} \Rightarrow \mathcal{F}(f_{\mathscr{A}}) \circ \mathrm{v}^2_{\mathscr{B}}$*.*

(*A4*) *Let us fix any pair of objects* $A_{\mathscr{A}}, B_{\mathscr{A}}$*, any pair of morphisms* $f^1_{\mathscr{A}}, f^2_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ *and any pair of 2-morphisms* $\gamma^1_{\mathscr{A}}, \gamma^2_{\mathscr{A}} : f^1_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}}$*. Moreover, let us fix any object* $A'_{\mathscr{B}}$ *and any morphism* $\mathrm{z}_{\mathscr{B}} : A'_{\mathscr{B}} \to \mathcal{F}(A_{\mathscr{A}})$ *in* $\mathbf{W}_{\mathscr{B}}$*. If* $\mathcal{F}(\gamma^1_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}} = \mathcal{F}(\gamma^2_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}$*, then there are an object* $A'_{\mathscr{A}}$ *and a morphism* $\mathrm{z}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ *in* $\mathbf{W}_{\mathscr{A}}$*, such that* $\gamma^1_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}} = \gamma^2_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}$*.*

(*A5*) *Let us fix any triple of objects* $A_{\mathscr{A}}, B_{\mathscr{A}}, A_{\mathscr{B}}$*, any pair of morphisms* $f^1_{\mathscr{A}}, f^2_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$*, any morphism* $\mathrm{v}_{\mathscr{B}} : A_{\mathscr{B}} \to \mathcal{F}(A_{\mathscr{A}})$ *in* $\mathbf{W}_{\mathscr{B}}$ *and any 2-morphism* $\alpha_{\mathscr{B}} : \mathcal{F}(f^1_{\mathscr{A}}) \circ \mathrm{v}_{\mathscr{B}} \Rightarrow \mathcal{F}(f^2_{\mathscr{A}}) \circ \mathrm{v}_{\mathscr{B}}$*. Then there are a pair of objects* $A'_{\mathscr{A}}, A'_{\mathscr{B}}$*, a triple of morphisms* $\mathrm{v}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ *in* $\mathbf{W}_{\mathscr{A}}$*,* $\mathrm{z}_{\mathscr{B}} : A'_{\mathscr{B}} \to \mathcal{F}(A'_{\mathscr{A}})$ *in* $\mathbf{W}_{\mathscr{B}}$ *and* $\mathrm{z}'_{\mathscr{B}} : A'_{\mathscr{B}} \to A_{\mathscr{B}}$*, a 2-morphism* $\alpha_{\mathscr{A}} : f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}$ *and an invertible 2-morphism* $\sigma_{\mathscr{B}} : \mathcal{F}(\mathrm{v}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}} \Rightarrow \mathrm{v}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}$*, such that* $\alpha_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}}$ *coincides with the following composition:*

$$
\begin{tikzcd}
& A_{\mathscr{B}} \arrow[r, "\mathrm{v}_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow (\sigma_{\mathscr{B}})^{-1}"] & \mathcal{F}(A_{\mathscr{A}}) \arrow[dr, "\mathcal{F}(f^1_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow \mathcal{F}(\alpha_{\mathscr{A}})"] & \\
A'_{\mathscr{B}} \arrow[ur, "\mathrm{z}'_{\mathscr{B}}"] \arrow[r, "\mathrm{z}_{\mathscr{B}}"] \arrow[dr, "\mathrm{z}'_{\mathscr{B}}"'] & \mathcal{F}(A'_{\mathscr{A}}) \arrow[ur, "\mathcal{F}(\mathrm{v}_{\mathscr{A}})"'] \arrow[dr, "\mathcal{F}(\mathrm{v}_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow \sigma_{\mathscr{B}}"] & & \mathcal{F}(B_{\mathscr{A}}). \\
& A_{\mathscr{B}} \arrow[r, "\mathrm{v}_{\mathscr{B}}"'] & \mathcal{F}(A_{\mathscr{A}}) \arrow[ur, "\mathcal{F}(f^2_{\mathscr{A}})"'] &
\end{tikzcd}
$$

If we want to apply Theorem 8.1 to $\mathcal{G}^{\,\mathrm{red}}$, then we need firstly to compute the class $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$ appearing in conditions (A1) and (A2) above. In this case, this amounts to computing the the right saturation of the class $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$ of all the refinements of $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. Using Definition 7.1 together with Lemmas 5.5, 5.6 and 7.4, we get easily that:

Lemma 8.2. *The right saturation* $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl},\mathrm{sat}}$ *is the class of all weak equivalences of reduced orbifold atlases.*

Then we have:

Theorem 8.3. *Assuming the axiom of choice, the pseudofunctor* $\mathcal{G}^{\,\mathrm{red}}$ *is an equivalence of bicategories.*

Proof. Let us verify condition (A1), so let us fix any $\mathscr{X}$ in $(\mathcal{PE\acute{E}}\,\mathscr{G}\mathbf{pd})$; using Lemma 5.7 there are a reduced orbifold atlas $\mathcal{X}$ and a weak equivalence $\psi : \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}) \to \mathscr{X}$. Therefore, (A1) holds if we choose the following set of data:

$$\mathcal{F}^{\,\mathrm{red}}(\mathcal{X}) \xleftarrow{\mathrm{id}_{\mathcal{F}^{\,\mathrm{red}}(\mathcal{X})}} \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}) \xrightarrow{\quad\psi\quad} \mathscr{X}.$$

Let us consider (A2), so let us fix any pair of reduced orbifold atlases $\mathcal{X}^1, \mathcal{X}^2$ and any $\mathscr{X}^3$ in $(\mathcal{PE\acute{E}}\,\mathscr{G}\mathbf{pd})$, together with any pair of weak equivalences as follows:

$$\mathcal{F}^{\,\mathrm{red}}(\mathcal{X}^1) \xleftarrow{\quad\psi^1\quad} \mathscr{X}^3 \xrightarrow{\quad\psi^2\quad} \mathcal{F}^{\,\mathrm{red}}(\mathcal{X}^2).$$

By Lemma 5.7 there are a reduced orbifold atlas $\mathcal{Y}$ and a weak equivalence $\phi : \mathcal{F}^{\,\mathrm{red}}(\mathcal{Y}) \to \mathscr{X}^3$. By Lemma 5.6 there is a weak equivalence $[\hat{\mathrm{v}}] : \mathcal{Y} \to \mathcal{X}^1$, such that $\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{v}}]) = \psi^1 \circ \phi$. Since $[\hat{\mathrm{v}}]$ is a weak equivalence, by Lemma 5.4 there are a reduced orbifold atlas $\mathcal{X}^4$ and a weak equivalence $[\hat{\mathrm{u}}] : \mathcal{X}^4 \to \mathcal{Y}$, such that the morphism $[\hat{\mathrm{w}}^1] = [\hat{\mathrm{v}}] \circ [\hat{\mathrm{u}}] : \mathcal{X}^4 \to \mathcal{X}^1$ is a refinement. We set $\xi := \mathcal{F}([\hat{\mathrm{u}}])$; this morphism is a weak equivalence by Lemma 5.5 and we have $\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}^1]) = \psi^1 \circ \phi \circ \xi$. By Lemma 5.6 there is a unique weak equivalence $[\hat{\mathrm{w}}^2] : \mathcal{X}^4 \to \mathcal{X}^2$, such that $\mathcal{F}^{\,\mathrm{red}}([\hat{\mathrm{w}}^2]) = \psi^2 \circ \phi \circ \xi$. By Lemma 8.2, we have that $[\hat{\mathrm{w}}^2]$ belongs to the right saturation of $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$. Then condition (A2) is satisfied by the following set of data

$$\begin{tikzcd}
& \mathscr{X}^3 \arrow[dl, "\psi^1"'] \arrow[dr, "\psi^2"] & \\
\mathcal{F}^{\mathrm{red}}(\mathcal{X}^1) & \mathcal{F}^{\mathrm{red}}(\mathcal{X}^4) \arrow[u, "\phi\circ\xi"'] \arrow[d, "\mathrm{id}_{\mathcal{F}^{\mathrm{red}}(\mathcal{X}^4)}"] \arrow[l, phantom, "\Downarrow\, i_{\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{w}}^1])}"] \arrow[r, phantom, "\Downarrow\, i_{\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{w}}^2])}"] & \mathcal{F}^{\mathrm{red}}(\mathcal{X}^2). \\
& \mathcal{F}^{\mathrm{red}}(\mathcal{X}^4) \arrow[ul, "\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{w}}^1])"] \arrow[ur, "\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{w}}^2])"'] &
\end{tikzcd}$$

Let us prove (A3), so let us fix any reduced orbifold atlas $\mathcal{Y}$, any object $\mathscr{X}$ in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ and any morphism $\phi : \mathscr{X} \to \mathcal{F}^{\mathrm{red}}(\mathcal{Y})$. By Lemma 5.7 there are a reduced orbifold atlas $\mathcal{X}$ and a weak equivalence $\psi : \mathcal{F}^{\mathrm{red}}(\mathcal{X}) \to \mathscr{X}$. By Lemma 3.16 there is a unique morphism $[\hat{f}] : \mathcal{X} \to \mathcal{Y}$, such that $\mathcal{F}^{\mathrm{red}}([\hat{f}]) = \phi \circ \psi$. Then (A3) is easily verified with $A'_{\mathscr{B}} := \mathcal{F}^{\mathrm{red}}(\mathcal{X})$, $\mathrm{v}^1_{\mathscr{B}} := \psi$ and $\mathrm{v}^2_{\mathscr{B}} := \mathrm{id}_{\mathcal{F}^{\mathrm{red}}(\mathcal{X})}$.

Let us prove also (A4), so let us fix any pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$, any pair of morphisms $[\hat{f}^1], [\hat{f}^2] : \mathcal{X} \to \mathcal{Y}$ and any pair of 2-morphisms $[\gamma^1], [\gamma^2] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. Moreover, let us fix any object $\mathscr{X}$ in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ and any weak equivalence $\psi : \mathscr{X} \to \mathcal{F}^{\mathrm{red}}(\mathcal{X})$, such that

$$\mathcal{F}^{\mathrm{red}}([\gamma^1]) * i_{\psi} = \mathcal{F}^{\mathrm{red}}([\gamma^2]) * i_{\psi}. \tag{8.1}$$

By Lemma 5.7 there are a reduced orbifold atlas $\mathcal{Z}$ and a weak equivalence $\phi : \mathcal{F}^{\mathrm{red}}(\mathcal{Z}) \to \mathscr{X}$. By Lemma 5.6 there is a unique weak equivalence $[\hat{\mathrm{u}}] : \mathcal{Z} \to \mathcal{X}$ such that $\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{u}}]) = \psi \circ \phi$. By Lemma 5.4 there are a reduced orbifold atlas $\mathcal{U}$ and a weak equivalence $[\hat{\mathrm{v}}] : \mathcal{U} \to \mathcal{Z}$, such that $[\hat{\mathrm{z}}] := [\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}]$ is a refinement. So:

$$\mathcal{F}^{\mathrm{red}}\left([\gamma^1] * i_{[\hat{\mathrm{z}}]}\right) = \mathcal{F}^{\mathrm{red}}([\gamma^1]) * i_{\psi\circ\phi\circ\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{v}}])} \stackrel{(8.1)}{=} \mathcal{F}^{\mathrm{red}}([\gamma^2]) * i_{\psi\circ\phi\circ\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{v}}])} = \mathcal{F}^{\mathrm{red}}\left([\gamma^2] * i_{[\hat{\mathrm{z}}]}\right).$$

By Lemma 3.17, this implies that $[\gamma^1] * i_{[\hat{\mathrm{z}}]} = [\gamma^2] * i_{[\hat{\mathrm{z}}]}$, so (A4) holds.

Lastly, let us prove (A5), so let us fix any pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$, any object $\mathscr{X}$ in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$, any pair of morphisms $[\hat{f}^1], [\hat{f}^2] : \mathcal{X} \to \mathcal{Y}$, any weak equivalence $\psi : \mathscr{X} \to \mathcal{F}^{\mathrm{red}}(\mathcal{X})$ and any natural transformation $\alpha : \mathcal{F}^{\mathrm{red}}([\hat{f}^1]) \circ \psi \Rightarrow \mathcal{F}^{\mathrm{red}}([\hat{f}^2]) \circ \psi$. By Lemma 5.7 there are a reduced orbifold atlas $\mathcal{Z}$ and a weak equivalence $\phi : \mathcal{F}^{\mathrm{red}}(\mathcal{Z}) \to \mathscr{X}$. By Lemma 5.6 there is a unique weak equivalence $[\hat{\mathrm{u}}] : \mathcal{Z} \to \mathcal{X}$ such that $\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{u}}]) = \psi \circ \phi$. By Lemma 5.4, there are a reduced orbifold atlas $\mathcal{X}'$ and a weak equivalence $[\hat{\mathrm{v}}] : \mathcal{X}' \to \mathcal{Z}$, such that $[\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}]$ is a refinement. Then let us consider the 2-morphism

$$\alpha * i_{\phi\circ\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{v}}])} : \mathcal{F}^{\mathrm{red}}([\hat{f}^1] \circ [\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}]) \Longrightarrow \mathcal{F}^{\mathrm{red}}([\hat{f}^2] \circ [\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}]). \tag{8.2}$$

By Lemma 3.17 there is a unique 2-morphism $[\delta] : [\hat{f}^1] \circ [\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}] \Rightarrow [\hat{f}^2] \circ [\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}]$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$, such that $\mathcal{F}^{\mathrm{red}}([\delta])$ is equal to (8.2). Then (A5) is satisfied if we choose $A'_{\mathscr{A}} := \mathcal{X}'$, $A'_{\mathscr{B}} := \mathcal{F}^{\mathrm{red}}(\mathcal{X}')$, $\mathrm{v}_{\mathscr{A}} := [\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}] : \mathcal{X}' \to \mathcal{X}$, $\mathrm{z}_{\mathscr{B}} := \mathrm{id}_{\mathcal{F}^{\mathrm{red}}(\mathcal{X}')}$, $\mathrm{z}'_{\mathscr{B}} := \phi \circ \mathcal{F}^{\mathrm{red}}([\hat{\mathrm{v}}]) : \mathcal{F}(\mathcal{X}') \to \mathscr{X}$ (this is a weak equivalence since it is a composition of weak equivalences), $\alpha_{\mathscr{A}} := [\delta]$, and if we define $\sigma_{\mathscr{B}}$ as the 2-identity of the morphism

$$\mathcal{F}^{\mathrm{red}}([\hat{\mathrm{u}}] \circ [\hat{\mathrm{v}}]) = \psi \circ \phi \circ \mathcal{F}^{\mathrm{red}}([\hat{\mathrm{v}}]).$$

□

As we mentioned in the introduction, a well-known way to define a 2-category of orbifolds is by exhibiting it as a full 2-subcategory of the 2-category of C^∞-stacks (these are called "differentiable stacks" in several papers, see for example [Pr]). For the Grothendieck topology used for such stacks, we refer to [J2, Definition 8.1]. A C^∞-stack $\mathfrak{X}$ is called an *orbifold* (see [J2, Definition 9.25]) if it is equivalent to the stack $[\mathscr{X}]$ associated to a proper, étale groupoid $\mathscr{X}$. In particular (see again [J2, Definition 9.25]) every orbifold is a separated, locally finitely presented Deligne-Mumford C^∞-stack. An orbifold $\mathfrak{X}$ is called *effective* or *reduced* (see [J1, Definition 1.9.4]) if and only if it is associated to a proper, étale, *effective* groupoid. According to [J2] we write $(\mathbf{Orb})$ and $(\mathbf{Orb}^{\mathbf{eff}})$ for the full 2-subcategories of orbifolds, respectively of effective orbifolds, in the 2-category of C^∞-stacks (or, equivalently, in the 2-category of Deligne-Mumford C^∞-stacks). Using [Pr, Corollary 43] and [J2, Theorem 9.26] there is an equivalence from the bicategory of fraction $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$ to the 2-category $(\mathbf{Orb}^{\mathbf{eff}})$. Composing with $\mathcal{G}^{\,\mathrm{red}}$, we conclude that:

Theorem 8.4. *Assuming the axiom of choice, there is an equivalence between the bicategory* $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ *and the* 2*-category* $(\mathbf{Orb}^{\mathbf{eff}})$.

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