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Necessary and sufficient conditions for discrete wavelet frames in $\mathbb{C}^N$

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ABSTRACT

We present necessary and sufficient conditions with explicit frame bounds for a discrete wavelet system of the form $\{D_a T_k \phi\}_{a\in U(N), k\in I_N}$ to be a frame for the unitary space $\mathbb{C}^N$. It is shown that the canonical dual of a discrete wavelet frame for $\mathbb{C}^N$ has the same structure. This is not true (well known) for canonical dual of a wavelet frame for $L^2(\mathbb{R})$. Several numerical examples are given to illustrate the results.

1. Introduction

The purpose of this paper is to analyze the discrete wavelet structure of the form $\{D_a T_k \phi\}_{a\in U(N), k\in I_N}$ in $\mathbb{C}^N$, where D_a and T_k are dilation and translation operators on $\mathbb{C}^N$, respectively and $\phi \in \mathbb{C}^N$. Of course, there is an extensive literature on wavelet frames for $L^2(\mathbb{R}^d)$ and for some special types of function spaces and it is impossible to give complete references; let us at least mention some [1–7]. The main contributions of this paper are as follows: Firstly, we present a necessary condition for discrete wavelet frames for $\mathbb{C}^N$ in terms of a series associated with the Fourier transform of the window function, see Theorem 3.2. It is observed that the necessary condition given in Theorem 3.2 is also a sufficient condition for discrete wavelet frames in $\mathbb{C}$ and $\mathbb{C}^2$, see Proposition 3.3. Theorem 3.6 provides a sufficient condition for a family of vectors of the form $\{D_a T_k \phi\}_{a\in U(N), k\in I_N}$ to be a frame for $\mathbb{C}^N$. Chui and Shi proved in [8] that the canonical dual of a wavelet frame for $L^2(\mathbb{R})$ need not have a wavelet structure. The situation is different for discrete wavelet frames for $\mathbb{C}^N$. More precisely, the canonical dual of a discrete wavelet frame for $\mathbb{C}^N$ has the same structure, see Theorem 3.10.

Frames are redundant building blocks which provide a series representation (not necessarily unique) for each vector in the space. Duffin and Schaeffer [9] in 1952, introduced the concept of frame in the context of nonharmonic Fourier series. Throughout, $\mathbb{C}^N$ will denote an N-dimensional complex separable Hilbert space with inner product $\langle ., . \rangle$. A family of vectors $\mathcal{F} = \{\phi_k\}_{k=1}^M$ in $\mathbb{C}^N$ is called a *frame* (or *Hilbert frame*) for $\mathbb{C}^N$ if there exist constants $0 < a_o \le b_o < \infty$ such that

$$a_o \|x\|^2 \le \sum_{k=1}^{M} |\langle x, \phi_k \rangle|^2 \le b_o \|x\|^2 \text{ for all } x \in \mathbb{C}^N.$$

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The numbers a_o and b_o are called *lower* and *upper frame bounds*, respectively. If it is possible to choose $a_o = b_o$, then we say that $\mathcal{F}$ is *tight*. If $\mathcal{F}$ is a frame for $\mathbb{C}^N$, the frame operator $S : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$Sx = \sum_{k=1}^{M} \langle x, \phi_k \rangle \phi_k, \ x \in \mathbb{C}^N$$

is a bounded, linear, positive and invertible operator on $\mathbb{C}^N$. Thus, each $x \in \mathbb{C}^N$ has the expansion

$$x = SS^{-1}x = \sum_{k=1}^{M} \langle S^{-1}x, \phi_k \rangle \phi_k = \sum_{k=1}^{M} \langle x, S^{-1}\phi_k \rangle \phi_k.$$

The scalars $\{\langle S^{-1}x, \phi_k \rangle\}$ are called *frame coefficients* of the vector $x \in \mathbb{C}^N$. The representation of f in the reconstruction formula need not be unique. Thus, frames allow each element in the space to be written as a linear combination of frame elements, where linear independence of frame elements is not required. Finite frames have potential applications in quantum mechanics [10,11]. Pfander studied Gabor frames on finite-dimensional complex vector spaces in [12]. Very recently, Deepshikha and Vashisht [13] discussed frame properties of a system of the form $\{T_k\phi\}_{k \in I_N}$ in $\mathbb{C}^N$. Thirulogasanthar and Bahsoun [14] discussed methods for constructing continuous, discrete and finite frames. They presented a method to obtain frames on fractals, by using a distance function. By using the iterated function systems (IFS), Thirulogasanthar and Bahsoun [14] obtained continuous and discrete frames, living on fractal sets, of both finite and infinite dimensional separable abstract Hilbert spaces. For more details about the link between frames and iterated function systems, we refer [15–17]. Discrete frames on a finite dimensional right quaternion Hilbert space were studied by Khokulan et al. in [18] (also see [19]). Application of frames in applied mathematics with different directions can be found in the books of Casazza and Kutyniok [10], Christensen [20,21], Daubechies [2], Han, Kornelson, Larson, and Weber [22] and Okoudjou [23].

2. Basic tools

We follow notations and definitions given in [12]. The symbol $\mathbb{C}$ will denote the set of complex numbers ; $\mathbb{Z}$ the set of all integers and N a positive integer. An arbitrary element x in the unitary space $\mathbb{C}^N$ is represented by $((x(0), x(1), \ldots, x(N-1))^T$, where $x(n)$ is the $(n+1)$th component of the column vector x and x^T denotes the transpose of x.

That is

$$\mathbb{C}^N = \Big\{ (x(0), x(1), \ldots, x(N-1))^T : x(i) \in \mathbb{C}, \ i \in I_N = \{0, 1, \ldots, N-1\} \Big\}.$$

An element $p \in I_N$ is called a unit in I_N if it has a multiplicative inverse in I_N, that is, if there exists $q \in I_N$ such that $p.q = 1$, where multiplication is over modulo N. The set of units in I_N is denoted by $U(N)$. By $\Phi(N)$ we denote the number of units in I_N.

We consider the following linear operators on I_N. For $k \in I_N$, the *translation operator* $T_k : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$T_k(x(0), x(1), \ldots, x(N-1))^T = (x(0-k), x(1-k), \ldots, x((N-1)-k))^T,$$

where subtraction is over modulo N.

For $l \in I_N$, the *modulation operator* $M_l : \mathbb{C}^N \to \mathbb{C}^N$ is defined as

$$M_l(x(0), x(1), \ldots, x(N-1))^T = (e^{2\pi i l 0/N}x(0), e^{2\pi i l 1/N}x(1), \ldots, e^{2\pi i l (N-1)/N}x(N-1))^T.$$

Let $a \in U(N)$. The *dilation operator* $D_a : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$D_a(x(0), x(1), \ldots, x(N-1))^T = (\ x(a.0), x(a.1), \ldots, x(a.(N-1))\ )^T,$$

where multiplication is over modulo N.

The dilation operator D_a is a unitary operator. Indeed, for all $x, y \in \mathbb{C}^N$, we have

$$\langle D_a x, y \rangle = \sum_{n=0}^{N-1} x(a.n)\overline{y(n)} = \sum_{n=0}^{N-1} x(n)\overline{y(a^{-1}.n)} = \langle x, D_{a^{-1}} y \rangle.$$

Therefore, $D_a^* = D_{a^{-1}}$.

Furthermore

$$\begin{aligned} D_a^* D_a x &= D_a^*(\ x(a.0), x(a.1), \ldots, x(a.(N-1))\ )^T \\ &= (\ x(a^{-1}.a.0), x(a^{-1}.a.1), \ldots, x(a^{-1}.a.(N-1))\ )^T \\ &= (\ x(0), x(1), \ldots, x(N-1)\ )^T \\ &= x \ \text{ for all } x \in \mathbb{C}^N. \end{aligned}$$

Hence the dilation operator D_a is unitary.

The *Fourier transform* $\mathcal{F}$ on $\mathbb{C}^N$ is given pointwise as follows (see [12, p. 196]):

$$\mathcal{F}x(m) = \widehat{x}(m) = \sum_{n=0}^{N-1} x(n)e^{-2\pi imn/N},\ m = 0, 1, \ldots, N-1.$$

One of the major properties of the Fourier transform are the *Fourier inversion formula* and the *Parseval–Plancherel formula*:

Theorem 2.1 *([12], p. 197).* *The normalized harmonics $\frac{1}{\sqrt{N}}e^{2\pi im(\bullet)/N}$, $m = 0, 1, \ldots, N-1$ form an orthonormal basis of $\mathbb{C}^N$ and, hence, we have*

$$x = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \widehat{x}(m)e^{2\pi im(\bullet)/N},\ x \in \mathbb{C}^N,$$

and

$$\langle x, y\rangle = \frac{1}{N}\langle \widehat{x}, \widehat{y}\rangle,\ x, y \in \mathbb{C}^N.$$

It is proved in [12] that $\widehat{M_l x} = T_l\widehat{x}$. In case of dilation, we have $\widehat{D_a x} = D_{a^{-1}}\widehat{x}$. Indeed, for any $x \in \mathbb{C}^N$, $a \in U(N)$, $m \in I_N$, we compute

$$\begin{aligned}\widehat{D_a x}(m) = \sum_{n=0}^{N-1} D_a x(n)e^{-2\pi imn/N} &= \sum_{n=0}^{N-1} x(a.n)e^{-2\pi imn/N}\\ &= \sum_{n=0}^{N-1} x(n)e^{-2\pi ima^{-1}.\, n/N}\\ &= \widehat{x}(a^{-1}.\, m) = D_{a^{-1}}\widehat{x}(m).\end{aligned}$$

Similarly we can show that $\widehat{T_k\phi} = M_{-k}\widehat{\phi}$.

In matrix notation, the Fourier transform is represented by the Fourier matrix given by

$$W_N = (\omega^{-rs})_{r,s=0}^{N-1},\ \text{where } \omega = e^{2\pi i/N}.$$

For example

$$W_1 = [1], \qquad W_2 = \begin{bmatrix}1 & 1\\ 1 & e^{-2\pi i1/2}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$W_3 = \begin{bmatrix}1 & 1 & 1\\ 1 & e^{-2\pi i1/3} & e^{-2\pi i2/3}\\ 1 & e^{-4\pi i1/3} & e^{-4\pi i2/3}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & \dfrac{-1-i\sqrt{3}}{2} & \dfrac{-1+i\sqrt{3}}{2}\\ 1 & \dfrac{-1+i\sqrt{3}}{2} & \dfrac{-1-i\sqrt{3}}{2}\end{bmatrix}.$$

The following lemma will be used in Example 3.4.

Lemma 2.2. *For any positive integer N, we have*

$$e^{-2\pi im/N} + e^{-4\pi im/N} \neq 0 \textit{ for all } m \in I_N.$$

Proof. Assume $e^{-2\pi im/N} + e^{-4\pi im/N} = 0$ for some $m \in I_N$. Then, we have

$$\begin{aligned}0 &= e^{-2\pi im/N} + e^{-4\pi im/N}\\ &= \cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{4\pi m}{N}\right) + i\left(-\sin\left(\frac{2\pi m}{N}\right) - \sin\left(\frac{4\pi m}{N}\right)\right).\end{aligned}$$

This gives $\cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{4\pi m}{N}\right) = 0$ and $-\sin\left(\frac{2\pi m}{N}\right) - \sin\left(\frac{4\pi m}{N}\right) = 0.$

Therefore

$$\cos\left(\frac{2\pi m}{N}\right) = -\cos\left(\frac{4\pi m}{N}\right) = \cos\left(\pi - \frac{4\pi m}{N}\right).$$

That is

$$\frac{2\pi m}{N} = 2n\pi + \left(\pi - \frac{4\pi m}{N}\right) \text{ or } \frac{2\pi m}{N} = 2n\pi - \left(\pi - \frac{4\pi m}{N}\right) \text{ for } n \in \mathbb{Z}.$$

Therefore, $m = \frac{N}{6}(2n+1)$ or $m = \frac{-N}{2}(2n-1)$ for $n \in \mathbb{Z}$.

Similarly by using $-\sin\left(\frac{2\pi m}{N}\right) - \sin\left(\frac{4\pi m}{N}\right) = 0$, we can show that

$$m = \begin{cases} \dfrac{-N}{2}n, & \text{if } n \text{ is an odd integer} \\ \dfrac{N}{6}n, & \text{if } n \text{ is an even integer.} \end{cases}$$

The following cases arise:

(I) If n is odd, then $m = \frac{N}{6}(2n+1)$ and $m = \frac{-N}{2}n$ is not possible simultaneously. Similarly, if $m = \frac{-N}{2}(2n-1) = \frac{-N}{2}n$, then $m = \frac{-N}{2}$ which is absurd since $m \in I_N$.

(II) If n is even, then $m = \frac{N}{6}(2n+1)$ and $m = \frac{N}{6}n$ is not possible simultaneously. Similarly, m cannot take the values $\frac{-N}{2}(2n-1)$ and $\frac{N}{6}n$ at the same time.

Hence we must have

$$\text{either } \cos\left(\frac{2\pi m}{N}\right) + \cos\left(\frac{4\pi m}{N}\right) \neq 0 \quad \text{or} \quad -\sin\left(\frac{2\pi m}{N}\right) - \sin\left(\frac{4\pi m}{N}\right) \neq 0.$$

Therefore, $e^{-2\pi i m/N} + e^{-4\pi i m/N} \neq 0$ for all $m \in I_N$. $\square$

The following theorem provides necessary and sufficient conditions for a family of vectors $\{f_k\}_{k=1}^{m} \subset \mathbb{C}^N$ to be a frame for $\mathbb{C}^N$.

Theorem 2.3 (*[21], p. 4*)**.** *A family of vectors $\{f_k\}_{k=1}^{m} \subset \mathbb{C}^N$ is a frame for $\mathbb{C}^N$ if and only if span$\{f_k\}_{k=1}^{m} = \mathbb{C}^N$.*

3. Discrete wavelet frames for $\mathbb{C}^N$

Definition 3.1. *Let $\phi \in \mathbb{C}^N$. A family of vectors $\{D_a T_k \phi\}_{a\in U(N), k\in I_N}$ in $\mathbb{C}^N$ is a* discrete wavelet frame *(in short, DWF) for $\mathbb{C}^N$ if there exist positive scalars $A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \sum_{a\in U(N)} \sum_{k\in I_N} |\langle D_a T_k \phi, x\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in \mathbb{C}^N.$$

We call ϕ a *window function* (or *scaling function*) for the *DWF*.

The following theorem gives a necessary condition for *DWF* for $\mathbb{C}^N$ in terms of an estimate of series associated with the Fourier transform of the window function.

Theorem 3.2. *Let $\phi \in \mathbb{C}^N$ and suppose $\{D_a T_k \phi\}_{a\in U(N), k\in I_N}$ is a DWF for $\mathbb{C}^N$ with frame bounds A and B. Then,*

$$A \leq |\widehat{\phi}(m)|^2 + \sum_{a\in U(N)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 \leq B \text{ for all } m \in I_N. \tag{3.1}$$

Proof. For any $x \in \mathbb{C}^N$, by using the Parseval–Plancherel formula, we compute

$$\begin{aligned} &N^2 \sum_{a\in U(N), k\in I_N} |\langle D_a T_k \phi, x\rangle|^2 \\ &= N^2 \sum_{a\in U(N), k\in I_N} \langle D_a T_k \phi, x\rangle \overline{\langle D_a T_k \phi, x\rangle} \\ &= \sum_{a\in U(N), k\in I_N} \langle \widehat{D_a T_k \phi}, \widehat{x}\rangle \overline{\langle \widehat{D_a T_k \phi}, \widehat{x}\rangle} \\ &= \sum_{a\in U(N), k\in I_N} \langle D_{a^{-1}} M_{-k} \widehat{\phi}, \widehat{x}\rangle \overline{\langle D_{a^{-1}} M_{-k} \widehat{\phi}, \widehat{x}\rangle} \\ &= \sum_{a\in U(N), k\in I_N} \langle M_{-k} \widehat{\phi}, D_a \widehat{x}\rangle \overline{\langle M_{-k} \widehat{\phi}, D_a \widehat{x}\rangle} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{a\in U(N),k\in I_N}\left[\sum_{n=0}^{N-1}\widehat{\phi}(n)e^{-2\pi ink/N}\overline{\widehat{x}(a.n)}\sum_{m=0}^{N-1}\overline{\widehat{\phi}(m)}e^{2\pi imk/N}\widehat{x}(a.m)\right]\\
&= \sum_{a\in U(N),k\in I_N}\left[\sqrt{N}\left\langle\widehat{\phi}\,\overline{D_a\widehat{x}},\frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle\sqrt{N}\overline{\left\langle\widehat{\phi}\,\overline{D_a\widehat{x}},\frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle}\right]\\
&= \sum_{a\in U(N),k\in I_N}N\left|\left\langle\widehat{\phi}\,\overline{D_a\widehat{x}},\frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle\right|^2\\
&= N\sum_{a\in U(N)}\sum_{k\in I_N}\left|\left\langle\widehat{\phi}\,\overline{D_a\widehat{x}},\frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle\right|^2\\
&= N\sum_{a\in U(N)}\left\|\widehat{\phi}\,\overline{D_a\widehat{x}}\right\|^2\\
&= N\sum_{a\in U(N)}\sum_{m=0}^{N-1}\left|\widehat{\phi}(m)\overline{\widehat{x}(a.m)}\right|^2\\
&= N\sum_{a\in U(N)}\sum_{m=0}^{N-1}\left|\widehat{\phi}(a^{-1}.\,m)\overline{\widehat{x}(m)}\right|^2\\
&= N\sum_{a\in U(N)}\sum_{m=0}^{N-1}\left|\widehat{\phi}(a.m)\overline{\widehat{x}(m)}\right|^2.
\end{aligned}
$$

This gives

$$
\sum_{a\in U(N),k\in I_N}|\langle D_aT_k\phi,x\rangle|^2=\frac{1}{N}\sum_{a\in U(N)}\sum_{m=0}^{N-1}\left|\widehat{\phi}(a.m)\overline{\widehat{x}(m)}\right|^2\quad\text{for all } x\in\mathbb{C}^N. \tag{3.2}
$$

For any $m'\in I_N$, choose $x\in\mathbb{C}^N$ such that $\widehat{x}(m)=0$ for $m\neq m'$ and $\widehat{x}(m)=1$ for $m=m'$. Then, by using (3.2) and lower frame inequality of *DWF* $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$, we have

$$
\begin{aligned}
A = A\|\widehat{x}\|^2 &= AN\|x\|^2\\
&\leq N\sum_{a\in U(N),k\in I_N}|\langle D_aT_k\phi,x\rangle|^2\\
&= \sum_{a\in U(N)}\sum_{m=0}^{N-1}\left|\widehat{\phi}(a.m)\overline{\widehat{x}(m)}\right|^2\\
&= \sum_{a\in U(N)}\left|\widehat{\phi}(a.m')\right|^2.
\end{aligned}
$$

Thus, we have

$$
A\leq\sum_{a\in U(N)}\left|\widehat{\phi}(a.m)\right|^2=\left|\widehat{\phi}(m)\right|^2+\sum_{a\in U(N)\setminus\{1\}}\left|\widehat{\phi}(a.m)\right|^2 \text{ for all } m\in I_N.
$$

The lower estimate is proved.

Next we prove upper inequality in (3.1) by contradiction method. Assume there exists a $m''\in I_N$ such that $\sum_{a\in U(N)}\left|\widehat{\phi}(a.m'')\right|^2>B$.

Choose $x\in\mathbb{C}^N$ such that $\widehat{x}(m)=0$ for $m\neq m''$ and $\widehat{x}(m)=1$ for $m=m''$.

Then, by using (3.2), we compute

$$
\begin{aligned}
\sum_{a\in U(N),k\in I_N}|\langle D_aT_k\phi,x\rangle|^2 &= \frac{1}{N}\sum_{a\in U(N)}\sum_{m=0}^{N-1}\left|\widehat{\phi}(a.m)\overline{\widehat{x}(m)}\right|^2\\
&= \frac{1}{N}\sum_{a\in U(N)}\left|\widehat{\phi}(a.m'')\overline{\widehat{x}(m'')}\right|^2\\
&= \frac{1}{N}\sum_{a\in U(N)}\left|\widehat{\phi}(a.m'')\right|^2
\end{aligned}
$$

$$> \frac{B}{N} = \frac{B}{N}\|\widehat{x}\|^2 = B\|x\|^2,$$

which is a contradiction.

Hence

$$|\widehat{\phi}(m)|^2 + \sum_{a\in U(N)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 = \sum_{a\in U(N)} |\widehat{\phi}(a.m)|^2 \leq B \text{ for all } m \in I_N.$$

This completes the proof. □

For $N \geq 3$, the condition given in Theorem 3.2 may not be sufficient, see Example 3.4. The situation is different for $N = 1, 2$. More precisely, condition (3.1) given in Theorem 3.2 is not only necessary but also sufficient for $\mathbb{C}$ and $\mathbb{C}^2$. This is given in the following proposition.

Proposition 3.3. *Let $\phi \in \mathbb{C}^N$ ($N = 1$ or 2). A family $\{D_a T_k \phi\}_{a\in U(N), k\in I_N}$ is a DWF for $\mathbb{C}^N$, provided condition (3.1) holds.*

Proof. First we prove the result for $N = 1$. Let $\phi = (\alpha) \in \mathbb{C}$ be such that

$$A \leq |\widehat{\phi}(m)|^2 + \sum_{a\in U(1)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 = |\widehat{\phi}(m)|^2 \leq B \text{ for all } m \in I_1.$$

Then, $|\alpha|^2 = |\widehat{\phi}(0)|^2 \geq A > 0$. This gives $\phi = (\alpha) \neq 0$. Hence $\{D_a T_k \phi\}_{a\in U(1), k\in I_1} = \{\phi\}$ is a *DWF* for $\mathbb{C}$.

Let $N = 2$ and let $\phi = (\alpha, \beta)^T \in \mathbb{C}^2$. Assume that there exist $A, B > 0$ such that

$$A \leq |\widehat{\phi}(m)|^2 + \sum_{a\in U(2)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 \leq B \text{ for all } m \in I_2. \tag{3.3}$$

Since

$$\widehat{\phi}(0) = \sum_{n=0}^{1} \phi(n) e^{-2\pi i 0n/2} = \alpha + \beta$$

and

$$\widehat{\phi}(1) = \sum_{n=0}^{1} \phi(n) e^{-2\pi i n/2} = \phi(0) + \phi(1) e^{-\pi i} = \alpha - \beta,$$

applying (3.3) for $m = 0, 1$, we get

$$0 < A \leq |\widehat{\phi}(0)|^2 + \sum_{a\in U(2)\setminus\{1\}} |\widehat{\phi}(a.0)|^2 = |\widehat{\phi}(0)|^2 = |\alpha + \beta|^2$$

$$0 < A \leq |\widehat{\phi}(1)|^2 + \sum_{a\in U(2)\setminus\{1\}} |\widehat{\phi}(a.1)|^2 = |\widehat{\phi}(1)|^2 = |\alpha - \beta|^2.$$

This gives $\alpha - \beta,\ \alpha + \beta \neq 0$ and hence $\alpha^2 \neq \beta^2$. Therefore, by the Cramer's rule the family of vectors $\{D_a T_k \phi\}_{a\in U(2), k\in I_2} = \{(\alpha, \beta)^T, (\beta, \alpha)^T\}$ is linearly independent and hence spans $\mathbb{C}^2$. Thus, by Theorem 2.3 the family of vectors $\{D_a T_k \phi\}_{a\in U(2), k\in I_2}$ is a *DWF* for $\mathbb{C}^2$. The proposition is proved. □

We now demonstrate by a concrete example that condition (3.1) given in Theorem 3.2 is not sufficient.

Example 3.4. Choose $N = 4$ and $\phi = (0, 1, 1, 0) \in \mathbb{C}^4$. Then, for each $m \in I_4$, by Lemma 2.2, we have

$$\widehat{\phi}(m) = \sum_{n=0}^{3} \phi(n) e^{-2\pi i m n/4} = e^{-2\pi i m 1/4} + e^{-2\pi i m 2/4} = e^{-2\pi i m/4} + e^{-4\pi i m/4} \neq 0.$$

This gives

$$\sum_{a\in U(4)} |\widehat{\phi}(a.m)|^2 = |\widehat{\phi}(m)|^2 + \sum_{a\in U(4)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 > 0 \text{ for all } m \in I_4.$$

Thus, there exist positive scalars A, B such that

$$A < |\widehat{\phi}(m)|^2 + \sum_{a\in U(4)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 < B \text{ for all } m \in I_4.$$

Therefore, condition (3.1) in Theorem 3.2 is satisfied.

But $\{D_aT_k\phi\}_{a\in U(4),k\in I_4} = \{(0, 1, 1, 0), (0, 0, 1, 1), (1, 0, 0, 1), (1, 1, 0, 0)\}$ is not a *DWF* for $\mathbb{C}^4$. Indeed, let $\{D_aT_k\phi\}_{a\in U(4),k\in I_4}$ be a frame for $\mathbb{C}^4$. Then, by using the fact that a spanning set of $\mathbb{C}^N$ with exactly N elements is linearly independent, the family of vectors $\{D_aT_k\phi\}_{a\in U(4),k\in I_4}$ is linearly independent, which is a contradiction. Hence $\{D_aT_k\phi\}_{a\in U(4),k\in I_4}$ is not a *DWF* for $\mathbb{C}^4$.

The following example gives an application of Theorem 3.2.

Example 3.5. Suppose $N \geq 3$ is odd. Choose $\phi = (1, -1, 1, -1, \dots, -2, 1) \in \mathbb{C}^N$.
We compute

$$\begin{aligned}\widehat{\phi}(0) &= \sum_{n=0}^{N-1} \phi(n) e^{-2\pi i 0n/N}\\ &= \sum_{n=0}^{N-1} \phi(n)\\ &= 1 + (-1) + 1 + (-1) + \cdots + (-2) + 1\\ &= \frac{(N+1)}{2} - \frac{(N+1)}{2}\\ &= 0.\end{aligned}$$

Therefore, for $m = 0$, we have

$$|\widehat{\phi}(m)|^2 + \sum_{a\in U(N)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 = \sum_{a\in U(N)} |\widehat{\phi}(a.0)|^2 = \sum_{a\in U(N)} |\widehat{\phi}(0)|^2 = 0.$$

Hence there is no positive real number A such that

$$A \leq |\widehat{\phi}(m)|^2 + \sum_{a\in U(N)\setminus\{1\}} |\widehat{\phi}(a.m)|^2 \quad \text{for all } m \in I_N.$$

Therefore, condition (3.1) in Theorem 3.2 is not satisfied. Thus, $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is not a *DWF* for $\mathbb{C}^N$.

Next theorem provides a sufficient condition for *DWF* in $\mathbb{C}^N$. A similar result for affine frames can be found in [2].

Theorem 3.6. *Let $\phi \in \mathbb{C}^N$. Assume that*

$$A = \inf_{n\in I_N}\left[\sum_{a\in U(N)} |\widehat{\phi}(a.n)|^2 - \sum_{k\in I_N\setminus\{0\}} |\widehat{\phi}(n)\widehat{\phi}(n-k)|\right] > 0. \tag{3.4}$$

Then, $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is a DWF for $\mathbb{C}^N$ with frame bounds A and $N\Phi(N)\|\phi\|^2$.

Proof. For any $x \in \mathbb{C}^N$, by using Theorem 2.1, we compute

$$\begin{aligned}&\sum_{a\in U(N)}\sum_{k\in I_N} |\langle D_aT_k\phi, x\rangle|^2\\ &\quad= \frac{1}{N^2}\sum_{a\in U(N)}\sum_{k\in I_N} \left|\langle D_{a^{-1}}M_{-k}\widehat{\phi}, \widehat{x}\rangle\right|^2\\ &\quad= \frac{1}{N^2}\sum_{a\in U(N)}\sum_{k\in I_N} \left|\sum_{n=0}^{N-1}\widehat{\phi}(a^{-1}.\, n)e^{-2\pi i a^{-1}.\, nk/N}\overline{\widehat{x}(n)}\right|^2\\ &\quad= \frac{1}{N^2}\sum_{a\in U(N)}\sum_{k\in I_N} \left|\sum_{n=0}^{N-1}\widehat{\phi}(n)e^{-2\pi i nk/N}\overline{\widehat{x}(a.n)}\right|^2\\ &\quad= \frac{1}{N}\sum_{a\in U(N)}\sum_{k\in I_N} \left|\left\langle \widehat{\phi}\,\overline{D_a\widehat{x}}, \frac{1}{\sqrt{N}}e^{2\pi i(\bullet)k/N}\right\rangle\right|^2\\ &\quad= \frac{1}{N}\sum_{a\in U(N)} \|\widehat{\phi}\,\overline{D_a\widehat{x}}\|^2\\ &\quad= \frac{1}{N}\sum_{a\in U(N)}\sum_{n\in I_N} |\widehat{\phi}(n)\overline{\widehat{x}(a.n)}|^2\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{N}\sum_{a\in U(N)}\sum_{n\in I_N}\left|\widehat{\phi}(a^{-1}.n)\right|^2|\widehat{x}(n)|^2 - \frac{1}{N}\sum_{n\in I_N}\sum_{k\in I_N\setminus\{0\}}|\widehat{x}(n)|^2\left|\widehat{\phi}(n-k)\widehat{\phi}(n)\right| \\
&= \frac{1}{N}\sum_{n\in I_N}|\widehat{x}(n)|^2\left(\sum_{a\in U(N)}\left|\widehat{\phi}(a^{-1}.n)\right|^2 - \sum_{k\in I_N\setminus\{0\}}\left|\widehat{\phi}(n-k)\widehat{\phi}(n)\right|\right) \\
&= \frac{1}{N}\sum_{n\in I_N}|\widehat{x}(n)|^2\left(\sum_{a\in U(N)}\left|\widehat{\phi}(a.n)\right|^2 - \sum_{k\in I_N\setminus\{0\}}\left|\widehat{\phi}(n-k)\widehat{\phi}(n)\right|\right) \\
&\geq \frac{A}{N}\sum_{n\in I_N}|\widehat{x}(n)|^2 \\
&= \frac{A}{N}\|\widehat{x}\|^2 \\
&= A\|x\|^2.
\end{aligned}
$$

This gives a lower frame bound for $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$.

For the upper frame inequality, by using Cauchy–Schwarz inequality we have

$$
\begin{aligned}
\sum_{a\in U(N)}\sum_{k\in I_N}|\langle D_aT_k\phi, x\rangle|^2 &\leq \sum_{a\in U(N)}\sum_{k\in I_N}\|D_aT_k\phi\|^2\|x\|^2 \\
&= \|x\|^2\sum_{a\in U(N)}\sum_{k\in I_N}\|D_aT_k\phi\|^2 \\
&= \left[N\Phi(N)\|\phi\|^2\right]\|x\|^2 \text{ for all } x\in\mathbb{C}^N.
\end{aligned}
$$

Hence $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is a *DWF* for $\mathbb{C}^N$ with desired frame bounds. □

Remark 3.7. One may observe that condition (3.4) given in Theorem 3.6 reduces to $|\widehat{\phi}(0)|^2 > 0$ in case of $\mathbb{C}$. That is, (3.4) is both a necessary and sufficient condition for *DWF* for $\mathbb{C}$.

The following example shows that condition (3.4) given in Theorem 3.6 is not necessary for $N \geq 2$.

Example 3.8. Choose $\phi = (1, 0, 0, \ldots, 0)^T \in \mathbb{C}^N$, where $N \geq 2$. Then, $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is a *DWF* for $\mathbb{C}^N$. Indeed, the family of N vectors $\{(1, 0, 0, \ldots, 0)^T, (0, 1, 0, \ldots, 0)^T, \ldots, (0, 0, 0, \ldots, 1)^T\} \subseteq \{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ and $\text{span}\left\{(0, 0, 0, ., \underbrace{1}_{i^{th}place}, ., 0)^T : 1 \leq i \leq N\right\} = \mathbb{C}^N$. Therefore, by Theorem 2.3, $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is a *DWF* for $\mathbb{C}^N$.

Next we show that condition (3.4) in Theorem 3.6 is not satisfied. By definition of the Fourier transform, we have

$$
\widehat{\phi} = W_N\phi = W_N(1, 0, 0, \ldots, 0)^T = (1, 1, \ldots, 1)^T.
$$

Using this we compute

$$
\begin{aligned}
A &= \inf_{n\in I_N}\left(\sum_{a\in U(N)}\left|\widehat{\phi}(a.n)\right|^2 - \sum_{k\in I_N\setminus\{0\}}\left|\widehat{\phi}(n-k)\widehat{\phi}(n)\right|\right) \\
&= \inf\Big\{\Phi(N) - \sum_{k\in I_N\setminus\{0\}}|\widehat{\phi}(0-k)\widehat{\phi}(0)|,\ \Phi(N) - \sum_{k\in I_N\setminus\{0\}}|\widehat{\phi}(1-k)\widehat{\phi}(1)|,\ \ldots, \\
&\quad \Phi(N) - \sum_{k\in I_N\setminus\{0\}}|\widehat{\phi}((N-1)-k)\widehat{\phi}(N-1)|\Big\} \\
&= \Phi(N) - (N-1) \ \leq 0 \ \left(\text{as } \Phi(N) \leq N-1 \ \text{ for } N \geq 2\right).
\end{aligned}
$$

Hence condition (3.4) in Theorem 3.6 is not satisfied.

Application of Theorem 3.6. Choose $\phi \in \mathbb{C}^3$ such that $\widehat{\phi} = (4, 1, 5)^T$.

We compute

$$\begin{aligned}
A &= \inf_{m\in I_3}\Big[\sum_{a\in U(3)} |\widehat{\phi}(a.m)|^2 - \sum_{k\in I_3\setminus\{0\}} |\widehat{\phi}(m-k)\widehat{\phi}(m)|\Big] \\
&= \inf\Big\{\sum_{a\in U(3)} |\widehat{\phi}(a.0)|^2 - \sum_{k\in I_3\setminus\{0\}} |\widehat{\phi}(0-k)\widehat{\phi}(0)|\ ,\ \sum_{a\in U(3)} |\widehat{\phi}(a.1)|^2 - \sum_{k\in I_3\setminus\{0\}} |\widehat{\phi}(1-k)\widehat{\phi}(1)|, \\
&\quad \sum_{a\in U(3)} |\widehat{\phi}(a.2)|^2 - \sum_{k\in I_3\setminus\{0\}} |\widehat{\phi}(2-k)\widehat{\phi}(2)|\Big\} \\
&= \inf\Big\{2|\widehat{\phi}(0)|^2 - |\widehat{\phi}(2)\widehat{\phi}(0)| - |\widehat{\phi}(1)\widehat{\phi}(0)|,\ |\widehat{\phi}(1)|^2 + |\widehat{\phi}(2)|^2 - |\widehat{\phi}(0)\widehat{\phi}(1)| - |\widehat{\phi}(2)\widehat{\phi}(1)|, \\
&\quad |\widehat{\phi}(2)|^2 + |\widehat{\phi}(1)|^2 - |\widehat{\phi}(0)\widehat{\phi}(2)| - |\widehat{\phi}(1)\widehat{\phi}(2)|\Big\} \\
&= \inf\{8\ ,\ 17\ ,\ 1\} \\
&= 1 > 0.
\end{aligned}$$

By Theorem 3.6, the family $\{D_aT_k\phi\}_{a\in U(3),k\in I_3}$ is a *DWF* for $\mathbb{C}^3$.

The following theorem gives a sufficient condition for a window function associated with a *DWF* for $\mathbb{C}^N$ in terms of the Fourier transform. One may observe that the lower frame bound given in the following theorem is different from the lower frame bound given in Theorem 3.6.

Theorem 3.9. *Let $\phi \in \mathbb{C}^N$. Assume that*

$$A = \inf_{n\in I_N} \left|\widehat{\phi}(n)\right|^2 > 0.$$

Then, $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is a DWF for $\mathbb{C}^N$ with frame bounds $\Phi(N)A$ and $N\Phi(N)\|\phi\|^2$.

Proof. Similar to the proof of Theorem 3.6. □

To conclude the paper, we discuss the canonical dual of a discrete wavelet frame for $\mathbb{C}^N$. First we recall that the dual of a frame $\mathcal{F} = \{\phi_k\}_{k=1}^M$ for $\mathbb{C}^N$ is a frame $\mathcal{G} = \{\psi_k\}_{k=1}^M$ for $\mathbb{C}^N$ satisfying

$$x = \sum_{k=1}^{M} \langle x, \psi_k\rangle \phi_k \text{ for all } x \in \mathbb{C}^N.$$

Let S be the frame operator associated with the frame $\mathcal{F}$. Then, there exists at least one dual frame $\{S^{-1}\phi_k\}_{k=1}^M$ which is called the *canonical dual* of $\mathcal{F}$. If $\mathcal{F}$ is a tight frame, then $\mathcal{F}$ has a dual of the form $\psi_k = C\phi_k$ for some constant $C > 0$. If $\mathcal{F}$ is a tight frame with frame bounds $A = B = 1$, then we can take $\psi_k = \phi_k$ and elements of $\mathbb{C}^N$ have representation of the form

$$x = \sum_{k=1}^{M} \langle x, \phi_k\rangle \phi_k \text{ for all } x \in \mathbb{C}^N.$$

Thus, each vector in the space has (possibly, infinitely many) representations with respect to the frame but it also has one natural representation given by the frame coefficients. For more details about dual of a frame and its applications, we refer to [10,21]. It is well known that the canonical dual frame of a wavelet frame for $L^2(\mathbb{R})$ need not be of the same wavelet structure, see [8] (also see, Example 12.1.1 of [20]). But the situation regarding the canonical dual of discrete wavelet frames for $\mathbb{C}^N$ is different, which is given in the following theorem.

Theorem 3.10. *Let $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ be a DWF for $\mathbb{C}^N$ with frame operator S. Then, the family $\{D_aT_kS^{-1}\phi\}_{a\in U(N),k\in I_N}$ is the canonical dual DWF of $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$. That is, the canonical dual of a discrete wavelet frame for $\mathbb{C}^N$ has the same structure.*

Proof. First we show that the frame operator S commutes with composition of dilation and translation operators. Let $\widetilde{\phi} \in \mathbb{C}^N$, $\widetilde{a} \in U(N)$, $\widetilde{k} \in I_N$ be arbitrary. We compute

$$\begin{aligned}
SD_{\widetilde{a}}T_{\widetilde{k}}\widetilde{\phi}(n) &= \sum_{a\in U(N)} \sum_{k\in I_N} \langle D_{\widetilde{a}}T_{\widetilde{k}}\widetilde{\phi}, D_aT_k\phi\rangle D_aT_k\phi(n) \\
&= \sum_{a\in U(N)} \sum_{k\in I_N} \sum_{\widetilde{n}\in I_N} \widetilde{\phi}(\widetilde{a}\widetilde{n} - \widetilde{k})\overline{\phi(a\widetilde{n} - k)}\phi(an-k) \\
&= \sum_{a\in U(N)} \sum_{k\in I_N} \sum_{\widetilde{n}\in I_N} \widetilde{\phi}(\widetilde{n})\overline{\phi(a(\widetilde{a}^{-1}(\widetilde{n}+\widetilde{k})) - k)}\phi(an-k)
\end{aligned}$$

$$= \sum_{a\in U(N)}\sum_{k\in I_N}\sum_{\widetilde{n}\in I_N}\widetilde{\phi}(\widetilde{n})\overline{\phi(a\widetilde{a}^{-1}\widetilde{n}+a\widetilde{a}^{-1}\widetilde{k}-k)}\phi(an-k)$$

$$= \sum_{a\in U(N)}\sum_{k\in I_N}\sum_{\widetilde{n}\in I_N}\widetilde{\phi}(\widetilde{n})\overline{\phi(a\widetilde{n}+a\widetilde{k}-k)}\phi(a\widetilde{a}n-k)$$

$$= \sum_{a\in U(N)}\sum_{k\in I_N}\sum_{\widetilde{n}\in I_N}\widetilde{\phi}(\widetilde{n})\overline{\phi(a\widetilde{n}-k)}\phi(a\widetilde{a}n-a\widetilde{k}-k)$$

$$= \sum_{a\in U(N)}\sum_{k\in I_N}\sum_{\widetilde{n}\in I_N}\widetilde{\phi}(\widetilde{n})\overline{\phi(a\widetilde{n}-k)}\phi\Big(a(\widetilde{a}n-\widetilde{k})-k\Big)$$

$$= \sum_{a\in U(N)}\sum_{k\in I_N}\langle\widetilde{\phi}, D_aT_k\phi\rangle D_{\widetilde{a}}T_{\widetilde{k}}D_aT_k\phi(n)$$

$$= D_{\widetilde{a}}T_{\widetilde{k}}\sum_{a\in U(N)}\sum_{k\in I_N}\langle\widetilde{\phi}, D_aT_k\phi\rangle D_aT_k\phi(n)$$

$$= D_{\widetilde{a}}T_{\widetilde{k}}S\widetilde{\phi}(n),\ \ n\in I_N.$$

This gives $S^{-1}D_aT_k\phi = D_aT_kS^{-1}\phi$ for all $a\in U(N)$, $k\in I_N$. Hence the canonical dual frame of $\{D_aT_k\phi\}_{a\in U(N),k\in I_N}$ is given by $\{S^{-1}D_aT_k\phi\}_{a\in U(N),k\in I_N} = \{D_aT_kS^{-1}\phi\}_{a\in U(N),k\in I_N}$. The result is proved. □

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