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Identifications of paths and curves under the plane similarity transformations and their applications to mechanics

İdris Ören [a,*], Djavvat Khadjiev [b], Ömer Pekşen [a]

[a] Departments of Mathematics, Karadeniz Technical University, 61080, Trabzon, Turkey
[b] Academy of Sciences of Uzbekistan, Institute of Mathematics 100125, Tashkent, Uzbekistan

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A B S T R A C T

In this paper, global differential G-invariants of paths in the two-dimensional Euclidean space E_2 for the similarity group $G = Sim(E_2)$ and the orientation-preserving similarity group $G = Sim^+(E_2)$ are investigated. A general form of a path in terms of its global G-invariants is obtained. For given two paths $\xi(t)$ and $\eta(t)$ with the common differential G-invariants, general forms of all transformations $g \in G$, carrying $\xi(t)$ to $\eta(t)$, are found. Similar results are given for curves. Moreover, analogous of the similarity groups in the three-dimensional space–time and in the four-dimensional space–time-mass are defined. Finally, applications to Newtonian mechanics of the above results are given.

1. Introduction

Let $Sim(E_2)$ be the group of all similarities of E_2 (see [4, p. 183]), $Sim^+(E_2)$ be the group of all orientation-preserving similarities of E_2, $LSim(E_2)$ be the group of all linear similarities of E_2 and $LSim^+(E_2)$ be the group of all orientation-preserving linear similarities of E_2. Similarity is important in many areas of mathematics, mechanics, physics, etc.

Definition of similarity of two flows is given in [9, p. 35–36]. The idea of the similarity of flows is used in the design of experimental models (see [24, p. 175], [33, p. 254]). Moreover, in [9, p. 36], [33, p. 252–254], [23, p. 423–425], definitions of "Geometric similarity", "Kinematic similarity" and "Dynamic similarity" are given.

The idea of "dynamic similarity" is commonly defined for fluid motions as follows (see [5, p. 99]):

"Two fluid motions u and v are called dynamically similar if they can be described by Newtonian coordinate systems which are related by transformation of space–time-mass, of the form

$$x_i' = \alpha x_i, \quad t' = \beta t, \quad m' = \gamma m, \tag{1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$".

Moreover, in [5, p. 101], Galilei–Newton group is defined and the importance of this group is given as follows:

"Theoretical Newtonian mechanics is invariant under this group, as well as under the group of transformations (1) of dynamic similitude. Experimentally, this principle has been verified in many different ways with very great precision, except at speeds comparable with that of light".

This principle will be called the principle of invariances of Newtonian mechanics.

* Corresponding author.
E-mail addresses: oren@ktu.edu.tr (İ. Ören), khadjiev@gmail.com (D. Khadjiev), peksen@ktu.edu.tr (Ö. Pekşen).

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Let $E_n \times \mathbb{R} \times \mathbb{R}^+$ be the n-dimensional space–time-mass, where E_n is the n-dimensional Euclidean space, elements of $\mathbb{R}$ are times and elements of $\mathbb{R}^+$ are masses. Denote by $Mech(n)$ the group of transformations of $E_n \times \mathbb{R} \times \mathbb{R}^+$ generated by Galilei–Newton group and transformations of the form (1), where $\alpha, \beta, \gamma \in \mathbb{R}^+$.

According to the principle of invariances of Newtonian mechanics, problems of an investigation of invariants of mechanical systems with respect to the groups $Mech(n)$, Galilei–Newton group and some subgroups of the group $Mech(n)$ appear.

We denote by $Sim(Gal, 2)$ the subgroup of the group $Mech(n)$ generated by the group $Sim(E_2)$ and the Galilean transformation of the space–time $E_2 \times \mathbb{R}$, by $Sim^+(Gal, 2)$ if $Sim(E_2)$ is replaced by $Sim^+(E_2)$.

Also, denote by $Sim(Gal, m, 2)$ the subgroup of the group $Mech(n)$ generated by the group $Sim(E_2)$, the Galilean transformation of the space–time $E_2 \times \mathbb{R}$ and the following transformation of a mass: $m' = \gamma m$, where $\gamma \in \mathbb{R}^+$, by $Sim^+(Gal, m, 2)$ if $Sim(E_2)$ is replaced by $Sim^+(E_2)$.

In present paper, we investigate G-invariants of paths and curves in the space E_2 with respect to groups $G = Sim(E_2)$, $Sim^+(E_2)$, in space–time $E_2 \times \mathbb{R}$ with respect to groups $Sim(Gal, 2)$, $Sim^+(Gal, 2)$ and in space–time-mass $E_2 \times \mathbb{R} \times \mathbb{R}^+$ with respect to groups $Sim(Gal, m, 2)$, $Sim^+(Gal, m, 2)$.

Furthermore, to identify similar objects in the field of pattern recognition and computer graphics, similarity of two plane curves and space curves are investigated. (See in some references [1,2,16,17,25,29,31,32]).

An algorithm and a method for determining similarity of two rational plane curves are presented in [1].

In the present paper, by a different method from [1], theorems for detecting whether two curves are similar with a similarity transformation are given.

Beside, the *local* differential invariants of curves for the group $Sim(E_2)$ is introduced in [6]. In [15], differential invariants of a regular curve with respect to the group of orientation preserving similarities of the n-dimensional Euclidean space E_n are obtained. Moreover, the uniqueness and existence theorems for a curve obtained only for the group $Sim^+(n)$. Thus invariant theory of curves in the similarity geometry in E_n was investigated only for the group $Sim^+(E_n)$. The method of the moving frame in the similarity geometry gives only *local* conditions of $Sim^+(E_n)$-similarity of curves. This theory for similarity of curves in the n-dimensional Minkowski space is investigated in [26].

As it is well known, *global* differential invariants are an important tool for many areas in sciences.

In the n-dimensional Euclidean space and the n-dimensional pseudo-Euclidean space of index p, invariant differential functions of paths and curves for the Euclidean motion groups $G = M(n), M^+(n)$ and for the pseudo-Euclidean motion groups $G = M(n, p), M^+(n, p)$ are obtained in papers [3,18,19,27]. These functions are called *global* differential invariants of paths and curves. In same papers, invariant parametrizations of non-degenerate curves are defined. By using this definition, conditions of *global* G-congruence(equivalence) of non-degenerate curves and non-degenerate paths are given.

The solutions of problems of *global* G-congruence of **all** Bézier curves without using *global* differential invariants of a Bézier curve for the groups $G = M^+(n), M(n)$ are given in [25].

Methods and results used in this paper are different from the results of above mentioned papers, books and the references therein.

Let $G = Sim(E_2)$ or $G = Sim^+(E_2)$. In order to make this paper more self-contained from a mathematical point of view, the structure of the present paper is the following. In Section 2, definitions of similarity groups in terms of complex numbers and *global* differential G-invariants of a regular plane path are introduced. In Section 3, definitions of completely degenerate and non-degenerate plane paths are expressed. The global G-similarity conditions of plane paths are obtained. For given two plane paths ξ and η with the common G-invariants, general forms of all transformations $g \in G$, carrying ξ to η, are found. In Section 4, the existence and rigidity theorems of paths are obtained. In Section 5, a short introduction to non-degenerate curves and type of curve as *global* invariant of non-degenerate curves are explained. We give the global G-similarity conditions of non-degenerate curves in terms of the type of curve for the group $Sim(E_2)$. In Section 6, applications to Newtonian mechanics of obtained results in other sections are introduced.

2. Preliminaries

Let $\mathbb{C}$ be the field of complex numbers. The product of two complex numbers u and v has the form

$$uv = (u_1 + iu_2)(v_1 + iv_2) = (u_1v_1 - u_2v_2) + i(u_1v_2 + u_2v_1)$$

Consider the complex number $u = u_1 + iu_2$ in the matrix form $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then, the complex number uv has the form

$$uv = \begin{pmatrix} u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_1 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2}$$

Denote by M_u the matrix $\begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$. Then $M_u : \mathbb{C} \to \mathbb{C}$ is a transformation and the equality (2) has the form

$$uv = M_u v. \tag{3}$$

for all $u, v \in \mathbb{C}$.

The field $\mathbb{C}$ can be used to represent E_2 with the scalar product $\langle u, v\rangle = u_1v_1+u_2v_2$ for all $u = u_1+iu_2$, $v = v_1+iv_2 \in \mathbb{C}$. Here, the quadratic form on E_2 is $\langle u, u\rangle = |u|^2$ for all $u \in \mathbb{C}$. The conjugate of u, denoted by $\overline{u}$, is defined as $\overline{u} = u_1 - iu_2$. Clearly, from definition we have $u + \overline{u} = 2u_1$, $u\overline{u} = |u|^2$, $|u| = |\overline{u}|$ and $\langle \overline{u}, \overline{v}\rangle = \langle u, v\rangle$. For $|u| \neq 0$, the inverse of u is defined as $\frac{1}{u} = \frac{\overline{u}}{|u|^2}$. Moreover, let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have $\overline{u} = \Lambda u$.

For complex numbers $u = u_1 + iu_2$, $v = v_1 + iv_2$, the matrix $\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ will be denoted by $\|u\ v\|$. Denote by $det(\|u\ v\|)$ the determinant of $\|u\ v\|$.

The following proposition is given in [20, Proposition 14].

Proposition 1. *Let $u, v \in \mathbb{C}$. Assume that $|u| \neq 0$. Then the element $\frac{v}{u}$ exists, the following equalities hold:*

$$\frac{v}{u} = \frac{\langle u, v\rangle}{|u|^2} + i\frac{det(\|u\ v\|)}{|u|^2}$$

and

$$M_{\frac{v}{u}} = \begin{pmatrix} \frac{\langle u,v\rangle}{|u|^2} & -\frac{det(\|u\ v\|)}{|u|^2} \\ \frac{det(\|u\ v\|)}{|u|^2} & \frac{\langle u,v\rangle}{|u|^2} \end{pmatrix}. \tag{4}$$

Let $LSim^+(E_2)$ and $LSim^-(E_2)$ be sets generated by all orientation-preserving and orientation-reserving linear similarities of E_2, respectively. Clearly, $LSim^+(E_2) \cap LSim^-(E_2) = \emptyset$. The set $LSim(E_2)$ of all linear similarities of E_2 can be written in the form $LSim(E_2) = LSim^+(E_2) \cup LSim^-(E_2)$.

Denote $\mathbb{C}^* = \mathbb{C} - \{0\}$. The following theorem is known from [4, p. 229].

Theorem 1.

(i) $LSim^+(E_2) = \{M_u | u \in \mathbb{C}^*\}$.
(ii) $LSim^-(E_2) = LSim^+(E_2)\Lambda = \{M_u\Lambda | u \in \mathbb{C}^*\}$.
(iii) $LSim(E_2) = LSim^+(E_2) \cup LSim^-(E_2)$.
(iv) $Sim^+(E_2) = \{F : E_2 \to E_2 | F(v) = M_uv + b, u \in \mathbb{C}^*, \forall v \in E_2, b \in E_2\}$.
(v) $Sim^-(E_2) = \{F : E_2 \to E_2 | F(v) = (M_u\Lambda)v + b, u \in \mathbb{C}^*, \forall v \in E_2, b \in E_2\}$.
(vi) $Sim(E_2) = Sim^+(E_2) \cup Sim^-(E_2)$.

Let $I = (a, b) \subseteq \mathbb{R}$. Throughout Sections 2–5, we consider the following path $\xi(t) = (\xi_1(t), \xi_2(t))$ such that

$$\xi : I \to E_2 \tag{5}$$

is a C^2-mapping. Here, a C^2-mapping ξ is called to be an I-path in E_2. The components $\xi_1(t), \xi_2(t)$ of $\xi(t)$ are real C^2-functions on I, and they are defined for all values of t in I. For shortly, in the expression $\xi(t)$ we will use ξ instead of $\xi(t)$.

Denote the first and the second derivatives of ξ by $\xi' = (\xi_1', \xi_2')$ and $\xi'' = (\xi_1'', \xi_2'')$, respectively.

Definition 1. A C^2-mapping $\xi : I \to E_2$ is called S-regular I-path if $\xi'(t) \neq 0$ for all $t \in I$.

For example, consider an $\mathbb{R}$-path $\xi(t) = (t, t^2)$. Then, $\xi'(t) = (1, 2t) \neq 0$ for all $t \in \mathbb{R}$. Hence, $\xi(t)$ is an S-regular $\mathbb{R}$-path.

Let $G = Sim(E_2)$ or $G = Sim^+(E_2)$.

Definition 2. Two paths $\xi, \eta : I \to E_2$ are called G-similar if there exists $F \in G$ such that $\eta(t) = F\xi(t)$ for all $t \in I$.

Proposition 2. *Let ξ and η be two I-paths. Then,*

(i) ξ and η are $Sim^+(E_2)$-similar if and only if ξ' and η' are $LSim^+(E_2)$-similar.
(ii) ξ and η are $Sim(E_2)$-similar if and only if ξ' and η' are $LSim(E_2)$-similar.

Proof.

(i) $\Rightarrow$: Let ξ and η be $Sim^+(E_2)$-similar. Then, by Theorem 1(iv), there is an orientation-preserving similarity transformation F such that $\eta(t) = F\xi(t) = M_u\xi(t) + b$ for some orientation-preserving linear similarity transformation M_u of E_2, the constant b in E_2 and all $t \in I$. This equality implies $\eta' = M_u\xi'$. Then, we obtain that ξ' and η' are $LSim^+(E_2)$-similar.

$\Leftarrow$: Let ξ' and η' be $LSim^{+}(E_2)$-similar. Then, by Theorem 1(i), there is orientation-preserving linear similarity transformation F such that $\eta'(t) = F\xi'(t) = M_u\xi'(t)$ for some $u \in \mathbb{C}^{*}$ and all $t \in I$. Since $\eta' = M_u\xi'$, there exists a constant $b \in \mathbb{C}$ such that $\eta = M_u\xi + b$. That is, ξ and η are $Sim^{+}(E_2)$-similar.

(ii) The proof is exactly the same as (i).

Let $\zeta_1, \zeta_2, \ldots, \zeta_n : I \to E_2$ are paths defined on the same interval.

Definition 3. A function $\psi(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is called G-invariant if $\psi(F\zeta_1, F\zeta_2, \ldots, F\zeta_n) = \psi(\zeta_1, \zeta_2, \ldots, \zeta_n)$ for all $F \in G$ and for all $t \in I$.

For the derivatives ξ', ξ'' of ξ in (5), the determinant of the matrix $\|\xi'\ \xi''\| = \begin{pmatrix} \xi_1' & \xi_1'' \\ \xi_2' & \xi_2'' \end{pmatrix}$ will be denoted by $det(\|\xi'\ \xi''\|)$.

Let $\xi(t)$ be an S-regular I-path. For shortness, we put $f_\xi(t) = \frac{\langle \xi', \xi'' \rangle}{|\xi'|^2}$, $g_\xi(t) = \frac{\langle \xi'', \xi'' \rangle}{|\xi'|^2}$, $h_\xi(t) = \frac{det(\|\xi'\ \xi''\|)^2}{|\xi'|^4}$ and $k_\xi(t) = \frac{det(\|\xi'\ \xi''\|)}{|\xi'|^2}$.

Proposition 3.

(i) The functions $f_\xi(t), g_\xi(t)$ and $h_\xi(t)$ are $Sim(E_2)$-invariant.

(ii) The function $k_\xi(t)$ is $Sim^{+}(E_2)$-invariant.

Proof. It is easy and similar to the proof of Proposition 13 in [20].

These functions are called global invariant functions of the groups $Sim(E_2)$ and $Sim^{+}(E_2)$.

3. Similarity of paths for the groups $Sim(E_2)$ and $Sim^{+}(E_2)$

Theorem 2. *Let $\xi, \eta : I \to E_2$ are S-regular paths. Then ξ and η are $Sim^{+}(E_2)$-similar if and only if*

$$\begin{cases} f_\xi(t) = f_\eta(t) \\ k_\xi(t) = k_\eta(t) \end{cases} \tag{6}$$

for all $t \in I$.

Furthermore, there is the unique orientation-preserving similarity transformation F of E_2 such that $\eta = F\xi = N\xi + b$, where the orientation-preserving linear similarity transformation N of E_2 and the constant b in E_2 can be written as

$$N = \begin{pmatrix} \frac{\langle \xi', \eta' \rangle}{|\xi'|^2} & -\frac{det(\|\xi'\ \eta'\|)}{|\xi'|^2} \\ \frac{det(\|\xi'\ \eta'\|)}{|\xi'|^2} & \frac{\langle \xi', \eta' \rangle}{|\xi'|^2} \end{pmatrix} \tag{7}$$

and

$$b = \eta - N\xi \tag{8}$$

for all $t \in I$, resp. Here, N and b are independent of the choice of t in I.

Proof. $\Rightarrow$: Let two S-regular I-paths ξ and η be $Sim^{+}(E_2)$-similar. Then, by Theorem 1(iv), there is an orientation-preserving similarity transformation F of E_2 such that $\eta = F\xi = M_z\xi + b$, where M_z is an orientation-preserving linear similarity transformation of E_2 and b is a constant in E_2. Using (3), we obtain $\eta = z\xi + b$. This implies that $\eta' = z\xi'$. Since ξ and η are S-regular, we have $|\xi'| \neq 0$ and $|\eta'| \neq 0$ for all $t \in I$. Then, $\frac{1}{\xi'}$ and $\frac{1}{\eta'}$ exist for all $t \in I$. In Proposition 1, we consider $u = \eta'$, $v = \eta''$. Then we have

$$\frac{\eta''}{\eta'} = \frac{\langle \eta', \eta'' \rangle + i det(\|\eta'\ \eta''\|)}{|\eta'|^2}. \tag{9}$$

and

$$\frac{\xi''}{\xi'} = \frac{\langle \xi', \xi'' \rangle + i det(\|\xi'\ \xi''\|)}{|\xi'|^2}. \tag{10}$$

The equalities $\eta' = z\xi'$ and $\eta'' = z\xi''$ imply $\frac{\eta''}{\eta'} = \frac{\xi''}{\xi'}$. From this equality with (9) and (10), we have (6).

$\Leftarrow$: Let the equalities (6) be hold. From (9), (10) and (6) we have

$$\frac{\eta''}{\eta'} = \frac{\xi''}{\xi'} \tag{11}$$

for all $t \in I$. By taking derivative we have

$$\frac{d}{dt}\left(\frac{\eta'}{\xi'}\right) = \frac{\eta''}{\xi'} - \frac{\eta'\xi''}{(\xi')^2} = \frac{\eta'}{\xi'}\left(\frac{\eta''}{\eta'} - \frac{\xi''}{\xi'}\right). \tag{12}$$

Using the equalities (11) and (12) we get $\frac{d}{dt}\left(\frac{\eta'}{\xi'}\right) = 0$ for all $t \in I$. Hence the function $\frac{\eta'}{\xi'}$ is constant on I. Put $g = \frac{\eta'}{\xi'}$. Since $|\xi'| \neq 0$ and $|\eta'| \neq 0$ for all $t \in I$, we obtain $g \neq 0$.

Moreover $\eta' = \frac{\eta'}{\xi'}\xi' = g\xi'$.

Using the equality (3), we have $\eta' = g\xi' = M_g\xi'$. By $g = \frac{\eta'}{\xi'} = \frac{\langle \xi', \eta' \rangle}{|\xi'|^2} + i\frac{det(\|\xi'\ \eta'\|)}{|\xi'|^2}$ and Proposition 1, M_g has the form (7) and $M_g = N$. Since g is a constant, N is independent of the choice of t in I Then $\eta' = N\xi' = M_g\xi'$. By Theorem 1(i), $N \in LSim^+(E_2)$. Hence ξ' and η' are $LSim^+(E_2)$-similar. From the equality $\eta' = M_g\xi' = N\xi'$, we have $(\eta - N\xi)' = 0$. Hence $\eta - N\xi$ is independent of the choice of t in I. Put $b = \eta - N\xi$. Then $b \in E_2$ and $\eta = N\xi + b$.

For uniqueness, assume that $H \in LSim^+(E_2)$ and $c \in E_2$ exist such that $\eta = H\xi + c$. Then, by this equality we have $\eta' = H\xi'$. Then, by the equality (3), Proposition 1 and Theorem 1(i), there exists the unique $h \in \mathbb{C}^*$ such that $H = M_h$. Hence we obtain $\eta' = M_h\xi'$. By the equality (3), we have $\eta' = h\xi'$. Since $|\xi'| \neq 0$, $\eta' = h\xi'$ implies that $h = \frac{\eta'}{\xi'} = g$. Hence $H = M_h = M_g = N$.

Consider $b = \eta - N\xi$. Then, by the uniqueness of N, we have $b = \eta - N\xi = c$. Hence the uniqueness of b and the uniqueness of F are proved.

Now, let us consider an example of detecting similarities between two S-regular paths under the orientation-preserving transformation.

Example 1. Suppose that the two $\mathbb{R}$-paths are given as follows: $\xi(t) = (t^2, e^t)$ and $\eta(t) = (2t^2 - 3e^t + 1, 3t^2 + 2e^t + 2)$. Clearly, they are S-regular and (6) hold. Then, using Theorem 2(ii), $\xi(t)$ and $\eta(t)$ are $Sim^+(E_2)$-similar. Further, using Theorem 2, we have $N = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ and $b = 1 + 2i$.

Definition 4.

(i) A completely S-degenerate I-path is a C^2-mapping $\xi : I \to E_2$ such that $det(\|\xi'(t)\ \xi''(t)\|) = 0$ for all $t \in I$.
(ii) An S-non-degenerate I-path is a C^2-mapping $\xi : I \to E_2$ such that $det(\|\xi'(t)\ \xi''(t)\|) \neq 0$ for all $t \in I$.

Let $G = Sim^+(E_2)$ or $G = Sim(E_2)$.

Theorem 3. *Let ξ and η be two completely S-degenerate S-regular I-paths. Then ξ and η are $Sim(E_2)$ -similar if and only if*

$$f_\xi(t) = f_\eta(t) \tag{13}$$

for all $t \in I$.

Furthermore, there are only two similarity transformations $F = F_1, F_2$ of E_2 such that $\eta = F_1\xi = N_1\xi + b_1$ or $\eta = F_2\xi = N_2\Lambda\xi + b_2$. Then

(i) in the case $\eta = F_1\xi = N_1\xi + b_1$, the orientation-preserving linear similarity transformation N_1 of E_2 and the constant b_1 in E_2 can be written as (7) and

$$b_1 = \eta - N_1\xi, \tag{14}$$

resp.

(ii) in the case $\eta = F_2\xi = N_2\Lambda\xi + b_2$, the orientation-preserving linear similarity transformation N_2 of E_2 and the constant b_2 in E_2 can be written as

$$N_2 = \begin{pmatrix} \frac{\langle \Lambda\xi', \eta' \rangle}{|\xi'|^2} & -\frac{det(\|\Lambda\xi'\ \eta'\|)}{|\xi'|^2} \\ \frac{det(\|\Lambda\xi'\ \eta'\|)}{|\xi'|^2} & \frac{\langle \Lambda\xi', \eta' \rangle}{|\xi'|^2} \end{pmatrix}. \tag{15}$$

and

$$b_2 = \eta - N_2\Lambda\xi \tag{16}$$

for all $t \in I$, resp. Here N_1, N_2, b_1 and b_2 are independent of the choice of t in I.

Proof. $\Rightarrow$: Let two completely S-degenerate S-regular I-paths ξ and η be $Sim(E_2)$-similar. Since $f_\xi(t)$ and $f_\eta(t)$ are $Sim(E_2)$-invariant, we obtain (13).

$\Leftarrow$: Let $f_\xi(t) = f_\eta(t)$ for all $t \in I$. For ξ and η, we have

$$det(\|\xi'\ \xi''\|) = det(\|\eta'\ \eta''\|) = 0. \tag{17}$$

From (13) and (17), we have (6). Then, using Theorem 2, there is the unique orientation-preserving similarity transformation F of E_2 such that $\eta(t) = F\xi = N_1\xi + b_1$, where N_1 is an orientation-preserving linear similarity transformation of E_2 and b_1 is a constant in E_2. Clearly, N_1 and b_1 have the forms (7) and (14), resp. By Theorem 2, N_1 and b_1 are independent of the choice of t in I.

Now we consider completely S-degenerate S-regular I-paths $\Lambda\xi(t)$ and $\eta(t)$. Since the scalar products $\langle \xi', \xi' \rangle$ and $\langle \xi', \xi'' \rangle$ are Λ-invariant, we have $\langle \Lambda\xi', \Lambda\xi' \rangle = \langle \xi', \xi' \rangle$, $\langle \Lambda\xi', \Lambda\xi'' \rangle = \langle \xi', \xi'' \rangle$, $\langle \Lambda\eta', \Lambda\eta' \rangle = \langle \eta', \eta' \rangle$ and $\langle \Lambda\eta', \Lambda\eta'' \rangle = \langle \eta', \eta'' \rangle$ for all $t \in I$. By (17) for all $t \in I$, we have

$$\begin{aligned} det(\|\Lambda\xi' \ \Lambda\xi''\|) &= (detW)det(\|\xi' \ \xi''\|) \\ &= -det(\|\xi' \ \xi''\|) = -det(\|\eta' \ \eta''\|) = 0. \end{aligned}$$

Using $\langle \Lambda\xi', \Lambda\xi' \rangle = \langle \xi', \xi' \rangle$, $\langle \Lambda\xi', \Lambda\xi'' \rangle = \langle \xi', \xi'' \rangle$, $det(\|\Lambda\xi' \ \Lambda\xi''\|) = det(\|\eta' \ \eta''\|)$, by (13) and (17), we obtain the equalities:

$$\begin{cases} f_{\Lambda\xi}(t) = f_\eta(t), \\ k_{\Lambda\xi}(t) = k_\eta(t). \end{cases}$$

for all $t \in I$. Then, from Theorem 2, there is the unique orientation-preserving similarity transformation F of E_2 such that $\eta = F(\Lambda\xi) = (N_2\Lambda)\xi + b_2$, where N_2 is an orientation-preserving linear similarity transformation of E_2 and b_2 is a constant in E_2. Here N_2 and b_2 have the forms (15) and (16), resp. As in Theorem 2, N_2 and b_2 are independent of the choice of t in I. Now assume that there is similarity transformation F of E_2 mapping ξ into η. Prove that $F\xi = N_1\xi + b_1$ or $F\xi = (N_2\Lambda)\xi + b_2$, where N_1, N_2 are two orientation-preserving linear similarity transformations of E_2 and b_1, b_2 are two constants in E_2. Let $\eta = F\xi = A\xi + b_3$ for some $A \in LSim(E_2)$, $b_3 \in E_2$. Then $A \in LSim^+(E_2)$ or $A \in LSim^-(E_2)$. First assume that $A \in LSim^+(E_2)$. Using of the uniqueness in Theorem 2, $A = N_1$ and $b_1 = b_3 = \eta - N_1\xi$. Let $A \in LSim^-(E_2)$. Then A and b_3 have the forms $A = B\Lambda$ and $b_3 = \eta - B\Lambda\xi$, where $B \in LSim^+(E_2)$. We have $\eta = (B\Lambda)\xi + b_3 = B(\Lambda\xi) + b_3$. Hence η and $\Lambda\xi$ are $Sim^+(E_2)$-similar. By the uniqueness in Theorem 2, $B = N_2$ and $b_2 = b_3 = \eta - (N_2\Lambda)\xi$ for all $t \in I$.

Theorem 4. *Let ξ and η be two S-non-degenerate I-paths in E_2. Assume that ξ and η are $Sim(E_2)$-similar. Then equalities*

$$\begin{cases} f_\xi(t) = f_\eta(t) \\ h_\xi(t) = h_\eta(t) \end{cases} \tag{18}$$

hold for all $t \in I$.

Conversely, assume that ξ and η such that equalities (18) hold for all $t \in I$. Then ξ and η are $Sim(E_2)$-similar. Furthermore, there is the unique similarity transformation F of E_2 such that $\eta = F\xi$. Only the following cases exist:

(i_1) *$det(\|\xi' \ \xi''\|) > 0$ and $det(\|\eta' \ \eta''\|) > 0$ for all $t \in I$.*
(i_2) *$det(\|\xi' \ \xi''\|) < 0$ and $det(\|\eta' \ \eta''\|) < 0$ for all $t \in I$.*
(ii_1) *$det(\|\xi' \ \xi''\|) > 0$ and $det(\|\eta' \ \eta''\|) < 0$ for all $t \in I$.*
(ii_2) *$det(\|\xi' \ \xi''\|) < 0$ and $det(\|\eta' \ \eta''\|) > 0$ for all $t \in I$.*

In the cases (i_1) and (i_2), F has the form $F\xi = N_1\xi + b_1$, where the orientation-preserving linear similarity transformation N_1 of E_2 and the constant b_1 in E_2 can be written as (7) and (14), resp.

In the cases (ii_1) and (ii_2), F has the form $F\xi = N_2\Lambda\xi + b_2$, where the orientation-preserving linear similarity transformation N_2 of E_2 and the constant b_2 in E_2 can be written as (15) and (16), resp.

Here N_1, N_2, b_1 and b_2 are independent of the choice of t in I.

Proof. $\Rightarrow$: Let ξ and η be $Sim(E_2)$-similar. Since $f_\xi(t)$ and $h_\xi(t)$ are $Sim(E_2)$-invariant, we obtain (18).

$\Leftarrow$: Let $f_\xi(t) = f_\eta(t), h_\xi(t) = h_\eta(t)$ for all $t \in I$. For ξ and η, from Definition 4, we have $det(\|\xi' \ \xi''\|) \neq 0$ and $det(\|\eta' \ \eta''\|) \neq 0$. Then the conditions $(i_1), (i_2), (ii_1), (ii_2)$ in theorem exist.

Using the conditions (i_1), (i_2) and the equality $h_\xi(t) = h_\eta(t)$ in (18), we obtain

$$k_\xi(t) = k_\eta(t). \tag{19}$$

Then, (18) and (19) imply (6). Then, by Theorem 2, we obtain that there is the unique similarity transformation F of E_2 such that $\eta(t) = F\xi = N_1\xi + b_1$, where N_1 is an orientation-preserving linear similarity transformation of E_2 and $b_1 = \eta - N_1\xi$ is a constant in E_2. Here, N_1 has the form (7).

Using the conditions (ii_1), (ii_2) and the equality $h_\xi(t) = h_\eta(t)$ in (18), we obtain

$$k_\xi(t) = -k_\eta(t). \tag{20}$$

Using (20), $|(\Lambda\xi)'| = |\xi'|$, $|\xi'|^2 > 0$ and $|\eta'|^2 > 0$, we have

$$\frac{det(\|\Lambda\xi' \ \Lambda\xi''\|)}{|(\Lambda\xi)'|^2} = \frac{det(\|\eta' \ \eta''\|)}{|\eta'|^2} \tag{21}$$

for all $t \in I$. From (21) and $\frac{\langle \Lambda\xi', \Lambda\xi''\rangle}{|(\Lambda\xi)'|^2} = \frac{\langle \xi', \xi''\rangle}{|\xi'|^2} = \frac{\langle \eta', \eta''\rangle}{|\eta'|^2}$ imply the equalities (6) for the paths $\Lambda\xi'$ and η. By Theorem 2, there exists the unique similarity transformation F of E_2 such that $F\xi = N_2\Lambda\xi + b_2$, where N_2 is an orientation-preserving linear similarity transformation of E_2 and $b_2 = \eta - N_2\Lambda\xi$ is a constant in E_2. Here, N_2 has the form (15). By Theorem 2, N_2 and b_2 are independent of the choice of t in I

Let the similarity transformation F such that $\eta = F\xi$. As in the proof of Theorem 3, we obtain that $F\xi = N_1\xi + b_1$ or $F\xi = (N_2\Lambda)\xi + b_2$, where N_1, N_2 are two orientation-preserving linear similarity transformations of E_2 and b_1, b_2 are constants in E_2.

Now, we consider an example of detecting similarities between two S-non-degenerate paths under the similarity transformation.

Example 2. Suppose two S-non-degenerate $\mathbb{R}$-paths are given as follows:$\xi = (t, e^t)$ and $\eta = (t + 3e^t, -3t + e^t + 2)$. It is easy to see that the equalities in (18) hold for ξ and η. Then, by Theorem 4(*ii*), ξ and η are $Sim(E_2)$-similar. Moreover, by Theorem 4(*ii*), we obtain that $N_1 = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$ and $b = 2i$.

The following lemma is known in [19, Lemma 13].

Lemma 1. *Let z_1, z_2, w_1, w_2 be vectors in E_2. Then $det(\|z_1\ z_2\|)det(\|w_1\ w_2\|) = \langle z_1, w_1\rangle\langle z_2, w_2\rangle - \langle z_1, w_2\rangle\langle z_2, w_1\rangle$.*

Theorem 5. *Let ξ and η be two S-non-degenerate I-paths in E_2. Assume that ξ and η are $Sim(E_2)$-similar. Then equalities*

$$\begin{cases} f_\xi(t) = f_\eta(t) \\ g_\xi(t) = g_\eta(t) \end{cases} \tag{22}$$

hold for all $t \in I$.

Conversely, assume that ξ and η such that equalities (22) hold for all $t \in I$. Then ξ and η are $Sim(E_2)$-similar. Furthermore, there is the unique similarity transformation F of E_2 such that $\eta = F\xi$. Then,

(i) In the cases (i_1) and (i_2) in Theorem 4, F has the form $F\xi = N_1\xi + b_1$, where the orientation-preserving linear similarity transformation N_1 of E_2 and the constant b_1 in E_2 have the forms (7) and (14), resp.

(ii) In the cases (ii_1) and (ii_2) in Theorem 4, F has the form $F\xi = N_2\Lambda\xi + b_2$, where the orientation-preserving linear similarity transformation N_2 of E_2 and the constant b_2 in E_2 have the forms (15) and (16), resp.

Here N_1, N_2, b_1 and b_2 are independent of the choice of t in I.

Proof. $\Rightarrow$: Let two S-non-degenerate I-paths ξ and η be $Sim(E_2)$-similar.

In Lemma 1, put $z_1 = w_1 = \xi', z_2 = w_2 = \xi''$, we obtain

$$det(\|\xi'\ \xi''\|)^2 = \langle \xi', \xi'\rangle\langle \xi'', \xi''\rangle - \langle \xi', \xi''\rangle^2. \tag{23}$$

Using (23) and $|\xi'|^2 = \langle \xi', \xi'\rangle$, we obtain

$$\frac{det(\|\xi'\ \xi''\|)^2}{|\xi'|^4} = \frac{\langle \xi'', \xi''\rangle}{|\xi'|^2} - \frac{\langle \xi', \xi''\rangle^2}{|\xi'|^4}. \tag{24}$$

From (24), we have

$$\frac{\langle \xi'', \xi''\rangle}{|\xi'|^2} = \frac{\langle \xi', \xi''\rangle^2}{|\xi'|^4} + \frac{det(\|\xi'\ \xi''\|)^2}{|\xi'|^4}. \tag{25}$$

From Proposition 3 and (25), we obtain $g_\xi(t)$ is $Sim(E_2)$-invariant. Since $f_\xi^2(t)$ and $g_\xi(t)$ are $Sim(E_2)$-invariants, for all $t \in I, f_\xi(t) = f_\eta(t)$ and $g_\xi(t) = g_\eta(t)$.

$\Leftarrow$: Let $f_\xi(t) = f_\eta(t)$ and $g_\xi(t) = g_\eta(t)$ for all $t \in I$. Using (22) and (24), we obtain (18). Hence the proof follows from Theorem 4.

4. Existence theorems of paths for the groups $Sim(E_2)$ and $Sim^+(E_2)$

Theorem 6. *Let $a_1(t)$ and $a_2(t)$ be arbitrary real continuous functions on I. Assume that an S-regular I-path ξ such that the following equalities*

$$\begin{cases} f_\xi(t) = a_1(t), \\ k_\xi(t) = a_2(t) \end{cases} \tag{26}$$

hold for all $t \in I$. Then it has the following form

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r}(a_1(u)+ia_2(u))du} dr + c_2, \tag{27}$$

where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2 is an arbitrary element of $\mathbb{C}$ and $t_0, r_0 \in I$.

Conversely, every I-path in the form (27), where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2 is an arbitrary element of $\mathbb{C}$, $t_0, r_0 \in I$, $a_1(t)$ and $a_2(t)$ are arbitrary real continuous functions on I, is an S-regular I-path and satisfies the equalities (26) for all $t \in I$.

Proof. $\Rightarrow$: Assume that an S-regular I-path ξ satisfies the equalities (26). From $\frac{v}{u} = \frac{\langle u,v\rangle}{|u|^2} + i\frac{det(\|u\ v\|)}{|u|^2}$ in Proposition 1, we have for the path ξ :

$$\frac{\xi''}{\xi'} = \frac{\langle \xi', \xi''\rangle}{|\xi'|^2} + i\frac{det(\|\xi'\ \xi''\|)}{|\xi'|^2}. \tag{28}$$

Then $a_1(t) = \frac{\langle \xi', \xi''\rangle}{|\xi'|^2}$, $a_2(t) = \frac{det(\|\xi'\ \xi''\|)}{|\xi'|^2}$ and (28) imply the following equality for ξ in $\mathbb{C}$:

$$\xi'' = (a_1(t) + ia_2(t))\xi'.$$

General solution of this equation is

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r}(a_1(t)+ia_2(t))du} dr + c_2,$$

where $c_1, c_2 \in \mathbb{C}$ and $t_0, r_0 \in I$. Since $\xi'(t) \neq 0$ for all $t \in I$, we have $c_1 \neq 0$.

$\Leftarrow$: Let an I-path ξ has the form (27). Then, by simple calculations, we have $\frac{\langle \xi', \xi''\rangle}{|\xi'|^2} = a_1(t)$ and $\frac{det(\|\xi'\ \xi''\|)}{|\xi'|^2} = a_2(t)$. Here, it is easy to see that ξ is an S-regular I-path.

Example 3. Suppose two real continuous functions on $\mathbb{R}$ are given as follows: $a_1(t) = \frac{4t}{4t^2+1}$ and $a_2(t) = \frac{2}{4t^2+1}$. Then the general solution of the equalities $\frac{\langle \xi', \xi''\rangle}{|\xi'|^2} = \frac{4t}{4t^2+1}$ and $\frac{det(\|\xi'\ \xi''\|)}{|\xi'|^2} = \frac{2}{4t^2+1}$ has the form:

$$\begin{aligned}
\xi(t) &= c_1 \int_{0}^{t} e^{\int_{0}^{r}(a_1(u)+ia_2(u))du} dr + c_2 \\
&= c_1 \int_{0}^{t} e^{\int_{0}^{r} a_1(u)du} \left[cos(\int_{0}^{r} a_2(u)du) + isin(\int_{0}^{r} a_2(u)du)\right] dr + c_2 \\
&= c_1 \int_{0}^{t} \sqrt{4r^2+1}\,[cos(arctan2r) + isin(arctan2r)]\, dv + c_2 \\
&= c_1 \int_{0}^{t} \sqrt{4r^2+1} \left[\frac{1}{\sqrt{4r^2+1}} + i\frac{2r}{\sqrt{4r^2+1}}\right] dr + c_2 \\
&= c_1(t + it^2) + c_2,
\end{aligned}$$

$\forall c_1 \in \mathbb{C}^*, \forall c_2 \in \mathbb{C}$ and $t_0 = 0 = r_0 \in \mathbb{R}$. Since $\xi'(t) = c_1(1 + 2it) \neq 0$, $\forall t \in \mathbb{R}$, $\xi(t)$ is an S-regular $\mathbb{R}$-path.

Corollary 1. *Let $a(t)$ be arbitrary real continuous function on I. Assume that a completely S-degenerate S-regular I-path ξ such that the following equality*

$$f_\xi(t) = a(t) \tag{29}$$

holds for all $t \in I$. Then it has the following form

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r} a(u)du} dr + c_2, \tag{30}$$

where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2 is an arbitrary element of $\mathbb{C}$ and $t_0, r_0 \in I$.

Conversely, every I-path in the form (30), where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2 is an arbitrary element of $\mathbb{C}$, $t_0, r_0 \in I$, $a_1(t)$ and $a_2(t)$ are arbitrary real continuous functions on I, is an S-regular I-path and satisfies the equality (29) for all $t \in I$ and it is a completely S-degenerate S-regular I-path.

Proof. The proof is given as a particular case of the proof of Theorem 6.

Assume that $\xi(t)$ is an S-non-degenerate I-path. Then $det(\|\xi'\ \xi''\|) > 0$ for all $t \in I$ or $det(\|\xi'\ \xi''\|) < 0$ for all $t \in I$

Corollary 2. *Let $a_1(t)$ and $a_2(t)$ be arbitrary real continuous functions on I. Assume that an S-non-degenerate I-path ξ such that the following equalities*

$$\begin{cases} f_\xi(t) = a_1(t), \\ (k_\xi(t))^2 = a_2(t) \end{cases} \tag{31}$$

hold for all $t \in I$. Then

(i) $a_2(t) > 0$ *for all* $t \in I$

(ii) In the case $det(\|\xi' \; \xi''\|) > 0$ *for all* $t \in I$, ξ *can be written as*

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r} (a_1(u)+i\sqrt{a_2(u)})du} dr + c_2, \tag{32}$$

where $\forall c_1 \in \mathbb{C}^*, \forall c_2 \in \mathbb{C}$ *and* $t_0, r_0 \in I$.

(iii) In the case $det(\|\xi' \; \xi''\|) < 0$ *for all* $t \in I$, ξ *can be written as*

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r} (a_1(u)-i\sqrt{a_2(u)})du} dr + c_2, \tag{33}$$

where $\forall c_1 \in \mathbb{C}^*, \forall c_2 \in \mathbb{C}$ *and* $t_0, r_0 \in I$.

Conversely, in the case $a_2(t) > 0$ for all $t \in I$, every path ξ of the forms (32) and (33) is an S-non-degenerate I-path satisfying equalities (31).

Proof. Let ξ be an S-non-degenerate I-path. Then, by Definition 4, we have $det(\|\xi' \; \xi''\|) \neq 0$ for all $t \in I$. This inequality and equalities (31) imply $a_2(t) > 0$ for all $t \in I$. Moreover, since ξ is an S-non-degenerate I-path, we have $det(\|\xi' \; \xi''\|) > 0$ or $det(\|\xi' \; \xi''\|) < 0$ for all $t \in I$.

Explicitly, for $det(\|\xi' \; \xi''\|) > 0$,

$$\frac{det(\|\xi' \; \xi''\|)^2}{|\xi'|^4} = a_2(t) \tag{34}$$

implies

$$\frac{det(\|\xi' \; \xi''\|)}{|\xi'|^2} = \sqrt{a_2(t)}. \tag{35}$$

Then, we obtain the following system

$$\begin{cases} \dfrac{\langle \xi', \xi'' \rangle}{|\xi'|^2} = a_1(t), \\ \dfrac{det(\|\xi' \; \xi''\|)}{|\xi'|^2} = \sqrt{a_2(t)}. \end{cases} \tag{36}$$

By Theorem 6, a general solution of (36) has the form (32).

Similarly, for $det(\|\xi' \; \xi''\|) < 0$, (34) implies

$$\frac{det(\|\xi' \; \xi''\|)}{|\xi'|^2} = -\sqrt{a_2(t)}.$$

Then, we obtain the following system

$$\begin{cases} \dfrac{\langle \xi', \xi'' \rangle}{|\xi'|^2} = a_1(t), \\ \dfrac{det(\|\xi' \; \xi''\|)}{|\xi'|^2} = -\sqrt{a_2(t)}. \end{cases} \tag{37}$$

By Theorem 6, a general solution of (37) has the form (33).

Conversely, let an I-path ξ have the forms (32) or (33). Then, by simple calculations, we obtain equalities (31). Here, since $a_2(t) \neq 0$ for all $t \in I$, it is easy to see that ξ is an S-non-degenerate I-path.

Corollary 3. *Let $a_1(t)$ and $a_2(t)$ be arbitrary real continuous functions on I. Assume that an S-non-degenerate I-path ξ such that the following equalities*

$$\begin{cases} f_\xi(t) = a_1(t), \\ g_\xi(t) = a_2(t) \end{cases} \tag{38}$$

hold for all $t \in I$. Then

(i) $a_2(t) - a_1^2(t) > 0$ *for all* $t \in I$

(ii) In the case $det(\|\xi' \ \xi''\|) > 0$ *for all* $t \in I$, ξ *can be written as*

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r}\left(a_1(u)+i\sqrt{a_2(u)-a_1^2(u)}\right)du} dr + c_2, \tag{39}$$

where $\forall c_1 \in \mathbb{C}^*, \forall c_2 \in \mathbb{C}$ *and* $t_0, r_0 \in I$.

(iii) In the case $det(\|\xi' \ \xi''\|) < 0$ *for all* $t \in I$, ξ *can be written as*

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r}\left(a_1(u)-i\sqrt{a_2(u)-a_1^2(u)}\right)du} dr + c_2, \tag{40}$$

where $\forall c_1 \in \mathbb{C}^*, \forall c_2 \in \mathbb{C}$ *and* $t_0, r_0 \in I$.

Conversely, in the case $a_2(t) - a_1^2(t) > 0$ *for all* $t \in I$, *every path* ξ *of the forms (39) and (40) is an* S*-non-degenerate* I*-path satisfying equalities (38).*

Proof. It is obvious from Theorem 5 and Corollary 2.

5. *G*-Similarity of *S*-non-degenerate curves for the groups $G = Sim(E_2)$ and $G = Sim^+(E_2)$

Definition 5 (*see [3]*)**.** An I_1-path $\xi(t), t \in I_1 = (a, b)$ is equivalent to an I_2-path $\eta(r), I_2 = (c, d)$, if a C^2-diffeomorphism $\psi : I_2 \to I_1$ exists such that $\psi'(r) > 0$ and $\eta(r) = \xi(\psi(r))$ for all $r \in I_2$. We define a curve Φ to be an equivalence class of these paths. A path $\xi \in \Phi$ is called a parametrization of Φ.

Let $G = Sim(E_2), Sim^+(E_2)$ and $\Phi = \{\nu_\tau, \tau \in \Pi\}$ be a curve, where ν_τ is a parametrization of Φ. Then $F\Phi = \{F\nu_\tau, \tau \in \Pi\}$ is a curve for all F in G.

Definition 6. Two curves Φ and Ψ are G -similar provided there exists some $F \in G$ such that $\Psi = F\Phi$.

Definition 7. A curve Φ is an S-non-degenerate curve provided Φ contains an S-non-degenerate path.

Proposition 4. *Let* Φ *be an* S*-non-degenerate curve. Then every* $\xi \in \Phi$ *is an* S*-non-degenerate path.*

Proof. It is obvious from Proposition 37 in [20].

Remark 1. Throughout this section, we consider paths and curves which are S-non-degenerate.

We define the arc length of the I-path ξ from $t = c$ to $t = d$ to be the number $\int_c^d \frac{|det(\|\xi'(t)\ \xi''(t)\|)|}{|\xi'(t)|^2} dt$, for $c, d \in I = (a, b) \subseteq \mathbb{R}$ and $c < d$. This number is denoted by $\ell_\xi(c, d)$. Then, there exist the limits $\lim_{c\to a} \ell_\xi(c, d) \leq +\infty$ and $\lim_{d\to b} \ell_\xi(c, d) \leq +\infty$. They are denoted by $\ell_\xi(a, d)$ and $\ell_\xi(c, b)$, respectively.

Now we define the type of I-path ξ for the group $Sim(E_2)$. This type is denoted by L_ξ. Firstly, we put $l = \ell_\xi(a, d) + \ell_\xi(c, b) - \ell_\xi(c, d)$, where $0 \leq l \leq +\infty$. Clearly, l is independent of the choice of c, d in I. Moreover, if $0 < \ell_\xi(a, d) < +\infty, 0 < \ell_\xi(c, b) < +\infty$ or $0 < \ell_\xi(a, d) < +\infty, \ell_\xi(c, b) = +\infty$, then $L_\xi = (0, l)$. If $0 < \ell_\xi(a, d) = +\infty, 0 < \ell_\xi(c, b) < +\infty$ or $0 < \ell_\xi(a, d) = +\infty, \ell_\xi(c, b) = +\infty$, then $L_\xi = (-\infty, 0)$ or $L_\xi = (-\infty, +\infty)$, respectively. So the type of I-path ξ for the group $Sim(E_2)$ are $(0, +\infty)$, $(-\infty, 0)$, $(-\infty, +\infty)$ and $(0, l)$, where $l < +\infty$.

We omit the easy proofs of the following Propositions.

Proposition 5.

(i) If ξ *and* η *are* $Sim(E_2)$*-similar, then* $L_\xi = L_\eta$.

(ii) If $\xi, \eta \in \Phi$, *then* $L_\xi = L_\eta$.

According to the group $Sim(E_2)$, the type of a path $\xi \in \Phi$ is called the type of the curve Φ and denoted by L_Φ.

Proposition 6. *If two curves* Φ *and* Ψ *are* $Sim(E_2)$*-similar, then* $L_\Phi = L_\Psi$.

For all types of the group $Sim(E_2)$, we define the function $s_\xi(t)$ for an I-path ξ, where $I = (a, b)$, as follows:

(i) $s_\xi(t) = \ell_\xi(a, t)$ for $L_\xi = (0, l)$, where $l \leq +\infty$.

(ii) $s_\xi(t) = -\ell_\xi(t, b)$ for $L_\xi = (-\infty, 0)$.

(iii) In each interval $I = (a, b)$ of the line R, we choose a fixed point and denote it by x_I. In the case $I = (-\infty, +\infty)$, we choose $x_I = 0$. We put $s_\xi(t) = \ell_\xi(x_I, t)$ for the interval I.

Since ξ is a S-non-degenerate path, $\frac{ds_\xi}{dt} > 0$. By a standard theorem of calculus, the function $s_\xi(t)$ has an inverse function $t_\xi(s)$. Clearly, the domain of $t_\xi(s)$ is L_ξ.

We omit the easy proofs of the following Propositions 7, 8, 9, 10.(see [3]).

Proposition 7. *Let $I=(a_1,b_1)$ and $J=(a_2,b_2)$. For I-path ξ and for all $F\in Sim(E_2)$, the following statements hold:*

(i) $s_{F\xi}(t)=s_\xi(t)$ and $t_{F\xi}(s)=t_\xi(s)$ for all $t\in I$, for all $s\in L_\xi$ and all $F\in Sim(E_2)$.
(ii) for any C^2-diffeomorphism with $\psi'(r)>0$ for all $r\in I$, the following equalities hold: $s_{\xi(\psi)}(r)=s_\xi(\psi(r))+a_0$, $\forall r\in I$, and $\psi(t_{\xi(\psi)}(s+a_0))=t_\xi(s)$, $\forall s\in L_\xi$. Here, $a_0=0$ for $L_\xi\neq(-\infty,+\infty)$ and $a_0=\ell_\xi(\psi(a_J),a_I)$ for $L_\xi=(-\infty,+\infty)$.

According to Proposition 7, we have $\xi(t_\xi(s))\in\Phi$.

Definition 8 (*see [3]*). $\xi(t_\xi(s))\in\Phi$ is called an invariant parametrization of Φ.

Denote P_Φ by the set of all invariant parametrizations of Φ.

Proposition 8. *Let $\xi\in\Phi$ and ξ be a I-path, where $I=L_\Phi$. Then the followings are equivalent:*

(i) $\xi\in\Phi$ is an invariant parametrization.
(ii) $\frac{|det(\|\xi'(s)\ \xi''(s)\|)|}{|\xi'(s)|^2}=1$, $\forall s\in L_\Phi$.
(iii) $s_\xi(s)=s$, $\forall s\in L_\Phi$.

In the case $s_\xi(s)=s$, $\forall s\in L_\Phi$, s will be called an invariant parameter of Φ.
Suppose I is one of the intervals $(0,l)$, $l<+\infty$; $(0,+\infty)$, $(-\infty,0)$ or $(-\infty,+\infty)$.

Theorem 7. *Let $\xi(s)\in P_\Phi$. Then $\xi(s)$ can be written in the form*

$$\xi(s)=c_1\int_{s_0}^{s} e^{\int_{r_0}^{r}(a(u)+i)du}dr+c_2, \tag{41}$$

or in the form

$$\xi(s)=c_1\int_{s_0}^{s} e^{\int_{r_0}^{r}(a(u)-i)du}dr+c_2, \tag{42}$$

where $c_1\in\mathbb{C}^*$, $c_2\in\mathbb{C}$ and $s_0,r_0\in I$ and $a(t)$ is a real continuous function on I.

Conversely, paths $\xi(s)$ of the forms (41) and (42) are invariant parametrizations of Φ for $\forall c_1\in\mathbb{C}^*$, $\forall c_2\in\mathbb{C}$, $\forall s_0,r_0\in I$ and arbitrary $a(t)$ real continuous functions on I.

Proof. $\Rightarrow$: Let $\xi(s)\in P_\Phi$. Then, by Proposition 8(*ii*) and Theorem 6, we obtain that $\xi(s)$ has the form (41) or the form (42).

$\Leftarrow$: Let $\xi(s)$ has the form (41) or the form (41), where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2 is an arbitrary element of $\mathbb{C}$, s_0,r_0 are arbitrary elements of $\in I$ and $a(t)$ is an arbitrary real continuous functions on I. Then, in Theorem 6, we have $|a_2(s)|=1$, $\forall s\in I$. Hence, by Theorem 6 and Proposition 8(*ii*), $\xi(s)$ is an invariant parametrization of Φ.

Proposition 9. *For the type $L_\Phi\neq(-\infty,+\infty)$, there exists the unique invariant parametrization of Φ.*

Remark 2. For $L_\Phi=(-\infty,+\infty)$, P_Φ is infinite and uncountable. Moreover, if $\xi(t)$ is a periodic path then $L_\xi=(-\infty,+\infty)$.

Proposition 10. *Let $\xi\in P_\Phi$ and $L_\Phi=(-\infty,+\infty)$. Then $P_\Phi=\{\eta:\eta(s)=\xi(s+u),u\in(-\infty,+\infty)\}$.*

Example 4. Suppose the S-nondegenerate $\mathbb{R}$-path is given as follows: $\xi(t)=(cost,sint)$ in E_2. Let $\xi(t)\in\Phi$. By Definition 7, Φ is an S-non-degenerate curve and $L_\Phi=(-\infty,+\infty)$. Using Proposition 8, we have $\xi(t)\in P_\Phi$. Consider the $\mathbb{R}$-path $\eta(t)=\xi(t+k)$, where $k\in\mathbb{R}$. The mapping $\psi:\mathbb{R}\to\mathbb{R}$, where $\psi(t)=t+k$ for all $t\in\mathbb{R}$, is a homeomorphism such that $\psi'(t)>0$ for all $t\in\mathbb{R}$. Hence, $\xi(t)$ and $\eta(t)$ are equivalent. So $\eta(t)=\xi(t+k)\in\Phi$ for all $k\in\mathbb{R}$. By Proposition 10, $\xi(t+k)\in P_\Phi$ for $k\in\mathbb{R}$.

Let $G=Sim(E_2)$ or $G=Sim^+(E_2)$. The proof of the following theorem is similar to the proof of [3, Theorem 1] for the group G.

Theorem 8. *Let Φ and Ψ are S-non-degenerate curves and $\xi\in P_\Phi$, $\eta\in P_\Psi$ are invariant parametrizations.*

(i) In the case $L_\Phi=L_\Psi\neq(-\infty,+\infty)$, Φ and Ψ are G-similar if and only if ξ and η are G-similar.
(ii) In the case $L_\Phi=L_\Psi=(-\infty,+\infty)$, Φ and Ψ are G-similar if and only if ξ and $\eta(\psi_x)$ are G-similar for some $x\in(-\infty,+\infty)$, where $\psi_x(s)=s+x$.

The importance of Theorem 8 is that it reduces the problem of G-similarity of the curves for the group G to that of paths for $L_\Phi = L_\Psi \neq (-\infty, +\infty)$.

Definition 9. $\mathbb{R}$-paths ξ and η are $[G, (-\infty, +\infty)]$-similar provided there exist $g \in G$ and $d \in \mathbb{R}$ such that $\eta = g\xi(t + d)$ for all $t \in \mathbb{R}$.

Let Φ and Ψ be two curves, where $L_\Phi = L_\Psi = (-\infty, +\infty)$. Then, Theorem 8 reduces the G-similarity of these curves to $[G, (-\infty, +\infty)]$-similarity of paths.

Now, we will give the conditions of the global G-similarity of S-non-degenerate curves in terms of the type and global differential G-invariants of an S-non-degenerate curve for the groups $G = Sim(E_2)$, $Sim^+(E_2)$.

By Theorem 8, G-similarity and uniqueness problems for curves are reduced to the same problems for invariant parametrizations of curves only for the case $L_\Phi = L_\Psi \neq (-\infty, +\infty)$. Below we use this reduction.

Let Φ be S-non-degenerate curves and $\xi \in P_\Phi$ be an invariant parametrization.

Then we denote the function $sgn(det(\|\xi' \ \xi''\|))$ by $\rho_\xi(s)$. We consider, for all $s \in L_\Phi$, the functions $f_\xi(s)$, $g_\xi(s)$, $h_\xi(s)$ and $k_\xi(s)$ in Proposition 3.

Theorem 9. *Let Φ and Ψ are S-non-degenerate curves such that $L_\Phi \neq (-\infty, +\infty)$, $L_\Psi \neq (-\infty, +\infty)$ and $\xi \in P_\Phi$, $\eta \in P_\Psi$ are invariant parametrizations. Then Φ and Ψ are $Sim^+(E_2)$-similar if and only if*

$$\begin{cases} L_\Phi = L_\Psi \\ f_\xi(s) = f_\eta(s) \\ \rho_\xi(s) = \rho_\eta(s) \end{cases} \tag{43}$$

for all $s \in L_\Phi$.

Furthermore, there is the unique orientation-preserving similarity transformation F such that $\Psi = F\Phi = N_1\Phi + b$, where the orientation-preserving linear similarity transformation N_1 of E_2 and the constant b in E_2 have the forms (7) and (8), *respectively. Here, N_1 and b are independent of the choice of $s \in L_\Phi$.*

Proof. $\Rightarrow$: Let Φ and Ψ be $Sim^+(E_2)$-similar. Using Proposition 6, we have $L_\Phi = L_\Psi$. Hence, by $L_\Phi = L_\Psi$ and Theorem 8(i), we have ξ and η are $Sim^+(E_2)$-similar. By Theorem 2, for all $s \in L_\Phi$, we obtain $f_\xi(s) = f_\eta(s)$ and $k_\xi(s) = k_\eta(s)$.

Since ξ and η are S-non-degenerate, with using $k_\xi(s) = k_\eta(s)$, we obtain $\rho_\xi(s) = \rho_\eta(s)$. So the equalities (43) hold.

$\Leftarrow$: Let $L_\Phi = L_\Psi$, $f_\xi(s) = f_\eta(s)$ and $\rho_\xi(s) = \rho_\eta(s)$ for all $s \in L_\Phi$. Since $\xi \in P_\Phi$, $\eta \in P_\Psi$, by Proposition 8(ii), we have $\frac{|det(\|\xi'(s)\ \xi''(s)\|)|}{|\xi'(s)|^2} = \frac{|det(\|\eta'(s)\ \eta''(s)\|)|}{|\eta'(s)|^2} = 1$ for all $s \in L_\Phi$. Using this equality and $\rho_\xi(s) = \rho_\eta(s)$, we have $k_\xi(s) = k_\eta(s)$ for all $s \in L_\Phi$. From $k_\xi(s) = k_\eta(s)$ and (43), we have (6). By Theorem 2, we obtain that ξ and η are $Sim^+(E_2)$-similar. Then, there exists the unique orientation-preserving similarity transformation F such that $\eta = F\xi = N_1\xi + b$, where N_1 is an orientation-preserving linear similarity transformation of E_2 and b is a constant in E_2. Then N_1 and b have the forms (7) and (8), respectively. Here N_1 and b are independent of the choice of s in L_Φ. From $\xi \in P_\Phi$, $\eta \in P_\Psi$, Theorem 8(i) and $\eta = F\xi$, we have $\Psi = F\Phi$.

Theorem 10. *Let Φ and Ψ are S-non-degenerate curves such that $L_\Phi = L_\Psi = (-\infty, +\infty)$ and $\xi \in P_\Phi$, $\eta \in P_\Psi$ are invariant parametrizations. Then Φ and Ψ are $Sim^+(E_2)$-similar if and only if there exists $s_1 \in L_\Phi$ such that the following equalities*

$$\begin{cases} f_\xi(s + s_1) = f_\eta(s) \\ \rho_\xi(s) = \rho_\eta(s) \end{cases} \tag{44}$$

hold for all $s \in L_\Phi$.

Furthermore, there is the unique orientation-preserving similarity transformation F such that $\Psi = F\Phi = M_1\Phi + b$, where the orientation-preserving linear similarity transformation M_1 of E_2 and the constant b in E_2 have the forms

$$M_1 = \begin{pmatrix} \frac{\langle \xi'(s+s_1), \eta'(s)\rangle}{|\xi'(s+s_1)|^2} & -\frac{det(\|\xi'(s+s_1)\ \eta'(s)\|)}{|\xi'(s+s_1)|^2} \\ \frac{det(\|\xi'(s+s_1)\ \eta'(s)\|)}{|\xi'(s+s_1)|^2} & \frac{\langle \xi'(s+s_1), \eta'(s)\rangle}{|\xi'(s+s_1)|^2} \end{pmatrix} \tag{45}$$

and

$$b = \eta - M_1\xi, \tag{46}$$

respectively. Here M_1 and b are independent of the choice of s in L_Φ.

Proof. $\Rightarrow$: Let Φ and Ψ be $Sim^+(E_2)$-similar. Using Proposition 6, we have $L_\Phi = L_\Psi$. Hence, by $L_\Phi = L_\Psi$ and Theorem 8(ii), there exists $s_1 \in (-\infty, +\infty)$ such that $\xi(s + s_1)$ and $\eta(s)$ are $Sim^+(E_2)$-similar. By Theorem 2(i), for all $s \in L_\Phi$, we obtain $f_\xi(s + s_1) = f_\eta(s)$ and $k_\xi(s + s_1) = k_\eta(s)$.

Since ξ and η are S-non-degenerate, with using $k_\xi(s+s_1) = k_\eta(s)$, we obtain $\rho_\xi(s) = \rho_\xi(s+s_1) = \rho_\eta(s)$. So the equalities (44) hold.

$\Leftarrow$: Let $L_\Phi = L_\Psi, f_\xi(s+s_1) = f_\eta(s)$ and $\rho_\xi(s+s_1) = \rho_\eta(s)$ for all $s \in L_\Phi$ and for some $s_1 \in L_\Phi$. Since $\xi \in P_\Phi, \eta \in P_\Psi$, by Proposition 8(*ii*), we have $\frac{|det(\|\xi'(s)\ \xi''(s)\|)|}{|\xi'(s)|^2} = \frac{|det(\|\xi'(s+s_1)\ \xi''(s+s_1)\|)|}{|\xi'(s+s_1)|^2} = \frac{|det(\|\eta'(s)\ \eta''(s)\|)|}{|\eta'(s)|^2} = 1$ for all $s \in L_\Phi$. Using this equality and $\rho_\xi(s) = \rho_\eta(s)$, we have $k_\xi(s+s_1) = k_\eta(s)$ for all $s \in L_\Phi$. From $k_\xi(s+s_1) = k_\eta(s)$ and (44), we have (6). By Theorem 2, we obtain that ξ and η are $Sim^+(E_2)$-similar. Then, there exists the unique orientation-preserving similarity transformation F such that $\eta(s) = F\xi(s+s_1) = M_1\xi(s+s_1) + b$, where M_1 is an orientation-preserving linear similarity transformation of E_2 and b is a constant in E_2. Then M_1 and b have the forms (45) and (46), respectively. Here M_1 and b are independent of the choice of $s \in L_\Phi$. From $\xi \in P_\Phi, \eta \in P_\Psi$, Theorem 8(*ii*) and $\eta(s) = F\xi(s+s_1)$, we have $\Psi = F\Phi$.

By Theorems 5, 8(*i*) and 9, we omit the easy proof of the following theorem.

Theorem 11. *Let Φ and Ψ are S-non-degenerate curves such that $L_\Phi \neq (-\infty, +\infty), L_\Psi \neq (-\infty, +\infty)$ and $\xi \in P_\Phi, \eta \in P_\Psi$ are invariant parametrizations. Then Φ and Ψ are $Sim(E_2)$-similar if and only if*

$$\begin{cases} L_\Phi = L_\Psi \\ f_\xi(s) = f_\eta(s) \end{cases} \tag{47}$$

for all $s \in L_\Phi$.

Furthermore, if Φ and Ψ are $Sim(E_2)$-similar, there is the unique similarity transformation F of E_2 such that $\Psi = F\Phi$, then

(i) in the case $\rho_\xi(s) = \rho_\eta(s)$, F has the form $F\xi = M_1\xi + b_1$, where the orientation-preserving linear similarity transformation M_1 and the constant b_1 in E_2 have the forms (7) and (14), resp.
(ii) in the case $\rho_\xi(s) = -\rho_\eta(s)$, F has the form $F\xi = M_2\Lambda\xi + b_2$, where the orientation-preserving linear similarity transformation M_2 and the constant b_2 in E_2 have the forms (15) and (16), resp.

Here M_1, M_2, b_1 and b_2 are independent of the choice of $s \in L_\Phi$.

By Theorems 5, 8(*ii*) and 10, we omit the easy proof of the following theorem.

Theorem 12. *Let Φ and Ψ are S-non-degenerate curves such that $L_\Phi = L_\Psi = (-\infty, +\infty)$ and $\xi \in P_\Phi, \eta \in P_\Psi$ are invariant parametrizations. Then Φ and Ψ are $Sim(E_2)$-similar if and only if there exists $s_1 \in L_\Phi$ such that the following equalities*

$$f_\xi(s+s_1) = f_\eta(s) \tag{48}$$

holds for all $s \in L_\Phi$.

Furthermore, if Φ and Ψ are $Sim(E_2)$-similar, there is the unique similarity transformation F of E_2 such that $\Psi = F\Phi$, then

(i) in the case $\rho_\xi(s+s_1) = \rho_\eta(s)$, F has the form $F\xi = M_1\xi + b_1$, where the orientation-preserving linear similarity transformation M_1 and the constant b_1 in E_2 have the forms (45) and $\eta(s) - M_1\xi(s)$, resp.
(ii) in the case $\rho_\xi(s+s_1) = -\rho_\eta(s)$, F has the form $F\xi = M_2\Lambda\xi + b_2$, where the orientation-preserving linear similarity transformation M_2 and the constant b_2 in E_2 have the forms

$$M_2 = \begin{pmatrix} \frac{\langle \Lambda\xi'(s+s_1), \eta'(s)\rangle}{|\xi'(s+s_1)|^2} & -\frac{det(\|\Lambda\xi'(s+s_1)\ \eta'(s)\|)}{|\xi'(s+s_1)|^2} \\ \frac{det(\|\Lambda\xi'(s+s_1)\ \eta'(s)\|)}{|\xi'(s+s_1)|^2} & \frac{\langle \Lambda\xi'(s+s_1), \eta'(s)\rangle}{|\xi'(s+s_1)|^2} \end{pmatrix}$$

and $b_2 = \eta(s) - M_2\Lambda\xi(s)$, resp.
Here M_1, M_2, b_1 and b_2 are independent of the choice of $s \in L_\Phi$.

6. Some applications to mechanics

6.1. Detecting G-similarity between two paths for groups $Sim(Gal, 2)$ and $Sim^+(Gal, 2)$

Let E_2 be the 2-dimensional Euclidean space and $E_2 \times \mathbb{R}$ be the Newton space–time.
In the Introduction, we have defined the following groups:
$Sim(Gal, 2) = \{F : E_2 \times \mathbb{R} \to E_2 \times \mathbb{R} \mid F(x,t) = (gx + bt, t), g \in Sim(E_2), b \in E_2, t \in \mathbb{R}\}$.
$Sim^+(Gal, 2) = \{F : E_2 \times R \to E_2 \times \mathbb{R} \mid F(x,t) = (gx + bt, t), g \in Sim^+(E_2), b \in E_2, t \in \mathbb{R}\}$.
Let $I = (a, b) \subseteq \mathbb{R}$. We consider the I-path $\xi(t) = (\xi_1(t), \xi_2(t))$ such that

$$\xi : I \to E_2 \tag{49}$$

is a C^3-mapping. The components $\xi_1(t), \xi_2(t)$ of $\xi(t)$ are real C^3-functions on I, and they are defined for all t in I.

Definition 10. Two paths $\xi, \eta : I \to E_2$ are $Sim(Gal, 2)$-similar provided there exists $F \in Sim(Gal, 2)$, where $F(\xi, t) = (H(\xi) + bt, t), H \in Sim(E_2), b \in E_2, \forall \xi \in E_2, \forall t \in I$, such that $\eta(t) = H\xi(t) + bt, \forall t \in I$.

Definition 11. Two paths $\xi, \eta : I \to E_2$ are $Sim^{+}(Gal, 2)$-similar provided there exists $F \in Sim^{+}(Gal, 2)$, where $F(\xi, t) = (H(\xi) + bt, t)$, $H \in Sim^{+}(E_2)$, $b \in E_2$, $\forall \xi \in E_2$, $\forall t \in I$, such that $\eta(t) = H\xi(t) + bt$, $\forall t \in I$.

Proposition 11. *Let ξ and η be two I-paths. Then, ξ and η are $Sim^{+}(Gal, 2)$-similar if and only if ξ' and η' are $Sim^{+}(E_2)$-similar.*

Proof. $\Rightarrow$: Let ξ and η be $Sim^{+}(Gal, 2)$-similar. Then there exists $F \in Sim^{+}(Gal, 2)$ such that $\eta = H\xi + bt$, $H \in Sim^{+}(E_2)$, $b \in E_2$, $\forall t \in I$. From this equality, we obtain $\eta' = H\xi' + b$, $H \in Sim^{+}(E_2)$, $b \in E_2$, $\forall t \in I$. That is, ξ' and η' are $Sim^{+}(E_2)$-similar.

$\Leftarrow$: Let ξ' and η' be $Sim^{+}(E_2)$-similar. Then there exists $H \in Sim^{+}(E_2)$ such that $\eta' = H\xi' = K\xi' + b$, $K \in LSim^{+}(E_2)$, $b \in E_2$, $\forall t \in I$. From this equality, we have $(\eta - K\xi)' = b$, $\forall t \in I$. Then there exist $c \in E_2$ such that $\eta = (K\xi + c) + bt$, $K \in LSim^{+}(E_2)$, $\forall t \in I$. This means that ξ and η are $Sim^{+}(Gal, 2)$-similar.

The following proposition is similar to Proposition 11.

Proposition 12. *Let ξ and η be two I-paths. Then ξ and η are $Sim(Gal, 2)$-similar if and only if ξ' and η' are $Sim(E_2)$-similar.*

Remark 3. Propositions 11 and 12 reduce the conditions of $Sim(Gal, 2)$-similarity and $Sim^{+}(Gal, 2)$-similarity of I-paths to the conditions of $Sim(E_2)$-similarity and $Sim^{+}(E_2)$-similarity of I-paths, respectively.

In this case, we can be give the following definitions.

Using the results of this article, the similarity conditions of motions of two fluid particles can be given. For example:

Definition 12. A C^3-regular I-path ξ followed by a fluid particle in E_2 is a C^3-mapping $\xi : I \to E_2$ such that $\xi''(t) \neq 0$ for all $t \in I$.

The following theorem is similar to Theorem 2.

Theorem 13. *$\xi, \eta : I \to E_2$ are C^3-regular I-paths followed by two fluid particles. Then ξ and η are $Sim^{+}(Gal, 2)$-similar if and only if*

$$\begin{cases} \dfrac{\langle \xi^{(2)}, \xi^{(3)} \rangle}{|\xi^{(2)}|^2} = \dfrac{\langle \eta^{(2)}, \eta^{(3)} \rangle}{|\eta^{(2)}|^2}, \\ \dfrac{det(\|\xi^{(2)}\ \xi^{(3)}\|)}{|\xi^{(2)}|^2} = \dfrac{det(\|\eta^{(2)}\ \eta^{(3)}\|)}{|\eta^{(2)}|^2}. \end{cases} \tag{50}$$

for all $t \in I$.

Furthermore, if ξ and η are $Sim^{+}(Gal, 2)$-similar, there exists the unique $F \in Sim^{+}(Gal, 2)$ such that $\eta = F\xi = R\xi + bt + c$, where $R \in LSim^{+}(E_2)$ has the form

$$R = \begin{pmatrix} \frac{\langle \xi^{(2)}, \eta^{(2)} \rangle}{|\xi^{(2)}|^2} & -\frac{det(\|\xi^{(2)}\ \eta^{(2)}\|)}{|\xi^{(2)}|^2} \\ \frac{det(\|\xi^{(2)}\ \eta^{(2)}\|)}{|\xi^{(2)}|^2} & \frac{\langle \xi^{(2)}, \eta^{(2)} \rangle}{|\xi^{(2)}|^2} \end{pmatrix}, \tag{51}$$

and $b, c \in E_2$ have the forms

$$b = \eta' - R\xi', c = \eta - R\xi - (\eta' - R\xi')t \tag{52}$$

for all $t \in I$.

Here, R, b and c are independent of the choice of t in I.

Definition 13.

(i) A completely C^3-degenerate path followed by a fluid particle is a C^3-mapping $\xi : I \to E_2$ such that $det(\|\xi^{(2)}(t)\ \xi^{(3)}(t)\|) = 0$ for all $t \in I$.

(ii) A C^3-non-degenerate path followed by a fluid particle is a C^3-mapping $\xi : I \to E_2$ such that $det(\|\xi^{(2)}(t)\ \xi^{(3)}(t)\|) \neq 0$ for all $t \in I$.

The following theorems are similar to Theorems 3, 4 and 6, respectively.

Theorem 14. *Let ξ and η be two completely C^3-degenerate C^3-regular I-paths in E_2 followed by two fluid particles. Then ξ and η are $Sim(Gal, 2)$-similar if and only if*

$$\left\{ \dfrac{\langle \xi^{(2)}, \xi^{(3)} \rangle}{|\xi^{(2)}|^2} = \dfrac{\langle \eta^{(2)}, \eta^{(3)} \rangle}{|\eta^{(2)}|^2} \right. \tag{53}$$

for all $t \in I$.

Furthermore, if ξ and η are $Sim(Gal,2)$-similar, there exists only two elements $F = F_1, F_2 \in Sim(Gal,2)$ such that $\eta = F_1\xi = R_1\xi + b_1 t + c_1$ and $\eta = F_2\xi = (R_2\Lambda)\xi + b_2 t + c_2$, where the orientation-preserving linear similarity transformation R_1 of E_2 has the form (51) and the orientation-preserving linear similarity transformation R_2 of E_2 has the form

$$R_2 = \begin{pmatrix} \frac{\langle \Lambda\xi^{(2)}, \eta^{(2)}\rangle}{|\Lambda\xi^{(2)}|^2} & -\frac{det(\|\Lambda\xi^{(2)}\ \eta^{(2)}\|)}{|\Lambda\xi^{(2)}|^2} \\ \frac{det(\|\Lambda\xi^{(2)}\ \eta^{(2)}\|)}{|\Lambda\xi^{(2)}|^2} & \frac{\langle \Lambda\xi^{(2)}, \eta'\rangle}{|\Lambda\xi^{(2)}|^2} \end{pmatrix}. \tag{54}$$

Here $b_1, b_2, c_1, c_2 \in E_2$ have the forms $b_1 = \eta' - R_1\xi'$, $b_2 = \eta' - R_2\xi'$, $c_1 = \eta - R_1\xi - (\eta' - R_1\xi')t$, $c_2 = \eta - R_2\xi - (\eta' - R_2\xi')t$ for all $t \in I$, respectively.

Here $R_1, R_2, b_1, b_2, c_1, c_2$ are independent of the choice of t in I.

Theorem 15. *Let ξ and η be two S-non-degenerate C^3-regular I-paths in E_2 followed by two fluid particles. Then ξ and η are $Sim(Gal,2)$-similar if and only if*

$$\begin{cases} \dfrac{\langle \xi^{(2)}, \xi^{(3)}(t)\rangle}{|\xi^{(2)}|^2} = \dfrac{\langle \eta^{(2)}, \eta^{(3)}\rangle}{|\eta^{(2)}|^2}, \\ \dfrac{det(\|\xi^{(2)}\xi^{(3)}\|)^2}{|\xi^{(2)}|^4} = \dfrac{det(\|\eta^{(2)}\eta^{(3)}\|)^2}{|\eta^{(2)}|^4}. \end{cases} \tag{55}$$

for all $t \in I$.

Furthermore, if ξ and η are $Sim(Gal,2)$-similar, there exists the unique $F \in Sim(Gal,2)$ such that $\eta = F\xi$. Then, Only the following cases exist:

(i_1) *$det(\|\xi^{(2)}\ \xi^{(3)}\|) > 0$ and $det(\|\eta^{(2)}\ \eta^{(3)}\|) > 0$ for all $t \in I$.*
(i_2) *$det(\|\xi^{(2)}\ \xi^{(3)}\|) < 0$ and $det(\|\eta^{(2)}\ \eta^{(3)}\|) < 0$ for all $t \in I$.*
(ii_1) *$det(\|\xi^{(2)}\ \xi^{(3)}\|) > 0$ and $det(\|\eta^{(2)}\ \eta^{(3)}\|) < 0$ for all $t \in I$.*
(ii_2) *$det(\|\xi^{(2)}\ \xi^{(3)}\|) < 0$ and $det(\|\eta^{(2)}\ \eta^{(3)}\|) > 0$ for all $t \in I$.*

(i) *In the cases (i_1) and (i_2), F has the form $F\xi = R_1\xi + b_1 t + c_1$, where the orientation-preserving linear similarity transformation R_1 of E_2 has the form (51) and $b_1, c_1 \in E_2$ have the forms $b_1 = \eta' - R_1\xi'$, $c_1 = \eta - R_1\xi - (\eta' - R_1\xi')t$ for all $t \in I$, respectively.*
(ii) *In the cases (ii_1) and (ii_2), F has the form $F\xi = R_2\Lambda\xi + b_2$, where the orientation-preserving linear similarity transformation R_2 of E_2 has the form (54) and $b_2, c_2 \in E_2$ have the forms $b_2 = \eta' - R_2\Lambda\xi'$, $c_2 = \eta - R_2\Lambda\xi - (\eta' - R_2\Lambda\xi')t$ for all $t \in I$, respectively.*

Here R_1, R_2, b_1, b_2, c_1 and c_2 are independent of the choice of t in I.

Theorem 16. *Let $a_1(t)$ and $a_2(t)$ be arbitrary real continuous functions on I. Assume that a C^3-regular I-path $\xi(t)$ in E_2 followed by a fluid particle such that the following equalities*

$$\begin{cases} \dfrac{\langle \xi^{(2)}, \xi^{(3)}\rangle}{|\xi^{(2)}|^2} = a_1(t), \\ \dfrac{det(\|\xi^{(2)}\ \xi^{(3)}\|)}{|\xi^{(2)}|^2} = a_2(t) \end{cases} \tag{56}$$

hold for all $t \in I$. Then it has the following form

$$\xi(t) = c_1 \int_{t_0}^{t} e^{\int_{r_0}^{r}(a_1(z)+ia_2(z))dz} dr + c_2 t + c_3, \tag{57}$$

where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2 and c_3 are arbitrary elements of $\mathbb{C}$ and $t_0, r_0 \in I$.

Conversely, every I-path in the form (57), where c_1 is an arbitrary element of $\mathbb{C}^*$, c_2, c_3 are arbitrary elements of $\mathbb{C}$, $t_0, r_0 \in I$, $a_1(t)$ and $a_2(t)$ are arbitrary real continuous functions on I, is a C^3-regular I-path $\xi(t)$ in E_2 followed by a fluid particle and satisfies the equalities (56) for all $t \in I$.

6.2. Detecting G-similarity between two paths for groups $Sim(Gal,m,2)$ and $Sim^+(Gal,m,2)$

In the Introduction, we have defined the following groups transformations of space–time-mass $E_2 \times \mathbb{R} \times \mathbb{R}^+$:
$Sim(Gal,m,2) = \{F : E_2 \times \mathbb{R} \times \mathbb{R}^+ \to E_2 \times \mathbb{R} \times \mathbb{R}^+ \mid F(x,t,m) = (gx + bt, t, \gamma m), g \in Sim(E_2), b \in E_2, t \in \mathbb{R}, \gamma \in \mathbb{R}^+\}$.
$Sim^+(Gal,m,2) = \{F : E_2 \times \mathbb{R} \times \mathbb{R}^+ \to E_2 \times \mathbb{R} \times \mathbb{R}^+ \mid F(x,t,m) = (gx + bt, t, \gamma m), g \in Sim^+(E_2), b \in E_2, t \in \mathbb{R}, \gamma \in \mathbb{R}^+\}$.

Let $I=(a,b)\subseteq\mathbb{R}$. We consider the I-path $(\xi(t),m(t))=(\xi_1(t),\xi_2(t),m(t))$ in space-mass $E_2\times\mathbb{R}^+$ such that

$$x: I\rightarrow E_2 \tag{58}$$

is a C^3-mapping and $m: I\rightarrow\mathbb{R}^+$ is a $C^{(1)}$-function. The components $\xi_1(t),\xi_2(t)$ of $\xi(t)$ are real C^3-functions on I.

Definition 14. Two paths $(\xi(t),m_1(t))$ and $(\eta(t),m_2(t))$ in the space-mass $E_2\times\mathbb{R}^+$ are $Sim(Gal,m,2)$-similar provided there exists $F\in Sim(Gal,m,2)$, where $F(\xi,t,m)=(H\xi(t)+bt,t,\gamma m)$, $H\in Sim(E_2)$, $b\in E_2$, $\gamma\in\mathbb{R}^+$, $\forall\xi(t)\in E_2$, $\forall t\in I$, such that $\eta(t)=H\xi(t)+bt$ and $m_2(t)=\gamma m_1(t)$, $\forall t\in I$.

Definition 15. Two paths $(\xi(t),m_1(t))$ and $(\eta(t),m_2(t))$ in the space-mass $E_2\times\mathbb{R}^+$ are $Sim^+(Gal,m,2)$-similar provided there exists $F\in Sim^+(Gal,m,2)$, where $F(\xi,t,m)=(H\xi(t)+bt,t,\gamma m)$, $H\in Sim^+(E_2)$,
$b\in E_2$, $\gamma\in\mathbb{R}^+$, $\forall\xi(t)\in E_2$, $\forall t\in I$, such that $\eta(t)=H\xi(t)+bt$ and $m_2(t)=\gamma m_1(t)$, $\forall t\in I$.

Definition 16. An I-path $(\xi(t),m(t))$ in the space-mass $E_2\times\mathbb{R}^+$ is called C^3-regular if $\xi(t)$ is C^3-mapping such that $\xi^{(2)}(t)\neq 0$ for all $t\in I$ and $m(t)\neq 0$ for all $t\in I$.

The proofs of the following theorems are similar to the proofs of Theorems 13 and 16, respectively.

Theorem 17. *Let $(\xi(t),m_1(t))$ and $(\xi(t),m_2(t))$ be C^3-regular I-paths in the space-mass $E_2\times\mathbb{R}^+$. Then $(\xi(t),m_1(t))$ and $(\xi(t),m_2(t))$ are $Sim^+(Gal,2)$-similar if and only if*

$$\begin{cases}\dfrac{\langle\xi^{(2)},\xi^{(3)}\rangle}{|\xi^{(2)}|^2}=\dfrac{\langle\eta^{(2)},\eta^{(3)}\rangle}{|\eta^{(2)}|^2},\\ \dfrac{det(\|\xi^{(2)}\ \xi^{(3)}\|)}{|\xi^{(2)}|^2}=\dfrac{det(\|\eta^{(2)}\ \eta^{(3)}\|)}{|\eta^{(2)}|^2},\\ \dfrac{m_1^{(1)}(t)}{m_1(t)}=\dfrac{m_2^{(1)}(t)}{m_2(t)}\end{cases} \tag{59}$$

for all $t\in I$.

Furthermore, if $(\xi(t),m_1(t))$ and $(\xi(t),m_2(t))$ are $Sim^+(Gal,2)$-similar, there exist the unique $F\in Sim^+(Gal,2)$ and the unique $\gamma\in\mathbb{R}^+$ such that $\eta(t)=F\xi(t)$, $m_2(t)=\gamma m_1(t)$ for all $t\in I$, where $F\xi(t)=R\xi(t)+bt+c$. In this case, the orientation-preserving linear similarity transformation R of E_2, $b,c\in E_2$ and γ have the forms (51), $b=\eta'-R\xi'$, $c=\eta-R\xi-(\eta'-R\xi')t\in E_2$ and $\gamma=\frac{m_2(t)}{m_1(t)}$ for all $t\in I$.

Here, R, b, c and γ do not depend on $t\in I$.

Theorem 18. *Let $a_1(t)$, $a_2(t)$ and $a_3(t)$ be arbitrary real continuous functions on I. Assume that a C^3-regular I-path $\xi(t)$ in E_2 followed by a fluid particle such that the following equalities*

$$\begin{cases}\dfrac{\langle\xi^{(2)},\xi^{(3)}\rangle}{|\xi^{(2)}|^2}=a_1(t),\\ \dfrac{det(\|\xi^{(2)}\ \xi^{(3)}\|)}{|\xi^{(2)}|^2}=a_2(t),\\ \dfrac{m^{(1)}(t)}{m(t)}=a_3(t)\end{cases} \tag{60}$$

hold for all $t\in I$. Then it has the following form

$$\begin{cases}\xi(t)=c_1\displaystyle\int_{t_0}^{t}e^{\int_{r_0}^{r}(a_1(z)+ia_2(z))dz}dr+c_2t+c_3,\\ m(t)=ke^{\int_p^t(a_3(z))dz},\end{cases} \tag{61}$$

where c_1 and k are arbitrary elements of $\mathbb{C}^*$, c_2 and c_3 are arbitrary elements of $\mathbb{C}$ and $t_0,r_0,p\in I$.

Conversely, every I-path in the form (61), where c_1 and k are arbitrary elements of $\mathbb{C}^*$, c_2, c_3 are arbitrary elements of $\mathbb{C}$, $t_0,r_0,p\in I$, $a_1(t)$, $a_2(t)$ and $a_3(t)$ are arbitrary real continuous functions on I, is a C^3-regular I-path $\xi(t)$ in E_2 followed by a fluid particle and satisfies the equalities (60) for all $t\in I$.

Remark 4. Obtained other results in this paper can be given for the groups $Sim(Gal,m,2)$, $Sim^+(Gal,m,2)$, $Sim(Gal,2)$ and $Sim^+(Gal,2)$.

7. Conclusion

The results of this paper may be useful in many areas of pure and applied mathematics, physics, similarity and dimensional methods in mechanics. The complete systems of invariants of mechanical systems for the group $Mech(n)$ are important in mechanics. (see references [1,2,8,10–13,15–17,21,22,26,28–30] and the references therein). Since the automatic identification of similar objects and two-dimensional similarity detection in the field of pattern recognition, computer vision and vision-based applications are one of the most important problems, results in the present paper may be used in mentioned fields. Moreover, similarity and fluids are also important in cosmology (see [7,14,34]). Methods, developed in the present paper, might be useful in the theory of fluids and relativistic fluids.

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