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Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case

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Abstract

We degenerate the finite gap solutions of the KdV equation from the general formulation given in terms of abelian functions when the gaps tends to points, to get solutions to the KdV equation given in terms of Fredholm determinants and wronskians. For this we establish a link between Riemann theta functions, Fredholm determinants and wronskians. This gives the bridge between the algebro-geometric approach and the Darboux dressing method.
We construct also multi-parametric degenerate rational solutions of this equation.

1 Introduction

We consider the KdV equation in the following normalization

$$u_t = 6uu_x - u_{xxx}, \tag{1}$$

where the subscripts x and t as usual denote partial derivatives.
This equation (1) was introduced for the first time in 1895 by Korteweg and de Vries [15]. This equation appears in a wide range of physical problems and describes the propagation of waves with weak dispersion in various nonlinear media.
A method of resolution was given in 1967 by Gardner et al. [11]. It was proven that this equation is a complete integrable system by Zakharov and Faddeev in 1971 [35].
Solutions were constructed by Hirota in 1971 by using the bilinear method [12]. Its and Matveev present solutions in terms of Riemann theta functions [13] in 1975. Lax gives in the same year the expressions of periodic and almost periodic solutions [18]. A lot of works have been realized in the following years. We can mention for example Airault et al. in 1977 [3], Adler and Moser in 1978 [2],

Ablowitz and Cornille in 1979 [1], Freeman and Nimmo in 1984 [5], Matveev in 1992 [23], Ma in 2004 [19], Kovalyov in 2005 [17] and more recently Ma in 2015 [20].

In the following, we are interested in the algebro-geometric approach given by Its and Matveev in 1975. We degenerate the solutions to the KdV equation given in terms of Riemann theta functions to get solutions in terms of Fredholm determinants. Then we give a representation in terms of wronskians. This gives the correspondence between the algebro-geometric approach and the Darboux dressing method.
Then we construct rational solutions in performing a passage to the limit. We obtain multi-parametric rational solutions to the KdV equation depending on $3N$ parameters at order N. We give explicit solutions in the simplest cases where $N = 1, 2, 3$.

2 The KdV equation and its solutions in terms of theta functions

We consider the Riemann surface Γ of the algebraic curve defined by

$$\omega^2 = \prod_{j=1}^{2g+1} (z - E_j),$$

with $E_j \neq E_k,\ j \neq k$. Let D be some divisor $D = \sum_{j=1}^{g} P_j,\ P_j \in \Gamma$. The so-called finite gap solution of the KdV equation

$$u_t = 6uu_x - u_{xxx} \tag{2}$$

can be expressed in the form [13]

$$u(x,t) = -2\partial_x^2 \left[\ln \theta(xg + tv + l)\right] + C. \tag{3}$$

We recall briefly, the notations. In (3), θ is the Riemann function defined by

$$\theta(z) = \sum_{k \in \mathbf{Z}^g} \exp\{\pi i(Bk|k) + 2\pi i(k|z)\}, \tag{4}$$

constructed from the matrix of the B-periods of the surface Γ, and the vectors g, v, l are defined by

$$g_j = 2ic_{j1}, \tag{5}$$

$$v_j = 8i(\frac{c_{j1}}{2} \sum_{k=1}^{2g+1} E_k + c_{j2}), \tag{6}$$

$$l_j = -\sum_{k=1}^{g} \int_{\infty}^{P_k} dU_j + \frac{j}{2} - \frac{1}{2} \sum_{k=1}^{g} B_{kj}, \tag{7}$$

$$C = \sum_{k=1}^{2g+1} E_k - 2\sum_{k=1}^{g} \int_{a_k} z dU_k, \tag{8}$$

the coefficients c_{jk} being relating with abelian differential dU_j by

$$dU_j = \frac{\sum_{k=1}^{g} c_{jk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+1}(z - E_k)}} dz, \tag{9}$$

and coefficients c_{jk} can be obtained by solving the system of linear equations

$$\int_{a_k} dU_j = \delta_{jk}, \quad 1 \leq j \leq g, \quad 1 \leq k \leq g.$$

3 Degeneracy of solutions

We suppose that E_j are real, $E_m < E_j$ if $m < j$ and try to evaluate the limits of all objects in formula (3) when E_{2m}, E_{2m+1} tends to $-\alpha_m$, $-\alpha_m = -\kappa_m^2$, $\kappa_m > 0$, for $1 \leq m \leq g$, and E_1 tends to 0 (these ideas were first presented by A. Its and V.B. Matveev, exposed for example in [4]).

3.1 Degeneracy of the components of the solution

3.1.1 Limit of $P(z) = \prod_{j=1}^{2g+1}(z - E_j)$

The limit of $P(z) = \prod_{j=1}^{2g+1}(z - E_j)$ is evidently equal to $\tilde{P}(z) = z\prod_{j=1}^{g}(z + \alpha_j)^2$

3.1.2 Limit of $dU_m = \frac{\sum_{k=1}^{g} c_{mk} z^{g-k}}{\sqrt{\prod_{k=1}^{2g+1}(z - E_k)}} dz$

The limit of dU_m is equal to $d\tilde{U}_m = \frac{\varphi_m(z)}{\sqrt{z}\prod_{j=1}^{g}(z + \alpha_j)} dz$, where $\varphi_m(z) = \sum_{k=1}^{g} \tilde{c}_{mk} z^{g-k}$. The normalization condition takes the form in the limit

$$\int_{a_k} dU_j \to \frac{2\pi i \varphi_j(-\alpha_k)}{-i\kappa_k \prod_{m\neq k}(-\alpha_k + \alpha_m)} = \delta_{kj}, \tag{10}$$

which proves that the numbers $-\alpha_m$, $m \neq k$ are the zeros of the polynomials $\varphi_k(z)$, and so $\varphi_k(z)$ can be written as $\varphi_k(z) = \tilde{c}_{k1}\prod_{m\neq k}(z + \alpha_m)$. By (10), we get in the limit

$$\tilde{c}_{k1} = -\frac{\kappa_k}{2\pi i}.$$

So

$$d\tilde{U}_k = -\frac{\kappa_k}{2\pi i\sqrt{z}(z + \alpha_k)} dz$$

3.1.3 Limit of v_k and g_k

By identification of the powers of z^{g-2} in (11)

$$\varphi_k(z) = \tilde{c_{k1}} \prod_{l \neq k} (z + \alpha_l) = \sum_{j=1}^{g} \tilde{c_{kj}} z^{g-j}, \tag{11}$$

we get in the limit

$$\tilde{c}_{k1} \sum_{l=1}^{g} \alpha_l = \tilde{c}_{k2}.$$

So we have the limit values of v_k and g_k :

$$\tilde{v}_k = \frac{4i\kappa_k^3}{\pi}$$

and

$$\tilde{g}_k = \frac{-i\kappa_k}{\pi}.$$

3.1.4 Limit of $U_k(P)$ and B_{mk}

For $\lambda_0 = -\alpha_m = -\kappa_m^2$, $I = \int_{\lambda_0}^{0} dU_k \to \frac{1}{2}\tilde{B}_{mk}$. The integral I can be easily evaluate along the real axis on the upper sheet of surface Γ and we get

$$I \to \frac{i}{2\pi} \ln \left| \frac{\kappa_m + \kappa_k}{\kappa_m - \kappa_k} \right|.$$

So we have the limit values of matrix B :

$$\tilde{B}_{mk} = \frac{i}{\pi} \ln \left| \frac{\kappa_m + \kappa_k}{\kappa_m - \kappa_k} \right|.$$

Therefore iB_{kk} tends to $-\infty$. As previously, we have

$$\int_{\infty}^{P} dU_j \to -\frac{i}{2\pi} \ln \left| \frac{\kappa_j - \sqrt{z_P}}{\kappa_j + \sqrt{z_P}} \right|. \tag{12}$$

3.1.5 Limit of argument of exponential in $\theta(p)$

Let us denote A_0 the argument of exponential in $\theta(p) = \sum_{k \in \mathbf{Z}^g} \exp\{\pi i(Bk|k) + 2\pi i(k|p)\}$.
A_0 can be rewritten in the form

$$A_0 = \pi i \sum_{j=1}^{g} B_{jj} k_j (k_j - 1) + 2\pi i \sum_{j>m} B_{mj} k_m k_j + \sum_{j=1}^{g} \pi i (2p_j + B_{jj}) k_j. \tag{13}$$

Using the inequality $k_j(k_j - 1) \geq 0$ for all $k \in \mathbf{Z}^g$ and the fact that iB_{kk} tends to $-\infty$, we can reduce the limit $\tilde{\theta}$ of $\theta(p)$ to a finite sum taken over vectors

$k \in \mathbf{Z}^g$ such that each k_j must be equal to 0 or 1.
So, if we denote A the argument of $\theta(xg+tv+l)$, it can be written in the form

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j-1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} k_j[2\pi i(g_j x + v_j t)$$

$$-\pi i(-j + 2\sum_{k=1}^{g}\int_{\infty}^{P_k} dU_j + \sum_{m\neq j} B_{mj})].$$

In other words

$$A = \pi i \sum_{j=1}^{g} B_{jj}k_j(k_j-1) + 2\pi i \sum_{j>m} B_{mj}k_m k_j + \sum_{j=1}^{g} k_j Q_j,$$

with

$$Q_j = 2\pi i(g_j x + v_j t) + \beta_j$$

and

$$\beta_j = -\pi i(-j + 2\sum_{k=1}^{g}\int_{\infty}^{P_k} dU_j + \sum_{m\neq j} B_{mj}).$$

The quantity β_j has a finite limit value $\tilde{\beta}_j$ independent from x and t.

3.1.6 Limit of $\theta(xg+vt+l)$

By means of the inequality $k_j(k_j-1) \geq 0$ for all $k \in \mathbf{Z}^{\mathbf{g}}$ and the previous relation iB_{kk} tends to $-\infty$, it turns out that the limit $\tilde{\theta}$ of $\theta(xg+tv+l)$ reduce to a finite sum taken over vectors $k \in \mathbf{Z}^{\mathbf{g}}$ with the property that each k_j must be equal to 0 or 1.

$$\tilde{\theta} = \sum_{k\in\mathbf{Z}^g,\, k_j=0\, or\, 1} \exp\{\sum_{m>j} 2\ln\left|\frac{\kappa_m-\kappa_j}{\kappa_m+\kappa_j}\right| k_m k_j + (\sum_{j=1}^{g} 2\kappa_j x - 8\kappa_j^3 t + 2\kappa_j x_j + \pi j i + \sum_{m\neq j}\ln\left|\frac{\kappa_m+\kappa_j}{\kappa_m-\kappa_j}\right|)k_j\},$$

with

$$x_j = \frac{1}{2\kappa_j}\sum_{k=1}^{g}\ln\left|\frac{\sqrt{z_k}-i\kappa_j}{\sqrt{z_k}+i\kappa_j}\right|.$$

It can be rewritten as

$$\tilde{\theta} = \sum_{J\subset\{1,\dots,g\}} \prod_{j\in J}(-1)^j \prod_{j\in J}\prod_{k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right| \exp\sum_{j\in J} 2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j). \quad (14)$$

3.1.7 Limit of the coefficient C

The coefficient C is defined in (8) by

$$C = \sum_{k=1}^{2g+1} E_k - 2\sum_{k=1}^{g} \int_{a_k} z dU_k = C_1 + C_2,$$

can be evaluated as follows.

$$C_2 = -2\sum_{k=1}^{g} \int_{a_k} z dU_k = -2\sum_{k=1}^{g} \int_{a_k} \frac{-\kappa_k z dz}{2\pi\sqrt{z}(z+\alpha_k)} = \sum_{k=1}^{g} \frac{\kappa_k}{\pi} \int_{a_k} \frac{\sqrt{z}dz}{(z+\alpha_k)}$$

$$= \sum_{k=1}^{g} \frac{\kappa_k}{\pi} 2i\pi(-i\kappa_k) = \sum_{k=1}^{g} 2\kappa_k^2 = 2\sum_{k=1}^{g} \alpha_k.$$

Thus when the gaps tends to points,, the coefficient C tends to $\tilde{C}$ equal to

$$\tilde{C} = 2\sum_{k=1}^{g} -\alpha_k + 2\sum_{k=1}^{g} \alpha_k = 0.$$

3.2 Degenerate solution to the KdV equation

Therefore, we have the following representation of the degenerate solution to the KdV equation

Theorem 3.1 *The fonction u defined by*

$$u(x,t) = -2\partial_x^2 \ln\left(\sum_{J\subset\{1,\dots,g\}} \prod_{j\in J}(-1)^j \prod_{j\in J}\prod_{k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right| \exp\left(\sum_{j\in J} 2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right)\right), \quad (15)$$

with κ_j, and x_j arbitrary real parameters, is a solution to the KdV equation (1).

4 From theta function to Fredholm determinant

4.1 The link between the degenerate solution and the Fredholm determinant

In a recent paper, Kirillov and Van Diejen [31] have given formulas in terms of determinants for zonal spherical functions on hyperboloids. In particular, they compute $\det(I+A)$, where I is the unit matrix and $A = (a_{jk})_{1\le j,k\le m}$ the matrix defined as :

$$a_{jk} = \frac{2\epsilon_j\kappa_j}{\kappa_j+\kappa_k} \exp(-2\kappa_j x) \prod_{l\neq j}\left|\frac{\kappa_l+\kappa_j}{\kappa_l-\kappa_j}\right|, \quad (16)$$

where $\epsilon_j \in \{-1;+1\}$ and $\kappa_j > 0$ for $1 \leq j \leq N$.
Then $\det(I+A)$ has the following form

$$\det(I+A) = \sum_{J\subset\{1,\dots,N\}} \exp\left(-2x\sum_{j\in J}\kappa_j\right)\prod_{j\in J}\epsilon_j \prod_{j\in J}\prod_{k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right|. \quad (17)$$

Using the same strategy, we can compute $\det(I+A)$ where $A = (a_{jk})_{1\leq j,k\leq m}$ is the matrix defined as :

$$a_{jk} = \frac{(-1)^j 2\kappa_j}{\kappa_j+\kappa_k}\exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right]\prod_{l\neq j}\left|\frac{\kappa_l+\kappa_j}{\kappa_l-\kappa_j}\right|,$$

x_j being an arbitrary parameter.
Then $\det(I+A)$ has the following form

$$\det(I+A) = \sum_{J\subset\{1,\dots,N\}} \exp\left(\sum_{j\in J}2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right)\prod_{j\in J}(-1)^j \prod_{j\in J}\prod_{k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right|, \quad (18)$$

By the previous section,

$$\tilde{\theta} = \sum_{J\subset\{1,\dots,g\}}\prod_{j\in J}(-1)^j \prod_{j\in J}\prod_{k\notin J}\left|\frac{\kappa_j+\kappa_k}{\kappa_j-\kappa_k}\right|\exp\left(\sum_{j\in J}(2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right). \quad (19)$$

If we compare the expression (18) to (19), we have clearly the equality with $g = N$

$$\tilde{\theta} = \det(I+A). \quad (20)$$

4.2 Solution to the KdV equation in terms of Fredholm determinant

So we have the following representation of the solutions to the KdV equation

Theorem 4.1 *The fonction u defined by*

$$u(x,t) = -2\partial_x^2 \ln(\det(I+A)), \quad (21)$$

with A the matrix defined by $A = (a_{jk})_{1\leq j,k\leq N}$

$$a_{jk} = \frac{(-1)^j 2\kappa_j}{\kappa_j+\kappa_k}\exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right]\prod_{l\neq j}\left|\frac{\kappa_l+\kappa_j}{\kappa_l-\kappa_j}\right|, \quad (22)$$

and κ_j, x_j arbitrary real parameters, is a solution to the KdV equation (1).

If we consider the matrix B defined by

$$b_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \prod_{l \neq k} \left| \frac{\kappa_l + \kappa_j}{\kappa_l - \kappa_k} \right|,$$

it is easy to verify that $\det(I + A) = \det(I + B)$, and so we can give another representation of the solutions to the KdV equation. We get the following statement :

Theorem 4.2 *The fonction u defined by*

$$u(x,t) = -2\partial_x^2 \ln(\det(I + B)), \tag{23}$$

with B the matrix defined by $B = (b_{jk})_{1 \leq j,k \leq m}$

$$b_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \prod_{l \neq k} \left| \frac{\kappa_l + \kappa_j}{\kappa_l - \kappa_k} \right|, \tag{24}$$

and κ_j, x_j arbitrary real parameters, is a solution to the KdV equation (1).

We can also consider the matrix C defined by

$$c_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \frac{\prod_{l \neq k} |\kappa_l + \kappa_j|}{\prod_{l \neq j} |\kappa_l - \kappa_j|}.$$

It is easy to check that $\det(I + A) = \det(I + C)$, and so we can give a third representation of the solutions to the KdV equation :

Theorem 4.3 *The fonction u defined by*

$$u(x,t) = -2\partial_x^2 \ln(\det(I + C)), \tag{25}$$

with C the matrix defined by $C = (c_{jk})_{1 \leq j,k \leq m}$

$$c_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \frac{\prod_{l \neq k} |\kappa_l + \kappa_j|}{\prod_{l \neq j} |\kappa_l - \kappa_j|}, \tag{26}$$

and κ_j, x_j arbitrary real parameters, is a solution to the KdV equation (1).

Another possibility is to choose the matrix D defined by

$$d_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \prod_{l \neq j} \left| \frac{\kappa_l + \kappa_k}{\kappa_l - \kappa_j} \right|.$$

It is also easy to check that $\det(I + A) = \det(I + D)$, and so we can give another representation of the solutions to the KdV equation :

Theorem 4.4 *The fonction u defined by*

$$u(x,t) = -2\partial_x^2 \ln(\det(I + D)), \tag{27}$$

with C the matrix defined by $D = (d_{jk})_{1\le j,k\le m}$

$$d_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \prod_{l\neq j} \left|\frac{\kappa_l + \kappa_k}{\kappa_l - \kappa_j}\right|, \tag{28}$$

and κ_j, x_j arbitrary real parameters, is a solution to the KdV equation (1).

It remains to find the link between this Fredholm determinant and a certain wronskian.

5 From Fredholm determinants to wronskians

5.1 Link between Fredholm determinants and wronskians

In this section, we consider the following functions

$$\phi_j(x) = \sinh(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j), \tag{29}$$

where κ_j are real numbers such that $\kappa_1 \le \ldots \le \kappa_N$, and x_j an arbitrary constant independent of x.
We use the following notations :
$\theta_j = (\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)$.
$W = W(\phi_j, \ldots, \phi_N)$ is the classical Wronskian $W = \det[(\partial_x^{j-1}\phi_i)_{i,\, j\in[1,\ldots,N]}]$.
We consider the matrix $A = (a_{jk})_{j,\, k\in[1,\ldots,N]}$ defined by

$$a_{jk} = (-1)^j \exp\left[2(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)\right] \prod_{l\neq j} \left|\frac{\kappa_l + \kappa_k}{\kappa_l - \kappa_j}\right|. \tag{30}$$

Then we have the following statement

Theorem 5.1

$$\det(I + A) = \frac{2^N(-1)^{\frac{N(N+1)}{2}} \exp(\sum_{j=1}^N \theta_j)}{\prod_{j=2}^N \prod_{i=1}^{j-1}(\kappa_j - \kappa_i)} W(\phi_1, \ldots, \phi_N) \tag{31}$$

Proof : We start to remove the factor $2^{-1}e^{\theta_j}$ in each row j in the Wronskian W for $1 \le \jmath \le N$.
Then

$$W = \prod_{j=1}^N e^{\theta_j} 2^{-N} \times W_1, \tag{32}$$

with

$$W_1 = \begin{vmatrix} (1-e^{-2\theta_1}) & \kappa_1(1+e^{-2\theta_1}) & \dots & (\kappa_1)^{N-1}(1+(-1)^N e^{-i\theta_1}) \\ (1-e^{-2\theta_2}) & \kappa_2(1+e^{-2\theta_2}) & \dots & (\kappa_2)^{N-1}(1+(-1)^N e^{-2\theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1-e^{-2\theta_N}) & \kappa_N(1+e^{-2\theta_N}) & \dots & (\kappa_N)^{N-1}(1+(-1)^N e^{-2\theta_N}) \end{vmatrix}$$

The determinant W_1 can be written as

$$W_1 = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where $\alpha_{jk} = (-1)^k(\kappa_j)^{k-1}$, $e_j = e^{-2\theta_j}$, and $\beta_{jk} = (\kappa_j)^{k-1}$.
Denoting $U = (\alpha_{ij})_{i,\, j\in[1,\dots,N]}$, $V = (\beta_{ij})_{i,\, j\in[1,\dots,N]}$, the determinant of U is clearly equal to

$$\det(U) = (-1)^{\frac{N(N+1)}{2}} \prod_{N\geq l>m\geq 1} (\kappa_l - \kappa_m). \tag{33}$$

Then we use the following Lemma

Lemma 5.1 *Let* $A = (a_{ij})_{i,\, j\in[1,\dots,N]}$, $B = (b_{ij})_{i,\, j\in[1,\dots,N]}$,
$(H_{ij})_{i,\, j\in[1,\dots,N]}$, *the matrix formed by replacing the jth row of A by the ith row of B*
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}\right) \tag{34}$$

Proof : For $\tilde{A} = (\tilde{a}_{ij})_{i,\, j\in[1,\dots,N]}$ the matrix of cofactors of A, we have the well known formula $A \times^t \tilde{A} = \det A \times I$.
So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.
The general term of the product $(c_{ij})_{i,j\in[1,..,N]} = (a_{ij}x_i + b_{ij})_{i,j\in[1,..,N]} \times (\tilde{a}_{ij})_{i,j\in[1,..,N]}$ can be written as
$c_{ij} = \sum_{s=1}^{N}(a_{is}x_i + b_{is}) \times \tilde{a}_{js}$
$= x_i \sum_{s=1}^{N} a_{is}\tilde{a}_{js} + \sum_{s=1}^{N} b_{is}\tilde{a}_{js}$
$= \delta_{ij}\det(A)x_i + \det(H_{ij})$.
We get
$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \dfrac{\det(H_{ij})}{\det(A)})$.
Thus $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \dfrac{\det(H_{ij})}{\det(A)})$.
□
Using the previous lemma (34), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}),$$

where $(H_{ij})_{i,\, j\in[1,\dots,N]}$ is the matrix formed by replacing the j-th row of U by the i-th row of V defined previously.
We compute $\det(H_{ij})$ and we get

$$\det(H_{ij}) = (-1)^{\frac{N(N+1)}{2}+1} \prod_{N\geq l>m\geq 1,\, l\neq j,\, m\neq j} (\kappa_l - \kappa_m) \prod_{l<j} (-\kappa_k - \kappa_l) \prod_{l>j} (\kappa_k + \kappa_l). \quad (35)$$

We can simplify the quotient $q = \dfrac{\det(H_{ij})}{\det(\alpha_{ij})}$:

$$q = \frac{(-1)^j \prod_{l\neq j} |\kappa_l + \kappa_k|}{\prod_{l\neq j} |\kappa_l - \kappa_j|}.$$

So $\det(\delta_{jk} e_j + \dfrac{\det(H_{jk})}{\det(\alpha_{jk})})$ can be expressed as

$$\det(\delta_{jk} e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{N} e^{-2\theta_j} \det(\delta_{jk} + (-1)^j \prod_{l\neq j} \left| \frac{\kappa_l + \kappa_k}{\kappa_l - \kappa_j} \right| e^{2\theta_j}).$$

and therefore

$$\det(\delta_{jk} e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{N} e^{-2\theta_j} \det(I + A).$$

The wronskian can be written as

$$W(\phi_1, \dots, \phi_N) = \prod_{j=1}^{N} e^{\theta_j} 2^{-N} (-1)^{\frac{N(N+1)}{2}} \prod_{j=2}^{N} \prod_{i=1}^{j-1} (\kappa_j - \kappa_i) \prod_{j=1}^{N} e^{-2\theta_j} \det(I + A)$$

It follows that

$$\det(I + A) = \frac{e^{\sum_{j=1}^{N} \theta_j} (2)^N (-1)^{\frac{N(N+1)}{2}}}{\prod_{j=2}^{N} \prod_{i=1}^{j-1} (\kappa_j - \kappa_i)} W(\phi_1, \dots, \phi_N) \quad (36)$$

$\square$

5.2 Solutions to the KdV equation in terms of wronskians

From the previous subsection, we can give the following wronskian representation of the solutions to the KdV equation.

Theorem 5.2 *The function u defined by*

$$u(x,t) = -2\partial_x^2 \left(\ln \left[W(\phi_1, \dots, \phi_N) \right] \right), \quad (37)$$

where $= W(\phi_1, \dots, \phi_N) = \det[(\partial_x^{j-1} \phi_i)_{i,\, j\in[1,\dots,N]}]$ *is the wronskian of the functions ϕ defined by $\phi_j(x,t) = \sinh(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)$, κ_j, x_j being real numbers, is a solution to the KdV equation* (1).

Proof : From the result of the previous subsection, the solution to the KdV equation u can be written as $u(x,t) = -2\partial_x^2(\det(I+A))$. From (31), we have

$$\det(I+A) = \frac{e^{\sum_{j=1}^{N}\theta_j}(2)^N(-1)^{\frac{N(N+1)}{2}}}{\prod_{j=2}^{N}\prod_{i=1}^{j-1}(\kappa_j-\kappa_i)}W(\phi_1,\dots,\phi_N).$$

We can conclude that u can be rewritten as

$$u(x,t) = -2\partial_x^2 \ln\left(\frac{e^{\sum_{j=1}^{N}\theta_j}(2)^N(-1)^{\frac{N(N+1)}{2}}}{\prod_{j=2}^{N}\prod_{i=1}^{j-1}(\kappa_j-\kappa_i)}W(\phi_1,\dots,\phi_N)\right) = -2\partial_x^2\left(\ln(W(\phi_1,\dots,\phi_N))\right).$$

□

It is relevant to note that we recover the result given by the Darboux dressing [23].

This realize the connection between the algebro-geometric approach and the Darboux dressing method.

Remark 5.1 *The choices of functions ϕ_j are not unique. For example, we can choose :*
$\phi_j(x) = \cosh(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)$, *or* $\phi_j(x) = \exp(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j)$, *or* $\phi_j(x) = \exp(-(\kappa_j x - 4\kappa_j^3 t + \kappa_j x_j))$ *or any combinations of these different last functions.*
We can also choose the following functions :
$\phi_j(x) = \sin(\kappa_j x + 4\kappa_j^3 t + \kappa_j x_j)$, *or* $\phi_j(x) = \cos(\kappa_j x + 4\kappa_j^3 t + \kappa_j x_j)$, *or* $\phi_j(x) = \exp(i(\kappa_j x + 4\kappa_j^3 t + \kappa_j x_j))$, *or* $\phi_j(x) = \exp(-i(\kappa_j x + 4\kappa_j^3 t + \kappa_j x_j))$ *or any combinations of these different last functions.*

6 Another approach and degenerate multi-parametric rational solutions to the KdV equation

6.1 Solutions to the KdV equation in terms of elementary exponentials

We consider the KdV equation (1)

$$u_t = 6uu_x - u_{xxx}.$$

We consider the following elementary functions :

$$f_{ij}(x,t) = (a_j e)^{i-1}\exp(a_j e x - 4(a_j e)^3 t + c_j e^{2N-1}) - (a_j e)^{i-1}\exp(-a_j e x + 4(a_j e)^3 t + d_j e^{2N-1}),$$
$$\text{for } 1 \le i \le N \quad (38)$$

Then, we have the following statement :

Theorem 6.1 *The function v defined by*

$$v(x,t) = -2\partial_x^2 \ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{39}$$

is a solution to the KdV equation (1) with e, a_j, c_j and d_j, $1 \leq j \leq N$ arbitrarily real parameters.

Proof : The corresponding Lax pair to the KdV equation (1) is

$$\begin{cases} -\phi_{xx} + u\phi = \lambda\phi, \\ \phi_t = -4\phi_{xxx} + 6u\phi_x + 3u_x\phi. \end{cases} \tag{40}$$

This system is covariant by the Darboux transformation. If $\phi_1, \ldots, \phi_N$ are solutions of the system (40), then $\phi[N]$ defined by $\phi[N] = \dfrac{W(\phi_1, \ldots, \phi_N, \phi)}{W(\phi_1, \ldots, \phi_N)}$ is another solution of this system (40) where u is replaced by $u[N] = u - 2\partial_x^2(\ln W(\phi_1, \ldots, \phi_N)$ [24].
We choose $u = 0$. Then the functions $\phi_j = f_{1j}$ verify the following system

$$\begin{cases} -\phi_{xx} = \lambda\phi, \\ \phi_t = -4\phi_{xxx}. \end{cases} \tag{41}$$

Then the solution of (1) can be written as $v(x,t) = -2\partial_x^2(\ln W(\phi_1, \ldots, \phi_N)$which is nothing else that (39) $v(x,t) = -2\partial_x^2 \ln(\det(f_{ij})_{(i,j)\in[1,N]})$.
□

6.2 Multi-parametric rational solutions to the KdV equation

To obtain rational solutions to the KdV equation, we are going to perform a limit when the parameter e tends to 0.

6.2.1 Rational solutions as a limit case

We get the following result :

Theorem 6.2 *The function v defined by*

$$v(x,t) = \lim_{e\to 0} -2\partial_x^2 \ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{42}$$

is a rational solution to the KdV equation (1) depending on $3N$ parameters a_j, c_j and d_j, $1 \leq j \leq N$.

Proof : Performing a passage to the limit when e tends to 0, it is an obvious consequence of the previous result.
□

6.2.2 Degenerate rational solutions

If we want a formulation without a limit, we consider the following functions g_{ij} and h_{ij} defined by

$$g_{ij}(x,t,e) = (a_j e)^{i-1}\exp(a_j ex - 4(a_j e)^3 t + c_j e^{2N-1}) - (a_j e)^{i-1}\exp(-a_j ex + 4(a_j e)^3 t - d_j e^{2N-1}),$$
$$h_{ij} = \frac{\partial^{2j-1} f_{ij}(x,t,0)}{\partial e^{2j-1}}, \text{for } 1 \le i \le N$$

Then get the following result :

Theorem 6.3 *The function v defined by*

$$v(x,t) = -2\partial_x^2 \ln(\det(h_{ij})_{(i,j)\in[1,N]}. \tag{43}$$

is a rational solution to the KdV equation (1).

Proof : It is sufficient to combine the columns of the determinant and to take a passage to the limit when e tends to 0 for each column different from the first one.
□
So we obtain an infinite hierarchy of rational solutions to the KdV equation depending on the integer N.
In the following we give some examples of rational solutions.
These results are consequences of the previous result (43).

6.3 First order rational solutions

We have the following result at order $N = 1$:

Proposition 6.1 *The function v defined by*

$$v(x,t) = \frac{8\,{a_1}^2}{(2\,a_1 x + c_1 - d_1)^2}, \tag{44}$$

is a solution to the KdV equation (1) with a_1, c_1, d_1 arbitrarily real parameters.

Remark 6.1 *This solution independent of t does not present any interest.*

6.4 Second order rational solutions

Proposition 6.2 *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{45}$$

with
$n(x,t) = 12\,a_2 a_1(-3\,{a_2}^2 + {a_1}^2)(3\,{a_2}^3 a_1 - {a_1}^3 a_2)x^4 + 12\,a_2 a_1(-3\,{a_2}^2 + {a_1}^2)(-72\,{a_2}^3 t a_1 +$

$24\,{a_1}^3ta_2+9\,c_2a_1+3\,d_1a_2-9\,d_2a_1-3\,c_1a_2)x$
$d(x,t)=(-6\,{a_2}^3a_1+2\,{a_1}^3a_2)x^3-72\,{a_2}^3ta_1+24\,{a_1}^3ta_2+9\,c_2a_1+3\,d_1a_2-9\,d_2a_1-3\,c_1a_2$

is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree 4 *in* x, 1 *in* t, *and denominator of degree* 6 *in* x, 2 *in* t.

In the case where all the parameters c_j and d_j are equal to 0 and the parameters a_j are equal to 1, the solution can be expressed as

$$u(x,t)=-6\,\frac{x\left(24\,t-x^3\right)}{\left(x^3+12\,t\right)^2}.$$

6.5 Rational solutions of order three

We get the following rational solutions given by :

Proposition 6.3 *The function* v *defined by*

$$v(x,t)=-2\,\frac{n(x,t)}{d(x,t)^2}, \tag{46}$$

with
$n(x,t)=(-7200\,{a_1}^4{a_2}^6{a_3}^8+14400\,{a_1}^4{a_2}^4{a_3}^{10}-4320\,{a_1}^6{a_2}^4{a_3}^8-5400\,{a_1}^2{a_2}^{10}{a_3}^6-216\,{a_1}^{10}{a_2}^2{a_3}^6+2160\,{a_1}^6{a_2}^6{a_3}^6-2400\,{a_1}^6{a_2}^2{a_3}^{10}+1440\,{a_1}^8{a_2}^2{a_3}^8-21600\,{a_1}^2{a_2}^6{a_3}^{10}+21600\,{a_1}^2{a_2}^8{a_3}^8)x^{10}+(72900\,a_1{a_2}^6{a_3}^6d_1+243000\,{a_1}^4{a_2}^6{a_3}^3c_3-243000\,{a_1}^4{a_2}^6{a_3}^3d_3-72900\,a_1{a_2}^6{a_3}^6c_1+729000\,{a_1}^2{a_2}^8{a_3}^3d_3-729000\,{a_1}^2{a_2}^8{a_3}^3c_3+243000\,{a_1}^4a_2{a_3}^8c_2-243000\,{a_1}^4a_2{a_3}^8d_2-72900\,{a_1}^6a_2{a_3}^6c_2-48600\,{a_1}^8{a_2}^2{a_3}^3c_3+48600\,{a_1}^8{a_2}^2{a_3}^3d_3-145800\,{a_1}^6{a_2}^4{a_3}^3d_3+72900\,{a_1}^6a_2{a_3}^6d_2+145800\,{a_1}^6{a_2}^4{a_3}^3c_3-364500\,{a_1}^2{a_2}^5{a_3}^6d_2+48600\,{a_1}^3{a_2}^2{a_3}^8d_1+162000\,{a_1}^6{a_2}^2{a_3}^5c_3-162000\,{a_1}^6{a_2}^2{a_3}^5d_3-48600\,{a_1}^3{a_2}^2{a_3}^8c_1-14580\,{a_1}^5{a_2}^2{a_3}^6d_1+14580\,{a_1}^5{a_2}^2{a_3}^6c_1-729000\,{a_1}^2{a_2}^3{a_3}^8c_2-972000\,{a_1}^4{a_2}^4{a_3}^5c_3+972000\,{a_1}^4{a_2}^4{a_3}^5d_3-1458000\,{a_1}^2{a_2}^6{a_3}^5d_3+729000\,{a_1}^2{a_2}^3{a_3}^8d_2+1458000\,{a_1}^2{a_2}^6{a_3}^5c_3+364500\,{a_1}^2{a_2}^5{a_3}^6c_2-145800\,a_1{a_2}^4{a_3}^8d_1+145800\,a_1{a_2}^4{a_3}^8c_1)x^5+(116640000\,{a_1}^2{a_2}^8t^2{a_3}^8-29160000\,{a_1}^2{a_2}^{10}t^2{a_3}^6-12960000\,{a_1}^6t^2{a_2}^2{a_3}^{10}-1166400\,{a_1}^{10}t^2{a_2}^2{a_3}^6-38880000\,{a_1}^4{a_2}^6t^2{a_3}^8-23328000\,{a_1}^6{a_2}^4{a_3}^8t^2+11664000\,{a_1}^6{a_2}^6t^2{a_3}^6+77760000\,{a_1}^4{a_2}^4{a_3}^{10}t^2+7776000\,{a_1}^8t^2{a_2}^2{a_3}^8-116640000\,{a_1}^2{a_2}^6{a_3}^{10}t^2)x^4+(62208000\,{a_1}^8{a_2}^2{a_3}^8t^3-103680000\,{a_1}^6{a_2}^2{a_3}^{10}t^3-233280000\,{a_1}^2{a_2}^{10}{a_3}^6t^3-933120000\,{a_1}^2{a_2}^6{a_3}^{10}t^3+622080000\,{a_1}^4{a_2}^4{a_3}^{10}t^3+93312000\,{a_1}^6{a_2}^6{a_3}^6t^3+933120000\,{a_1}^2{a_2}^8{a_3}^8t^3-9331200\,{a_1}^{10}{a_2}^2{a_3}^6t^3-311040000\,{a_1}^4{a_2}^6{a_3}^8t^3-186624000\,{a_1}^6{a_2}^4{a_3}^8t^3)x+364500\,c_1{a_2}^4{a_3}^3a_1d_3-182250\,c_1a_2{a_3}^6a_1d_2-364500\,c_1{a_2}^4{a_3}^3a_1c_3+1822500\,{a_1}^2d_3{a_2}^3d_2{a_3}^3-1822500\,{a_1}^2d_2{a_3}^3c_3{a_2}^3-182250\,d_1a_2{a_3}^6a_1c_2-121500\,d_1{a_2}^2{a_3}^3{a_1}^3c_3+121500\,d_1{a_2}^2{a_3}^3{a_1}^3d_3-364500\,d_1{a_2}^4{a_3}^3a_1d_3+182250\,a_1c_2{a_3}^6c_1a_2-1822500\,{a_1}^2c_2{a_3}^3d_3{a_2}^3+1822500\,{a_1}^2c_2{a_3}^3c_3{a_2}^3+121500\,{a_1}^3c_3{a_2}^2c_1{a_3}^3+607500\,{a_1}^4c_3a_2d_2{a_3}^3-121500\,{a_1}^3d_3{a_2}^2c_1{a_3}^3-607500\,{a_1}^4d_3a_2d_2{a_3}^3-455625\,{a_1}^2{c_2}^2{a_3}^6-1822500\,{a_1}^2{c_3}^2{a_2}^6+182250\,d_1a_2{a_3}^6a_1d_2+364500\,d_1{a_2}^4{a_3}^3a_1c_3-607500\,{a_1}^4c_2{a_3}^3c_3a_2+607500\,{a_1}^4c_2{a_3}^3d_3a_2-1822500\,{a_1}^2{d_3}^2{a_2}^6-18225\,{d_1}^2{a_2}^2{a_3}^6+1215000\,{a_1}^4{d_3}^2{a_2}^4-202500\,{a_1}^6{d_3}^2{a_2}^2+36450\,d_1{a_2}^2{a_3}^6c_1+911250\,{a_1}^2c_2{a_3}^6d_2+405000\,{a_1}^6c_3{a_2}^2d_3-2430000\,{a_1}^4c_3{a_2}^4d_3+3645000\,{a_1}^2d_3{a_2}^6c_3-18225\,{c_1}^2{a_2}^2{a_3}^6-455625\,{a_1}^2{d_2}^2{a_3}^6+1215000\,{a_1}^4{c_3}^2{a_2}^4-202500\,{a_1}^6{c_3}^2{a_2}^2$

$d(x,t)=(-6\,{a_1}^5a_2{a_3}^3-60\,a_1{a_2}^3{a_3}^5+20\,{a_1}^3a_2{a_3}^5+30\,a_1{a_2}^5{a_3}^3)x^6+(1200\,{a_1}^3ta_2{a_3}^5-3600\,a_1{a_2}^3t{a_3}^5-360\,{a_1}^5t{a_3}^3a_2+1800\,a_1{a_3}^3t{a_2}^5)x^3+(-675\,a_1d_2{a_3}^3+135\,d_1a_2{a_3}^3+675\,a_1c_2{a_3}^3+450\,{a_1}^3c_3a_2-450\,{a_1}^3d_3a_2-135\,c_1a_2{a_3}^3-1350\,a_1c_3{a_2}^3+1350\,a_1d_3{a_2}^3)x-$

$14400\, {a_1}^3 t^2 a_2 {a_3}^5 + 43200\, a_1 {a_2}^3 t^2 {a_3}^5 + 4320\, {a_1}^5 t^2 {a_3}^3 a_2 - 21600\, a_1 {a_3}^3 t^2 {a_2}^5$

is a rational solution to the KdV equation (1), quotient of two polynomials with the numerator of order 10 *in* x, 3 *in* t, *the denominator of degree* 12 *in* x, 4 *in* t.

In the case where all the parameters c_j and d_j are equal to 0, and the parameters a_j equal to 1 the solution can be expressed as

$$u(x,t) = 12\, \frac{x(x^9 + 43200\, t^3 + 5400\, t^2 x^3)}{(-x^6 - 60\, tx^3 + 720\, t^2)^2}.$$

6.6 Further orders

We can go on, and calculate different solutions of the hierarchy. The solutions becoming very complexes. In the case of order 4, the numerator contains 1658 terms, the denominator 2396 terms; for order 5, the numerator contain 22200 terms and the denominator 31260 terms. So, we give only the solutions with the parameters c_j and d_j equal to 0, and the parameters a_j equal to 1, at order 4 and 5 to shorten the paper.

Proposition 6.4 *The solution* v *of order* 4 *can be expressed as*

$$v(x,t) = \frac{20\, x^{18} + 2880\, tx^{15} + 453600\, t^2 x^{12} - 42336000\, t^3 x^9 - 3048192000\, t^4 x^6 + 182891520000\, t^6}{(x^{10} + 180\, tx^7 + 302400\, t^3 x)^2}.$$

v *is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree* 18 *in* x, 6 *in* t, *and denominator of degree* 20 *in* x, 6 *in* t.

Proposition 6.5 *The solution* v *of order* 5 *is given by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)^2}, \tag{47}$$

with

$n(x,t) = 30\, x^{28} + 15120\, tx^{25} + 3628800\, t^2 x^{22} + 43436736000\, t^4 x^{16} + 15362887680000\, t^5 x^{13} + 530019624960000\, t^6 x^{10} + 5530639564800000\, t^7 x^7 + 5807171543040000000\, t^8 x^4 - 4645737234432000000\, t^9 x$

$d(x,t) = -x^{15} - 420\, tx^{12} - 25200\, t^2 x^9 - 2116800\, t^3 x^6 + 254016000\, t^4 x^3 + 1524096000\, t^5$

v *is a rational solution to the KdV equation (1), quotient of two polynomials with numerator of degree* 28 *in* x, 9 *in* t, *and denominator of degree* 30 *in* x, 10 *in* t.

7 Conclusion

In this paper, we succeed to construct different types of representations of the solutions to the KdV equation. First, it was essential to express the degenerate

θ function into an explicit Fredholm determinant. The second step was to get the transformation of the Fredholm determinant into a wronskian.
I have to mention a paper of Whitham [33] in connection with this work, and I would like to thank the referee about this information. The article [33] deals with equations as the KdV equation and the representation of solutions as sum of solitons, and also the relation of these solutions with Riemann theta functions in particular. It can be compared with the solutions given in the present work expressed in terms of Fredholm determinant and wronskians.

In another approach, we have given solutions to the KdV equation in terms of elementary exponential functions. In particular, performing a passage to the limit when one parameter goes to 0 we get rational solutions to the KdV equation. So we obtain an infinite hierarchy of multi-parametric families of rational solutions to the KdV equation as a quotient of a polynomials depending on $3N$ real parameters.
But, unlike other equations such as the NLS equation [6, 7, 8, 9, 10] there are no specific structures that emerge as depending on parameters.

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