Journal of Geometry and Physics 59 (2009) 400–408

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Singularity formation in general relativistic dynamics of homogeneous scalar fields

Roberto Giambò *, Andrea Stimilli

Dipartimento di Matematica e Informatica, Università di Camerino, Italy

ARTICLE INFO

Article history:
Received 8 February 2008
Received in revised form 13 November 2008
Accepted 25 November 2008
Available online 3 December 2008

MSC:
83C75
83F05

Keywords:
General relativity
Scalar fields
Singularity
Black hole

ABSTRACT

Collapsing dynamics of a wide class of self-interacting, self-gravitating homogeneous scalar field models is analyzed. The assumptions made on the potential satisfy some general conditions allowing to show that the generic evolution is divergent in a finite time. Combining results shown here with the ones from [R. Giambó, F. Giannoni, G. Magli, J. Math. Phys. 49 (2008) 042504], dealing with sub-exponential growing potentials, allows us to obtain the same results of singularity formation for more general potentials. Moreover it turns out that these models can be completed to find radiating collapsing star models of the Vaidya type, where blackholes are generically formed.

1. Introduction and formulation of the problem

The study of singularities has always played a central role in General Relativity, since theorems by Hawking and Penrose predicted formation of singularities under some particular assumptions as formation of trapped surfaces and energy conditions [1]. Nonetheless, in general, when one imposes regular initial condition on a spacelike hypersurface and lets it evolve under laws of General Relativity is not always able to predict whether the evolution leads to a singularity, or not, both in the future and/or in the past. Indeed, on the one side, cosmological models have always been investigated to study the evolution of early universe and in this context past singularities from these examples are interpreted as models of the big-bang. On the other side, there are many interesting models of gravitational collapse in the literature that have been used by Relativists to test singularity theorems, in particular connection with Penrose Cosmic Censorship conjecture [2], which roughly asserts that a singularity arising from dynamical collapse is always hidden inside a trapped region and produces a black hole. A number of counterexamples to the conjecture have been found out since its formulation, so that now the debate mostly concerns on other features of these counterexamples yielding the so-called *naked* singularities, such as their stability with respect to perturbation of the spacetime (see for instance [3–5]). For instance, it is well known that naked singularities are stable with respect to regular initial data leading to complete gravitational collapse of spherical dust [6,7]. A generalization of this result also exists, that deals with a wider class of elastic-solid matter solutions obeying a particular equation of state [8], but the behavior in all these cases is not known if, for example, one removes the spherical symmetry assumption.

* Corresponding author.
E-mail address: roberto.giambo@unicam.it (R. Giambò).
URL: http://www.cs.unicam.it/giambo (R. Giambò).

0393-0440/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2008.11.014

In this paper we deal with scalar field spacetimes which are used in the literature both as cosmological models [9–13], and as model of gravitational collapse [14–19], and are therefore of great importance for both relativistic cosmology and astrophysics. In particular, we will consider flat Robertson–Walker metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{1}$$

where gravity minimally couples with a scalar field $\phi(t)$ under the action of a potential $V(\phi)$. The evolution of the unknown functions $a(t)$, $\phi(t)$ are driven by Einstein Field Equations that in this case take the following form (hereafter geometrized units will be used so that $8\pi G = 1$):

$$\frac{3\dot{a}^2}{a^2} = (\dot{\phi}^2 + 2V(\phi)), \tag{2}$$

$$\frac{\dot{a}^2 + 2a\ddot{a}}{a^2} = -(\dot{\phi}^2 - 2V(\phi)), \tag{3}$$

and so the weak energy condition is satisfied throughout the evolution. Moreover, these equations imply the Bianchi identity

$$T^{\mu}_{0;\mu} = -2\dot{\phi}\left(\ddot{\phi} + V'(\phi) + 3\frac{\dot{a}}{a}\dot{\phi}\right) = 0. \tag{4}$$

The system expands or collapses if $a(t)$ increases or decreases, respectively, and it is precisely the form of the potential $V(\phi)$ that strongly influences the evolution. In this paper, we will find results about singularity formation for some classes of potentials, without solving explicitly the equations, but rather through a qualitative study of Eq. (4). In particular, we will be concerned with solutions that do not exhibit a trivial evolution in any subinterval $[t_1, t_2]$, which means that there is no such interval where $\phi(t)$ is constant. In this case, it can be proven [20, Proposition 2.3] that if (a, ϕ) is a C^2 solution of (2) and (3) on some interval $[a, b]$ with ϕ nonconstant on such interval then

$$\ddot{\phi} + V'(\phi) + 3\frac{\dot{a}}{a}\dot{\phi} = 0 \tag{5}$$

holds on $[a, b]$.

Remark 1.1. It is easy to see that, if $\dot{a} \neq 0$, then (2) and (5) imply (3). Therefore, in view of (2), and denoted by ϵ the *energy density* given by

$$\epsilon = \dot{\phi}^2 + 2V(\phi), \tag{6}$$

we would like to exclude that ϵ vanishes for some $t > 0$. Since we will study situations starting from an initial collapsing state, i.e.

$$\dot{a}(0) < 0, \tag{7}$$

it can be easily proved that (7) implies, for C^2 solutions of (2) and (3) with ϕ nonconstant on any subinterval, that

$$\epsilon(t) > 0, \qquad \dot{a}(t) < 0, \qquad \forall t > 0. \tag{8}$$

Indeed, let $t_1 > 0$ such that $\epsilon(t_1) = 0$, $\epsilon(t) > 0\, \forall t \in]0, t_1[$. Using (4) we have

$$\dot{\epsilon} = -6\frac{\dot{a}}{a}\dot{\phi}^2, \tag{9}$$

so (7) ensures that ϵ starts increasing and then $\exists t_2 \in]0, t_1[$ local maximum for $\epsilon(t)$. Therefore (2) would give $\epsilon(t_2) = 0$, a contradiction. Then this kind of solutions will never re-expand.

By the aforesaid we will consider the system (2) and (5) and taking the square root in the second equation we arrive at the form

$$\ddot{\phi} + V'(\phi) = \sqrt{3}\sqrt{\epsilon}\,\dot{\phi}, \tag{10}$$

$$\frac{\dot{a}}{a} = -\sqrt{\frac{\epsilon}{3}}, \tag{11}$$

where the first equation is decoupled from the scale factor. Using (9) and (11) we also have the relation

$$\dot{\epsilon} = 2\sqrt{3}\sqrt{\epsilon}\dot{\phi}^2. \tag{12}$$

We will make the assumption (7) on the initial data, together of course with $\epsilon(0) \geq 0$. Our task will be to determine whether this collapse is defined for all times $t > 0$, or a singularity develops at a finite amount of time $t = t_s$. We will found out that, for the situation considered, the latter is always the case.

The paper is organized as follows. In Section 2 the general sufficient condition on $V(\phi)$ for singularity formation is stated, and it will be also used in the examples contained in the same section. In Section 3 a collapsing model is built, performing a junction of the scalar field interior spacetime with a suitable choice for the exterior metric. Final section is devoted to the discussion, and in particular it is observed how to deal with more general potentials starting from the classes considered in Section 2.

2. Singularity formation

In this section we are going to study forward-in-time evolution of solutions of (10) and (11) such that $\chi(0) < 0$. Let us make the positions $\phi_0 = \phi(0)$, $\dot{\phi}_0 = \dot{\phi}(0)$. Eq. (12) implies that the (monotonically increasing) energy density function is such that $\epsilon(t) \geq \epsilon_0 := \dot{\phi}_0^2 + 2V(\phi_0)$.

Remark 2.1. In the above section we needed to impose the condition $\epsilon_0 \geq 0$ on the derived system (10) and (11). Therefore, if $V(\phi)$ is not strictly positive, one can choose $\epsilon_0 = 0$ in principle. In this case, Eq. (10) is nonregular at the initial data, and this may give rise to loss of uniqueness of solutions. However, a local result of existence and uniqueness of solutions of (10) such that $\epsilon_0 = 0$ and $\epsilon(t) > 0$ as $t > 0$ can be proven [21]. Because of this fact we will henceforth suppose $\epsilon_0 > 0$.

Here we are going to state a sufficient condition for singularity formation, that will be applied in the examples that we are going to review next.

Proposition 2.2. *Let $\phi(t)$ be a solution of (10) such that $\epsilon_0 > 0$, and let $\mathbb{I}$ be its maximal interval of definition. Call $t_s = \sup \mathbb{I} \in]0, +\infty]$, and suppose that*

$$\limsup_{t \to t_s^-} \frac{2V(\phi(t))}{\dot{\phi}(t)^2} < +\infty. \tag{13}$$

Then, the solution cannot be prolonged indefinitely on the right, i.e. $t_s \in \mathbb{R}$, and

$$\lim_{t \to t_s} \epsilon(t) = +\infty. \tag{14}$$

Moreover, suppose that

$$\liminf_{t \to t_s^-} \frac{2V(\phi(t))}{\dot{\phi}(t)^2} > -1, \tag{15}$$

$$\dot{\phi}(t) \neq 0 \quad \textit{eventually true}, \tag{16}$$

$$\lim_{t \to t_s^-} |\phi(t)| = +\infty. \tag{17}$$

Then, the function $a(t)$ solution of (11) is such that

$$\lim_{t \to t_s^-} a(t) = 0. \tag{18}$$

Proof. To prove (14), let us call

$$\rho(t) = \frac{2V(\phi(t))}{\dot{\phi}(t)^2}. \tag{19}$$

Then (12) implies that

$$\dot{\epsilon} = 2\sqrt{3}\epsilon^{3/2} \frac{1}{1+\rho}, \tag{20}$$

and if (13) holds then there exists a positive constant ℓ such that $\dot{\epsilon} > \ell\epsilon^{3/2}$, that means that $\epsilon(t)$ is not defined for all $t > 0$, and moreover (14) must hold.

To prove (18), observe that (15) implies $\epsilon(t) = \dot{\phi}(t)^2(1+\rho) \geq k^2\dot{\phi}(t)^2$, for some positive constant k, from which $\sqrt{\epsilon(t)} \geq k|\dot{\phi}(t)|$. Integrating this relation and using (16) and (17) we obtain

$$\lim_{t \to t_s^-} \int_0^t \sqrt{\epsilon(\tau)} \mathrm{d}\tau = +\infty. \tag{21}$$

Therefore (11) implies

$$\ln \frac{a(t)}{a(0)} = \int_0^t \frac{\dot{a}(\tau)}{a(\tau)} \mathrm{d}\tau = -\frac{1}{\sqrt{3}} \int_0^t \sqrt{\epsilon}(\tau)\, \mathrm{d}\tau,$$

from which (18) follows. □

Condition (13) basically states that the solution asymptotically behaves as in the free massless case, i.e. when $V \equiv 0$. In the following, we are going to see some examples when conditions to apply Proposition 2.2 will be proved to hold. Notice that one can prove (13) without knowing that $t_s < +\infty$, which comes out as a consequence. Moreover, in some of the examples (some of the) conditions (15) and (17) will be necessary to prove (13) later.

2.1. Increasing and positive potential

Assumption 2.3. We consider a potential $V(\phi) \in \mathcal{C}^1(]-\infty,\alpha],\mathbb{R})$ such that

$$V(\phi) \geq 0, \quad V'(\phi) \geq 0, \qquad \forall \phi \leq \alpha \tag{22}$$

where we take the initial data for the scalar field as follows:

$$\phi_0 \leq \alpha, \quad \dot\phi_0 = -v_0 < 0. \tag{23}$$

We are going to show that this situation leads to singularity formation in a finite amount of comoving time.

Lemma 2.4. *Let $\mathbb{I}$ be the (possibly unbounded) maximal interval of definition of (10) with initial data (23). Then, $\forall t \in \mathbb{I}$, $\phi(t) \leq \phi_0$, $\dot\phi(t) \leq -v_0 < 0$, $\ddot\phi(t) < 0$.*

Proof. Since $\dot\phi_0$ and $V'(\phi_0) > 0$, from (10), $\ddot\phi(0) < 0$ and then $\dot\phi$ is decreasing in a right neighborhood of $t = 0$, yielding a negative $\dot\phi$. Using the same argument it is easy to show that the same happens along the whole solution until its maximal extension on the right.

So $\ddot\phi(t) < 0 \quad \forall t \in \mathbb{I}$, that is, $\dot\phi(t)$ decreasing $\forall t \in \mathbb{I}$; but then $\dot\phi(t) \leq -v_0 < 0 \quad \forall t \in \mathbb{I}$ and so $\phi(t)$ decreasing $\forall t \in \mathbb{I}$. So we can conclude that $\phi(t) \leq \phi_0$, $\forall t \in \mathbb{I}$. □

Before stating next result, we need to recall the following definition.

Definition 2.5. Given the ordinary differential equation (ODE) $y'(t) = F(t, y(t))$, we say that $v(t)$ is a *supersolution* of the ODE if $\dot v(t) \geq F(t, v(t))$.

A standard comparison result in ODE theory ensures that if $v(t)$ is a supersolution of the ODE $y'(t) = F(t, y(t))$ for $t > 0$, and $y(t)$ is a solution such that $v(0) \geq y(0)$ then $v(t) \geq y(t)$ for $t \geq 0$.

The following crucial lemma shows that ϕ satisfies the sufficient conditions for singularity formation stated in Proposition 2.2.

Lemma 2.6. *There exists $t_s \in \mathbb{R}$ such that*

$$\lim_{t\to t_s} \dot\phi(t) = -\infty.$$

In particular, (13), (15) and (16) hold.

Proof. Let $v(t) := -\dot\phi(t)$. Then, by Lemma 2.4, $v(t) > 0$, $\dot v(t) > 0$, and using (6), (10) and (22), it is

$$\dot v(t) = -\ddot\phi(t) = V'(\phi(t)) - \sqrt{3\epsilon(t)}\dot\phi(t) = V'(\phi(t)) + \sqrt{3}\sqrt{v^2(t) + 2V(\phi(t))}\,v(t) \geq \sqrt{3}v^2(t).$$

So $v(t)$ is a supersolution of the ODE $\dot y(t) = \sqrt{3}y^2(t)$ and then considering the Cauchy problem

$$\begin{cases} \dot y(t) = \sqrt{3}y^2(t) \\ y(0) = v_0 = -\dot\phi_0 \end{cases}$$

it must be

$$v(t) \geq \frac{1}{\frac{1}{v_0} - \sqrt{3}t} \Rightarrow -\dot\phi(t) \geq \frac{1}{-\frac{1}{\dot\phi_0} - \sqrt{3}t} \Rightarrow \dot\phi(t) \leq \dot\phi_0 \frac{1}{\sqrt{3}\dot\phi_0 t + 1} = \dot\phi_0 \frac{1}{1 - \sqrt{3}v_0 t}.$$

The above fact shows that $t_s = \sup \mathbb{I} \leq \frac{1}{\sqrt{3}v_0}$. Moreover, $\dot\phi$ must diverge as $t \to t_s^-$, otherwise it could be prolonged on a right neighborhood of t_s, against the assumption of maximality. In particular, since $V(\phi(t))$ must be bounded, it follows that $\lim_{t\to t_s^-} \rho(t) = 0$. □

From the above lemma, and first part of Proposition 2.2, it follows that a singularity forms in a finite amount of comoving time.

Lemma 2.7. *The solution $\phi(t)$ satisfies (17).*

Proof. Let us suppose by contradiction:

$$\lim_{t \to t_s^-} \phi(t) = \bar{\phi} \in \mathbb{R}.$$

Then $\phi(t)$ takes values in a compact K, so $V(\phi)$ also takes values in a compact K', that is, $V(\phi(t))$ is bounded in a left neighborhood of t_s; from (6) there must exists a positive constant M such that:

$$\sqrt{\epsilon(t)} \le |\dot{\phi}(t)| + M = -\dot{\phi}(t) + M,$$

from which, integrating, we have

$$\int_0^t \sqrt{\epsilon(\tau)}\mathrm{d}\tau \le \int_0^t -\dot{\phi}(\tau)\mathrm{d}\tau + Mt = -\phi(t) + \phi(0) + Mt \le -\phi(t) + \phi_0 + Mt_s.$$

If we show that the integral on the left-hand side diverges as $t \to t_s^-$ we have a contradiction.

Equation (10) implies:

$$\sqrt{3}\int_0^t \sqrt{\epsilon(\tau)}\mathrm{d}\tau = \int_0^t \frac{\ddot{\phi}(\tau)}{\dot{\phi}(\tau)}\mathrm{d}\tau + \int_0^t \frac{V'(\phi(\tau))}{\dot{\phi}(\tau)}\mathrm{d}\tau = \ln\frac{\dot{\phi}(t)}{\dot{\phi}_0} + \int_0^t \frac{V'}{\dot{\phi}}\mathrm{d}\tau.$$

By Lemma 2.7, the first term on the right-hand side above diverges to $+\infty$. Moreover, since we are supposing that $\phi(t)$ has a finite limit for $t \to t_s^-$, also $V'(\phi(t))$ is bounded in a right neighborhood of t_s; since $\dot{\phi}(t)$ diverges, it must be

$$\frac{V'}{\dot{\phi}} \to 0 \quad \text{if } t \to t_s^- \Rightarrow \int_0^t \frac{V'}{\dot{\phi}}\mathrm{d}\tau < +\infty,$$

so:

$$\lim_{t \to t_s^-} \sqrt{3}\int_0^t \sqrt{\epsilon}\mathrm{d}\tau = +\infty \Rightarrow \lim_{t \to t_s^-} \int_0^t \sqrt{\epsilon}\mathrm{d}\tau = +\infty.$$

But this is a contradiction, so:

$$\bar{\phi} \notin \mathbb{R} \Rightarrow \lim_{t \to t_s^-} \phi(t) = -\infty. \quad \square$$

From the results of Lemma above, we can apply the second part of Proposition 2.2 to find that $a(t)$ converges to zero as $t \to t_s^-$.

2.2. *Bounded and positive potential*

Now let us focus on the case given by the following assumption.

Assumption 2.8. We consider a nonzero potential $V(\phi) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$0 \le V(\phi) \le \sup_{\mathbb{R}} V(\phi) =: V^* \tag{24}$$

and we take initial data $\phi_0, \dot{\phi}_0$ for the scalar field such that

$$\epsilon_0 \ge 2V^*. \tag{25}$$

Under the above hypotheses, we can again show that $\dot{\phi}(t)$ diverges in a finite amount of comoving time.

Let us define the function $J(t)$ as

$$J(t) = \int_0^t \dot{\phi}^2(\tau)\mathrm{d}\tau. \tag{26}$$

Then, from (6) and (11), $J(t)$ solves the following Cauchy problem:

$$\begin{cases} \dot{J}(t) = [\sqrt{3}J(t) + \sqrt{\epsilon_0}]^2 - 2V(\phi(t)) \\ J(0) = 0. \end{cases} \tag{27}$$

We will now show a similar result to Lemma 2.6 of previous case.

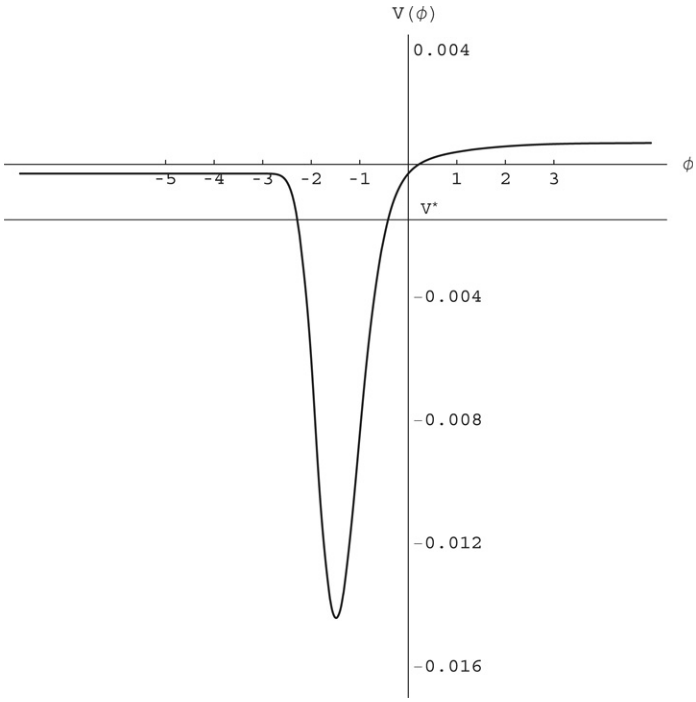

Fig. 1. Cyclic potential corresponding to parameters $V_0 = 10^{-3}$, $m_1 = 0.2$, $m_2 = 20/21$.

Lemma 2.9. *There exists $t_s > 0$ such that*

$$\lim_{t \to t_s^-} J(t) = +\infty. \tag{28}$$

In particular, $\lim_{t\to t_s^-} |\dot{\phi}(t)| = +\infty$ *and so* (13), (15) *and* (16) *hold.*

Proof. Using (24) in (27), the nonnegative function $J(t)$ is a supersolution of the Cauchy problem:

$$\begin{cases} \dot{y}(t) = [\sqrt{3}y(t) + \sqrt{\epsilon_0}]^2 - 2V^* \\ y(0) = 0 \end{cases} \tag{29}$$

Denoting $Q := \frac{\sqrt{\epsilon_0} - \sqrt{2V^*}}{\sqrt{\epsilon_0} + \sqrt{2V^*}}$, then

$$J(t) \geq \frac{Q(\sqrt{\epsilon_0} + \sqrt{2V^*})\mathrm{e}^{2t\sqrt{6V^*}} - (\sqrt{\epsilon_0} - \sqrt{2V^*})}{\sqrt{3}(1 - Q\mathrm{e}^{2t\sqrt{6V^*}})}. \tag{30}$$

Moreover, (24) and (25) hold, then $0 < Q < 1$, so the right-hand side function in (30) diverges for $t^* = -\frac{1}{2\sqrt{6V^*}} \log Q > 0$, that is, there exists $t_s \in]0, t^*[$ such that $J(t)$ satisfies (28). □

Also in this case we can state that a singularity forms in a finite amount of comoving time. Now arguments from the above Section, in particular Lemma 2.7, can be almost repeated, only paying attention to the sign of $\dot{\phi}(t)$, to show also (17), and so, again, $a(t) \to 0$ as $t \to t_s^-$.

2.3. Cyclic potential

Cyclic potentials are of relevant physical interest in cosmology; they are named after the possibility of a cyclic universe proposed in [22] (see also the references cited therein), where the universe undergoes a periodic sequence of expansion and contraction. Loop quantum gravity modifications in this framework have been considered in [23], where the following potential is considered: $V(\phi) = V_0(1 - \mathrm{e}^{-\phi/m_1})\mathrm{e}^{\mathrm{e}^{-\phi/m_2}}$, with V_0, m_1, m_2 positive constants (Fig. 1). In this section we will consider cyclic potentials as an extension of the case analyzed in [21]. In that work the potential V satisfies the following conditions:

1. $V(\phi)$ is a C^2 function bounded from below;

2. The critical points of V are isolated. They are either minimum points or nondegenerate maximum points;
3. $\exists V^*$ positive constant such that:
   - the set $B := \{\phi \in \mathbb{R} : V(\phi) \leq V^*\}$ is bounded;
   - $\phi \geq \sup B \Rightarrow V'(\phi) > 0, \phi \leq \inf B \Rightarrow V'(\phi) < 0$;
4. The function $u(\phi) := \frac{V'(\phi)}{2\sqrt{3}V(\phi)}$ satisfies the following conditions:
   - $\limsup_{\phi\to\pm\infty} |u(\phi)| < 1$;
   - $\exists \lim_{\phi\to\pm\infty} u'(\phi)(= 0)$.

Under the above hypotheses, in [21] it is shown that the set of initial data of (10) and (11), except for a null set, is such that a singularity forms in a finite amount of comoving time.

Let us now consider the following extension of the case just described. The first two conditions on the potential V are the same, while the other two conditions are replaced with the following ones:

3′. $\exists V^* \in \mathbb{R}$ (possibly negative) constant such that:
   - the set $B := \{\phi \in \mathbb{R} : V(\phi) \leq V^*\}$ is bounded;
   - $\phi \geq \sup B \Rightarrow V'(\phi) > 0, \phi \leq \inf B \Rightarrow V'(\phi) < 0$;

4′. If conditions on $u(\phi)$ are not satisfied if $\phi \to -\infty$, the potential V must satisfy the following conditions:
   - $\lim_{\phi\to-\infty} V(\phi) = V_\infty \leq 0$;
   - $\exists \lim_{\phi\to-\infty} V'(\phi)\ (= 0)$.

Remark 2.10. Same situation in 4′ can happen if $\phi \to +\infty$; for simplicity we will not consider this further extension.

Lemma 2.11. *The solution $\phi(t)$ satisfies (16) and (17).*

Proof. The first two conditions on potential are enough, and the same arguments of [21] can be used, to prove that $\phi(t)$ is unbounded, that is, $\limsup_{t\to t_s^-} |\phi(t)| = +\infty$. We will actually show that $\phi(t)$ admits limit if $t \to t_s^-$. Indeed, let us suppose by contradiction that $\phi(t)$ is not monotonic in its interval of definition, that is, $\phi(t)$ has an oscillatory character. That implies that $\exists t_k$ sequence of time instants such that $\dot\phi(t_k) = 0$. So we have that $V(\phi(t_k)) > 0$, indeed:

$$0 < \epsilon_0 < \epsilon(t_k) = \dot\phi^2(t_k) + 2V(\phi(t_k)) = 2V(\phi(t_k)).$$

Moreover, we can consider V^* a negative constant (otherwise we can refer to case in [21]), and so it is:

$$2V(\phi(t_k)) > 0 > 2V^* \Rightarrow V(\phi(t_k)) > V^* \Rightarrow \phi(t_k) \notin B.$$

Then, recalling that $V(\phi)$ is eventually negative as $\phi \to +\infty$, we can conclude that $\phi(t_k) \geq sup\, B$, from which $V'(\phi(t_k)) > 0$ follows. Using (10), we also have: $\ddot\phi(t_k) = -V'(\phi(t_k)) + \sqrt{3}\sqrt{\epsilon(t_k)}\dot\phi(t_k) = -V'(\phi(t_k)) < 0$. This means that all time instants t_k are maximum points of $\phi(t)$, but this is a contradiction.

Then $\dot\phi(t)$ is eventually nonzero and this implies that there exists $\lim_{t\to t_s^-} |\phi(t)|$ and it is $+\infty$.

Lemma 2.12. *The solution $\phi(t)$ satisfies (13) and (15). In particular, a singularity forms in a finite amount of comoving time t_s, and $a(t) \to 0$ as $t \to t_s^-$.*

Proof. The case in which $\lim_{\phi\to t_s} \phi(t) = +\infty$ is analyzed in [21], then here we consider the following situation:

$$\dot\phi(t) < 0 \quad \text{and} \quad \lim_{t\to t_s^-} \phi(t) = -\infty.$$

We have two different cases according to limit in 4′.

- If $V_\infty = 0$, recalling $0 < \epsilon_0 < \epsilon(t) = \dot\phi^2(t) + 2V(\phi(t))$, then $\dot\phi(t)^2$ is strictly bounded away from zero, and so $\lim_{t\to t_s^-} \rho(t) = 0$.
- If $V_\infty < 0$, we can prove the same result. First of all let us remark that $\ddot\phi(t)$ is eventually negative; indeed, since $\epsilon(t)$ is increasing and $\dot\phi$ is negative, we have:

  $$\dot\phi^2 = \epsilon - 2V > \epsilon_0 - 2V \Rightarrow |\dot\phi| > \sqrt{\epsilon_0 - 2V} \Rightarrow -\dot\phi > \sqrt{\epsilon_0 - 2V} \Rightarrow \dot\phi < -\sqrt{\epsilon_0 - 2V};$$

  this implies:

  $$\ddot\phi = -V'(\phi) + \sqrt{3}\sqrt{\epsilon}\dot\phi < -V'(\phi) - \sqrt{3}\sqrt{\epsilon_0}\sqrt{\epsilon_0 - 2V}.$$

  Recalling 4′, the right-hand side of the inequality above goes to $-\sqrt{3}\sqrt{\epsilon_0}\sqrt{\epsilon_0 - 2V} < 0$ if $t \to t_s^-$, so $\ddot\phi(t)$ is eventually smaller than zero. But then $\dot\phi(t)$ is eventually monotonically decreasing, so $\exists \lim_{t\to t_s^-} \dot\phi = m < 0$.

  Let us suppose that $m \in \mathbb{R}$; then: $\lim_{t\to t_s^-} \epsilon(t) = m^2 + 2V_\infty \in \mathbb{R}^+$. On the other hand we have:

  $$\frac{\dot\phi^2}{\epsilon} \xrightarrow{t\to t_s^-} \frac{m^2}{m^2 + 2V_\infty} > 0 \Rightarrow \exists k > 0 : \frac{\dot\phi^2}{\epsilon} > k.$$

But, since it is $\dot\epsilon = 2\sqrt{3}\sqrt{\epsilon}\,\dot\phi^2 = 2\sqrt{3}\sqrt{\epsilon}\,\epsilon(\dot\phi^2/\epsilon) > 2\sqrt{3}k\epsilon^{3/2}$, using comparison theorems for ODE it easily follows that $\lim_{t\to t_s^-}\epsilon(t) = +\infty$, in contradiction with the aforesaid. So we can conclude that $m = -\infty$ and that also in this case $\rho = \frac{2V}{\dot\phi^2} \to 0$ if $t \to t_s^-$. $\square$

3. Gravitational collapse models

We now show how models of collapsing objects composed by homogeneous scalar fields can be constructed, matching the interior solution (1) with the so-called *generalized Vaidya solution*

$$\mathrm{d}s^2_{\mathrm{ext}} = -\left(1 - \frac{2M(U,Y)}{Y}\right)\mathrm{d}U^2 - 2\mathrm{d}Y\mathrm{d}U + Y^2\mathrm{d}\Omega^2, \tag{31}$$

where M is an arbitrary (positive) function (we refer to [24] for a detailed physical discussion of this spacetime, which is essentially the spacetime generated by a radiating fluid). The matching is performed along the hypersurface Σ defined by $r = r_b$ constant, where $r^2 = x^2 + y^2 + z^2$ in the coordinate system of the interior metric (1). The Israel junction conditions at the matching hypersurface read as follows:

$$M(U(t), Y(t)) = \frac{1}{2}r_b^2 a(t)\dot a(t)^2, \tag{32}$$

$$\frac{\partial M}{\partial Y}(U(t), Y(t)) = \frac{1}{2}r_b^3(\dot a(t)^2 + 2a(t)\ddot a(t)), \tag{33}$$

where the functions $(Y(t), U(t))$ satisfy

$$Y(t) = r_b a(t), \qquad \frac{\mathrm{d}U}{\mathrm{d}t}(t) = \frac{1}{1 + \dot a(t) r_b}. \tag{34}$$

The two equations (32) and (33) are equivalent to require that Misner–Sharp mass M is continuous and $\frac{\partial M}{\partial U} = 0$ on the junction hypersurface Σ.

The endstate of the collapse of these models is analyzed in the following theorem.

Proposition 3.1. *Let $\phi(t)$ be a solution of (10) that forms a singularity in a finite amount of comoving time (i.e. there exists $t_s \in \mathbb{R}$ such that the solution can be prolonged up to t_s and not beyond, and $\lim_{t\to t_s}\epsilon(t) = +\infty$). If*

$$\limsup_{t\to t_s^-}\rho(t) < 2 \tag{35}$$

then the scalar field model matched with the generalized Vaidya solution (31) collapses to a black hole.

Proof. We will make use of the result in [16, Theorem 5.2], that states that if, in the approach to the singularity, the quantity $\dot a^2 = a^2\epsilon$ is bounded, then the apparent horizon cannot form, and the singularity is naked. It is indeed easy to check that the equation of the apparent horizon for the metric (1) is given by $r^2\dot a(t)^2 = 1$. If $\dot a$ is bounded, one can choose the junction surface $r = r_b$ sufficiently small such that $\dot a^2(t) < \frac{1}{r^2}$, $\forall (t, r) \in [0, t_s] \times [0, r_b]$, and so $\left(1 - \frac{2M}{R}\right)$ is bounded away from zero near the singularity. As a consequence, one can find in the exterior portion of the spacetime (31), null radial geodesics which meet the singularity in the past, and therefore the singularity is naked. Otherwise, if $\dot a^2$ is unbounded, the trapped region forms and the collapse ends into a black hole. Now, field equations imply

$$\ddot a = -\frac{\dot a^2}{a}\left(\frac{2-\rho}{1+\rho}\right), \tag{36}$$

and therefore, recalling (35) and weak energy condition (which implies $1 + \rho \geq 0$), there exists a positive constant C such that $\ddot a \leq -C\frac{\dot a^2}{a}$ eventually holds in a left neighborhood of t_s, say $[t_0, t_s[$, where $\dot a$ is decreasing. It follows that $\forall t \in]t_0, t_s[$, $\dot a(t) \leq \dot a(t_0)$, which is negative by hypotheses. Then

$$\dot a(t) - \dot a(t_0) = \int_{t_0}^t \ddot a(\tau)\mathrm{d}\tau \leq -\int_{t_0}^t \frac{\dot a(\tau)^2}{a(\tau)}\mathrm{d}\tau \leq -\dot a(t_0)\int_{t_0}^t \frac{\dot a(\tau)}{a(\tau)}\mathrm{d}\tau,$$

that diverges to $-\infty$, and so $\dot a(t)$ is unbounded. $\square$

Remark 3.2. All the examples from Sections 2.1–2.3 shows that $\rho(t) \to 0$ as $t \to t_s^-$, and so the above Proposition applies, to find that the model exhibited here collapses completely to a black hole.

4. Conclusions

In this paper, we have studied collapsing evolution of scalar field under some conditions on the potential function $V(\phi)$. In the cases studied, conditions on the initial data were also imposed to ensure that the scalar field diverges either negatively or positively. For instance, in the example sketched in Section 2.1 conditions (23) ensure that $\phi \le \phi_0$ and therefore we do not need to impose conditions on $V(\phi)$ as $\phi \to \phi_0$. However, we can combine the results obtained here with the ones studied in [21] to obtain the same results dropping the assumptions on the initial data – except, of course, $\epsilon_0 > 0$ which ensures weak energy condition. To give an example, consider for instance the potential $V(\phi) = \lambda^2 + e^{\kappa^2 \phi}$. If $\dot{\phi}(t_0) < 0$ for some t_0 one can set $\alpha = \phi_0 = \phi(t_0)$ and apply results from Section 2.1. Otherwise $\dot{\phi}(t) > 0 \forall t$, which allow us to use results from [21]: indeed, provided $k^2 < 2\sqrt{3}$, the potential satisfies condition (4) stated at the beginning of Section 2.3. Therefore, up to a zero measured set of initial data, singularity always forms in a finite time for this potential.

Observe that the situation studied here can also be interpreted in terms of expanding solutions, where one wants to examine *time backwards* evolution. In other words, the particular form of (10) and (11) allows us to reverse time and re-read the future evolution of a collapsing solution as the past evolution of an expanding solution: then, one can conclude that for the class of potential studied in the present paper expanding scalar fields are always generated by an initial big-bang. Notice that this result cannot be deduced simply by ordinary singularity theorems in cosmology, since the solution under exam do not satisfy in principle the strong energy condition, but only the weak energy condition.

References

[1] S.W. Hawking, R. Penrose, Proc. Roy. Soc. Lond. A 314 (1970) 529.
[2] R. Penrose, Nuovo Cimento, (1969) (reprinted Gen. Rel. Grav. 34, 1141 (2002)).
[3] F. Giannoni, On gravitational collapse in spherical symmetry, in: Proceedings of Workshop Dynamics and Thermodynamics of Blackholes and Naked Singularities, online version on http://www.mate.polimi.it/bh, Politecnico di Milano, 2004.
[4] P.S. Joshi, Gravitational Collapse and Spacetime Singularities, Cambridge University Press, 2007.
[5] G. Magli, AIP Conf. Proc. 751 (2005) 101–107.
[6] P.S. Joshi, I.H. Dwivedi, Phys. Rev. D 47 (1993) 5357.
[7] B. Nolan, F. Mena, Class Quantum Grav. 18 (2001) 4531.
[8] R. Giambò, F. Giannoni, G. Magli, P. Piccione, Commun. Math. Phys. 235 (2003) 545.
[9] S. Foster, Class. Quant. Grav. 15 (1998) 3485–3504.
[10] J. Miritzis, Class. Quantum Grav. 20 (14) (2003) 2981–2990.
[11] J. Miritzis, J. Math. Phys. 46 (2005) 082502.
[12] A.D. Rendall, Class. Quant. Grav. 21 (2004) 2445–2454.
[13] A.D. Rendall, Class. Quant. Grav. 24 (2007) 667–678.
[14] D. Christodoulou, Ann. Math. 140 (1994) 607.
[15] D. Christodoulou, Ann. Math. 149 (1999) 183.
[16] R. Giambò, Class. Quantum Grav. 22 (2005) 1–11.
[17] R. Goswami, P.S. Joshi, P. Singh, Phys. Rev. Lett. 96 (2006) 031302.
[18] T. Hertog, G.T. Horowitz, K. Maeda, Phys. Rev. Lett. 92 (2004) 131101.
[19] T. Hertog, G.T. Horowitz, K. Maeda, arXiv:gr-qc/0405050v2.
[20] R. Giambò, F. Giannoni, G. Magli, J. Math. Phys. 47 (2006) 112505.
[21] R. Giambò, F. Giannoni, G. Magli, J. Math. Phys. 49 (2008) 042504.
[22] P.J. Steinhardt, N. Turok, Phys. Rev. D 65 (2002) 126003.
[23] M. Bojowald, R. Maartens, P. Singh, Phys. Rev. D 70 (2004) 083517.
[24] A. Wang, Y. Wu, Gen. Rel. Grav. 31 (1999) 107.