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A gauge-theoretic description of μ-prolongations, and μ-symmetries of differential equations

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ABSTRACT

We consider generalized (possibly depending on fields as well as on space–time variables) gauge transformations and gauge symmetries in the context of general – that is, possibly non variational nor covariant – differential equations. In this case the relevant principal bundle admits the first jet bundle (of the phase manifold) as an associated bundle, at difference with standard Yang–Mills theories. We also show how in this context the recently introduced operation of μ-prolongation of vector fields (which generalizes the λ-prolongation of Muriel and Romero), and hence μ-symmetries of differential equations, arise naturally. This is turn suggests several directions for further development.

1. Introduction

The analysis and use of symmetry properties of differential equations [1–4] is by now recognized as the most powerful general method to attack nonlinear problems, and widely used not only in Physics (where this theory was first extensively applied) but also in Applied Mathematics and Engineering.

The original theory of Lie-point symmetries was over the years generalized in several directions [1–5]. All these make use of the fact that once we know how a vector field acts on independent as well as on dependent variables (i.e. fields), we also know how it acts on field derivatives; the lift of a transformation from the extended phase manifold (space–time variables and fields; with due account of the relevant side – e.g. boundary – conditions, this is the phase bundle) to its action on field derivatives is known in the mathematical literature as the *prolongation* operation.

Despite its success, the symmetry theory of differential equations was until recently not able to cope with certain very simple problems which could be explicitly integrated, yet seemed to have no symmetry underlying this integrability (see e.g. [6,3] and references therein).

Muriel and Romero [6] were able to solve this puzzle in analytical terms by considering a modified prolongation operation, and thus a new kind of symmetries, for scalar ODEs and then also for systems of these [7,8] (see also [9] in this respect). These depend on the choice of a $\mathcal{C}^\infty$ function, denoted λ in their papers; referring to this fact the new kind of symmetries are known as $\mathcal{C}^\infty$-symmetries, or λ-*symmetries*.

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The geometrical meaning of λ-prolongations was then clarified in [10], by means of the classical theory of characteristics of vector fields. A different geometrical characterization, in the language of Cartan exterior differential ideals (ideals of differential forms), was proposed in [11].

This opened the way for generalizing λ-prolongations and λ-symmetries to the framework of (single, or systems of) PDEs [11]; in this case the central object is a matrix-valued differential one-form $\mu = \Lambda_i \mathrm{d}x^i$ (the Λ_i being $\mathcal{C}^\infty$ matrix functions satisfying the horizontal Maurer-Cartan equation), and these are therefore called *μ-prolongations* and *μ-symmetries*.

It was then realized that for each vector field Y obtained as the μ-prolongation of some vector field X, there is a vector field $\widetilde{Y}$, locally (and globally under certain conditions) gauge-equivalent to Y, obtained as the standard prolongation of a vector field $\widetilde{X}$, locally (and globally under certain conditions) gauge equivalent to X; see [12] for details.[1]

This result calls for a more complete geometrical understanding of its origin, and shows that gauge transformations also play a role out side of the well-known framework of Yang–Mills theories [15–20], and actually also for *non-variational problems* and for *non-invariant equations* (or *non-covariant equations* in physical language). The first task is to extend the formalism so to fully include the gauge variables, not just leaving them to the role of external parameters.[2]

In this note we provide a formalism including gauge variables, i.e. set the problem in an augmented bundle; and give a precise formulation of the μ-prolongation operation in terms of such an augmented bundle and correspondingly an enlarged set of variables.

The reader should be warned that this geometrical understanding does *not* – at the present stage – correspond to a substantial computational advantage; thus the geometric construction presented here has – at the present stage – interest only *per se*, i.e. for the understanding of the Geometry behind twisted prolongations. On the other hand, our work also suggests how to extend the applications of μ-symmetries, and how to further generalize them; these matters will however only be shortly mentioned in our final discussion, see Section 9, deferring the implementation of such suggestions to a later time.

We will assume the reader to be familiar with basic jet-theoretic material and with the standard theory of symmetry of differential equations, as given e.g. in [1–4]. We will freely use standard multi-index notation; thus a multi-index $J = (j_1, \ldots, j_k)$, where $j_i \in \mathbf{N}$, will have order $|J| = j_1 + \cdots + j_k$; and for the same multi-index J we will have $D_J = D_1^{j_1} \cdots D_k^{j_k}$.

We use the formalism of evolutionary representatives of vector fields [1–4], which provide an action on sections of a bundle via (generalized) vertical vector fields describing the action of general (proper) vector fields in the bundle. A discussion without resorting to evolutionary representatives would be equivalent, but would require more involved computations; on the other hand, our discussion immediately extends (with the standard cautions [1–4] and obvious modifications) to the full class of generalized vertical vector fields.

2. Underlying geometry

In this section we will first set some notation regarding standard constructions, and then introduce the **gauge bundles**, which are the essential part of our construction.

Given a fiber bundle $\mathcal{P}$, the space of sections in it will be denoted as $\Gamma(\mathcal{P})$. The algebra of vector fields in $\mathcal{P}$ will be denoted as $\mathcal{X}(\mathcal{P})$, and that of vertical (with respect to the bundle projection) vector fields as $\mathcal{X}_v(\mathcal{P})$.

2.1. Bundles, differential equations, and symmetry

When dealing with differential equations, the independent variables will be denoted as $x \in B$; dependent variables as $u \in U$. Here B and U are smooth manifolds. We will use local coordinates $\{x^1, \ldots, x^m\}$ in B and $u = \{u^1, \ldots, u^n\}$ in U. We stress that all of our considerations will be local.

We will then consider the bundle (M, π, B) with fiber $\pi^{-1}(x) = U$; thus its total space is $M \simeq B \times U$. The bundle M will be our **phase bundle**.[3]

Differential equations Δ of order k identify a submanifold in the total space of the Jet bundle $J^k M$, the *solution manifold* $S_\Delta \subset J^k M$.

Sections of M are naturally prolonged to sections of $J^k M$; the function $u = f(x)$ is a solution to Δ if and only if the prolongation of the corresponding section $\sigma_f = (x, f(x))$ in $\Gamma(M)$ to a section in $\Gamma(J^k M)$, call it $\sigma_f^{(k)}$, is a submanifold of S_Δ.

Similarly, given a vector field X acting in M, there is a natural prolongation of X to a vector field $X^{(k)}$ in $J^k M$. The vector field X is a symmetry of Δ if and only if it maps solutions into solutions; equivalently, if $X^{(k)} : S_\Delta \to \mathrm{T}S_\Delta$.

We stress that a different fibred structure is also possible for the jet manifolds $M^{(k)} = J^k M$. These can also be seen as bundles over M; we will denote the bundle maps for these structures as σ_k, so we have $(J^k M, \sigma_k, M)$. The compatibility between these two structures is given by $\pi_k = \pi \circ \sigma_k$.

1 This also has some interesting consequences in the frame of variational problems: it is possible to extend Noether theory to λ and μ symmetries; see [13,14].

2 In order to avoid any confusion, we stress that here we consider gauge transformations more general than those considered in standard Yang–Mills theory: (1) these may depend on the field themselves and not only on space–time variables; (2) we allow nonlinear actions on the fields. See the discussion later on in this paper.

3 Physically, B should be thought as a region of space–time, and U as a manifold (possibly a space) in which the field φ takes values; the global structure of the bundle also carries information on the boundary conditions the fields are subject to at the boundary (if any) ∂B of B.

Remark 1. As already mentioned, one can consider – besides the natural prolongation operation for vector fields mentioned above – some "twisted" (or "deformed") prolongation operations, known in the literature as "λ-prolongation" (when the deformation is related to a point-dependent scale factor) or "μ-prolongation" (when the deformation is related to a general point-dependent linear map). These were first introduced by Muriel and Romero [6], and quite surprisingly turn out to be "as useful as the natural ones" in analyzing differential equations. The purpose of this paper is to investigate the geometrical structures behind this seemingly "unreasonable effectiveness of twisted prolongations". □

2.2. Prolongation of vector fields in J^kM

The prolongation of vector fields from M to J^kM goes essentially through consideration of partial derivatives of functions corresponding to a general section in M and its transformed under the action of the vector field X. As well known, if $\sigma_f = \{(x, u) : u = f(x)\}$, then the one-parameter group generated by $X = \varphi^a(x, u)(\partial/\partial u^a) + \xi^i(x, u)(\partial/\partial x^i)$ maps σ_f into $\sigma_{\widetilde{f}}$ with $\widetilde{f}^a(x) = f^a(x) + \varepsilon[\varphi^a(x, f(x)) - \xi^i(x, f(x))u^a_i]$. Properties of transformation for partial derivatives are readily derived from this expression.

In the case of a vertical vector field (including evolutionary representatives of general vector fields) the prolongation formula is specially simple: in multi-index notation, the vector field $X = \eta^a(\partial/\partial u^a)$ is prolonged to $Y = \eta^a_J(\partial/\partial u^a_J)$, with $\eta^a_J = D_J\eta^a$. In particular we have in recursive form

$$\eta^a_{J,i} = D_i\,\eta^a_J. \tag{1}$$

2.3. μ-prolongations in J^kM

Let us briefly recall how μ-prolongations are defined, restricting again to vertical vector fields for the sake of simplicity (see [11,12] for the general case). We stress that here we work in M and J^kM, and *not* in the augmented (jet) bundle to be defined below.

Vertical vector fields in M are prolonged to vector fields in J^kM via a modified procedure based on a horizontal one form μ with values in a representation of a Lie algebra $\mathcal{G}$ [15,17,21,22]. The form μ should satisfy the horizontal Maurer–Cartan equation

$$D\mu + \frac{1}{2}[\mu, \mu] = 0. \tag{2}$$

In terms of the local coordinates introduced above, we have

$$\mu = \Lambda_i(x, u, u_x)\mathrm{d}x^i \tag{3}$$

with Λ_i some $n \times n$ matrices (these are related to the Lie algebra $\mathcal{G}$ of a Lie group G acting in U, see [11,12] for details on this relation). With this notation, the horizontal Maurer–Cartan equation (2) reads

$$D_i\,\Lambda_j - D_j\,\Lambda_i + [\Lambda_i, \Lambda_j] = 0. \tag{4}$$

We write vertical vector fields in M as $X_0 = \eta^a(x, u, u_x)(\partial/\partial u^a)$, and the corresponding vertical vector fields in J^kM as $Y_0 = \eta^a_J(\partial/\partial u^a_J)$. The μ-prolongation formula for vertical vector fields is then, in recursive form (cf. (1) above),

$$\eta^a_{J,i} = D_i\eta^a_J + (\Lambda_i)^a_{\ b}\,\eta^b_J. \tag{5}$$

The condition (4) guarantees the η^a_J are well defined, see [11,12].

It was shown in [12] that (locally, and possibly globally as well) μ can always be written as $\mu = g^{-1}\mathrm{D}g$, and correspondingly μ-prolonged vector fields are related to standardly prolonged ones via a gauge transformation. We refer to [12] (in particular Theorem 1 in there) for details.

3. The gauge bundles

Let G be a Lie group and $\epsilon : G \to e$ the operator mapping the whole Lie group G into its identity element. We assume G acts on U via a (possibly nonlinear) representation $T : G \times U \to U$; at a point $p \in U$ the action on $V := \mathrm{T}_pU$ (this is the relevant action when discussing how G acts on vector fields) is described by the linearization $\Psi = DT$ of T. This also induces an action of the Lie algebra $\mathcal{G}$ of G in $V = \mathrm{T}_pU$ via the linear representation $\psi = D\Psi$.

Let $(\ell_1, \ldots, \ell_r)$ be a basis of left-invariant vector fields in $\mathcal{G}$; we write $L_i = \psi(\ell_i)$ for their representation. Any element $\xi \in \mathcal{G}$ can be written[4] as $\xi = \alpha^m\ell_m$ and acts in V via the vector field $\psi(\xi) = \alpha^mL_m$.

[4] With $\alpha^m = \langle \ell_m, \xi\rangle$, where $\langle ., .\rangle$ is the scalar product in $\mathcal{G}$. The α^m are natural coordinates in $\mathcal{G}$, and using these ℓ_m is given by $\ell_m = \partial/\partial\alpha^m$.

3.1. The basic gauge bundles

We introduce a principal bundle (P_G, ϵ, M) over M with bundle map ϵ, fiber $\epsilon^{-1}(p) = G$, and total space $P_G \simeq M \times G = \widetilde{M}$. This will be called the **global gauge bundle**. Sections $\gamma \in \Gamma(P_G)$ are described in local coordinates by $g = g(x, u)$.

The total space $P_G = M \times G$ can also be given the structure of a fiber bundle $(\widetilde{M}, \widetilde{\pi}, B)$ over B with projection $\widetilde{\pi} = \pi \times \epsilon$. The compatibility between these two structures is given by $\widetilde{\pi} = \pi \circ \varepsilon$.

Let us consider a reference section $\varpi \in \Gamma(P_G)$; in a tubular neighborhood $\widehat{G} \simeq M \times G_0 \simeq M \times \mathcal{G}_0$ of ϖ (with $G_0 \simeq \mathcal{G}$ a neighborhood of zero in G, and $\mathcal{G}_0$ a neighborhood of zero in $\mathcal{G}$) we can use local coordinates (x, u, α), where $\alpha = 0$ identifies the section ϖ. We will, for ease of notation and discussion, work in $M \times \mathcal{G}$; it should be kept in mind that our results will be local in $\mathcal{G}$ (and hence *a fortiori* in G). The projection from $\mathcal{G}$ to $\{0\} \in \mathcal{G}$ will be denoted as ρ.

The manifold $M \times \mathcal{G}$ can also be given two different fiber bundle structures: we can consider it as a bundle $(\widehat{M}, \widehat{\pi}, B)$ over B with total space $\widehat{M} = M \times \mathcal{G}$, projection $\widehat{\pi} = \pi \times \rho$ and fiber $\widehat{\pi}^{-1}(x) = U \times \mathcal{G}$; or as a bundle $(\mathcal{G}\mathcal{B}, \rho, M)$ over M with total space $\mathcal{G}\mathcal{B} = \widehat{M} = M \times \mathcal{G}$, projection ρ and fiber $\rho^{-1}(p) = \mathcal{G}$. The compatibility between these two structures is given by $\widehat{\pi} = \pi \circ \rho$.

We will denote $\widehat{M} = (M \times \mathcal{G}, \widehat{\pi}, B)$ as the **augmented phase bundle**, and $\mathcal{G}\mathcal{B} = (M \times \mathcal{G}, \rho, M)$ as the **local gauge bundle**. In the following we will also call these the **augmented bundle** and the **gauge bundle** for short.

Remark 2. We stress that M is *not* an associated bundle for this principal fiber bundle, contrary to what happens in Yang–Mills theories (where $g = g(x)$ and does not depend on u); this is the reason for some features which could appear odd to readers familiar with standard Yang–Mills theory. On the other hand, (J^1M, σ_1, M), *is* an associated bundle for (P_G, ϵ, M), which acts on fibers $\epsilon^{-1}(m)$ via the representation $\psi = (D\Psi)$. □

3.2. Higher order gauge bundles

We will also introduce gauge bundles associated with higher jet spaces J^kM. Thus we consider the order k augmented jet bundle $\widehat{M}^{(k)} = (J^kM \times \mathcal{G}, \widehat{\pi}_k, B)$ with fiber $\widehat{\pi}_k^{-1}(x) = U^{(k)} \times \mathcal{G}$; and the order k jet gauge bundle $J^k\mathcal{G}\mathcal{B} = (J^kM \times \mathcal{G}, \rho_k, J^kM)$ with fiber $\rho_k^{-1} = \mathcal{G}$. (Similarly, higher order bundles with total space $\widetilde{M}^{(k)} \simeq J^kM \times G$ could be defined. We will not enter into such details.)

Moreover, recall that J^kM can be seen as a bundle over M with projection σ_k, and correspondingly $J^k\widehat{M}$ can be seen as a bundle over $\widehat{M}$ with projection $\widehat{\sigma}_k$.

The situation is summarized in the following diagram, which also embodies the different double fibrations considered above.

$$\begin{tikzcd}
J^k\widehat{M} \arrow[rr, "\rho_k"] \arrow[dr, "\widehat{\pi}_k"] \arrow[dd, "\widehat{\sigma}_k"] & & J^kM \arrow[dl, "\pi_k"'] \arrow[dd, "\sigma_k"] \\
& B & \\
\widehat{M} \arrow[rr, "\rho"] \arrow[ur, "\widehat{\pi}"] & & M \arrow[ul, "\pi"']
\end{tikzcd} \tag{6}$$

3.3. Total derivative operators in gauge jet bundles

As well known, the prolongation operation is usually performed by applying the total derivative operators in J^kM; see Section 2.2 above. As the gauge jet bundles we are considering have a peculiar structure (that is, they are order k jet bundles for what concerns the u variables, not for what concerns the α ones), we should discuss what are the total derivative operators to be considered in this case.

Jet spaces are equipped with a contact structure $\mathcal{C}$, defined by the contact forms $\chi_J^a := \mathrm{d}u_J^a - u_{J,i}^a \mathrm{d}x^i$; note that there are no contact forms associated to gauge variables, as we are not considering jets of these. The total derivative operators D_i can then be defined in geometric terms as the vector fields (with a component $\partial/\partial x^i$) annihilating all the contact forms in $\mathcal{C}$; it is immediate to check that this requirement yields just the usual total derivative operators (associated to the u variables alone) $D_i = (\partial/\partial x^i) + u_{J,i}^a(\partial/\partial u_J^a)$. We stress that one should *not* add also components "along the gauge variables", i.e. of the form $\alpha_i^m(\partial/\partial\alpha^m)$.

Thus, *the prolongation operation leading from $\widehat{M}$ to $J^k\widehat{M}$ should be based on the usual total derivative operators D_i*, and hence does *not* involve derivation with respect to the gauge variables.[5]

This fact shows that a substantial difference exists between the gauge bundle and the bundle obtained by simply adding new dependent variables α^m.

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5 It may be worth stressing, just to avoid any possible misunderstanding, that albeit a vector field in $\widetilde{M}$ (respectively, in $\widehat{M}$) will have components both in the M and in the G (respectively, $\mathcal{G}$) directions, the prolongation operation should be applied *only* to the M components, as obvious from the definition of $J^k\widetilde{M}$ and $J^k\widehat{M}$ above.

4. Prolongation of vector fields in $\widetilde{M}$ and in $\widehat{M}$

Given a vector field in M, this is naturally *prolonged* (or lifted) to a vector field in J^kM. The same applies for vector fields in $\widetilde{M}$ and $\widehat{M}$, which are naturally prolonged to vector fields respectively in $J^k\widetilde{M}$ and $J^k\widehat{M}$.

We will denote by $\mathtt{Pr}^{(k)}[\mathcal{P}]$ the operator of prolongation of vector fields in a bundle $\mathcal{P}$ to vector fields in the jet bundle $J^k\mathcal{P}$, and omit the indication of the bundle $\mathcal{P}$ (i.e. just write $\mathtt{Pr}^{(k)}$) when there is no risk of misunderstanding.

4.1. Prolongation of general vector fields

We will give some explicit formulas in the local coordinates (x, u, α) introduced above. With the (x, u) coordinates in M, any vector field in $\widetilde{M}$ is written as $\widetilde{X} = \xi^i(\partial/\partial x^i) + \varphi^a(\partial/\partial u^a) + B^m \ell_m$, with $\{\xi^i, \varphi^a, B^m\}$ depending on (x, u, g). Passing to the restriction $\widehat{X}$ of $\widetilde{X}$ to $\widehat{\mathcal{G}}$, i.e. its expression in $\widehat{M}$, and introducing also the local coordinates α in $\mathcal{G}$ (recall $\ell_m = \partial/\partial\alpha^m$), we have $\widehat{X} = \xi^i(x, u, \alpha)(\partial/\partial x^i) + \varphi^a(x, u, \alpha)(\partial/\partial u^a) + B^m(x, u, \alpha)(\partial/\partial\alpha^m)$.

As usual in considerations involving vector fields on jet bundles, it will be convenient to work with evolutionary representatives [1–4]; we will consistently use these. The evolutionary representative of $\widehat{X}$ is

$$X \equiv \widehat{X}_v = Q^a \frac{\partial}{\partial u^a} + P^m \frac{\partial}{\partial\alpha^m}, \tag{7}$$

where $Q^a := \varphi^a - u^a_i \xi^i$, $P^m := B^m$.

The coordinate expression of the prolongation $X^{(k)} \in \mathcal{X}(J^k\widehat{M})$ of X is given by the (standard) prolongation formula [1–4]. (As implied by the discussion in Section 3.3 above, no prolongation of α^m components appear). For the evolutionary representative $Y := (X^{(k)})_v = X^{(k)}_v$ we get, with $Q^a_J = D_J Q^a$,

$$Y = Q^a_J \frac{\partial}{\partial u^a_J} + P^m \frac{\partial}{\partial\alpha^m}. \tag{8}$$

4.2. Prolongation of gauged vector fields

We are specially interested in a particular class of vector fields in $\mathcal{X}_v(\widetilde{M})$, i.e. those for which

$$Q^a(x, u, g; u_x) = [\Psi(g)]^a_b\, \Theta^b(x, u; u_x). \tag{9}$$

In the following we will refer to these as **gauged vector fields**.

Restricting gauged vector fields to $\widehat{M}$, and using local coordinates (x, u, α), Eq. (9) becomes

$$Q^a(x, u, \alpha; u_x) = [K(\alpha)]^a_b\, \Theta^b(x, u; u_x); \tag{10}$$

here $K(\alpha)$ is the representation of the group element $g(\alpha) = \exp(\alpha)$, i.e. $K(\alpha) = \Psi[\exp(\alpha)]$. (We stress that (9) and (10) do not constrain in any way the components P^m of the vector fields along the α^m variables; this will be of use below.)

Keeping in mind our discussion above about the total derivative operators in $\widehat{M}^{(k)}$, see Section 3.3, we obtain immediately that

$$Q^a_J = D_J Q^a = [K(\alpha)]^a_b\, D_J \Theta^b. \tag{11}$$

This implies that – writing $\Theta_J = D_J \Theta$ – the prolongation Y of the vector field X, see (8), is given by

$$Y = [K(\alpha)]^a_b\, \Theta^b_J \frac{\partial}{\partial u^a_J} + P^m \frac{\partial}{\partial\alpha^m}. \tag{12}$$

Remark 3. Let $X_0 = \rho_* X$ and $Y_0 = \rho^{(k)}_* Y$ be the projection of the vector fields X and Y to the bundles, respectively, M and J^kM. Then we can state formally that Y_0 *is the* μ*-prolongation of* X_0 *for a suitable* μ. In fact, we have

$$X_0 = \rho_* X = [K(\alpha)]^a_b\, \Theta^b \frac{\partial}{\partial u^a}; \qquad Y_0 = \rho^{(k)}_* Y = [K(\alpha)]^a_b\, \Theta^b_J \frac{\partial}{\partial u^a_J}. \tag{13}$$

Thus X_0 and Y_0 are the gauge transformed – via the same gauge transformation – of vector fields $\overline{X}_0$ and $\overline{Y}_0$ such that $\overline{Y}_0$ is the ordinary prolongation of $\overline{X}_0$; By Proposition 1 above, Y_0 is the μ-prolongation of X_0 for a suitable one-form μ. Note this statement is only formal, as the α variables have no meaning when we work in M and J^kM; in order to make this into a real theorem, we will need to "fix the gauge", as discussed below. Note also the relation between X_0 and Y_0 depends substantially on the assumption X is a gauged vector field. □

Remark 4. The reason for the name "gauged" of vector fields considered here is quite clear: the horizontal (for the gauge fibration) component $Q^a \partial_a$ of these corresponds to vector fields in M on which we operate with an element of the Lie group G, element which may vary for varying x and u.[6] $\square$

5. Standard prolongation of vector fields in $\widehat{M}$ and μ-prolongation of vector fields in M

We showed above that the natural prolongation in gauge bundles is naturally related (via the results in [12]) to a μ-prolongation. However, λ- and μ-prolongations are usually [6–9,11,12,14,24] defined with no use of auxiliary gauge variables; in our language this will correspond to a gauge fixing.

In this section we will discuss how gauge fixing affects the prolongation operation, and the relation between gauge-fixed prolongation and μ-prolongations.

5.1. Sub-bundles defined by sections of the gauge bundle

Earlier on we considered the augmented bundle $\widehat{M}$ and correspondingly $J^k\widehat{M}$. Here we want to consider the subbundle $\widehat{M}_\gamma := \gamma(M) \subset \widehat{M}$ defined by a section $\gamma \subset \Gamma(\mathcal{G}\mathcal{B})$ of the gauge bundle (hence by a section of the global gauge bundle close to the reference section ϖ); we will also consider $M_\gamma^{(k)} := \gamma(J^kM) \subset J^k\widehat{M}$.

In the (x, u, α) coordinates, $\widehat{M}_\gamma$ is the set of points (x, u, α) with $\alpha = A(x, u)$; similarly $\widehat{M}_\gamma^{(k)}$ is the set of points $(x, u, \alpha, u^{(1)}, \dots, u^{(k)})$ again with $\alpha = A(x, u)$. Note that $\widehat{M}_\gamma \simeq M$, and correspondingly $\widehat{M}_\gamma^{(k)} \simeq J^kM$.

These submanifolds of the gauge bundle and of the jet gauge bundle have a natural structure of fiber bundles (over B) themselves, and can be seen as sub-bundles of $\widehat{M}$ and $J^k\widehat{M}$[7]; we will thus also write $J^k\widehat{M}_\gamma$ for $\widehat{M}_\gamma^{(k)}$. It will thus make sense to speak of vertical vector fields in $\widehat{M}_\gamma$ and $J^k\widehat{M}_\gamma$ (referring implicitly to these fiber bundle structures).

Given a section $\gamma \in \Gamma(\mathcal{G}\mathcal{B})$, we will denote by $\omega^{(\gamma)}$ the operator of restriction from $\widehat{M}$ to $\widehat{M}_\gamma$, and by $\rho^{(\gamma)}$ the restriction of the projection $\rho : \widehat{M} \to M$ to $\widehat{M}_\gamma$. We also denote by $\omega_k^{(\gamma)} : J^k\widehat{M} \to \widehat{M}_\gamma^{(k)}$ and by $\rho_k^{(\gamma)} : \widehat{M}_\gamma^{(k)} \to J^kM$ the lift of the maps $\omega^{(\gamma)}$ and $\rho^{(\gamma)}$ to maps between corresponding jet spaces of order k. Note that while ρ is of course not invertible, it follows from $M_\gamma \simeq M$ that $\rho^{(\gamma)}$ is invertible, with $(\rho^{(\gamma)})^{-1} = \gamma$. similarly, $\rho^{(\gamma)}$ is invertible.

We will summarize relations and maps between relevant fiber bundles in the following diagram:

$$\begin{array}{ccccc} \widehat{M} & \xrightarrow{\omega^{(\gamma)}} & \widehat{M}_\gamma & \xrightarrow{\rho^{(\gamma)}} & M \\ \Big\downarrow{\scriptstyle j^k[\widehat{M}]} & & \Big\downarrow{\scriptstyle j^k[\widehat{M}_\gamma]} & & \Big\downarrow{\scriptstyle j^k[M]} \\ J^k\widehat{M} & \xrightarrow{\omega_k^{(\gamma)}} & \widehat{M}_\gamma^{(k)} & \xrightarrow{\rho_k^{(\gamma)}} & J^kM \end{array} \tag{14}$$

Remark 5. In physical terms, passing to consider $\widehat{M}_\gamma$ rather than the full $\widehat{M}$, and $\widehat{M}_\gamma^{(k)}$ rather than the full $\widehat{M}^{(k)}$, corresponds to a **gauge fixing**. $\square$

5.2. Prolongations and gauge fixing

When dealing with M_γ, i.e. working in a fixed gauge, we should think the gauge variables α as explicit functions of x and u (given by $A(x, u)$ identifying γ); note this means $D_i\alpha^m$ will now read as $D_iA^m(x, u)$ and thus will in general give a nonzero function. This entails the jet structure of $\widehat{M}_\gamma^{(k)}$ is not the one inherited from the jet structure of $\widehat{M}^{(k)}$ (which we denote by $j_k[\widehat{M}_\gamma]$, the associated prolongation operator being $\mathtt{Pr}^{(k)}[\widehat{M}_\gamma]$).

Let us now consider vector fields. Consider $X \in \mathcal{X}_v(\widehat{M})$ written as in (7), and $X_\gamma = \omega_*^{(\gamma)}X$ be its restriction to $\widehat{M}_\gamma \subset \widehat{M}$. Then X_γ is

$$X_\gamma = Q_\gamma^a\,(\partial/\partial u^a) + P_\gamma^m(\partial/\partial\alpha^m) \tag{15}$$

where the coefficients Q_γ, P_γ are of course given by

$$Q_\gamma^a = [Q^a]_{\alpha=A(x,u)}, P_\gamma^m = [P^m]_{\alpha=A(x,u)}. \tag{16}$$

It should be noted that for arbitrary X and γ, the submanifold γ is is general not invariant under X_γ. More precisely, with the notation Eq. (16), we have:

6 Note that if we operate on $\mathbf{e}$ by a x-dependent change of frame, i.e. pass to a frame $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_n)$ with $\mathbf{f}_a = T_a^{\ b}\mathbf{e}_b$ where $T = (K^T)^{-1}$, then we get $\Phi = \varphi^a\mathbf{e}_a = \varphi^a(T^{-1})_a^{\ b}\mathbf{f}_b := \widetilde{\varphi}^a\mathbf{f}_a$, i.e. the components of Φ in the new frame are $\widetilde{\varphi}^a = K_b^a\varphi^b$. This point of view is discussed elsewhere [23].

7 We stress this refers to the structure of bundles over B, i.e. – referring to diagram (6) – to the $\widehat{\pi}_k$ projections; more care will be needed for what concerns the structure of bundles over $\widehat{M}$, see Section 5.2.

Lemma 1. *Let X be in the form (7). Then the submanifold $\widehat{M}_\gamma \subset \widehat{M}$ identified by $\alpha^m = A^m(x,u)$ is invariant under X_γ if and only if*

$$P^m_\gamma = \left(\partial A^m/\partial u^a\right)\, Q^a_\gamma. \tag{17}$$

Proof. By standard computation. Note that using the notation in Section 4.1, this also reads as $P^m = X(A)$ on $\widehat{M}_\gamma$. $\diamond$

Corollary 1. *Given arbitrary smooth functions $Q^a(x,u,\alpha)$, and an arbitrary section $\gamma \in \Gamma(\widehat{M})$, there is always a vector field $X_v \in \mathcal{X}_v(\widehat{M})$ of the form $X_v = Q^a\partial_a + P^m\partial_m$ and such that X_v leaves $\widehat{M}_\gamma$ invariant.*

Proof. In view of (12), the condition (17) reads also $P^m = (\partial A^m/\partial u^a)Q^a$; for any given γ the vector fields leaving $\widehat{M}_\gamma$ invariant, have arbitrary Q^a and P^m given by $P^m = (\partial A^m/\partial u^a)Q^a + \delta P^m$ with δP^m arbitrary functions vanishing on $\alpha = A(x,u)$. $\diamond$

Remark 6. The vector field X_γ projects in turn to a vector field $W \in \mathcal{X}_v(M)$, $W = Q^a(x,u)(\partial/\partial u^a)$; and conversely any such $W \in \mathcal{X}_v(M)$ lifts to a $W^\gamma = X_\gamma \in \mathcal{X}_v(\widehat{M}_\gamma)$, $W^\gamma = Q^a(x,u)[(\partial/\partial u^a) + ((\partial A^m/\partial u^a)(\partial/\partial\alpha^m))]$. $\square$

The set of vector fields $X \in \mathcal{X}_v(\widehat{M})$ which leave $\widehat{M}_\gamma$ invariant, i.e. satisfy (17), will be denoted as $\mathcal{X}_v^{(\gamma)}(\widehat{M})$. If $X \in \mathcal{X}_v^{(\gamma)}(\widehat{M})$, then X_γ is actually a vector field on $\widehat{M}_\gamma$, and $X_\gamma^{(k)}$ a vector field on $J^k\widehat{M}_\gamma$.

The diagram (14) has a counterpart for these vector fields. We will use for graphic convenience a simplified notation with $\widehat{\mathcal{X}} := \mathcal{X}_v^{(\gamma)}(\widehat{M})$, $\widehat{\mathcal{X}}_\gamma := \mathcal{X}_v(\widehat{M}_\gamma)$, $\mathcal{X} := \mathcal{X}_v(M)$; $\widehat{\mathcal{X}}^{(k)} := \mathcal{X}_v^{(\gamma)}(\widehat{M}^{(k)})$, $\widehat{\mathcal{X}}^{(k)}_\gamma := \mathcal{X}_v(\widehat{M}^{(k)}_\gamma)$, $\mathcal{X}^{(k)} := \mathcal{X}_v(M^{(k)})$. With this, (14) yields:

$$\begin{array}{ccccc} \widehat{\mathcal{X}} & \xrightarrow{\omega_*^{(\gamma)}} & \widehat{\mathcal{X}}_\gamma & \xrightarrow{\rho_*^{(\gamma)}} & \mathcal{X} \\ \big\downarrow \mathtt{Pr}^{(k)}[\widehat{M}] & & \big\downarrow \mathtt{Pr}^{(k)}[\widehat{M}_\gamma] & & \big\downarrow \mathtt{Pr}^{(k)}[M] \\ \widehat{\mathcal{X}}^{(k)} & \xrightarrow{(\omega_k^{(\gamma)})_*} & \widehat{\mathcal{X}}^{(k)}_\gamma & \xrightarrow{(\rho_k^{(\gamma)})_*} & \mathcal{X}^{(k)} \end{array} \tag{18}$$

The operators $\mathtt{Pr}^{(k)}[\widehat{M}]$ and $\mathtt{Pr}^{(k)}[\widehat{M}_\gamma]$ should be understood with the discussion of Section 3.3 in mind.

The diagram (18) is in general *not* commutative. We will now discuss how it can be made into a commutative one by replacing $\mathtt{Pr}^{(k)}[\widehat{M}_\gamma]$ and $\mathtt{Pr}^{(k)}[M]$ by, respectively, suitable operators $\widehat{\mathtt{P}}^{(k)}_\gamma : \mathcal{X}_v(\widehat{M}_\gamma) \to \mathcal{X}_v(\widehat{M}^{(k)}_\gamma)$ and $\mathtt{P}^{(k)}_\gamma : \mathcal{X}_v(M) \to \mathcal{X}_v(M^{(k)})$. That is, we want to identify $\widehat{\mathtt{P}}^{(k)}_\gamma$ and $\mathtt{P}^{(k)}_\gamma$ yielding, for a given $X \in \mathcal{X}_v^{(\gamma)}(\widehat{M})$,

$$\begin{array}{ccccc} X & \xrightarrow{\omega_*^{(\gamma)}} & X_\gamma & \xrightarrow{\rho_*^{(\gamma)}} & W \\ \big\downarrow \mathtt{Pr}^{(k)}[\widehat{M}] & & \big\downarrow \widehat{\mathtt{P}}^{(k)}_\gamma & & \big\downarrow \mathtt{P}^{(k)}_\gamma \\ X^{(k)} & \xrightarrow{(\omega_k^{(\gamma)})_*} & X^{(k)}_\gamma & \xrightarrow{(\rho_k^{(\gamma)})_*} & Y \end{array} \tag{19}$$

5.3. Twisted differential operators in $\widehat{M}_\gamma$

Let us first discuss the left-hand side of the diagram (19). We will write K_γ for $K(\alpha)$ computed on $\alpha = A(x,u)$.

In general we obtain different results by changing the order in which the prolongation and the gauge fixing operations are performed.

Remark 7. This is immediately seen by considering a vector field in the form (7) and (10). We have of course $X_\gamma = Q^a_\gamma(\partial/\partial u^a) + P^m_\gamma(\partial/\partial\alpha^m)$; as $Q_\gamma = K_\gamma\Theta$, the prolongation of X_γ is $(X_\gamma)^{(k)} = \overline{\psi}^a_J(\partial/\partial u^a_J) + P^m_\gamma(\partial/\partial\alpha^m)$, with $\overline{\psi}^a_J = (D_J Q^a_\gamma) = D_J[(K_\gamma)^a{}_b\Theta^b]$. On the other hand, the prolongation of X is $X^{(k)} = (D_J Q^a)(\partial/\partial u^a_J) + P^m_\gamma(\partial/\partial\alpha^m)$ with $D_J Q^a = D_J(K^a{}_b\Theta^b) = K^a{}_b D_J(\Theta^b)$; hence gauge fixing after prolongation yields $(X^{(k)})_\gamma = \psi^a_J(\partial/\partial u^a_J) + P^m_\gamma(\partial/\partial\alpha^m)$ with $\psi^a_J = (K_\gamma)^a{}_b D_J(\Theta^b)$. Needless to say, in general $\psi^a_J \neq \overline{\psi}^a_J$. $\square$

Denote by δ_γ the operator fixing the gauge to γ. We will look for "twisted differential operators" $\nabla_i^{(\gamma)}$ (we will also write ∇_i for short) such that

$$\delta_\gamma\,[D_J(Q^a)] = \nabla_J^{(\gamma)}\left[\delta_\gamma(Q^a)\right] \tag{20}$$

when Q is of the form (10). The operator $\widehat{\mathtt{P}}_\gamma$ will then be the "twisted prolongation" obtained by replacing D_J with $\nabla_J^{(\gamma)}$.

Let us define matrices $R_i^{(\gamma)}$ by

$$R_i^{(\gamma)}(x,u,u_x) = (D_i K_\gamma)\, K_\gamma^{-1}. \tag{21}$$

Lemma 2. *The matrices $R_i^{(\gamma)}(x,u,u_x)$ satisfy the horizontal Maurer–Cartan equation*

$$D_i\, R_j^{(\gamma)} \;-\; D_j\, R_i^{(\gamma)} + [R_i^{(\gamma)}, R_j^{(\gamma)}] = 0. \tag{22}$$

Proof. It follows from (21) that $D_i R_j^{(\gamma)} = (D_i D_j S) S^{-1} - R_i^{(\gamma)} R_j^{(\gamma)}$. Recalling $[D_i, D_j] = 0$, we get $D_i R_j^{(\gamma)} - D_j R_i^{(\gamma)} = -[R_i^{(\gamma)}, R_j^{(\gamma)}]$. $\diamond$

We will define the operators $\nabla_i^{(\gamma)}$ as

$$\nabla_i^{(\gamma)} \;:=\; D_i \;-\; R_i^{(\gamma)}. \tag{23}$$

It follows from Lemma 2 that $[\nabla_i, \nabla_j] = 0$. For a multiindex $J = (j_1, \ldots, j_m)$, the operators $\nabla_J^{(\gamma)}$ are defined as $\nabla_J = \nabla_1^{j_1} \ldots \nabla_m^{j_m}$; this is well defined in view of $[\nabla_i, \nabla_j] = 0$.

Lemma 3. *Let $Q^a = K^a_b \Theta^b$. The operators $\nabla_i^{(\gamma)}$ defined in (23) satisfy*

$$\delta_\gamma\, [D_i(Q^a)] = \nabla_i^{(\gamma)}\, [Q^a_\gamma] \tag{24}$$

for all i, and hence (20) for all multi-indices J.

Proof. We can proceed by direct computation; we will omit indices in intermediate formulas for ease of notation. For $Q = K\Theta$ we have immediately $D_i Q = K[D_i(\Theta)]$ and hence

$$\delta_\gamma\, [D_i(Q^a)] = (K_\gamma)^a_{\ b}\, D_J \Theta^b. \tag{25}$$

On the other hand, $\nabla_i(Q_\gamma) = D_i[(K_\gamma)\Theta] - [R_i^{(\gamma)}](K_\gamma)\Theta = K_\gamma D_i \Theta + [D_i K_\gamma]\Theta - [R_i^{(\gamma)}](K_\gamma)\Theta$. Recalling the definition of $R_i^{(\gamma)}$, we have

$$[R_i^{(\gamma)}](K_\gamma)\Theta = (D_i K_\gamma) K_\gamma^{-1} K_\gamma \Theta = (D_i K_\gamma)\Theta,$$

and hence

$$\nabla_i\, [Q^a_\gamma] = (K_\gamma)^a_{\ b}\, [D_i(\Theta^b)]; \tag{26}$$

comparing this and (25) we obtain (24).

In order to see this implies (20) it suffices to note we have actually proved

$$\nabla_i\, [K_\gamma\, \Theta] = K_\gamma\, [D_i \Theta]; \tag{27}$$

applying this repeatedly we obtain (20). Alternatively, one can explicitly compute $(\nabla_i \nabla_j Q_\gamma)$ and check this is equal to $\delta_\gamma(D_i D_j Q)$. $\diamond$

As anticipated, the operator $\widehat{\mathrm{P}}_\gamma$ will be the "twisted prolongation" obtained by replacing D_J with ∇_J.

Lemma 4. *Let $\widehat{\mathrm{P}}_\gamma$ be the (twisted prolongation) operator associating to any vector field $X_\gamma = Q^a_\gamma(\partial/\partial u^a) + P^m_\gamma(\partial/\partial \alpha^m)$ in $\mathcal{X}_v(\widehat{M}_\gamma)$ the vector field $Y_\gamma = \eta^a_J(\partial/\partial u^a_J) + P^m(\partial/\partial \alpha^m)$ in $\mathcal{X}_v(\widehat{M}^{(k)}_\gamma)$ with coefficients $\eta^a_J := \nabla_J^{(\gamma)} Q^a_\gamma$. Then the left-hand side of diagram (19) is commutative.*

Proof. See the explicit computations in the proof to Lemma 3. $\diamond$

Lemma 5. *Let P_γ be the (twisted prolongation) operator associating to any vector field $W = Q^a_\gamma(\partial/\partial u^a)$ in $\mathcal{X}_v(M)$ the vector field $Y = \eta^a_J(\partial/\partial u^a_J)$ in $\mathcal{X}_v(M^{(k)})$ with coefficients $\eta^a_J := \nabla_J^{(\gamma)} Q^a_\gamma$. Then the right-hand side of diagram (19) is commutative.*

Proof. This P_γ is nothing else than the restriction of $\widehat{\mathrm{P}}_\gamma$ to the components along $M^{(k)}$. It is obvious (by construction) that $\mathrm{P}_\gamma \circ \rho_*^{(\gamma)} = (\rho_k^{(\gamma)})_* \circ \widehat{\mathrm{P}}_\gamma$. $\diamond$

5.4. The main results

We are now ready to state and prove our main results, which will actually just collect results appearing in the previous Lemmas; these will make the formal statement in Remark 3 into precise ones.

We will introduce, in order to state our result in a compact form, operators $\tau^{(\gamma)} := \rho^{(\gamma)} \circ \omega^{(\gamma)}$, $\tau^{(\gamma)} : \widehat{M} \to M$; and correspondingly $\tau_k^{(\gamma)} := \rho_k^{(\gamma)} \circ \omega_k^{(\gamma)}$, $\tau_k^{(\gamma)} : \widehat{M}^{(k)} \to M^{(k)}$.

Theorem 1. *The twisted prolongation operator $\mathtt{P}_\gamma^{(k)}$ is uniquely defined by the requirement that $(\tau_k^{(\gamma)})_* \circ (\mathtt{Pr}^{(k)}[\widehat{M}]) = \mathtt{P}_\gamma^{(k)} \circ \tau_*^{(\gamma)}$.*

Moreover, $\mathtt{P}_\gamma^{(k)}$ corresponds to the μ-prolongation operator of order k with $\mu = [\mathrm{d}\Psi(\gamma)]\Psi(\gamma^{-1})$. With the local coordinates (x, u, α), this corresponds to $\mu = \Lambda_i \mathrm{d}x^i$ where $\Lambda_i = -R_i^{(\gamma)} = -(D_i K_\gamma) K_\gamma^{-1} = K_\gamma (D_i K_\gamma^{-1})$.

Proof. The first part of the statement just summarizes the discussion in Section 5.3. As for the second part, it follows at once considering the definitions of $\mathtt{P}_\gamma^{(k)}$, of $\nabla_i^{(\gamma)}$ and of $R_i^{(\gamma)}$, and comparing with the μ-prolongation formula (5). $\diamond$

Theorem 2. *Let $Y \in \mathcal{X}_v(J^k M)$ be the k-th μ-prolongation of $W \in \mathcal{X}_v(M)$, with $\mu \in \Lambda^1(J^1 M, \psi(\mathcal{G}))$ given in coordinates by $\mu = \Lambda_i(x, u, u_x)\mathrm{d}x^i$; let $\mathcal{G}\mathcal{B}$ be the local gauge bundle over M with fiber $\mathcal{G}$. Then:*

(i) *there is a section $\gamma \in \Gamma(\mathcal{G}\mathcal{B})$ such that $Y = \mathtt{P}_\gamma(W)$.*
(ii) *there is $X \in \mathcal{X}_v(\widehat{M})$ such that (19) applies.*
(iii) *The matrix function $K_\gamma(x, u, u_x)$ satisfies $D_i K_\gamma = -\Lambda_i K_\gamma$.*

Proof. The theorem states that any μ-prolongation can be obtained locally through the construction considered here. Part (i) is obvious given the identification of μ-prolongation and the $\mathtt{P}_\gamma$ operators, see Lemma 5. Part (ii) follows at once from Lemmas 4 and 5. Part (iii) follows from the relation $\Lambda_i = -R_i^{(\gamma)} = -(D_i K_\gamma) K_\gamma^{-1}$ already considered above (note K_γ is surely invertible as it is the representation of an element of the Lie group G). $\diamond$

It is maybe worth commenting on the geometrical meaning of our main results.

We have shown that μ-symmetries in the phase bundle can be understood as ordinary symmetries in the augmented phase bundle, restricted to a section of the gauge bundle and mapped to the standard phase bundle.

In other words, we have obtained that the μ-prolongation operator appears if we are insisting in restricting our analysis to the phase bundle M (or to the subbundle $\widehat{M}_\gamma \subset \widehat{M}$ seen as an image of M under the gauge map γ embedding it into $\widehat{M}$) rather than to the full gauge bundle $\widehat{M}$.

The fact we are considering projections of vector fields in $\widehat{M}_\gamma \subset \widehat{M}$ and $\widehat{M}_\gamma^{(k)} \subset \widehat{M}^{(k)}$ to vector fields in M and $M^{(k)}$ makes that the relation between basic vector fields and prolonged ones is not the natural one, described by the prolongation operator, but is the "twisted" one described by the μ-prolongation operator. See also the discussion in the Appendix.

We would also like to stress that gauge fields and hence variables can also be seen – like in standard gauge theories – as indexing reference frames. The restriction to a section of the gauge bundle corresponds thus to fixing a gauge and hence a reference frame (not equivalent to the original, "natural", one if the section is nontrivial); projection to the phase bundle corresponds to loosing track of the change of reference frame. See also [23].

Finally we mention that our construction in the augmented phase bundle corresponds, when working in the phase bundle alone, to the fact that (locally, and globally when we deal with topologically trivial phase bundles) μ-symmetries can be transformed into standard ones by a gauge transformation [12].

6. Examples I. Abelian groups

In this section we will provide some very simple examples illustrating our results; we will consider $B = \mathbf{R}$ and $U = \mathbf{R}^2$ (see e.g. [12] for the reasons making the case $U = R^1$ less interesting in this context); we will consider one-parameter (hence abelian; see next section for non-abelian examples) Lie groups G, its Lie algebra being $\mathcal{G}$ with generator ℓ. The gauge variable will be denoted as α. We will use the notation (u, v) for the coordinates (u^1, u^2) in U; we denote by L the representation $\psi(\ell)$ of the generator ℓ.

We will consider second order prolongations (i.e. $k = 2$), and write the vector field Y, see (19), in the form

$$Y = \eta^a(\partial/\partial u^a) + \eta_x^a(\partial/\partial u_x^a) + \eta_{xx}^a(\partial/\partial u_{xx}^a); \tag{28}$$

the required relation between its coefficients will then be

$$\eta_x = D_x\eta + \Lambda\eta, \qquad \eta_{xx} = D_x\eta_x + \Lambda\eta_x. \tag{29}$$

Examples 1 through 4 concern Theorem 1, while Examples 5 and 6 deal with Theorem 2.

6.1. Example 1

Let us first consider the case where $G = R$ acts in U via

$$T(g) = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}.$$

In this case we have immediately

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad K = T[e^{\alpha L}] = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}; \qquad K^{-1} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

Let us consider the gauged vector field X of the form (10) with $K(\alpha)$ as above and $\Theta = (2u, v)$; the gauge section γ to be considered will be the one identified by

$$\alpha = A(x, u, v) := u.$$

The invariance of $\widehat{M}_\gamma$ requires then, see (17) above, $P = [X(A)] = (2u + \alpha v)$. We will thus be considering the vector field

$$X = (2u + \alpha v)(\partial/\partial u) + v\,(\partial/\partial v) + (2u + \alpha v)\,(\partial/\partial \alpha).$$

The restriction of X to $\widehat{M}_\gamma$ is

$$X_\gamma = (2u + uv)(\partial/\partial u) + v\,(\partial/\partial v) + (2u + uv)\,(\partial/\partial \alpha),$$

and the projection of X_γ to M is simply

$$W = (2u + uv)(\partial/\partial u) + v\,(\partial/\partial v).$$

As for second prolongations, it follows from general formulas that

$$X^{(2)} = X + (2u_x + \alpha v_x)(\partial/\partial u_x) + v_x\,(\partial/\partial v_x) + (2u_{xx} + \alpha v_{xx})(\partial/\partial u_{xx}) + v_{xx}\,(\partial/\partial v_{xx}).$$

Restricting this to $\widehat{M}_\gamma^{(2)}$ and projecting to $M^{(2)}$ yields

$$Y = W + (2u_x + uv_x)(\partial/\partial u_x) + v_x\,(\partial/\partial v_x) + (2u_{xx} + uv_{xx})(\partial/\partial u_{xx}) + v_{xx}\,(\partial/\partial v_{xx}).$$

Writing this in the form (28) we have

$$\eta = \begin{pmatrix} 2u + uv \\ v \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} 2u_x + uv_x \\ v_x \end{pmatrix}, \qquad \eta_{xx} = \begin{pmatrix} 2u_{xx} + uv_{xx} \\ v_{xx} \end{pmatrix}.$$

According to our general theorem, these should satisfy the recurrence formula (29) with $\Lambda = -R_x^{(\gamma)} = -(D_x K_\gamma) K_\gamma^{-1}$. With our choices we have

$$\Lambda = -R_x^{(\gamma)} = -\begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -u_x \\ 0 & 0 \end{pmatrix}.$$

It is immediate to check the η_J^a satisfy (29) with this Λ.

6.2. Example 2

Let us now the case where $G = R$ acts in U as above, and γ is also as above (so that the K_γ, $R^{(\gamma)}$ and Λ are the same as before), but $\Theta = (-v, u)$. Now the condition $P = X(A)$ yields $P = (-v + \alpha u)$ That is, we have

$$X = (-v + \alpha u)(\partial/\partial u) + u\,(\partial/\partial v) + (\alpha u - v)\,(\partial/\partial \alpha).$$

The restriction of X to $\widehat{M}_\gamma$ and its projection to M are

$$X_\gamma = (u^2 - v)(\partial/\partial u) + u\,(\partial/\partial v) + (u^2 - v)\,(\partial/\partial \alpha);$$

$$W = (u^2 - v)(\partial/\partial u) + u\,(\partial/\partial v).$$

Let us consider again second prolongations. Now

$$\begin{aligned} X^{(2)} = \; & X + (-v_x + \alpha u_x)(\partial/\partial u_x) + u_x\,(\partial/\partial v_x) \\ & + (-v_{xx} + \alpha u_{xx})(\partial/\partial u_{xx}) + u_{xx}\,(\partial/\partial v_{xx}). \end{aligned}$$

Restricting this to $\widehat{M}_\gamma^{(2)}$ and projecting to $M^{(2)}$ yields Y; with the notation (28) this corresponds to

$$\eta = \begin{pmatrix} u^2 - v \\ u \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} uu_x - v_x \\ u_x \end{pmatrix}, \qquad \eta_{xx} = \begin{pmatrix} uu_{xx} - v_{xx} \\ u_{xx} \end{pmatrix}.$$

The matrix Λ is as above, and one checks easily the required recursion relations (29) are satisfied.

6.3. Example 3

Let us now consider $G = SO(2) \simeq S^1$ acting in $U = R^2$ through its standard representation,

$$T(g) = \begin{pmatrix} \cos(g) & -\sin(g) \\ \sin(g) & \cos(g) \end{pmatrix}.$$

In this case

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \qquad K(\alpha) = e^{\alpha L} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

We will once again consider γ identified by $\alpha = u$; it follows by the usual formulas that

$$\Lambda = -R_x^{(\gamma)} = -(D_x K_\gamma) K_\gamma^{-1} = \begin{pmatrix} 0 & u_x \\ -u_x & 0 \end{pmatrix}.$$

We will consider the gauged vector field with $Q = K\Theta$ and the same Θ considered in Example 1. This yields $Q = (2u\cos\alpha - v\sin\alpha, v\cos\alpha + 2u\sin\alpha)$, and $P = X(A)$ yields $P = 2u\cos\alpha - v\sin\alpha$. With the by now usual computations, we get $W = [2u\cos(u) - v\sin(u)](\partial/\partial u) + [v\cos(u) + 2u\sin(u)](\partial/\partial v)$; and working in the usual way with second prolongations we get

$$\eta = \begin{pmatrix} 2u\cos(u) - v\cos(u) \\ v\cos(u) + 2u\cos(u) \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} 2u_x\cos(u) - v_x\cos(u) \\ v_x\cos(u) + 2u_x\cos(u) \end{pmatrix},$$
$$\eta_{xx} = \begin{pmatrix} 2u_{xx}\cos(u) - v_{xx}\cos(u) \\ v_{xx}\cos(u) + 2u_{xx}\cos(u) \end{pmatrix}.$$

It is again immediate to check that these satisfy the required recursion relation with the Λ computed above.

6.4. Example 4

We consider the same G-action and gauge section as above, so that the K, R_x and Λ are the same as in Example 3; but choose now $Q = K\Theta$ with the Θ considered in Example 2, $\Theta = (-v, u)$. In this case

$$\eta = \begin{pmatrix} -v\cos(u) - u\cos(u) \\ u\cos(u) - v\cos(u) \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} -v_x\cos(u) - u_x\cos(u) \\ u_x\cos(u) - v_x\cos(u) \end{pmatrix},$$
$$\eta_{xx} = \begin{pmatrix} -v_{xx}\cos(u) - u_{xx}\cos(u) \\ u_{xx}\cos(u) - v_{xx}\cos(u) \end{pmatrix}.$$

It is again immediate to check that (29) are satisfied.

6.5. Example 5

Consider the vector field Y in the form (28) with

$$\eta = \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} u_x v \\ v_x \end{pmatrix}, \qquad \eta_{xx} = \begin{pmatrix} u_{xx}v + 2u_x v_x \\ v_{xx} \end{pmatrix}.$$

It is immediate to see these satisfy (29) with

$$\Lambda = \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix};$$

we already know from Examples 1 and 2 the form of the K giving such a Λ, but let us pretend we do not. We have then to solve $\Lambda = -(D_x K_\gamma)K_\gamma^{-1}$ as an equation for K_γ; recall we look for solutions such that K_γ belongs to a one-parameter (hence abelian) Lie group. Note that we can write $\Lambda = D_x P$ for a matrix P. We can rewrite the equation linking Λ and K as $D_x P = -D_x(\log K_\gamma)$, with solution $K_\gamma = \exp(-P)$ (the arbitrary constant matrix for K_γ can be embodied in the arbitrary constant matrix for P). In our case we get easily K_γ; it is immediate to see the resulting K_γ as the restriction to $\alpha = u$ of a matrix $K(\alpha)$,

$$K_\gamma = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}; \qquad K(\alpha) = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

Note these allow to build the diagram (19) by working backwards; that is, we act on the components η^a_J of Y by the matrix KK_γ^{-1}, thus obtaining $X^{(2)} = \psi^a_J(\partial/\partial u^a_J) + P(\partial/\partial\alpha)$ with $P = X(u) = [(u-\alpha)v]$ and

$$\psi = \begin{pmatrix} (u-\alpha)v \\ v \end{pmatrix}, \qquad \psi_x = \begin{pmatrix} u_x v + (u-\alpha)v_x \\ v_x \end{pmatrix},$$
$$\psi_{xx} = \begin{pmatrix} u_{xx}v + 2u_x v_x + (u-\alpha)v_{xx} \\ v_{xx} \end{pmatrix}.$$

It is immediate to check this is the standard second prolongation in $\widehat{M}^{(2)}$ of $X = [(u-\alpha)v](\partial/\partial u) + v(\partial/\partial v) + [(u-\alpha)v](\partial/\partial\alpha)$.

6.6. Example 6

Consider the vector field Y in the form (28) with

$$\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} 0 \\ v_x \end{pmatrix}, \qquad \eta_{xx} = \begin{pmatrix} 0 \\ v_{xx} \end{pmatrix}.$$

It is immediate to see these satisfy (29) with

$$\Lambda = \begin{pmatrix} 0 & 0 \\ v_x & 0 \end{pmatrix}.$$

Proceeding as above, and choosing as γ the section $\alpha = -v$ of the gauge bundle, we get

$$K_\gamma = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}, \qquad K(\alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 0 \end{pmatrix}.$$

We thus obtain $X^{(2)} = \psi^a_J(\partial/\partial u^a_J) + P(\partial/\partial\alpha)$ with $P = -(\alpha+v)$ and

$$\psi = \begin{pmatrix} 1 \\ \alpha+v \end{pmatrix}, \qquad \psi_x = \begin{pmatrix} 0 \\ v_x \end{pmatrix}, \qquad \psi_{xx} = \begin{pmatrix} 0 \\ v_{xx} \end{pmatrix};$$

i.e. the second prolongation of $X = (\partial/\partial u) + (\alpha+v)(\partial/\partial v) - (\alpha+v)(\partial/\partial\alpha)$.

7. Examples II. Non-abelian groups: SU(2)

The examples considered so far all concern very simple actions of a one-generator, hence abelian, Lie group. It would not be difficult to extend these to actions of k-generators abelian groups; but for physical applications one should rather consider non-abelian Lie groups such as e.g. the rotation or unitary groups.

In this section we will consider the group $G = SU(2)$ acting[8] in $R^4 \simeq C^2$; we will consider very simple vector fields and section, but still will have to set down rather complex formulas. We apologize to the reader for such unavoidable complexities; computations were performed in `Mathematica`, which also shows that our formalism can be readily implemented with a symbolic manipulation program.

Examples 7 and 8 concern Theorem 1, while Examples 9 and 10 deal with Theorem 2.

7.1. SU(2) algebra and group action; lambda matrices

We will consider generators

$$L_1 = T(\ell_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad L_2 = T(\ell_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$L_3 = T(\ell_3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

These satisfy the $su(2)$ relations $L_iL_j = \varepsilon_{ijk}L_k - \delta_{ij}I$; here I is the four dimensional identity matrix, and summation over repeated indices is implicit here and in the following. These relations also imply $[L_i, L_j] = 2\varepsilon_{ijk}L_k$, $\{L_i, L_j\} = -2\delta_{ij}I$.

In the following we will need to compute the group element g corresponding to $g = \exp(\ell)$ for ℓ an element of the algebra. Consider a generic element of the algebra $\ell = \alpha^k\ell_k$, and correspondingly a generic matrix

$$L = \alpha^1L_1 + \alpha^2L_2 + \alpha^3L_3$$

(in the following we write all indices as lower ones in order to avoid confusion with exponents in the computations). This is written explicitly as

$$L = \begin{pmatrix} 0 & \alpha_1 & \alpha_3 & \alpha_2 \\ -\alpha_1 & 0 & \alpha_2 & -\alpha_3 \\ -\alpha_3 & -\alpha_2 & 0 & \alpha_1 \\ -\alpha_2 & \alpha_3 & -\alpha_1 & 0 \end{pmatrix}$$

and its square is given by $L^2 = -|\alpha|^2I$, with $|\alpha| := \sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}$. Thus higher powers of L satisfy

$$L^{2k} = (-1)^k\,|\alpha|^{2k}\,I;\quad L^{2k+1} = (-1)^k\,|\alpha|^{2k}\,L. \tag{30}$$

We have to compute $e^L := \sum_{k=0}^\infty (L^k/k!)$; it follows from (30) that

$$e^L = \sum_{k=0}^\infty \left[\frac{(-1)^k|\alpha|^{2k}}{(2k)!}\,I + \frac{(-1)^k|\alpha|^{2k}}{(2k+1)!}\,L\right];$$

recognizing the Taylor expansions of trigonometric functions, this reads

$$e^L = \cos(|\alpha|)\,I + |\alpha|^{-1}\,\sin(|\alpha|)\,L.$$

8 We have so far used real vector spaces; in order to avoid converting at this point to complex ones, we will use a real representation of $G = SU(2)$, acting in R^4 rather than in C^2.

Recalling the definitions of $|\alpha|$ and L, we have

$$\begin{aligned} K(\alpha) \;:=\;& T[\exp(\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3)] \\ =\;& \cos(|\alpha|)\, I + \sin(|\alpha|)\, [(\alpha_1/|\alpha|)L_1 + (\alpha_2/|\alpha|)L_2 + (\alpha_3/|\alpha|)L_3]. \end{aligned}$$

We will introduce the matrix $J := [(\alpha_1/|\alpha|)L_1 + (\alpha_2/|\alpha|)L_2 + (\alpha_3/|\alpha|)L_3] = |\alpha|^{-1}L$; note $J^2 = -I$. With this,

$$\begin{aligned} K(\alpha) &= \cos(|\alpha|)\, I + \sin(|\alpha|)\, J; \\ K^{-1}(\alpha) &= \cos(|\alpha|)\, I - \sin(|\alpha|)\, J. \end{aligned} \tag{31}$$

These formulas allow to give the general form of gauged vector fields for this G action: under the action of $K(\alpha)$, a vector field of components Θ^a is transformed into a vector field of components $\Phi^a = K^a_b \Theta^b$ given by

$$\Phi = \cos(|\alpha|)\, \Theta + \frac{\sin(|\alpha|)}{|\alpha|} \begin{pmatrix} \alpha_1\theta_2 + \alpha_3\theta_3 + \alpha_2\theta_4 \\ -\alpha_1\theta_1 + \alpha_2\theta_3 - \alpha_3\theta_4 \\ -\alpha_3\theta_1 - \alpha_2\theta_2 + \alpha_1\theta_4 \\ -\alpha_2\theta_1 + \alpha_3\theta_2 - \alpha_1\theta_3 \end{pmatrix}. \tag{32}$$

This also fully describes the set of vector fields in $\mathbf{R}^4$ which are gauged vector fields under the action of the presently considered representation of $SU(2)$.

Our formalism also requires to consider matrices $\Lambda = -(D_x K_\gamma) K_\gamma^{-1}$. It is possible, with standard but rather tedious computations, to obtain the explicit form of these starting from (31), and restricting to the section γ identified by $\alpha_m = A_m(x, u, v, w, z)$; using the notation $\omega = |\alpha|_\gamma = \sqrt{A_1^2 + A_2^2 + A_3^2}$, the final result turns out to be

$$\Lambda = \cos^2(\omega)\, M_1 + \sin^2(\omega)\, M_2 + \sin(\omega)\, \cos(\omega)\, M_3,$$

where the M_i are four-dimensional matrices. These can be written in terms of the matrix

$$\mathcal{L} = \begin{pmatrix} 0 & -A_1 & -A_3 & -A_2 \\ A_1 & 0 & -A_2 & A_3 \\ A_3 & A_2 & 0 & -A_1 \\ A_2 & -A_3 & A_1 & 0 \end{pmatrix},$$

of the matrix $D_x\mathcal{L}$ with entries $(D_x\mathcal{L})_{ij} = D_x[(\mathcal{L})_{ij}]$, and of the skew-symmetric matrix $\mathcal{M}$ with entries

$$\begin{aligned} &\mathcal{M}_{12} = A_3 D_x A_2 - A_2 D_x A_3, \quad \mathcal{M}_{13} = A_2 D_x A_1 - A_1 D_x A_2, \\ &\mathcal{M}_{14} = A_1 D_x A_3 - A_3 D_x A_1, \quad \mathcal{M}_{23} = A_1 D_x A_3 - A_3 D_x A_1, \\ &\mathcal{M}_{24} = A_1 D_x A_2 - A_2 D_x A_1, \quad \mathcal{M}_{34} = A_3 D_x A_2 - A_2 D_x A_3, \end{aligned}$$

in the form

$$\begin{aligned} M_1 &= \omega^{-1}\, (D_x\omega)\, \mathcal{L}; \\ M_2 &= \omega^{-2}\, \mathcal{M} + \omega^{-1}\, (D_x\omega)\, \mathcal{L}; \\ M_3 &= -\,\omega^{-2}\, (D_x\omega)\, \mathcal{L} + \omega^{-1}\, (D_x\mathcal{L}). \end{aligned}$$

7.2. Example 7

We can now consider a concrete example, i.e. a specific vector field to be μ-prolonged and a specific section γ.

We will use coordinates (u, v, w, z) for the space $U = \mathbf{R}^4$ of dependent variables, and restrict to the subset $|u| < 1$. We choose a vector field X_0 depending on α and acting in U. This will be

$$X_0 = -u\cos(|\alpha|)\partial_u + (\sin(|\alpha|)/|\alpha|)\ \left(\alpha^1 u\partial_v\ +\ \alpha^3 u\partial_w\ +\ \alpha^2 u\partial_z\right),$$

which is obtained from (32) for $\Theta = (-u, 0, 0, 0)$.

As for the section γ we choose the one identified by

$$A^1 = Bu,\ A^2 = 0,\ A^3 = B\sqrt{1-u^2}, \tag{33}$$

with $B \neq 0$ an arbitrary real constant; in the following we write $\rho = \sqrt{1-u^2}$. There is nothing special about these choices, except that we use rather simple ones in order to keep the resulting formulas simple enough; for the same reason we will choose $B = \pi/2$.

The vector field X_0 in U can then be completed to a vector field X in $\widehat{M}$ by the prescription $P^m = X_0(A^m)$. This yields

$$P^1 = -Bu\cos(|\alpha|), \qquad P^2 = 0, \qquad P^3 = Bu^2\rho^{-1}\ \cos(|\alpha|);$$

note that on γ we have $|\alpha| = B$, so that with our choice $B = \pi/2$ one gets simply $P^m = 0$, $m = 1, 2, 3$. The vector field X is thus

$$X = -u\cos(|\alpha|)\partial_u + |\alpha|^{-1}\sin(|\alpha|)\,\left(\alpha^1 u\partial_v + \alpha^3 u\partial_w + \alpha^2 u\partial_z\right) - Bu\cos(|\alpha|)\partial_1 + Bu^2\rho^{-1}\cos(|\alpha|)\partial_3.$$

It is immediate to check, see (17) above, that the manifold $\widehat{M}_\gamma$ corresponding to γ given by (33) is invariant under X (we recall this holds by construction). On this manifold, $|\alpha| = B = \pi/2$ and X reduces to

$$X_\gamma = u^2\,\partial_v + u\,\rho\,\partial_w. \tag{34}$$

Finally, the projection of this to a vector field in M is simply $W = X_\gamma$.

With this, we have completely described the upper row of the diagram (19). Let us now consider the lower one. First of all we have to compute $X^{(2)}$, which turns out to be

$$\begin{aligned} X^{(2)} = \; & -\cos(|\alpha|)\,(u(\partial/\partial u) + u_x(\partial/\partial u_x) + u_{xx}(\partial/\partial u_{xx})) \\ & + |\alpha|^{-1}\sin(|\alpha|)[\alpha_1 u(\partial/\partial v) + \alpha_1 u_x(\partial/\partial v_x) + \alpha_1 u_{xx}(\partial/\partial v_{xx})] \\ & + |\alpha|^{-1}\sin(|\alpha|)[\alpha_3 u(\partial/\partial w) + \alpha_3 u_x(\partial/\partial w_x) + \alpha_3 u_{xx}(\partial/\partial w_{xx})] \\ & + |\alpha|^{-1}\sin(|\alpha|)[\alpha_2 u(\partial/\partial z) + \alpha_2 u_x(\partial/\partial z_x) + \alpha_2 u_{xx}(\partial/\partial z_{xx})] \\ & - Bu\cos(|\alpha|)(\partial/\partial\alpha_1) + (Bu^2\rho^{-1})\cos(|\alpha|)(\partial/\partial\alpha_3). \end{aligned}$$

The restriction $X^{(2)}_\gamma$ of this to $\widehat{M}^{(2)}_\gamma$ is just

$$X^{(2)}_\gamma = u^2(\partial/\partial v) + \rho u(\partial/\partial w) + uu_x(\partial/\partial v_x) + \rho u_x(\partial/\partial w_x) + uu_{xx}(\partial/\partial v_{xx}) + \rho u_{xx}(\partial/\partial w_{xx});$$

the projection Y of this to $M^{(2)}$ is of course $Y = X^{(2)}_\gamma$.

Thus we get, using the notation (28) for Y,

$$\eta = -\begin{pmatrix} 0 \\ u^2 \\ \rho u \\ 0 \end{pmatrix}, \qquad \eta_x = -\begin{pmatrix} 0 \\ uu_x \\ \rho u_x \\ 0 \end{pmatrix}, \qquad \eta_{xx} = -\begin{pmatrix} 0 \\ uu_{xx} \\ \rho u_{xx} \\ 0 \end{pmatrix}.$$

We should then check that these satisfy the relations (29) with a suitable Λ; more precisely, in view of Theorem 1, with $\Lambda = -(D_x K_\gamma)K_\gamma^{-1}$ with K_γ the restriction of $K(\alpha)$ to the section γ.

Such a Λ can be computed using the general formulas given above, or more simply from (31). On γ we have $|\alpha| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} = B$; as for the matrix L, on γ this is

$$L_\gamma = B(uL_1 + \rho L_3) = B\begin{pmatrix} 0 & u & \rho & 0 \\ -u & 0 & 0 & -\rho \\ -\rho & 0 & 0 & u \\ 0 & \rho & -u & 0 \end{pmatrix};$$

hence on γ the matrix $J = |\alpha|^{-1}L$ is just the square matrix appearing in the formula above. It follows that K_γ and K_γ^{-1} are given by

$$K_\gamma = \cos(B)\,I + \sin(B)\begin{pmatrix} 0 & u & \rho & 0 \\ -u & 0 & 0 & -\rho \\ -\rho & 0 & 0 & u \\ 0 & \rho & -u & 0 \end{pmatrix},$$

$$K_\gamma^{-1} = \cos(B)\,I - \sin(B)\begin{pmatrix} 0 & u & \rho & 0 \\ -u & 0 & 0 & -\rho \\ -\rho & 0 & 0 & u \\ 0 & \rho & -u & 0 \end{pmatrix},$$

as follows from (31). These formulas are simplified by the choice $B = \pi/2$, yielding

$$K_\gamma = \begin{pmatrix} 0 & u & \rho & 0 \\ -u & 0 & 0 & -\rho \\ -\rho & 0 & 0 & u \\ 0 & \rho & -u & 0 \end{pmatrix}, \qquad K_\gamma^{-1} = -K_\gamma.$$

As for Λ, this is immediately computed from the above and $\Lambda = -(D_x K_\gamma) K_\gamma^{-1}$, yielding for $B = \pi/2$ (we omit the more complex formulas for the general case of arbitrary B)

$$\Lambda = \frac{1}{\sqrt{1-u^2}} \begin{pmatrix} 0 & 0 & 0 & -u_x \\ 0 & 0 & -u_x & 0 \\ 0 & u_x & 0 & 0 \\ u_x & 0 & 0 & 0 \end{pmatrix}.$$

One can then easily check that the $\{\eta, \eta_x, \eta_{xx}\}$ given above satisfy the prescribed relations, i.e. $\eta_x = D_x\eta + \Lambda\eta$, $\eta_{xx} = D_x\eta_x + \Lambda\eta_x$.

7.3. Example 8

Let us consider a different examples for the same action of $SU(2)$, now in the full $\mathbf{R}^4$ space. The section γ will now correspond to

$$\alpha^1 = z, \qquad \alpha^2 = 2z, \qquad \alpha^3 = 5z. \tag{35}$$

We will moreover choose $\Theta = (u, v, -z, w)$; thus the vector field to be gauged is a scaling in the (u, v) plane and a rotation in the (w, z) one. The corresponding gauged vector field in U is

$$\begin{aligned} X_0 = {} & \cos(\omega)(u\partial_u + v\partial_v - z\partial_w + w\partial_z) + \omega^{-1}\sin(\omega)[(\alpha_1 v + \alpha_2 w - \alpha_3 z)\partial_u - (\alpha_1 u + \alpha_3 w + \alpha_2 z)\partial_v \\ & - (\alpha_3 u + \alpha_2 v - \alpha_1 w)\partial_w - (\alpha_2 u - \alpha_3 v - \alpha_1 z)\partial_z]. \end{aligned}$$

Here we have written $\omega = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. With the usual method, we get

$$\begin{aligned} P^1 &= w\cos(\omega) - (\alpha_2 u - \alpha_3 v - \alpha_1 z)\omega^{-1}\sin(\omega), \\ P^2 &= 2[w\cos(\omega) - (\alpha_2 u - \alpha_3 v - \alpha_1 z)\omega^{-1}\sin(\omega)], \\ P^3 &= 5[w\cos(\omega) - (\alpha_2 u - \alpha_3 v - \alpha_1 z)\omega^{-1}\sin(\omega)]. \end{aligned}$$

We will thus consider the vector field in $\widehat{M}$ given, with again $\partial_i := (\partial/\partial\alpha^i)$, by

$$\begin{aligned} X = {} & \cos(\omega)[u\partial_u + v\partial_v - z\partial_w + w\partial_z + w(\partial_1 + 2\partial_2 + 5\partial_3)] \\ & + \omega^{-1}\sin(\omega)[(\alpha_1 v + \alpha_2 w - \alpha_3 z)\partial_u - (\alpha_1 u + \alpha_3 w + \alpha_2 z)\partial_v \\ & - (\alpha_3 u + \alpha_2 v - \alpha_1 w)\partial_w - (\alpha_2 u - \alpha_3 v - \alpha_1 z)\partial_z - (\alpha_2 u - \alpha_3 v - \alpha_1 z)(\partial_1 + 2\partial_2 + 5\partial_3)]. \end{aligned}$$

The manifold $\widehat{M}_\gamma$ is invariant under this, and the restriction of X to $\widehat{M}_\gamma$ is

$$\begin{aligned} X_\gamma = {} & \cos(\Omega)[u\partial_u + v\partial_v - z\partial_w + w\partial_z + w(\partial_1 + 2\partial_2 + 5\partial_3)] \\ & + \Omega^{-1}\sin(\Omega)[(vz + 2wz - 5z^2)\partial_u - (uz + 5wz + 2z^2)\partial_v \\ & - (5uz + 2vz - wz)\partial_w - (2uz - 5vz - z^2)\partial_z - (2uz - 5vz - z^2)(\partial_1 + 2\partial_2 + 5\partial_3)]; \end{aligned}$$

note that now ω has been replaced by $\Omega = \sqrt{30}z$. The projection of this vector field to M is

$$\begin{aligned} W = {} & \cos(\Omega)[u\partial_u + v\partial_v - z\partial_w + w\partial_z + w(\partial_1 + 2\partial_2 + 5\partial_3)] \\ & + \Omega^{-1}\sin(\Omega)[(vz + 2wz - 5z^2)\partial_u - (uz + 5wz + 2z^2)\partial_v \\ & - (5uz + 2vz - wz)\partial_w - (2uz - 5vz - z^2)\partial_z]. \end{aligned}$$

Let us now consider prolongations; the second prolongation of X in $\widehat{M}^{(2)}$ is computed by standard algebra, giving a rather long formula which we omit.

The restriction $X_\gamma^{(2)}$ of this to $\widehat{M}_\gamma^{(2)}$ is readily obtained via the substitution $\omega \to \Omega$ and those given by (35); as for the projection Y of $X_\gamma^{(2)}$ to $M^{(2)}$, in which we are mostly interested, in the notation (28), this corresponds to

$$\eta = \cos(\Omega)\begin{pmatrix} u \\ v \\ -z \\ w \end{pmatrix} + (1/\sqrt{30})\sin(\Omega)\begin{pmatrix} v + 2w - 5z \\ -(u + 5w + 2z) \\ -(5u + 2v - w) \\ -2u + 5v + z \end{pmatrix};$$

$$\eta_x = \cos(\Omega)\begin{pmatrix} u_x \\ v_x \\ -z_x \\ w_x \end{pmatrix} + (1/\sqrt{30})\sin(\Omega)\begin{pmatrix} v_x + 2w_x - 5z_x \\ -(u_x + 5w_x + 2z_x) \\ -(5u_x + 2v_x - w_x) \\ -2u_x + 5v_x + z_x \end{pmatrix};$$

$$\eta_{xx} = \cos(\Omega)\begin{pmatrix} u_{xx} \\ v_{xx} \\ -z_{xx} \\ w_{xx} \end{pmatrix} + (1/\sqrt{30})\sin(\Omega)\begin{pmatrix} v_{xx} + 2w_{xx} - 5z_{xx} \\ -(u_{xx} + 5w_{xx} + 2z_{xx}) \\ -(5u_{xx} + 2v_{xx} - w_{xx}) \\ -2u_{xx} + 5v_{xx} + z_{xx} \end{pmatrix}.$$

We should now check that relations (29) are satisfied for a suitable matrix $\Lambda = -(D_x K_\gamma)K_\gamma^{-1}$. In our case

$$K_\gamma = \cos(\Omega)\, I + \frac{\sin(\Omega)}{\sqrt{30}} \begin{pmatrix} 0 & 1 & 5 & 2 \\ -1 & 0 & 2 & -5 \\ -5 & -2 & 0 & 1 \\ -2 & 5 & -1 & 0 \end{pmatrix}.$$

It follows that

$$\Lambda = -\, z_x \begin{pmatrix} 0 & 1 & 5 & 2 \\ -1 & 0 & 2 & -5 \\ -5 & -2 & 0 & 1 \\ -2 & 5 & -1 & 0 \end{pmatrix}.$$

One can easily check that this satisfies indeed the required relations

$$\eta_x = D_x \eta + \Lambda \eta, \quad \eta_{xx} = D_x \eta_x + \Lambda \eta_x.$$

7.4. Example 9

We will now turn – still using the same real representation of $G = SU(2)$ – to examples dealing with Theorem 2.
Let us consider the vector field Y corresponding, in the notation (28) and writing $\rho := \sqrt{v^2 + z^2}$, to

$$\eta = u \left[\cos(\rho) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\sin(\rho)}{\rho} \begin{pmatrix} 0 \\ (v-z) \\ 0 \\ (v+z) \end{pmatrix} \right], \qquad \eta_x = \frac{u_x}{u}\eta, \qquad \eta_{xx} = \frac{u_{xx}}{u}\eta. \tag{36}$$

This is a μ-prolonged vector field, with $\mu = \Lambda \mathrm{d}x$ identified by

$$\Lambda = \rho^{-2} B_0 + (1/2)\rho^{-3} \sin(2\rho)\, B_1 + \rho^{-2} \sin^2(\rho)\, B_2,$$

where the matrices B_i are given by

$$B_0 = (v v_x + z z_x) \begin{pmatrix} 0 & -v & 0 & -z \\ v & 0 & -z & 0 \\ 0 & z & 0 & -v \\ z & 0 & v & 0 \end{pmatrix},$$

$$B_1 = (v_x z - v z_x) \begin{pmatrix} 0 & -z & 0 & v \\ z & 0 & v & 0 \\ 0 & -v & 0 & -z \\ -v & 0 & z & 0 \end{pmatrix},$$

$$B_2 = (v_x z - v z_x) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This Λ is in the general form given above provided γ is identified by

$$\alpha^1 = v, \; \alpha^2 = z, \; \alpha^3 = 0.$$

Having identified γ allows in turn to identify the gauging matrix K_γ as

$$K_\gamma = \cos(\rho)\, I + \rho^{-1} \sin(\rho) \begin{pmatrix} 0 & v & 0 & z \\ -v & 0 & z & 0 \\ 0 & -z & 0 & v \\ -z & 0 & -v & 0 \end{pmatrix};$$

hence the vector field $X_\gamma^{(2)}$ is also determined and – with the procedure described in Section 5.4 – the full $X^{(2)}$ is readily obtained as well (we omit the involved explicit formula); this is the standard prolongation of

$$X \;=\; \cos(\omega)\,(u\partial_u + u\partial_w) \;-\; \omega^{-1} \sin(\omega) u \left[\alpha_3 \partial_u + (\alpha_2 - \alpha_1)\partial_v - \alpha_3 \partial_w + (\alpha_2 - \alpha_1)\partial_z + \partial_1 - \partial_3 \right].$$

This leaves γ invariant, and its components along U correspond to a gauged vector field, being of the form $\varphi^a = [K(\alpha)]^a_{\ b} \Theta^b$ with $\Theta = (1, 0, 0, 0)$.

7.5. Example 10

We will now consider the vector field Y in $M^{(2)}$ given (writing $\Omega = \sqrt{30} z$) by

$$Y = \cos(\Omega)\,(\partial_u + \partial_w) + \frac{\sin(\Omega)}{\sqrt{30}}\,(5\partial_u + \partial_v - 5\partial_w - 3\partial_z);$$

that is, with the notation (28) we have

$$\eta = \cos(\Omega) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{z \sin(\Omega)}{\Omega} \begin{pmatrix} 5 \\ 1 \\ -5 \\ -3 \end{pmatrix}, \qquad \eta_x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \eta_{xx} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is a μ-prolonged vector field, with

$$\Lambda = z_x \begin{pmatrix} 0 & -1 & -5 & -2 \\ 1 & 0 & -2 & 5 \\ 5 & 2 & 0 & -1 \\ 2 & -5 & 1 & 0 \end{pmatrix};$$

this in turn corresponds, see our general formulas above, to γ identified by

$$\alpha_1 = z, \qquad \alpha_2 = 2z, \qquad \alpha_3 = 5z.$$

We can in this way identify $X_\gamma^{(2)}$ and $X^{(2)}$; the latter turns out to be

$$\begin{aligned} X^{(2)} = \; & \cos(\omega)\ (\partial_u + \partial_v) + [\sin(\omega)/\omega]\ (\alpha_3 \partial_u + (\alpha_2 - \alpha_1)\partial_v - \alpha_3 \partial_w \\ & - \ (\alpha_1 + \alpha_2)\partial_z \ - \ (\alpha_1 + \alpha_2)(\partial_1 + 2\partial_2 + 5\partial_3)) \,. \end{aligned}$$

The vector field X just coincides with $X^{(2)}$; its components along U, given by $\varphi = \cos(\omega)(1, 1, 0, 0) + \omega^{-1} \sin(\omega)(\alpha_3, (\alpha_2 - \alpha_1), -\alpha_3, -(\alpha_1 + \alpha_2))$, correspond to a gauged vector field obtained for the choice $\Theta = (1, 0, 1, 0)$. The gauging matrix $K(\alpha)$ can be derived either by this or noticing that the Λ given above corresponds to

$$K_\gamma = \cos(\Omega)\, I;$$

with the usual prescription this yields

$$K(\alpha) = \cos(\omega)\, I + \frac{\sin(\omega)}{\omega} \begin{pmatrix} 0 & a_1 & a_3 & a_2 \\ -a_1 & 0 & a_2 & -a_3 \\ -a_3 & -a_2 & 0 & a_1 \\ -a_2 & a_3 & -a_1 & 0 \end{pmatrix}.$$

The same result is obtained comparing X and the Θ given above.

8. Examples III. Non abelian groups: SO(3)

In this section we will consider the group $G = SO(3)$ acting in R^3 by its natural representation; once again we will consider very simple vector fields and section. Coordinates in U will be denoted by (u, v, w). Example 11 deals with Theorem 1, while Example 12 with Theorem 2.

8.1. SO(3) algebra and group action; lambda matrices

We will consider generators $L_i = T(\ell_i)$,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The group element g corresponding to $g = \exp(\ell)$ for ℓ a generic element of the algebra is readily computed. Consider a generic matrix $L = \alpha^i L_i$; this is written explicitly as

$$L = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}.$$

Some easy computations show that higher powers of L satisfy $(k \geq 0)$

$$L^{2k+1} = (-1)^k \omega^{2k} L, \qquad L^{2(k+1)} = (-1)^k \omega^{2k} L^2,$$

where we used

$$\omega = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}.$$

Using the Taylor expansions of trigonometric functions, it follows that $K(\alpha) = \exp(L)$ and its inverse can be written as

$$\begin{aligned} K(\alpha) &= I \ + \ \omega^{-1} \sin(\omega)\, L + \omega^{-2} [1 - \cos(\omega)]\, L^2 \\ K^{-1}(\alpha) &= I \ - \ \omega^{-1} \sin(\omega)\, L + \omega^{-2} [1 - \cos(\omega)]\, L^2. \end{aligned}$$

We will not give the general expression of matrices $\Lambda = -(D_x K_\gamma) K_\gamma^{-1}$ corresponding to K, K^{-1} given above, as the formula – which can be readily derived with the help of a symbolic manipulation program – is quite involved; its general shape is $\Lambda = M_0 + \sin(\omega) M_1 + \cos(\omega) M_2$, where the M_i are three-dimensional matrices.[9]

The general form of gauged vector fields in U is easily obtained applying $K(\alpha)$ on $\Theta = (\theta^1, \theta^2, \theta^3)$; conversely, given a gauged vector field with components $\Phi = (\varphi^1, \varphi^2, \varphi^3)$, the corresponding Θ is given by $\Theta = [K^{-1}(\alpha)]\Phi$.

8.2. Example 11

We will consider $\mathbf{R}^3$ with cartesian coordinates (u, v, w) and $U \subset \mathbf{R}^3$ defined by $u < 1$. The vector field X_0 in U will be

$$X_0 = u\omega^{-2} \left[\left(\alpha_1^2 + (\alpha_2^2 + \alpha_3^2) \cos(\omega) \right) \partial_u + \left(\alpha_1 \alpha_2 (1 - \cos(\omega)) + \alpha_3 \omega \sin(\omega) \right) \partial_v \right.$$
$$\left. + \left(\alpha_1 \alpha_3 (1 - \cos(\omega)) - \alpha_2 \omega \sin(\omega) \right) \partial_w \right],$$

where as usual $\omega = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. This is a gauged vector field, corresponding to the choice $\Theta = (u, 0, 0)$.

We choose the section γ identified by

$$\alpha^1 = \beta u, \ \alpha^2 = 0, \ \alpha^3 = \beta \sqrt{1 - u^2},$$

where β is an arbitrary real constant; formulas are simpler with the choice $\beta = \pi/2$. The vector field X_0 is completed to a vector field $X = X_0 + P^m \partial_m$ in $\widehat{M}$, leaving γ invariant, with the choice $P^m = X_0(A^m)$.

Using the notation $\rho = \sqrt{1 - u^2}$, the restriction of X to γ is

$$X_\gamma = \left(u + u^3 (1 - \cos\beta) \right) \partial_u + (u\rho \sin\beta) \partial_v + \left(2u^2 \rho \sin^2(\beta/2) \right) \partial_w$$
$$+ \beta u \left(u^2 + (1 - u^2) \cos\beta \right) \partial_1 - \beta u^2 \rho^{-1} \left(u^2 + (1 - u^2) \cos\beta \right) \partial_3;$$

the projection of this to M is easily computed to be

$$W = \left(u + u^3 (1 - \cos\beta) \right) \partial_u + (u\rho \sin\beta) \partial_v + \left(2u^2 \rho \sin^2(\beta/2) \right) \partial_w.$$

We can compute with standard procedure the second prolongation of X, restrict it to $\widehat{M}^{(2)}_\gamma$ and project to $M^{(2)}$. The final result is in the form ((28)) with

$$\eta = u \begin{pmatrix} u^2 + (1 - u^2) \cos\beta \\ \rho \sin\beta \\ 2\rho u \sin^2(\beta/2) \end{pmatrix}, \qquad \eta_x = \frac{u_x}{u} \eta, \qquad \eta_{xx} = \frac{u_{xx}}{u} \eta.$$

These should be checked to satisfy the relations ((29)) with Λ given by Theorem 1. In the present case,

$$K_\gamma = \begin{pmatrix} u^2 + (1 - u^2) \cos\beta & -\rho \sin\beta & \rho u (1 - \cos\beta) \\ \rho \sin\beta & \cos\beta & -u \sin\beta \\ \rho u (1 - \cos\beta) & u \sin\beta & (1 - u^2) + u^2 \cos\beta \end{pmatrix};$$

$$K_\gamma^{-1} = \begin{pmatrix} u^2 + (1 - u^2) \cos\beta & \rho \sin\beta & \rho u (1 - \cos\beta) \\ -\rho \sin\beta & \cos\beta & u \sin\beta \\ \rho u (1 - \cos\beta) & -u \sin\beta & (1 - u^2) + u^2 \cos\beta \end{pmatrix}.$$

With standard computations, we get first $(D_x K_\gamma)$ and then

$$\Lambda = -(D_x K_\gamma) K_\gamma^{-1} = \frac{u_x}{\rho} \begin{pmatrix} 0 & -u & -1 \\ u \sin\beta & 0 & \rho \sin\beta \\ 2 \sin^2(\beta/2) & -\rho \sin\beta & 0 \end{pmatrix}.$$

It is easily checked that indeed ((29)) are satisfied.

In the simple case $\beta = \pi/2$, we are reduced to

$$\eta = u \begin{pmatrix} u^2 \\ \rho \\ \rho u \end{pmatrix}, \qquad \eta_x = \left(\frac{u_x}{u} \right) \eta, \qquad \eta_{xx} = \frac{u_{xx}}{u} \eta; \qquad \Lambda = \frac{u_x}{\rho} \begin{pmatrix} 0 & -u & -1 \\ u & 0 & \rho \\ 1 & -\rho & 0 \end{pmatrix}.$$

8.3. Example 12

Let us consider the vector field Y given, in the notation ((28)) and using the conventions set in the previous Example, by

$$\eta = u \begin{pmatrix} -\rho w \\ 0 \\ uw \end{pmatrix}, \qquad \eta_x = u_x \begin{pmatrix} -\rho w \\ 0 \\ uw \end{pmatrix}, \qquad \eta_{xx} = u_{xx} \begin{pmatrix} -\rho w \\ 0 \\ uw \end{pmatrix}.$$

9 We stress this Λ depends on the three arbitrary smooth functions $A_i(x, u, v, w, z)$ and their derivatives. Thus if we want to identify a given three-dimensional matrix with Λ, this yields a system of PDEs for the three functions A_i.

These satisfy ((29)) with

$$\Lambda = \rho \begin{pmatrix} 0 & -uu_x & -u_x \\ uu_x & 0 & \rho u_x \\ u_x & -\rho u_x & 0 \end{pmatrix}.$$

In order to associate this with a gauging matrix K_γ, one can either proceed by massive computations using the general form of Λ in terms of the functions $A^i(x, u, v, w)$, or observe that only the u variable appears in Λ, and proceed by trial and error to determine

$$K_\gamma = \begin{pmatrix} u^2 & -\rho & \rho u \\ \rho & 0 & -u \\ \rho u & u & 1-u^2 \end{pmatrix} \text{ with } K_\gamma^{-1} = \begin{pmatrix} u^2 & \rho & \rho u \\ -\rho & 0 & u \\ \rho u & -u & 1-u^2 \end{pmatrix}.$$

At this point it suffices to use K_γ^{-1} to transform the $(\eta, \eta_x, \eta_{xx})$ into $\theta = K_\gamma^{-1}\eta$ etc.; we get

$$\theta = \begin{pmatrix} 0 \\ u^2 w + u(1-u^2)w \\ 0 \end{pmatrix}, \qquad \theta_x = u_x\,\theta, \qquad \theta_{xx} = u_{xx}\,\theta.$$

It is easy to check that $\theta_x = D_x\theta$ and $\theta_{xx} = D_x\theta_x$, i.e. the vector field

$$\theta^a(\partial/\partial u^a) + \theta^a_x(\partial/\partial u^a_x) + \theta^a_{xx}(\partial/\partial u^a_{xx})$$

is the standard second prolongation of

$$X_0 = \left(u^2 w + u(1-u^2)w\right)\,(\partial/\partial v).$$

In order to complete this to a vector field in $\widehat{M}$, we compare K_γ and the general expression for $K(\alpha)$, and observe that we obtain such a K_γ by choosing γ identified by

$$\alpha^1 = (\pi/2)u,\ \alpha^2 = 0, \qquad \alpha^3 = (\pi/2)\sqrt{1-u^2}.$$

Applying X_0 on the functions $X^i(x, u, v, w)$ defined by these relations, we get $P^1 = (\pi/2)u$, $P^2 = 0$, $P^3 = (\pi/2)\sqrt{1-u^2}$.

9. Discussion

We have thus shown that μ-prolongation can be understood as a standard prolongation in the gauge jet bundle $\widehat{M}^{(k)}$, restricted to a section of the bundle and then projected to the $M^{(k)}$ bundle.

With this point of view on μ-prolongations – and on the discussion in the present work – there are some observations to be made and points to be stressed; some of these suggest in turn further developments.

(1) We were able to conduct our discussion within the framework of gauged vector fields; acting by a gauge transformation $\gamma \in \Gamma(P_G)$ in $\widehat{M}^{(k)}$, the bundle J^kM, which in this context should be seen as a submanifold of $\widehat{M}^{(k)}$, is mapped into $\widehat{M}^{(k)}_\gamma$, and prolonged vector fields are mapped into vector fields which, if projected back to J^kM, appear as μ-prolonged ones. By acting with the inverse transformation, the μ-prolonged vector field can be transformed back into a standard prolonged vector field.

(2) Roughly speaking, this shows that the possibility to apply μ-prolongations and μ-symmetries of differential equations with the same effectiveness as standard prolongations and symmetries can be understood as a consequence of the fact that the differential equations under consideration are written in terms of ordinary rather than covariant derivatives; this in turn makes that the equations are set (with the language employed in this note) in the ordinary bundle J^kM rather than in the augmented bundle $J^k\widehat{M}$.

(3) In this way, a gauge transformation maps the differential equation Δ under study into a different equation $\widetilde{\Delta}$, which can admit (ordinary) symmetries not admitted by the original equation as ordinary symmetries, albeit they are admitted as μ-symmetries. Thus one could say that the approach devised by Muriel and Romero is somehow opposite to the one which is standard in field theory: rather than promoting PDEs to covariant equations (that is, write them in terms of covariant derivatives), one keeps non-covariant equations and uses gauge transformation to maps them to different equations. This procedure represents an advantage if the gauge orbit of the considered equation Δ contains an equation $\widetilde{\Delta}$ with a higher symmetry.

(4) This point of view also suggests an obvious approach to apply μ-symmetries for the reduction of general systems of differential equations by differential invariants (so far a procedure for this is known only when the Λ_i satisfy some additional conditions [7–9]; it is not known if these are only sufficient or also necessary). That is, work in the whole $J^k\widehat{M}$, where differential invariants of higher orders can be obtained from lower order ones by the familiar recursive procedure, and then restrict to the relevant sub-bundle $J^k\widehat{M}_\gamma$. Such an approach is however too simple to work; the reason is that, due to

the term $P^m(\partial/\partial\alpha^m)$, the standard recursive procedure [3] does not in general give new invariants[10] and should be modified accordingly. This will be considered in a separate contribution.

(5) Note also, making free use of the notation introduced in the proof to Theorem 1, that the reduction from vector fields in $\widehat{M}$ to vector fields in $\widehat{M}_\gamma$ was natural for the diagram (18) provided the relation $\vartheta^m = (\partial A^m/\partial u^a)\varphi^a$ was satisfied, and this independently of the condition (10) for X. On the other hand, the operator $\widehat{\mathrm{P}}_\gamma$ and hence P_γ (which then turned out to be the μ-prolongation operator with suitable μ) was at first defined as the operator making the left-hand side of (18) commutative. Thus, it may also be defined independently of (10). In principles, this could give a generalization of the μ-prolongation operation. This problem will also be considered elsewhere.

(6) It should also be mentioned that here we worked with sections of $\mathcal{GB}$ as basic objects, corresponding to the requirement $g = g(x, u)$. One could start from sections of $J^s\mathcal{GB}$, corresponding to gauge transformations with $g = g(x, u^{(s)})$, i.e. depending not only on the space–time point x and on the values of fields u^a at x, but also on the values of field derivatives up to order s at x.

(7) The point (3) of the present discussion suggests that in order to deal with μ-symmetries of differential equations set in terms of standard partial derivatives, it might be convenient to rewrite them in terms of covariant derivatives (that is, write partial derivatives u_i as $u^a_i = \nabla_i u^a - (\Lambda_i)^a_{\ b} u^b$, and the like for higher derivatives), extending them to covariant equations in the $\widehat{M}$ space – allowing of course also changes in the Λ_i matrices and the reference $\mathcal{GB}$ section – in order to take full advantage of the gauge formalism.

(8) As for physical relevance, the analysis here (and that in [23], respectively) considered above show that with the formalism of μ-symmetries one can be able to detect symmetries and hence conserved quantities even if working in a non-convenient gauge (respectively, reference frame). This opens the interesting possibility of applying symmetry analysis and Noether's theorem (see [13] in this respect) also when working in a gauge in which the equations are not manifestly symmetric.

(9) The point of view embodied in this work, based on the gauge bundle, has several points of contact with those explored in recent works by other authors: P. Morando considered a gauging of the exterior derivative and showed how this leads to μ-symmetries [25]; D. Catalano-Ferraioli considered auxiliary variables in the context of the theory of *coverings*, showing intriguing relations between (local) λ-symmetries and nonlocal standard symmetries [26]; see also [24]. This relations are also used in connection to *solvable structures* in [27]. Relation with this approach is briefly discussed in the Appendix.

(10) Finally, we recall that – as discussed in [12] – the case of a scalar ODE, originally considered by Muriel and Romero [6], is degenerate in several ways. These degenerations hide the rich geometrical structure displayed in the general case of system of PDEs, i.e. passing from λ to μ-prolongations, and make that the case considered at first is actually the most difficult one. Needless to say, this makes the work by Muriel and Romero even more remarkable.

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Appendix. Gauge variables as auxiliary variables

When we fix a gauge, i.e. set $\alpha^m = A^m(x, u)$, we are actually prescribing a correspondence between functions $u^a = f^a(x)$ and expressions of α as functions of the x themselves, via $\alpha^m = F^m(x) := A^m(x, f(x))$. In this sense, gauge fixing is equivalent to a constraint relating the x, u and the α seen as auxiliary dependent variables. This description of gauge variables as auxiliary dependent variables is reminiscent of the approach to λ-symmetries via the formalism of coverings [26], and we would thus like in this Appendix to discuss how our construction can be modified to take this point of view into account.

Let us consider "fully augmented" bundles $\mathcal{M} = M \times \mathcal{G}$, in which the gauge variables α should be seen as new dependent variables. Thus, in the corresponding jet bundles $J^k\mathcal{M} = \mathcal{M}^{(k)}$ will also appear variables α^m_J corresponding to derivatives of the α, and the contact structure in $\mathcal{M}^{(k)}$ will also include forms $\Xi^m_J = \mathrm{d}\alpha^m_J - \alpha^m_{J,i}\mathrm{d}x^i$. The total derivatives operators $\mathcal{D}_i$ in $\mathcal{M}^{(k)}$ will thus be

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + u^a_{J,i}\,\frac{\partial}{\partial u^a_J} + \alpha^m_{J,i}\,\frac{\partial}{\partial \alpha^m_J} = D_i + Z_i;$$

here we have of course defined $Z_i = \alpha^m_{J,i}(\partial/\partial\alpha^m_J)$ and denoted again by D_i the usual total derivative operator in $M^{(k)}$.

Now the gauge fixing operator δ_γ corresponds to introducing the constraint γ given by $\alpha^m - A^m(x, u) = 0$. For any function $F(x, u, \alpha)$ we have immediately

$$\delta_\gamma\,[D_i F] = \left[(\partial F/\partial x^i)\right]_\gamma + u^a_i\,\left[(\partial F/\partial u^a)\right]_\gamma .$$

[10] This is related to the fact we now have, for general vector fields and with standard notation, $[X^{(k)}, D_x] = -(D_x\xi)D_x - (D_x P^m)(\partial/\partial\alpha^m)$ rather than just $[X^{(k)}, D_x] = -(D_x\xi)D_x$.

On the other hand,

$$D_i\,[\delta_\gamma F] = \left[(\partial F/\partial x^i)\right]_\gamma + u^a_i\,\left[(\partial F/\partial u^a)\right]_\gamma + \left[D_i A^m(x,u)\right]\,\left[(\partial F/\partial \alpha^m)\right]_\gamma .$$

Comparing these two expressions we get $D_i[\delta_\gamma F] = \delta_\gamma[(D_i + Z_i)(F)]$, which can also be written as

$$\delta_\gamma\,(\mathcal{D}_i F) = D_i\,(\delta_\gamma F), \tag{A.1}$$

or equivalently

$$\delta_\gamma\,(D_i F) = D_i\,(\delta_\gamma F) - \delta_\gamma\,(Z_i F). \tag{A.2}$$

The relation (A.1) describes how total differential operators should be modified if applied before or after gauge fixing.

In particular, if we apply (A.2) on $Q^a(x,u,\alpha,u_x) = [K(\alpha)]^a_{\ b}\Theta^b(x,u,u_x)$, we have $Z_i Q^a = (Z_i K^a_{\ b})\Theta^b = [(Z_i K)K^{-1}]^a_{\ b} Q^b$. It is easy to see that

$$\delta_\gamma[(Z_i K)K^{-1}] = (D_i K_\gamma)K_\gamma^{-1} \;:=\; R_i^{(\gamma)},$$

where as before we wrote $K_\gamma = \delta_\gamma(K)$ and $R_i^{(\gamma)}$ is also defined as above.

That is, if we first act with D_i and then fix the gauge we obtain the same result as by first fixing the gauge and then acting with $\nabla_i^{(\gamma)} = D_i - R_i^{(\gamma)}$. We can then identify the twisted prolongation obtained by acting with operators $\nabla_i^{(\gamma)}$ (rather than D_i) with a μ-prolongation, proceeding in the same way – and with the same μ – as in the main text.

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