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On double Poisson structures on commutative algebras

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Abstract

Double Poisson structures (à la Van den Bergh) on commutative algebras are considered. The main result shows that there are no non-trivial such structures on polynomial algebras of Krull dimension greater than one. For an arbitrary commutative algebra A, this places significant restrictions on possible double Poisson structures. Exotic double Poisson structures are exhibited by the case of the polynomial algebra on a single generator, previously considered by Van den Bergh.

Keywords: Double Poisson algebra, polynomial algebra, commutative algebra, multi-derivation
2010 MSC: 17B63

1. Introduction

The notion of a double Poisson structure on an associative R-algebra (for R a commutative unital ring) was introduced by Van den Bergh as a form of non-commutative Poisson structure [VdB08]; the structure is defined by a double bracket, which is an R-linear map $A^{\otimes 2} \to A^{\otimes 2}$ satisfying antisymmetry and non-commutative derivation conditions. For a double Poisson structure, the double bracket also satisfies the double Jacobi relation (see Sections 2 and 3).

The naïve relationship with (commutative) Poisson structures is as follows: composing with the multiplication map of A gives a bracket $A \otimes A \to A$ which induces a Poisson structure on the abelianization of A. More generally, a double Poisson structure induces a Poisson structure on the associated representation schemes [VdB08]. There is a related, but weaker, version of non-commutative Poisson structure, due to Crawley-Boevey [CB11]; this is sufficient to induce a Poisson structure on the representation schemes.

The notion of double Poisson structure is very rigid; nevertheless, interesting examples are known, for example those related to non-commutative symplectic structures. Moreover, a classification of certain double Poisson structures on free associative algebras (tensor algebras) has been given in small rank [ORS13,

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Sok13]; however, a double Poisson structure on a non-commutative algebra does not in general induce a double Poisson on its abelianization.

It is natural to consider what happens when the algebra A is already commutative. For example, Van den Bergh stated a classification of (homogeneous) double Poisson structures on the polynomial algebra $\Bbbk[t]$ over a field: up to scalar, there are only two non-trivial (homogeneous) structures. A proof of the corresponding result (over a commutative ring R on which squaring is injective) is given here as Proposition A.1, also giving the (essentially unique) non-homogeneous example. This provides important *exotic* examples.

For higher Krull dimension the situation is more dramatic; the following is Theorem 5.7 below:

Theorem 1. *Let R be a commutative ring on which the squaring map $x \mapsto x^2$ is injective, then there is no non-trivial double Poisson structure on $A := R[t_1, \dots, t_d]$ for $d \geq 2$.*

The result is a simple consequence of a general structure result on multi-derivations (see Theorem 4.3); these multi-derivations (defined in Section 2) correspond to the n-brackets of Van den Bergh, except that the 'anti-equivariance' condition with respect to the action of the cyclic group $\mathbb{Z}/n$ is not imposed.

This highlights the fact that, on commutative algebras, the axioms of a double Poisson structure are highly restrictive and provides further evidence that the notion of double Poisson structure should be relaxed, considering weaker structures such as Crawley-Boevey's non-commutative Poisson structures.

Other authors have observed that it is useful to relax the axioms of double Poisson algebras (see [Art15], for example); it is however desirable (from the computational viewpoint) to retain the multi-derivation property, so the general structure result,Theorem 4.3, applies in this setting. Corollary 4.7 shows that, in the polynomial case (of Krull dimension greater than one), this relaxation is not sufficient to be able to construct non-trivial non-commutative Poisson structures (in the sense of [CB11]).

Section 6 considers the general case of double Poisson structures on a commutative algebra. These are either *standard*, arising from double brackets on polynomial algebras, or are *exotic*. The results for polynomial algebras give a reasonable understanding of the standard double Poisson structures; the exotic case is illustrated by the results for $\Bbbk[t]$, as indicated above. Further consequences will be considered elsewhere.

Various other lines of investigation are possible. For instance, the work of Berest, Ramadoss et al. [BKR13, BCER12] suggests that double Poisson structures for algebras should be studied in the derived setting.

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2. Multi-derivations

Fix a commutative unital ring R and a unital, associative R-algebra A; all tensor products are taken over R. For $2 \leq n \in \mathbb{N}$, the symmetric group $\mathfrak{S}_n$ acts by place permutations on the tensor product $A^{\otimes n}$ ($\sigma(a_1 \otimes \ldots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)}$) and, hence, by conjugation on $\mathrm{Hom}_R(A^{\otimes n}, A^{\otimes n})$ via $\varphi \mapsto \sigma \cdot \varphi := \sigma \circ \varphi \circ \sigma^{-1}$, so that a linear map φ is $\mathfrak{S}_n$-equivariant if and only if it is fixed under this action. The group $\mathbb{Z}/n$ is considered as a subgroup of $\mathfrak{S}_n$, hence the above action restricts to $\mathbb{Z}/n$.

The R-module of double derivations $\mathbb{D}\mathrm{er}(A)$ is by definition the submodule

$$\mathrm{Der}(A, A^{\otimes 2}) \subset \mathrm{Hom}_R(A, A^{\otimes 2})$$

of derivations, where $A^{\otimes 2}$ is equipped with the outer bimodule structure; explicitly $\psi \in \mathrm{Hom}_R(A, A^{\otimes 2})$ belongs to $\mathbb{D}\mathrm{er}(A)$ if and only if, for all $a, b \in A$, $\psi(ab) = (a \otimes 1)\psi(b) + \psi(a)(1 \otimes b)$, using the product in $A^{\otimes 2}$. (See [Gin05], for example.)

Example 2.1. The double derivation $d_A \in \mathbb{D}\mathrm{er}(A)$ is the R-linear map $a \mapsto a \otimes 1 - 1 \otimes a$. This induces the universal derivation, $A \to \Omega^{nc}_A$, where the bimodule Ω^{nc}_A of non-commutative differentials is identified as the kernel of the multiplication $A \otimes A \xrightarrow{\mu} A$.

Lemma 2.2. *Let A be a commutative R-algebra. Multiplication at the codomain $A^{\otimes 2}$ induces a morphism of R-modules:*

$$\mathbb{D}\mathrm{er}(A) \otimes A^{\otimes 2} \to \mathbb{D}\mathrm{er}(A).$$

In particular, the double derivation d_A gives rise to the morphism of R-modules:

$$\Pi : A^{\otimes 2} \to \mathbb{D}\mathrm{er}(A)$$

sending $\Theta \in A^{\otimes 2}$ to $a \mapsto (a \otimes 1 - 1 \otimes a)\Theta$.

Proof. Straightforward. □

Remark 2.3. This result does not require that A is a commutative and corresponds to the usual A-bimodule structure on $\mathbb{D}\mathrm{er}(A)$ provided by the inner bimodule structure of $A^{\otimes 2}$. This formulation is given for ease of comparison with Lemma 2.7 (where commutativity is required).

By analogy with the case of double derivations, $\varphi \in \mathrm{Hom}_R(A^{\otimes n}, A^{\otimes n})$ is said to be a derivation with respect to the last variable if, $\forall a, b \in A$ and $\forall \alpha \in A^{\otimes n-1}$:

$$\varphi(\alpha \otimes ab) = (a \otimes 1^{\otimes n-1})\varphi(\alpha \otimes b) + \varphi(\alpha \otimes a)(1^{\otimes n-1} \otimes b),$$

using the product of $A^{\otimes n}$. This allows the following definition of multi-derivations, where the $\mathbb{Z}/n$-action is used to define the relevant bimodule structures.

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Definition 2.4. For $2 \leq n \in \mathbb{N}$, the R-module of multi-derivations

$$\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}) \subset \mathrm{Hom}_R(A^{\otimes n}, A^{\otimes n})$$

is the submodule of morphisms φ such that $\sigma \cdot \varphi$ is a derivation with respect to the last variable, for every $\sigma \in \mathbb{Z}/n$.

Let

$$\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})^{\mathrm{sgn}} \subset \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})$$

denote the sub R-module of multi-derivations φ such that $\sigma \cdot \varphi = (-1)^{\mathrm{sgn}(\sigma)}\varphi$, $\forall \sigma \in \mathbb{Z}/n$.

The following is clear from the definition:

Lemma 2.5. *The sub R-modules*

$$\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})^{\mathrm{sgn}} \subset \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}) \subset \mathrm{Hom}_R(A^{\otimes n}, A^{\otimes n})$$

are stable under the action of $\mathbb{Z}/n$.

Remark 2.6. For $2 \leq n \in \mathbb{N}$, $\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})^{\mathrm{sgn}}$ is the R-module of n-brackets (in the terminology of [VdB08, Definition 2.2.1]). In particular, for $n = 2$, this gives the definition of a double bracket, namely an *anti-symmetric* bi-derivation and, for $n = 3$, triple brackets are multi-derivations which are cyclically invariant.

The following results provide analogues of Lemma 2.2.

Lemma 2.7. *Let A be a commutative R-algebra and $2 \leq n \in \mathbb{N}$. Multiplication in the codomain induces a morphism of $R[\mathbb{Z}/n]$-modules*

$$\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}) \otimes A^{\otimes n} \to \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}),$$

where the left hand side is equipped with the diagonal $\mathbb{Z}/n$-action.

Proof. Straightforward. □

Proposition 2.8. *Let A be a commutative R-algebra and $2 \leq n \in \mathbb{N}$. The map $\varphi_n \in \mathrm{Hom}_R(A^{\otimes n}, A^{\otimes n})$ defined by*

$$\varphi_n(a_1 \otimes \ldots \otimes a_n) := \prod_{\sigma \in \mathbb{Z}/n} \sigma(a_{\sigma(n)} \otimes 1^{\otimes n-1} - 1^{\otimes n-1} \otimes a_{\sigma(n)})$$

(where the product is formed in $A^{\otimes n}$) is a $\mathbb{Z}/n$-equivariant multi-derivation (that is $\varphi_n \in \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})^{\mathbb{Z}/n}$).

In particular, φ_n together with the map of Lemma 2.7 induce a $\mathbb{Z}/n$-equivariant map:

$$\Pi_n : A^{\otimes n} \to \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}).$$

Proof. That φ_n is $\mathbb{Z}/n$-equivariant is clear from the construction. The proof that it is a multi-derivation is analogous to the proof that d_A is a double derivation. The final statement is then an immediate consequence of Lemma 2.7. □

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Remark 2.9. For $n = 2$, $\varphi_2(x\otimes y) = (x\otimes 1-1\otimes x)(y\otimes 1-1\otimes y)$ is clearly invariant under exchange of x and y. This reflects the fact that, when A is commutative, $A^{\otimes 2}$ has a canonical bimodule structure given by the algebra structure.

For $n = 3$, $\varphi_{a,b,c} := \varphi_3(a \otimes b \otimes c)$ is given explicitly by:

$$\begin{aligned}\varphi_{a,b,c} &= (c \otimes 1 \otimes 1 - 1 \otimes 1 \otimes c)(1 \otimes 1 \otimes b - 1 \otimes b \otimes 1)(1 \otimes a \otimes 1 - a \otimes 1 \otimes 1)\\ &= ac \otimes b \otimes 1 - ac \otimes 1 \otimes b - c \otimes ab \otimes 1 + c \otimes a \otimes b - a \otimes b \otimes c\\ &\quad +a \otimes 1 \otimes bc + 1 \otimes ab \otimes c - 1 \otimes a \otimes bc,\end{aligned}$$

where the terms have been arranged using the left lexicographical order for the partial order corresponding to the number of terms in a monomial in a, b, c of A. (Observe that there is a unique term of maximal lexicographical order, namely $ac \otimes b \otimes 1$.)

The expression for $\varphi_3(a\otimes b\otimes c)$ is normalized (up to sign) by the choice of the bimodule structure of $A^{\otimes 3}$, corresponding to the factor $(c \otimes 1 \otimes 1 - 1 \otimes 1 \otimes c)$. Although φ_3 is invariant under the action of $\mathbb{Z}/3$, φ_3 evaluated on $b \otimes c \otimes a$ clearly gives a different expression, contrary to the behaviour for $n = 2$.

2.1. Graded algebras

When A is an $\mathbb{Z}$-graded R-algebra it is natural to consider the graded components of multi-derivations.

Remark 2.10. The grading is not taken into account in the symmetric monoidal structure on graded R-modules.

Lemma 2.11. *For A a $\mathbb{Z}$-graded R-algebra which is finitely-generated as a graded algebra and $2 \leq n \in \mathbb{N}$, there is a $R[\mathbb{Z}/n]$-equivariant decomposition into homogeneous components:*

$$\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}) \cong \bigoplus_t \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})^t$$

where $\mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})^t$ is the submodule of morphisms of degree t.

In particular, any element $\varphi \in \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})$ can be written in terms of homogeneous components $\varphi = \sum_{t\in\mathbb{Z}} \varphi^t$, where $\varphi^t = 0$ for $|t| \gg 0$.

Proof. The multi-derivation property and the fact that A is assumed to be finitely-generated implies that an element $\varphi \in \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})$ is determined by its restriction to $(V)^{\otimes n}$ for V a finitely-generated graded R-submodule of A, so that $(V)^{\otimes n}$ is a finitely-generated R-module. The proof is then straightforward. □

Notation 2.12. For $A, \varphi \neq 0$ as in Lemma 2.11, write $\varphi^{\min}$ and $\varphi^{\max}$ respectively for the non-trivial homogeneous components of minimal and maximal degrees.

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3. Recollections on double Poisson algebras

Definition 3.1. For $\varphi \in \mathrm{Hom}_R(A^{\otimes 2}, A^{\otimes 2})$, the double Jacobiator $\mathfrak{Jac}(\varphi)$ is

$$\mathfrak{Jac}(\varphi) := \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot \big((\varphi \otimes 1_A) \circ (1_A \otimes \varphi)\big) \in \mathrm{Hom}_R(A^{\otimes 3}, A^{\otimes 3})^{\mathbb{Z}/3}.$$

A basic fact is the following (using the terminology of n-brackets recalled in Remark 2.6).

Proposition 3.2. *[VdB08, Proposition 2.3.1] If φ is a double bracket on A, then $\mathfrak{Jac}(\varphi)$ is a triple bracket.*

Definition 3.3. [VdB08] A double Poisson structure on A is a double bracket $\{\!\{,\}\!\} : A \otimes A \to A \otimes A$, $a \otimes b \mapsto \{\!\{a, b\}\!\}$, such that the double Jacobiator $\{\!\{,,\}\!\} := \mathfrak{Jac}\big(\{\!\{,\}\!\}\big)$ is zero (the *double Jacobi relation*).

Remark 3.4. For A, $\{\!\{,\}\!\}$ a double Poisson algebra, the bracket

$$\{,\} : A^{\otimes 2} \to A$$

defined as the composite of $\{\!\{,\}\!\}$ with the product of A is a left Leibniz algebra, by [VdB08, Corollary 2.4.4]. Moreover, [VdB08, Proposition 1.4] implies that, if A is commutative, $\{,\}$ defines a Poisson algebra structure on A.

3.1. Graded algebras

As in Section 2.1, let A be a $\mathbb{Z}$-graded R-algebra which is finitely-generated as a graded algebra.

Definition 3.5. A double Poisson structure $\{\!\{,\}\!\}$ on the graded algebra A is homogeneous if $\{\!\{,\}\!\} = \{\!\{,\}\!\}^t$ for some $t \in \mathbb{Z}$.

As in Notation 2.12, the following notation is adopted:

Notation 3.6. For A as above and $\{\!\{,\}\!\}$ a double Poisson structure on A, write $\{\!\{,\}\!\}^{\min}$ and $\{\!\{,\}\!\}^{\max}$ for the components of minimal (respectively maximal) degree.

Lemma 3.7. *For A, $\{\!\{,\}\!\}$ as above, $\{\!\{,\}\!\}^{\min}$ and $\{\!\{,\}\!\}^{\max}$ define homogeneous double Poisson structures on A.*

Proof. Straightforward. □

4. Multi-derivations for polynomial algebras

In this section, A is taken to be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$ and R is a commutative unital ring. Hence $A^{\otimes 2}$ is a polynomial algebra on $2d$ generators, and the elements $t_i \otimes 1 - 1 \otimes t_i$ are algebraically independent and can be extended to a set of algebra generators of $A^{\otimes 2}$. In particular, the elements $t_i \otimes 1 - 1 \otimes t_i$ are regular elements of $A^{\otimes 2}$.

A key observation is the following:

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Lemma 4.1. *Let A be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$. The morphism of R-modules of Lemma 2.2*

$$\Pi : A^{\otimes 2} \to \mathbb{D}\mathrm{er}(A)$$

is an isomorphism.

Proof. It is straightforward to see that Π is a monomorphism of R-modules, hence it suffices to check surjectivity. Consider $\varphi \in \mathbb{D}\mathrm{er}(A)$ and $x, y \in A$, the commutativity relation $xy = yx$ gives in $A^{\otimes 2}$

$$\varphi(xy) = (x \otimes 1)\varphi(y) + \varphi(x)(1 \otimes y) = (y \otimes 1)\varphi(x) + \varphi(y)(1 \otimes x),$$

thus

$$(x \otimes 1 - 1 \otimes x)\varphi(y) = (y \otimes 1 - 1 \otimes y)\varphi(x). \tag{1}$$

The double derivation property of φ implies that φ is determined by its restriction to the R-module generated by the t_i and, by R-linearity, by the elements $\varphi(t_i)$, $i \in \{1, \ldots, d\}$. Taking $x = t_i$ and $y = t_j$ for $i \neq j$ (recall that $d \geq 2$ by hypothesis), equation (1) implies that

$$\varphi(t_\alpha) = (t_\alpha \otimes 1 - 1 \otimes t_\alpha)\Theta$$

for $\alpha \in \{i, j\}$ and for some $\Theta \in A^{\otimes 2}$. Hence this equation holds for all $\alpha \in \{1, \ldots, d\}$, showing that $\varphi = \Pi(\Theta)$, as required. □

Remark 4.2. The above argument extends to treat the map

$$\Pi_M : M \to \mathrm{Der}(A, M),$$

when M is a free $A^{\otimes 2}$-module (the module structure giving the A-bimodule structure).

Theorem 4.3. *Let A be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$. For $2 \leq n \in \mathbb{N}$ the morphism of R-modules of Proposition 2.8,*

$$\Pi_n : A^{\otimes n} \quad \to \quad \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n}),$$

is an isomorphism of $R[\mathbb{Z}/n]$-modules.

Proof. It clearly suffices to prove that Π_n is an isomorphism of R-modules. It is straightforward to check that Π_n is a monomorphism, thus it suffices to show that Π_n is surjective.

First consider the case $n = 2$ and take $\varphi \in \mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2})$; then, for fixed $a \in A$, the map $\varphi(a \otimes -) : A \to A^{\otimes 2}$ belongs to $\mathbb{D}\mathrm{er}(A)$, hence Lemma 4.1 implies that

$$\varphi(a \otimes b) = (b \otimes 1 - 1 \otimes b)\Theta_a$$

for some $\Theta_a \in A^{\otimes 2}$ that is independent of b.

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Now take $b = t_1$, so that $b \otimes 1 - 1 \otimes b$ is a regular element of $A^{\otimes 2}$. It follows that $a \mapsto \Theta_a$ defines a double derivation of $\mathbb{D}\mathrm{er}(A)$. Again by Lemma 4.1, Θ_a can be written as $(a \otimes 1 - 1 \otimes a)\Theta$, for some $\Theta \in A^{\otimes 2}$ that is independent of a, so that

$$\varphi(a \otimes b) = (a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)\Theta$$

for *any* $a, b \in A$, as required.

For $n > 2$, the above argument is modified in the obvious way, by appealing to Remark 4.2. For example, given $\varphi \in \mathbb{M}\mathrm{Der}(A^{\otimes n}, A^{\otimes n})$, fix $\alpha \in A^{\otimes n-1}$ and consider the map $\varphi(\alpha \otimes -)$ as belonging to $\mathrm{Der}(A, A^{\otimes n})$, where $A^{\otimes n}$ is the free bimodule with respect to the outer bimodule structure. As above, one deduces that

$$\varphi(\alpha \otimes b) = (b \otimes 1^{\otimes n-1} - 1^{\otimes n-1} \otimes b)\Theta_\alpha$$

where Θ_α is independent of b. The argument is then repeated recursively, starting as above by analysing Θ_α, at each step reducing the number of dependencies. □

Remark 4.4. The argument for the case $n = 2$ (and, by extension, the general case) depends on the fact that each $t_i \otimes 1 - 1 \otimes t_i$ is a regular element. Clearly the argument fails in general for A an arbitrary commutative ring; even the injectivity of Π_n need not hold.

Example 4.5. For $A = \Bbbk[t]$, with $\Bbbk$ a field, there is a double bracket defined by

$$t \otimes t \mapsto t \otimes 1 - 1 \otimes t$$

(see Section A). This is clearly not in the image of Π_2.

Remark 4.6. For the free associative algebra $T(V)$ on a free R-module V and $2 \leq n \in \mathbb{N}$, any morphism $V^{\otimes n} \to T(V)^{\otimes n}$ extends uniquely to an element of $\mathbb{M}\mathrm{Der}(T(V)^{\otimes n}, T(V)^{\otimes n})$ (and clearly every multi-derivation is determined by its restriction to $V^{\otimes n}$). The corresponding result is false in the commutative case; Theorem 4.3 provides an analogous (but much stronger) result.

For A a commutative R-algebra, the multiplication $\mu : A^{\otimes 2} \to A$ induces an R-linear map $\mathrm{Hom}_R(A^{\otimes 2}, A^{\otimes 2}) \to \mathrm{Hom}_R(A^{\otimes 2}, A)$ which restricts to a map $\mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2}) \to \mathrm{Hom}_R(A^{\otimes 2}, A)$.

Corollary 4.7. *Let A be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$. Then the morphism of R-modules*

$$\mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2}) \to \mathrm{Hom}_R(A^{\otimes 2}, A)$$

is trivial.

Proof. By inspection, the map φ_2 is sent to zero, whence the result, by Theorem 4.3, using the definition of Π_2. □

Remark 4.8. Corollary 4.7 shows that no Poisson structure on $R[t_1, \ldots, t_d]$ is induced by a multi-derivation. This shows that the weakening of the notion of double Poisson structure proposed in [Art15] does not provide further non-trivial examples of non-commutative Poisson structures (in the sense of [CB11]).

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5. Double Poisson structures on polynomial algebras

Let R be a commutative unital ring and A be a commutative R-algebra.

Notation 5.1. Write

1. $\Lambda^2(A) \subset A^{\otimes 2}$ for the sub R-module of anti-commutative elements (namely the kernel of $\mathrm{id} + \tau : A^{\otimes 2} \circlearrowleft$, where τ transposes the tensor factors);
2. $(A^{\otimes 3})^{\mathbb{Z}/3} \subset A^{\otimes 3}$ for the sub R-module of cyclically invariant elements.

Remark 5.2. In characteristic two the above does not give the usual definition of $\Lambda^2(A)$.

Proposition 5.3. *Let $A = R[t_1, \ldots, t_d]$, where $d \geq 2$.*

1. *The isomorphism $\Pi_2 : A^{\otimes 2} \xrightarrow{\cong} \mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2})$ restricts to an isomorphism of R-modules*
$$\Lambda^2(A) \xrightarrow{\cong} \mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2})^{\mathrm{sgn}},$$
where $\mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2})^{\mathrm{sgn}}$ is the R-module of double brackets on A.
2. *The isomorphism $\Pi_3 : A^{\otimes 3} \xrightarrow{\cong} \mathbb{M}\mathrm{Der}(A^{\otimes 3}, A^{\otimes 3})$ restricts to an isomorphism of R-modules*
$$(A^{\otimes 3})^{\mathbb{Z}/3} \xrightarrow{\cong} \mathbb{M}\mathrm{Der}(A^{\otimes 3}, A^{\otimes 3})^{\mathrm{sgn}},$$
where $\mathbb{M}\mathrm{Der}(A^{\otimes 3}, A^{\otimes 3})^{\mathrm{sgn}}$ is the R-module of triple brackets on A.

Proof. An immediate consequence of the definitions and Theorem 4.3, using the $R[\mathbb{Z}/n]$-equivariance in the cases $n = 2$ and $n = 3$. □

Remark 5.4. Explicitly, for $\Psi \in \Lambda^2(A) \subset A^{\otimes 2}$ (so that $\tau\Psi = -\Psi$), the associated double bracket is
$$\{\!\{a, b\}\!\}_\Psi = (a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)\Psi.$$

Notation 5.5. Let $(-)_{23}$ denote the R-linear map $A^{\otimes 2} \to A^{\otimes 3}$, $a \otimes b \mapsto 1 \otimes a \otimes b$ and $(-)_{13}$ the map $a \otimes b \mapsto a \otimes 1 \otimes b$.

Proposition 5.6. *Let $A = R[t_1, \ldots, t_d]$, where $d \geq 2$. Under the isomorphisms of Proposition 5.3, the set map induced by the double Jacobiator*
$$\left\{\textit{double brackets on } A\right\} \xrightarrow{\mathfrak{Jac}} \left\{\textit{triple brackets on } A\right\}$$
(cf Proposition 3.2) identifies with the (non-linear) map
$$\begin{aligned} \mathfrak{J} : \Lambda^2(A) &\to (A^{\otimes 3})^{\mathbb{Z}/3} \\ \Psi &\mapsto \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot (\Psi_{13}\Psi_{23}) \end{aligned}$$
where the product $\Psi_{13}\Psi_{23}$ is formed in $A^{\otimes 3}$.

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Proof. By Proposition 5.3, it suffices to identify $\mathfrak{Jac}\{\!\{,\}\!\}_\Psi$ in the image of Π_3. It is clear that the expression must be a $\mathbb{Z}/3$-invariant quadratic expression in Ψ. The result follows by direct calculation, using the anti-symmetry $\tau\Psi = -\Psi$. (The calculation may be simplified by using the proof of [VdB08, Proposition 2.3.1].) □

Theorem 5.7. *Let R be a commutative ring on which the squaring map $x \mapsto x^2$ is injective, then there is no non-trivial double Poisson structure on $A := R[t_1, \ldots, t_d]$ for $d \geq 2$.*

Proof. It follows from the identification given in Remark 2.9 that, for any $i, j, k \in \{1, \ldots, d\}$, the element $\varphi_3(t_i, t_j, t_k)$ is a regular element of $A^{\otimes 3}$ (this does not require the hypothesis upon R). Hence, by Proposition 5.6, to prove the result it suffices to show that $\Psi \in A^{\otimes 2}$ is zero if and only if $\sum_{\sigma\in\mathbb{Z}/3} \sigma\cdot(\Psi_{13}\Psi_{23}) \in A^{\otimes 3}$ is zero (anti-symmetry of Ψ plays no rôle here).

The latter fact is seen by exploiting the natural grading of A (placing the generators in degree one, so the grading coincides with the length grading), together with the induced left lexicographical ordering on $A^{\otimes 2}$ and $A^{\otimes 3}$. Namely, if Ψ is non-zero, the terms of maximal lexicographical order in Ψ contribute to a non-zero term of maximal lexicographical order in $\sum_{\sigma\in\mathbb{Z}/3} \sigma \cdot (\Psi_{13}\Psi_{23})$ (Cf. Remark 2.9).

Explicitly, writing $\Psi = \sum_m \alpha_m \otimes m$ in terms of the monomial basis of A, one considers the contributions

$$(\alpha_m)^2 \otimes m \otimes m$$

in $\Psi_{13}\Psi_{23}$ to the terms of maximal lexicographical order in $\sum_{\sigma\in\mathbb{Z}/3} \sigma\cdot(\Psi_{13}\Psi_{23})$. Finally, the hypothesis upon R implies that $(\alpha_m)^2 \otimes m \otimes m$ is zero if and only if α_m is zero. □

6. Double Poisson structures on commutative algebras

In this section, A denotes a commutative R-algebra.

Definition 6.1. A double bracket on A is *standard* if it lies in the image of the morphism of R-modules

$$\Pi_2 : \Lambda^2(A) \to \mathbb{M}\mathrm{Der}(A^{\otimes 2}, A^{\otimes 2})^{\mathrm{sgn}}$$

induced by Π_2 (as in Proposition 5.3) and is *exotic* otherwise, so that the R-module of exotic double brackets is the cokernel of the above morphism.

Remark 6.2.

1. Exotic double brackets exist: cf. Example 4.5. However, these cannot be classified easily (cf. the case $A = R[t]$ in Section A).
2. The restriction to $\Lambda^2(A)$ is not severe. For example, if 2 is invertible in R, the inclusion $\Lambda^2(A) \hookrightarrow A^{\otimes 2}$ admits the retract $x \otimes y \mapsto \frac{1}{2}(x \otimes y - y \otimes x)$.

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Observe that the set map $\mathfrak{J} : \Lambda^2(A) \to (A^{\otimes 3})^{\mathbb{Z}/3}$ of Proposition 5.6 can be defined for any commutative algebra A.

Theorem 6.3. *Let $\Psi \in \Lambda^2(A)$ and consider the associated (standard) double bracket $\{\!\{,\}\!\}_\Psi := \Pi_2(\Psi)$. Then:*

1. *the associated bracket $\{,\} : A^{\otimes 2} \to A$ is trivial;*
2. *$\{\!\{,\}\!\}_\Psi$ defines a double Poisson structure on A if and only if $\Pi_3(\mathfrak{J}(\Psi))$ is zero in $\mathbb{M}\mathrm{Der}(A^{\otimes 3}, A^{\otimes 3})^{\mathrm{sgn}}$.*

Proof. The first statement follows as for Corollary 4.7. The fact that $\{\!\{,\}\!\}_\Psi$ is a standard double bracket implies that the calculation of Proposition 5.6 is universal, the only difference being that the triple brackets on A cannot be identified with $(A^{\otimes 3})^{\mathbb{Z}/3}$ via Π_3. □

Remark 6.4. Theorem 6.3 provides a recipe for constructing examples of non-trivial double Poisson structures on commutative algebras: for any $\Psi \in \Lambda^2(R[t_i])$ ($R[t_i]$ a polynomial algebra) it suffices to pass to a quotient A of $R[t_i]$ for which $\Pi_3(\mathfrak{J}(\Psi))$ is trivial in $\mathbb{M}\mathrm{Der}(A^{\otimes 3}, A^{\otimes 3})^{\mathrm{sgn}}$. Note that, in all cases, the associated bracket (as in Corollary 4.7) is trivial.

A. Double Poisson structures on $R[t]$

In [VdB08, Example 2.3.3], Van den Bergh stated a classification of the (homogeneous) double Poisson structures on the polynomial algebra $\Bbbk[t]$, for $\Bbbk$ a field. A proof over a more general ring, also considering non-homogeneous structures, is given here.

Proposition A.1. *Let R be a commutative ring on which $x \mapsto x^2$ is injective, then the only homogeneous double Poisson structures on $A := R[t]$ are scalar multiples of the double Poisson brackets determined by*

$$\begin{aligned} \{\!\{t,t\}\!\}^1 &= t \otimes 1 - 1 \otimes t \\ \{\!\{t,t\}\!\}^3 &= t^2 \otimes t - t \otimes t^2 = (t \otimes 1 - 1 \otimes t)(t \otimes t), \end{aligned}$$

where the suffix corresponds to the degree of the element $\{\!\{t,t\}\!\}$.

In general, for $\lambda, \mu, \nu \in R$,

$$\{\!\{t,t\}\!\} = \lambda \{\!\{t,t\}\!\}^1 + \mu(t^2 \otimes 1 - 1 \otimes t^2) + \nu \{\!\{t,t\}\!\}^3$$

defines a double Poisson structure if and only if $\lambda\nu - \mu^2 = 0$ and any double Poisson structure on A is of this form.

Proof. A double Poisson structure on $R[t]$ is determined by $\{\!\{t,t\}\!\}$. It is straightforward to verify that $\{\!\{t,t\}\!\}^1$ and $\{\!\{t,t\}\!\}^3$ define homogeneous double Poisson structures on $R[t]$.

The derivation property (using induction upon $n \geq 1$) implies that

$$\{\!\{t,t^n\}\!\} = \Big(\sum_{i+j=n-1} t^i \otimes t^j \Big) \{\!\{t,t\}\!\},$$

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where the product is formed in the algebra $A^{\otimes 2}$.

Anti-symmetry implies that a homogeneous double bracket $\{\!\{,\}\!\}^N$ is an R-linear combination of terms of the form $(t^{N-i}\otimes t^i - t^i\otimes t^{N-i})$, for N corresponding to the homogeneous degree and $0\leq i<N/2$.

First consider the case where $\{\!\{t,t\}\!\}^N = \lambda(t^{N-i}\otimes t^i - t^i\otimes t^{N-i})$ for some i; this is subdivided into two cases:

1. $\{\!\{t,t\}\!\}^N = \lambda(t^{a+1}\otimes t^a - t^a\otimes t^{a+1})$, for $a\in\mathbb{N}$ (so that $N=2a+1$) and $\lambda\in R$. The cases $a\in\{0,1\}$ correspond to the two cases given above, hence suppose that $a>1$ (which implies that $2a>a+1$).
   Consider the coefficient of $t^{2a}\otimes t^{a+1}\otimes t^a$ in $\{\!\{t,t,t\}\!\}$. Write Φ for the element $(\{\!\{,\}\!\}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\{\!\{,\}\!\})(t\otimes t\otimes t)$. Thus the double Jacobiator $\{\!\{t,t,t\}\!\}$ is the sum of the cyclic permutations of Φ. Hence it is necessary to consider the coefficients of $t^{2a}\otimes t^{a+1}\otimes t^a$, $t^a\otimes t^{2a}\otimes t^{a+1}$ and $t^{a+1}\otimes t^a\otimes t^{2a}$ in Φ. The coefficient of the first is zero (the two contributions cancel) and the second has coefficient $-\lambda^2$; the hypothesis on a ensures that the third cannot occur. Thus $\{\!\{,,\}\!\}=0$ implies that $-\lambda^2=0$, so that $\{\!\{,\}\!\}^N=0$.
2. $\{\!\{t,t\}\!\}^N = \lambda(t^{N-a}\otimes t^a - t^a\otimes t^{N-a})$, with $N-a>a+1$. Consider the coefficient of $t^{2(N-a-1)}\otimes t^{a+1}\otimes t^a$ in $\{\!\{t,t,t\}\!\}$. In this case, the coefficient of $t^{2(N-a-1)}\otimes t^{a+1}\otimes t^a$ in Φ is λ^2. If $2(N-a-1)>N-a$ then the coefficients of $t^a\otimes t^{2(N-a-1)}\otimes t^{a+1}$ and $t^{a+1}\otimes t^a\otimes t^{2(N-a-1)}$ in Φ are both trivial. Hence (in this case) the condition $\{\!\{,,\}\!\}=0$ implies that $\lambda^2=0$ and again $\{\!\{,\}\!\}^N=0$.
   The inequality $2(N-a-1)>N-a$ is equivalent to $N>a+2$; since $N>2a+1$, by hypothesis, this is satisfied if $a\geq 1$ or if $a=0$ and $N>2$. In the remaining case, $N=2$ and $a=0$, it can be checked directly that $\{\!\{,\}\!\}^N=0$.

To complete the proof, one considers the case where $\{\!\{t,t\}\!\}^N$ has at least two non-trivial coefficients with respect to the basis $\{t^{N-i}\otimes t^i - t^i\otimes t^{N-i} | 0\leq i< N/2\}$. Thus one can write

$$\begin{aligned}\{\!\{t,t\}\!\}^N &= \lambda(t^{N-a}\otimes t^a - t^a\otimes t^{N-a}) + \mu(t^{N-b}\otimes t^b - t^b\otimes t^{N-b})\\ &\quad + \sum_{b<k<N/2}\nu_k(t^{N-k}\otimes t^k - t^k\otimes t^{N-k})\end{aligned}$$

where $\lambda\neq 0$, $\mu\neq 0$ and $0\leq a<b<N/2$ (hence $N>2$).

Consider the coefficient of

$$t^{2N-b-a-1}\otimes t^b\otimes t^a$$

in $\{\!\{t,t,t\}\!\}$. As above using the notation Φ,

1. the coefficient of $t^{2N-b-a-1}\otimes t^b\otimes t^a$ in Φ is $\lambda^2+\lambda\mu$;
2. the coefficient of $t^a\otimes t^{2N-b-a-1}\otimes t^b$ in Φ is $-\lambda\mu$ (the sign arises from antisymmetry);
3. the term $t^b\otimes t^a\otimes t^{2N-b-a-1}$ cannot arise in Φ, since $2N-b-a-1>N-a$ (the difference is $N-1-b$ and the latter is positive by the hypotheses).

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It follows that the coefficient of $t^{2N-b-a-1} \otimes t^b \otimes t^a$ in $\{\!\{t,t,t\}\!\}$ is λ^2, thus $\lambda = 0$, contradicting the hypothesis that $\lambda \neq 0$.

Finally, consider the non-homogeneous case. Here, by Lemma 3.7, the only non-trivial possibility is

$$\{\!\{t,t\}\!\} = \lambda(t \otimes 1 - 1 \otimes t) + \mu(t^2 \otimes 1 - 1 \otimes t^2) + \nu(t^2 \otimes t - t \otimes t^2)$$

where, if $\mu \neq 0$, then both λ and ν are non zero.

The associated double Jacobiator $\{\!\{t,t,t\}\!\}$ in principle has terms in degrees 1, 2, 3, 4 and 5; since $\{\!\{,\}\!\}^1$ and $\{\!\{,\}\!\}^3$ give double Poisson structures, the terms in degrees 1 and 5 vanish (as already observed in Lemma 3.7). A straightforward calculation also shows that the terms in degrees 2 and 4 vanish.

Finally, one finds that

$$\{\!\{t,t,t\}\!\} = (\lambda\nu - \mu^2)\big(\overline{1 \otimes t \otimes t^2} - \overline{1 \otimes t^2 \otimes t}\big)$$

where $\overline{1 \otimes t \otimes t^2}$ and $\overline{1 \otimes t^2 \otimes t}$ denote the respective $\mathbb{Z}/3$-orbit sums. Hence the double bracket defines a double Poisson structure if and only if $\lambda\nu = \mu^2$. □

Remark A.2.

1. The transformation given by [VdB08, Example 2.3.3] associated to the change of variables $t \mapsto t^{-1}$ (after extending to $\Bbbk[t^{\pm 1}]$) acts by $\lambda \mapsto -\nu$, $\nu \mapsto -\lambda$ and $\mu \mapsto -\mu$, as expected.
2. Over a field $\Bbbk$, up to scalar multiplication and the action of $\Bbbk^*$, considered as automorphisms of $\Bbbk[t]$ via $\alpha : t \mapsto \alpha t$, this gives the single non-homogeneous example

$$\{\!\{t,t\}\!\} := (t \otimes 1 - 1 \otimes t) + (t^2 \otimes 1 - 1 \otimes t^2) + (t^2 \otimes t - t \otimes t^2).$$

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