



Quotients of complex algebraic supergroups

R. Fiorese^{a,*}, S.D. Kwok^b, D.W. Taylor^c

^a Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato, 5, 40126 Bologna, Italy

^b Department of Biostatistics, UCLA, Los Angeles, CA 90095-1772, USA

^c Department of Mathematics, UCLA, CA 90095-1555, USA

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To the memory of V.S. Varadarajan

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ABSTRACT

In this paper we prove that the étale sheafification of the functor arising from the quotient of an algebraic supergroup by a closed subsupergroup is representable by a smooth superscheme.

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1. Introduction

The purpose of this paper is to provide a construction of the quotient of a complex algebraic supergroup by a closed subsupergroup. This construction is already available in a more general setting in the literature (see [17]), however here we present a different and more geometric proof, that is closer to the original approach by Chevalley (see [2] Ch. II).

We start by reviewing the ordinary construction. Suppose G is a complex algebraic group and H a closed subgroup. Then, G/H admits a unique algebraic variety structure, compatible with the group multiplication. In fact, there exists a rational representation of G in a finite dimensional vector space V and a line L in V whose stabilizer is H . Hence, we have an action of G on the projective space $\mathbb{P}(V)$ and H is the stabilizer subgroup of the point $[L]$ in $\mathbb{P}(V)$. We can thus identify set-theoretically the quotient G/H with the orbit Y of the point $[L]$; Y being an orbit is also an algebraic variety, because of Chevalley's theorem. The uniqueness of this structure is obtained by the universal property of the quotient (see [2] Ch. II).

We want to replicate this geometric construction in the super setting. There are two major obstructions: the \mathbf{C} -points of a supervariety do not carry enough information on its geometry, as it happens for the ordinary counterpart. Also, quotients of supergroups may not admit a projective embedding. We overcome the first difficulty by making use of the functor of points of superschemes and introducing étale coverings and étale sections, which mimic in some sense the differential approach to the construction of quotients (see [13,1]). As for the latter problem, we replace the projective superspace with Grassmannian superschemes. In supergeometry the projective superspace appears somehow too rigid and it is necessary to allow for more general structures, as the Grassmannians. In this way we can realize an embedding of an orbit of a supergroup action into a suitable Grassmannian, hence identifying it with a smooth superscheme. In this sense, our proof

* Corresponding author.

E-mail addresses: rita.fiorese@UniBo.it (R. Fiorese), sdkwok2@gmail.com (S.D. Kwok), dwtaylor@math.ucla.edu (D.W. Taylor).

will also provide a variation of the ordinary construction of quotients of complex algebraic groups and goes beyond a mere translation of the known recipe into the super context.

Our main result is the following.

Theorem 1.1. *Let G be a complex algebraic supergroup, H a closed subsupergroup. Then, the sheafification in the étale topology of the functor $T \rightarrow G(T)/H(T)$, T a superscheme, is representable in the category of superschemes, by a smooth superscheme.*

We shall prove this result in several steps. In Sec. 2 we give some preliminaries and notation on algebraic supervarieties and superschemes, while in Sec. 3 we establish some results on smoothness. In Sec. 4 we prove the representability of the étale sheafification of the functor $T \rightarrow G(T)/H(T)$, when H is the stabilizer of a point for an action of G on a superscheme. Finally, in Sec. 5 we give our main result, Theorem 5.5 and a comparison with [17] and the definition by Brundan in [3]. In the end we provide a section with some examples.

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2. Supervarieties and superschemes

In this section we collect some facts of supergeometry. For more details see [6,16,4,20,15].

Let \mathbf{C} be our ground field. Let (salg) be the category of commutative superalgebras and let $A = A_0 \oplus A_1 \in (\text{salg})$. Let us consider a non-zero $f \in A_0$ and A_f the localization of the A_0 -module A at f . The assignment:

$$U_f := \{x \in |\text{Spec}(A_0)| \mid f(x) \neq 0\} \rightarrow A_f \quad (1)$$

defines a \mathcal{B} -sheaf on $|\text{Spec}(A_0)|$. Hence, there exists a unique sheaf of superalgebras \mathcal{O}_A on $|\text{Spec}(A_0)|$ such that $\mathcal{O}_A|_{U_f} = A_f$.

Definition 2.1. We define *affine superscheme* X associated with A the pair $X = (|X|, \mathcal{O}_A)$, where $|X|$ is the spectrum $|\text{Spec}(A_0)|$ of the ordinary algebra A_0 , while \mathcal{O}_A is the sheaf described above. The *reduced scheme* X_r underlying X is the ordinary scheme associated with $A_r = A/J_A$, where J_A is the ideal of the odd elements in A .

We shall also denote with \mathcal{O}_X the sheaf of the superscheme X and with $\mathcal{O}(X)$ the superalgebra A .

A *morphism* $f : X \rightarrow Y$ of affine superschemes is a pair $(|f|, f^*)$, where $|f| : |X| \rightarrow |Y|$ is a continuous map and $f^* : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a map of sheaves of superalgebras, such that $f_p^* : \mathcal{O}_{Y,|f|(p)} \rightarrow \mathcal{O}_{X,p}$ is a local morphism for all p in $|X|$.

We define *superscheme* a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf of superalgebras \mathcal{O}_X , which is locally isomorphic to an affine superscheme.

Definition 2.2. Let X be an affine superscheme, $\mathcal{O}(X)$ the corresponding superalgebra. We say that S is a *subscheme* of X , if S is the affine superscheme corresponding to the superalgebra $\mathcal{O}(X)/I$ for I ideal in $\mathcal{O}(X)$. If $X = (|X|, \mathcal{O}_X)$ is a superscheme, we say $S = (|S|, \mathcal{O}_S)$ is a *subscheme* of X , if:

1. $|S|$ is a closed subspace of $|X|$.
2. $\mathcal{O}_S = \mathcal{O}_X/\mathcal{I}$, where \mathcal{I} is an ideal sheaf.
3. For an affine cover $\{U_i\}$ of X , $\mathcal{I}(U_i)$ is an ideal in $\mathcal{O}_X(U_i)$, $\mathcal{O}_S(U_i) = \mathcal{O}_X(U_i)/\mathcal{I}(U_i)$;
4. The sheaf $\mathcal{O}_S|_{U_i}$ is obtained starting from the superalgebra $\mathcal{O}_S(U_i)$ as in (1).

We now come to the functor of points.

Definition 2.3. Let S and T be superschemes. A T -point of S is a morphism $u : T \rightarrow S$. We denote the set of all T -points by $S(T)$. We define the *functor of points* of the superscheme S as the functor:

$$S : (\text{sschemes})^o \rightarrow (\text{sets}), \quad T \mapsto S(T), \quad S(\phi)(f) = f \circ \phi,$$

where (sschemes) denotes the category of superschemes, (sets) the category of sets and the index o as usual refers to the opposite category.

By a common abuse of notation the superscheme S and the functor of points of S are denoted with the same letter.

Definition 2.4. Let A be a commutative superalgebra, J_A the ideal generated by the odd elements. We say that A is an *affine superalgebra*, if A_0 is a finitely generated superalgebra, such that its reduced associated algebra $A_r = A/J_A$ is an affine algebra (i.e. finitely generated and with no nilpotents) and A_1 is a finitely generated A_0 -module.

We say that X is an *affine supervariety*, if $X = (|\mathrm{Spec} A|, \mathcal{O}_A)$ and A_r is an integral domain, i.e. X_r is an ordinary affine variety. A *supervariety* X is a superscheme which is locally isomorphic to an affine supervariety.

Remark 2.5. We are also interested in the functor of points of algebraic supervarieties, which are a subcategory of the category of superschemes. The category of affine superschemes is equivalent to the category of commutative superalgebras (see [4] Ch. 10), moreover the functor of points of a superscheme is determined by its behaviour on affine superschemes. We can then regard the functor of points of an algebraic supervariety (or superscheme) X as starting from the category of commutative superalgebras, that is $X : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$, $X(\phi)(f) = \phi \circ f$.

3. Smooth morphisms

We now introduce the notion of smooth morphism of relative dimension. For the ordinary setting see [19] Ch. 5.

Definition 3.1. A morphism of affine superschemes

$$f : \mathrm{Spec} A[x_1 \dots x_{m+r}, \xi_1, \dots, \xi_{n+s}] / (f_1, \dots, f_r, \phi_1, \dots, \phi_s) \longrightarrow \mathrm{Spec} A$$

is *smooth* of relative dimension $m|n$ at a topological point x , if the rank of the Jacobian is maximal, i.e.

$$\mathrm{rk} \frac{\partial(f_i, \phi_j)}{\partial(x_k, \xi_l)}(x) = r|s$$

($1 \leq i \leq r, 1 \leq j \leq s, 1 \leq k \leq m+r, 1 \leq l \leq n+s$).

We say that a morphism of superschemes $f : X \longrightarrow Y$ is *smooth* at $x \in |X|$ of relative dimension $m|n$, if there exists two affine neighbourhoods $U \subset X$ and $V \subset Y$ such that:

$$\begin{array}{ccc} U & \xrightarrow{\quad} & \mathrm{Spec} A[x_1 \dots x_{m+r}, \xi_1, \dots, \xi_{n+s}] / (f_1, \dots, f_r, \phi_1, \dots, \phi_s) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\quad} & \mathrm{Spec} A \end{array}$$

and the rank of the Jacobian is maximal, i.e.

$$\mathrm{rk} \frac{\partial(f_i, \phi_j)}{\partial(x_k, \xi_l)}(x) = r|s$$

($1 \leq i \leq r, 1 \leq j \leq s, 1 \leq k \leq m+r, 1 \leq l \leq n+s$). f is *smooth* of relative dimension $m|n$, if it is smooth of relative dimension $m|n$ at all $x \in |X|$.

We say that a morphism of superschemes is *etale*, if it is smooth of relative dimension $0|0$.

We say that $x \in |X|$ is a *smooth point*, if the corresponding morphism $X \longrightarrow \mathbf{C}$ is smooth ($|X|$ is identified with $X(\mathbf{C})$, see [4] Ch. 10, 10.6.4). The superscheme X is *smooth*, if all $x \in |X|$ are smooth.

This notion of smoothness of a superscheme X is equivalent to the one in [9] and [17,18].

Proposition 3.2. A morphism of superschemes $f : X \longrightarrow Y$ is smooth of relative dimension $m|n$ at $x \in |X|$ if and only if there exist an open $V \subset Y$, $U = f^{-1}(V) \subset X$ ($x \in |U|$) such that $f = \pi \circ g$,

$$U \xrightarrow{g} V \times \mathbf{C}^{m|n} \xrightarrow{\pi} V$$

where π is the projection and g is etale.

Proof. One direction is clear, since the composition of smooth morphisms is smooth. Since the question is local, we can look at superalgebra maps, that is $f^* : A \longrightarrow A[x_1, \dots, x_m, \dots, x_{m+r}, \xi_1, \dots, \xi_n, \dots, \xi_{n+s}] / (f_1, \dots, f_r, \phi_1, \dots, \phi_s)$. We can write:

$$\begin{aligned} A &\xrightarrow{\pi^*} A[x_1, \dots, x_m, \xi_1, \dots, \xi_n] \longrightarrow \\ &\xrightarrow{g^*} A[x_1, \dots, x_m, \dots, x_{m+r}, \xi_1, \dots, \xi_n, \dots, \xi_{n+s}] / (f_1, \dots, f_r, \phi_1, \dots, \phi_s) \end{aligned}$$

with

$$\mathrm{rk} \frac{\partial(f_i, \phi_j)}{\partial(x_k, \xi_l)}(x) = r|s$$

by the very definition of g etale, the result follows immediately. \square

Lemma 3.3. *Let $f : X \rightarrow Y$ be a smooth morphism of superschemes of relative dimension $m|n$. Then, for any morphism $Y' \rightarrow Y$ we have that $\mathrm{pr}_2 : X \times_Y Y' \rightarrow Y'$ is smooth of relative dimension $m|n$.*

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{\mathrm{pr}_2} & Y' \\ \downarrow \mathrm{pr}_1 & & \downarrow \\ X & \xrightarrow{\quad} & Y \end{array}$$

In particular, if $f : X \rightarrow Y$ is etale, also $\mathrm{pr}_2 : X \times_Y Y' \rightarrow Y'$ is etale.

Proof. Since the question is local, we can assume to be in the affine case.

$$\begin{array}{ccc} A[x_i, \xi_j]/(f_k, \phi_l) \otimes_A S & \longleftarrow & S \\ \uparrow \mathrm{pr}_1^* & & \uparrow \\ A[x_i, \xi_j]/(f_k, \phi_l) & \longleftarrow & A \end{array}$$

Since $A[x_i, \xi_j]/(f_k, \phi_l) \otimes_A S \cong S[x_i, \xi_j]/(f_k, \phi_l)$ we obtain the result. \square

We now make some observations on Grothendieck topologies. For more details see [21] for the ordinary setting and [12] for the supergeometric one.

Observation 3.4. Let us consider the category (sschemes) of superschemes and define coverings of a superscheme U to be collections of etale maps whose images cover U . This is a Grothendieck topology, because of the existence and the properties of the fibered product in (sschemes), together with Lemma 3.3. This topology defines the *super Etale site*. Similarly, we can define another Grothendieck topology by taking Zariski coverings, i.e. collections of open embeddings, and obtain the *super Zariski site* (see [12]). Notice that if $U_i \rightarrow U$ is a Zariski covering of a superscheme, then it is also an etale covering. Hence the etale topology is finer than the Zariski one. By the previous observation, we immediately have that a sheaf on the etale topology is a sheaf in the Zariski one, but not vice-versa (see [12] Sec. 2, Prop. 2.5).

As in the ordinary setting, any etale morphism will admit an *etale section*; this fact is essential for our construction of quotients.

Proposition 3.5. *Let $f : X \rightarrow Y$ be a morphism of superschemes smooth of relative dimension $m|n$ at $p \in |X|$. Then there exist open $V \subset Y$, $U \subset X$, $p \in |U|$, an etale cover $\phi : W \rightarrow V$ and a morphism $W \rightarrow U$ making the following diagram commute:*

$$\begin{array}{ccc} U & \subset & X \\ \uparrow & & \downarrow \\ W & \xrightarrow{\phi} & V \subset Y \end{array}$$

Proof. By Proposition 3.2, we have that there exists U open in X and V open in Y such that

$$U \rightarrow V \times \mathbb{C}^{m|n} \rightarrow V$$

We can write immediately a section s for the projection, $s : V \rightarrow V \times \mathbb{C}^{m|n}$.

$$\begin{array}{ccc} W = U \times_{V \times \mathbb{C}^{m|n}} V & \longrightarrow & V \\ \downarrow & & \downarrow s \\ U & \xrightarrow{g} & V \times \mathbb{C}^{m|n} \end{array}$$

By Lemma 3.3, since g is etale, we have that $\mathrm{pr}_2 : W = U \times_{V \times \mathbb{C}^{m|n}} V \rightarrow V$ is also etale. \square

The morphism $W \rightarrow U$ is called a *local etale section* of $f : X \rightarrow Y$.

We conclude this section with a characterization of smooth morphisms (see [16] Ch. 4 and [4] Ch. 10 for the notion of tangent space and differential).

Proposition 3.6. Let $f : X \longrightarrow Y$ be a morphism of smooth superschemes of finite type, X an algebraic variety. If $|f|$ is surjective, $(df)_x : T_x X \longrightarrow T_{f(x)} Y$ is surjective and $\dim T_x X - \dim T_{f(x)} Y = m|n$ for all $x \in |X|$, then f is smooth of relative dimension $m|n$.

Proof. The proof follows closely Cor. 5.4.6, Ch. V in [19]. We briefly recap here the main steps. The statement is local, so let $x \in |X|$. We can factor f as:

$$\begin{array}{ccc} U & \xrightarrow{\quad} & Y \times \mathbb{C}^{m|n} \\ & \searrow f & \downarrow p \\ & & Y \end{array}$$

where $U \subset X$ is open, $x \in |U|$. In terms of superalgebra maps this diagram reads:

$$\begin{array}{ccc} A[x_1, \dots, x_{m+r}, \xi_1, \dots, \xi_{n+s}] / (f_i, \varphi_j) & \longleftarrow & A[x_1, \dots, x_{m+r}, \xi_1, \dots, \xi_{n+s}] \\ & \nwarrow f^* & \uparrow p^* \\ & & A \end{array}$$

with $i = 1, \dots, r$, $j = 1, \dots, s$. Furthermore (f_i, φ_j) can be chosen such that

$$\text{rk} \frac{\partial(f_i, \varphi_j)}{\partial(x_k, \xi_l)} = r|s$$

This is because we can choose such f_i, φ_j so that their images $\bar{f}_i, \bar{\varphi}_j$ in $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ are linearly independent.

Since $(df)_x : T_x X \longrightarrow T_{f(x)} Y$ is surjective, we have an embedding $\mathfrak{m}_{Y,f(x)}/\mathfrak{m}_{Y,f(x)}^2 \subset \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. Using elementary facts of linear algebra, we have that $\bar{f}_i, \bar{\varphi}_j$ are independent also in $\mathfrak{m}_{Z,x}/(\mathfrak{m}_{Z,x}^2 + \mathfrak{m}_{X,x}\mathcal{O}_{x,Z})$ for $Z = Y \times \mathbb{C}^{m|n}$. This latter condition gives the independence of the differentials $df_i, d\varphi_j$, hence the result. \square

4. Etale sections and quotients

In this section we examine supergroup actions and homogeneous superspaces.

Definition 4.1. An *affine supergroup* is an affine supervariety whose functor of points is group valued, that is to say, it associates a group to each superalgebra.

If G is an affine supergroup, then G is a closed subgroup of $\text{GL}(m|n)$ and the superalgebra $\mathcal{O}(G)$ has a natural Hopf superalgebra structure (see [4] Ch. 11). Furthermore, G is smooth (see [9]).

Definition 4.2. Let V be a super vector space. We define *linear representation* of G in V a group morphism $\rho : G \longrightarrow \text{End}(V)$ where $\text{End}(V)$ are the endomorphism of V . We will also say that G acts on V .

Let Y be a superscheme. We say that G acts on Y if we have a morphism of superschemes: $a : G \times Y \longrightarrow Y$, $g, x \mapsto a_T(g, x) := g \cdot x$, $x \in Y(T)$, $g \in G(T)$, such that:

1. $1 \cdot x = x$, $\forall x \in Y(T)$
2. $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$, $\forall x \in Y(T)$, $\forall g_1, g_2 \in G(T)$.

For $p \in |Y|$ we define the *orbit map* $a_p : G \longrightarrow Y$ by $a_p, T(g) = g \cdot p \ \forall g \in G(T)$.

Let Y be smooth. We say that the action a is *transitive*, if there exists a $p \in |Y|$ such that $|a_p|$ and $(da_p)_{1_G}$ are surjective. In this case we call Y an *homogeneous superspace*.

Notice that according to Proposition 3.6, this is equivalent to ask that a_p is smooth of relative dimension $m|n = \ker(da_p)_{1_G}$.

Proposition 4.3. Let G be an affine supergroup acting transitively on a smooth superscheme Y . Then there exists an etale cover $\{W_i \longrightarrow Y\}$ making the following diagram commute:

$$\begin{array}{ccc} U_i & \subset & G \\ \uparrow & & \downarrow \\ W_i & \xrightarrow{\phi_i} & Y \end{array}$$

where the U_i are open and cover G .

Proof. Immediate from Proposition 3.5. \square

Lemma 4.4. *Let the notation be as above. Let $\alpha \in Y(T)$, $T \in (\text{sschemes})$. Then there exists an étale covering $\{\phi_i : T_i \rightarrow T\}$ and elements $\beta_i \in G(T_i)$ such that*

$$\begin{array}{ccc} G(T_i) & \longrightarrow & Y(T_i) \\ \beta_i & \mapsto & \alpha_i = \alpha \circ \phi_i \end{array}$$

Proof. Let $f_i : W_i \rightarrow V_i$ be the étale covering of Y described in Proposition 4.3 and $\sigma_i : W_i \rightarrow U_i \subset G$ the corresponding étale sections. Let $\alpha : T \rightarrow Y$ be a T -point of Y . Define $T_i := W_i \times_Y T$. We have the diagram:

$$\begin{array}{ccc} T_i = W_i \times_Y T & \xrightarrow{g_i} & T \\ \text{pr}_1 \downarrow & & \downarrow \\ W_i & \xrightarrow{f_i} & Y \end{array}$$

Since the f_i are étale, we have that the g_i are étale. Take $\beta_i := \sigma_i \circ \text{pr}_1 : T_i \rightarrow G$; by the very construction $a_{p,T_i}(\beta_i) = \alpha_i$, where $a_{p,T_i} : G(T_i) \rightarrow Y(T_i)$. \square

Let H be the stabilizer functor of $p \in |Y|$, that is $H(T) := \{g \in G(T) | g \cdot p = p\}$. This is representable by a closed subgroup of G (see [4] Ch. 11). We can define the functor:

$$G/H : (\text{sschemes})^0 \rightarrow (\text{sets}), \quad (G/H)(T) = G(T)/H(T)$$

the definition on the arrows being clear.

The morphism a_p induces a natural transformation $G/H \rightarrow Y$, with $G(T)/H(T) \rightarrow Y(T)$ injective for all T . In general, it will not be surjective, however we have the following (see [21,12] for the notion of sheafification in this context).

Theorem 4.5. *Let the notation be as above. The sheafification $\widetilde{G/H}$ in the étale topology of the functor $T \rightarrow G(T)/H(T)$ is isomorphic to Y and it is the functor of points of a superscheme.*

Proof. By our previous observation, we have a natural transformation: $\psi : G/H \rightarrow Y$, that factors as:

$$G/H \rightarrow \widetilde{G/H} \xrightarrow{\psi} Y$$

We want to show that ψ is an isomorphism. We only need to show it is surjective. Let $\alpha \in Y(T)$. Then by Lemma 4.4 there exists an étale cover $\phi_i : T_i \rightarrow T$ and elements $\beta_i \in G(T_i)$ such that

$$\begin{array}{ccc} a_{p,T_i} : G(T_i) & \longrightarrow & Y(T_i) \\ \beta_i & \mapsto & \alpha_i = \alpha \circ \phi_i \end{array}$$

Let β'_i be the projections of the β_i onto $G(T_i)/H(T_i)$. We have the commutative diagram

$$\begin{array}{ccc} T_i \times_{G/H} T_j & \longrightarrow & T_j \\ \downarrow & & \downarrow \beta'_j \\ T_i & \xrightarrow{\beta'_i} & G/H \end{array}$$

Hence, the β'_i correspond to a unique $\beta \in \widetilde{G/H}(T)$, so this shows that $\widetilde{G/H}(T) \cong Y(T)$. \square

5. Quotients

In this section we prove our main result.

Proposition 5.1. *Let the G be an affine algebraic supergroup and H a closed subsupergroup. Then, there exists a finite dimensional representation ρ of G in V and a subspace $W \subset V$, such that:*

$$H(T) = \{g \in G(T) | \rho(g)W = W\},$$

Proof. See [4], 11.7.11. \square

Once we fix suitable coordinates, the subsuperspace $W \subset V$ corresponds to a point $p \in |\text{Gr}|$, where Gr is the Grassmannian of $r|s$ subsuperspaces of $\mathbf{C}^{m|n}$, where $r|s = \dim W$ and $m|n = \dim V$ (see [4] Ch. 10 for the definition of Gr as superscheme and our Appendix A). So we have an action $a = \rho|_G : G \times \text{Gr} \rightarrow \text{Gr}$, where $H = \text{Stab } p$, and the corresponding

orbit map $a_p : G \longrightarrow \text{Gr}$, $a_{p,T}(g) = g \cdot p$, for all $g \in G(T)$. Notice that both G and Gr are smooth algebraic supervarieties; a_p is of finite type. By Chevalley's theorem $|a_p|(|G|)$ is open in its closure, hence it defines a superscheme that we denote by $G \cdot p$ and call the *orbit* of p . We have then the following commutative diagram:

$$\begin{array}{ccc} \text{GL}(m|n) & \xrightarrow{\rho_p} & \text{Gr} \\ \uparrow & & \uparrow \\ G & \xrightarrow{a_p} & G \cdot p \end{array} \quad (2)$$

Notice that, since Proposition 5.1 applies replacing G with $\text{GL}(m|n)$ and both G and H are closed subsupergroups of $\text{GL}(m|n)$ (Cor. 11.7.10 in [4]), we can assume V to be a $\text{GL}(m|n)$ representation (besides a G one), so that the vertical arrows of the above diagram are indeed injections.

Without loss of generality, choose $p \in \text{Gr}$ as the subsuperspace $\langle e_1, \dots, e_r, \epsilon_{n-s}, \dots, \epsilon_n \rangle$. So its stabilizer in $\text{GL}(m|n)$ is:

$$P(A) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \alpha_{13} & \alpha_{14} \\ 0 & a_{22} & \alpha_{23} & 0 \\ 0 & \alpha_{32} & a_{33} & 0 \\ \alpha_{41} & 0 & 0 & a_{44} \end{pmatrix} \right\} \subset \text{GL}(m|n)(A)$$

where a_{11}, a_{44} are $r \times r, s \times s$ matrices with entries in A_0 , while α_{14} is $r \times s$, α_{41} is $s \times r$ matrix with entries in A_1 (similarly for the others).

Observation 5.2. Let

$$g = \begin{pmatrix} g_{11} & g_{12} & \gamma_{13} & \gamma_{14} \\ g_{21} & g_{22} & \gamma_{23} & \gamma_{34} \\ \gamma_{31} & \gamma_{32} & g_{33} & g_{34} \\ \gamma_{41} & \gamma_{42} & g_{34} & g_{44} \end{pmatrix} \in \text{GL}(m|n)(A)$$

with g_{11} and g_{44} invertible.

In the equivalence class $gP(A) \in G(A)/H(A)$, we can choose a unique representative of the form:

$$\begin{pmatrix} I_r & 0 & 0 & 0 \\ u & I_{m-r} & 0 & \eta \\ \xi & 0 & I_{n-s} & v \\ 0 & 0 & 0 & I_s \end{pmatrix}$$

where I_t denotes the identity matrix of rank t .

This is a straightforward calculation coming from the fact that the system:

$$\begin{pmatrix} g_{11} & g_{12} & \gamma_{13} & \gamma_{14} \\ g_{21} & g_{22} & \gamma_{23} & \gamma_{34} \\ \gamma_{31} & \gamma_{32} & g_{33} & g_{34} \\ \gamma_{41} & \gamma_{42} & g_{34} & g_{44} \end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 & 0 \\ u & I_{m-r} & 0 & \eta \\ \xi & 0 & I_{n-s} & v \\ 0 & 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \alpha_{13} & \alpha_{14} \\ 0 & a_{22} & \alpha_{23} & 0 \\ 0 & \alpha_{32} & a_{33} & 0 \\ \alpha_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

has a unique solution. It is given by:

$$\begin{aligned} a_{11} &= g_{11}, & a_{12} &= g_{12}, & \alpha_{13} &= \gamma_{13}, & \alpha_{14} &= \gamma_{14}, & \alpha_{41} &= \gamma_{41}, & a_{44} &= g_{44} \\ a_{22} &= g_{22} - u g_{12}, & \alpha_{23} &= \gamma_{23} - u \gamma_{13}, & \alpha_{32} &= \gamma_{32} - \xi g_{12}, & a_{33} &= g_{33} - \xi \gamma_{13} \\ \eta &= (\gamma_{24} - u \gamma_{14}) g_{44}^{-1}, & \eta &= (\gamma_{31} - u \gamma_{41}) g_{11}^{-1}, \\ u &= (g_{21} - \gamma_{24} g_{44}^{-1} \gamma_{41}) (g_{11} - \gamma_{14} g_{44}^{-1} \gamma_{41})^{-1}, \\ v &= (g_{34} - \gamma_{31} g_{11}^{-1} \gamma_{14}) (g_{44} - \gamma_{41} g_{11}^{-1} \gamma_{14})^{-1} \end{aligned}$$

Lemma 5.3. The superscheme $G \cdot p$ is smooth.

Proof. It is enough to prove smoothness at p . Let N be the closed subsupergroup of $\text{GL}(m|n)$ defined via functor of points by:

$$N(A) = \left\{ \begin{pmatrix} I_r & 0 & 0 & 0 \\ x & I_{m-r} & 0 & v \\ \mu & 0 & I_{n-s} & y \\ 0 & 0 & 0 & I_s \end{pmatrix} \right\}$$

Let $|U|$ be the open subset in $|\mathrm{GL}(m|n)|$ defined by the open condition g_{11} and g_{44} invertible (see Obs 5.2) and $\pi(|U|)$ its projection on $|\mathrm{Gr}|$, $\pi : |\mathrm{GL}(m|n)| \rightarrow |\mathrm{Gr}| = |\mathrm{GL}(m|n)|/|P|$. Since $|U|$ and $\pi(|U|)$ are open respectively in $|\mathrm{GL}(m|n)|$ and $|\mathrm{Gr}|$, they define superschemes, that we denote with U and $\pi(U)$.

Then, we have a functorial bijection:

$$\rho_{p,A} : N(A) \rightarrow \pi(U)(A) \subset \mathrm{Gr}(A)$$

so N and $\pi(U)$ are isomorphic supervarieties.

Let N_G be the closed subsupergroup of G defined as $N_G(A) = N(A) \cap G(A)$. We have $N_G(A) = \pi(U)(A) \cap (G \cdot p)(A)$, hence N_G is isomorphic to the open subscheme $\pi(U) \cap G \cdot p$. \square

Proposition 5.4. *The action $a : G \times G \cdot p \rightarrow G \cdot p$ is transitive.*

Proof. By the previous lemma, we know that $G \cdot p$ is smooth and since G is an algebraic supergroup, by [9] it is smooth. It is enough to show that $|a_p|$ is surjective (obvious) and $(da_p)_{1_G}$ is surjective. By the previous lemma this is clear. \square

Now we prove our main result.

Theorem 5.5. *Let the G be an affine algebraic supergroup and H a closed subsupergroup. The etale sheafification of the functor*

$$T \rightarrow G(T)/H(T), \quad T \in (\text{sschemes}) \quad (3)$$

is representable in the category of superschemes, by a smooth superscheme.

Proof. By Proposition 5.4, G acts transitively on the smooth superscheme $G \cdot p$. Hence, by Proposition 3.2, a_p is a smooth morphism of relative dimension $m|n$ (for suitable m, n). By Proposition 5.1, we have that H is the stabilizer of a point, so that we can apply Theorem 4.5 and obtain the result. \square

We conclude with a comparison with the results in [17] and the definition in [3].

Observation 5.6.

1. The functor of points of a superscheme is a sheaf in the following Grothendieck topologies: Zariski, etale and fppf (see [12,17]). Theorem 5.5 asserts that the etale sheafification of the functor (3) is representable by a superscheme G/H , hence also its fppf sheafification has the same property. This is because G/H is already a sheaf in the fppf topology and the sheafification construction is unique up to isomorphism (see [21]).
2. Our realization of quotients satisfies the properties (Q1)-(Q3) in [3] Sec. 2. Properties (Q1), (Q2) are clear from our construction. As for property (Q3), notice that, in diagram (2), ρ_p is an affine morphism, and the embedding of G into $\mathrm{GL}(m|n)$ is also an affine morphism. The Grassmannian Gr is covered by affine open subsets $V_i = \pi(U_i)$, $V_i \cap G \cdot p$ is a closed subscheme of V_i hence affine and open in $G \cdot p$. By the commutativity of (2), $a_p^{-1}(V_i \cap G \cdot p)$ is affine.

6. Examples

In this section we present some examples of the theory developed so far.

Example 6.1. Quotients of Chevalley Supergroups and flag superschemes. Let \mathfrak{g} be a complex contragredient Lie superalgebra (see [14]), with Cartan subalgebra \mathfrak{h} and root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $\Delta = \Delta_0 \amalg \Delta_1$. Let $\mathcal{B} = \{H_i, X_\alpha\}_{\alpha \in \Delta, 1 \leq i \leq \ell}$ be a Chevalley basis (see [10], Ch. 3) and let $G : (\text{sschemes}) \rightarrow (\text{sets})$ be the Chevalley supergroup obtained as in [10], Ch. 5, Def. 5.9, with ground field $k = \mathbb{C}$ (so $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ in 5.2 in [10]):

$$T \mapsto G(T) := \langle g_0 g_1, g_0 \in G_0(T_0), g_1 = \prod_{\alpha \in \Delta_1} (1 + \xi_\alpha X_\alpha) \rangle, \quad T \in (\text{sschemes}) \quad (4)$$

$G(\phi)$ is defined in a natural way for a morphism ϕ , since by the very construction, Chevalley supergroups are matrix supergroups (see [10]). By Thm 5.19 in [10], we can express the Chevalley supergroup functor G in (4) also as:

$$G(T) = G_1^{-, <}(T) G_0(T) G_1^{+, <}(T), \quad T \in (\text{sschemes}) \quad (5)$$

$$G_1^{\pm, <}(T) := (1 + \xi_{\alpha_1} X_{\alpha_1}) \dots (1 + \xi_{\alpha_n} X_{\alpha_n}), \quad \alpha_i < \alpha_{i+1}, \quad \alpha_j \in \Delta_1^{\pm}$$

for a total ordering $<$ on Δ^{\pm} and a choice of positive system Δ^+ , $\Delta = \Delta^- \amalg \Delta^+$, $\Delta^- = -\Delta^+$. As a consequence G is representable by a group superscheme. For example, when $g \in \text{SL}(m|n)(T)$ (the special linear supergroup), the decomposition (5) takes the familiar form:

$$g = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ \nu & I_n \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} I_m & \mu \\ 0 & I_n \end{pmatrix},$$

$$\begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \in \text{SL}(m|n)_0(T) = \text{SL}(m)(T) \times \text{SL}(n)(T) \times \mathbb{C}^{\times}(T)$$

where we take as Cartan subalgebra \mathfrak{h} the diagonal matrices and we fix a positive system so that the root spaces with roots in Δ_1^+ are in block μ , while the root spaces with roots in Δ_1^- are in block ν (see [4] Ch. 1).

Let Δ_P be a parabolic set of roots (see [7] Sec. 1 and [8] for more details):

$$\Delta_P = \Delta^0 + \Delta^n, \quad \Delta = -\Delta^n \amalg \Delta^0 \amalg \Delta^n.$$

By the ordinary theory, we have that $\mathfrak{p}_0 := \mathfrak{h} \oplus \sum_{\alpha \in \Delta_{P,0}} \mathfrak{g}_{\alpha}$ is a parabolic subalgebra of \mathfrak{g}_0 and P_r the corresponding ordinary parabolic groupscheme, i.e. $\mathfrak{p}_0 = \text{Lie}(P_r)$. Δ_P defines the supergroup functor:

$$T \mapsto P(T) := \langle g_0 g_1, g_0 \in P_r(T), g_1 = \prod_{\alpha \in \Delta_{P,1}} (1 + \xi_{\alpha} X_{\alpha}) \rangle$$

Since $\Delta_{P,0} + \Delta_{P,1} \subset \Delta_{P,1}$, and we can choose a basis for \mathfrak{p} as a subset of the Chevalley basis \mathcal{B} , following the arguments as in Thm 5.19 in [10], we have that P is representable by an affine supergroup scheme. P_r is a closed subgroup of G_r by the ordinary theory. We call P a *parabolic subgroup* of G . Theorem 5.5 gives then immediately the following proposition (see Obs. 5.6 for notation).

Proposition 6.2. *Let G be a Chevalley supergroup, P a parabolic subgroup. The etale sheafification of the functor:*

$$T \longrightarrow G(T)/P(T), \quad T \in (\text{sschemes}) \quad (6)$$

is representable by a smooth superscheme.

This shows the representability of flag supervarieties G/P for any supergroup, either belonging to one of the families A, B, C, D or to the exceptional $G(2), F(4)$ and provides an explicit way to express them. To our knowledge this is novel for exceptional superalgebras, while the type A was extensively studied in [16,11] (see also the references therein) and the types B, C, D were briefly considered in [5] in the analytic category. Notice that the expression (6) gives an explicit way to describe a super flag, with a manifest action of the supergroup G and gives its representability avoiding gluing arguments.

We now give another example to elucidate our construction and relate it with [5], where it is instrumental to the construction of super Harish-Chandra representations.

Example 6.3. Lagrangian Superscheme and Siegel Superspace. We now focus on $G = \text{Osp}(m|2n)$, the complex orthosymplectic supergroup, (see [4] Sec. 11.8 for its definition). We define the closed subsupergroup $P : (\text{sschemes}) \longrightarrow (\text{sets})$ of G as:

$$P(T) = \left\{ \begin{pmatrix} a & 0 & \alpha_2 \\ b_{11}\alpha_2^t a & b_{11} & b_{12} \\ 0 & 0 & (b_{11}^t)^{-1} \end{pmatrix} \mid \begin{cases} a^t a = 1 \\ (b_{11}^{-1} b_{12}) - (b_{11}^{-1} b_{12})^t = \alpha_2^t \alpha_2 \end{cases} \right\} \quad (7)$$

where $a \in \text{GL}(m)(T)$, $b_{11} \in \text{GL}(n)(T)$, $b_{12} \in \text{M}(n)(T)$, $\alpha_2 \in \text{M}(m|0, 0|n)(T)$, (GL, M denoting respectively the general linear (super)group and the (super)matrices).

Proposition 6.4. *Let the notation be as above. The etale sheafification of the functor:*

$$T \mapsto \text{Osp}(m|2n)(T)/P(T)$$

is representable by a smooth superscheme \mathcal{L} , whose functor of points is:

$$T \mapsto \mathcal{L}(T) = \left\{ \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \mid \zeta^t \zeta + z^t - z = 0 \right\} \quad (8)$$

Proof. The first statement is an immediate consequence of our Theorem 5.5. As for the explicit expression of the quotient, since we already know the representability, we may work on local superalgebras, where we do not need to take sheafification. The calculation is then the same as in [5] Sec. 4. \square

We call \mathcal{L} as the *Lagrangian superscheme*, since it contains the maximally isotropic superspaces, with respect to the standard symplectic form J (see [5]). The Lagrangian \mathcal{L} was extensively studied in [5] as analytic supermanifold; its open subsupermanifold \mathcal{S} defined by the condition $\text{red}(z) > 0$, is the Siegel superspace, $\text{red}(z)$ denoting the image of z by the projection with respect to the ideal sheaf J_T generated by the odd elements in \mathcal{O}_T . \mathcal{S} is an example of hermitian superspace: Harish-Chandra representations of real forms of $\text{Osp}(m|n)$ are constructed in the superspace of holomorphic sections on a vector bundle on \mathcal{S} determined by a character of the parabolic subgroup P (see [5]).

We conclude observing that in (8), \mathcal{L} is realized as an orbit in the grassmannian supervariety $\text{Gr}(m|n, m|2n)$ of $m|n$ subspaces into $m|2n$ dimensional superspace (see Sec. 5).

Appendix A. The Grassmannian superscheme

In this appendix we provide, for the reader convenience, the definition of the Grassmannian superscheme. For more details see [16], [4] Ch. 10.

Consider the functor $\text{Gr} : (\text{salg}) \rightarrow (\text{sets})$, where for any superalgebra A , $\text{Gr}(A)$ is the set of projective A -submodules of rank $r|s$ of $A^{m|n}$. When A is a local superalgebra, $\text{Gr}(A)$ consists of the free A -submodules of dimension $r|s$ of the free module $A^{m|n}$.

Equivalently, $\text{Gr}(A)$ is defined as the set $\text{Gr}(A) = \{\alpha : A^{m|n} \rightarrow L\}$, where α is a surjective morphism, L is a projective A -module of rank $r|s$, modulo the following equivalence relation: $\alpha \equiv \alpha'$ if and only if they have the same kernel. Notice that in this definition L also varies.

We now specify Gr on morphisms.

Given a morphism $\psi : A \rightarrow B$ of superalgebras, and the element of $\text{Gr}(A)$, $f : A^{m|n} \rightarrow L$, we define an element of $\text{Gr}(B)$ as follows:

$$\text{Gr}(\psi)(f) : B^{m|n} = A^{m|n} \otimes_A B \rightarrow L \otimes_A B$$

where $L \otimes_A B$ is clearly a projective B -module of rank $r|s$. In order to show that Gr is the functor of points of a superscheme, we show that it admits a cover of open affine subfunctors and that it is local, by the Representability Criterion (see [4] Thm 10.3.7).

Consider the multiindex $I = (i_1, \dots, i_r | \mu_1, \dots, \mu_s)$ and the map $\phi_I : A^{r|s} \rightarrow A^{m|n}$ where $\phi_I(x_1, \dots, x_r | \xi_1, \dots, \xi_s)$ with $1 \leq i_1 < \dots < i_r \leq m$, $1 \leq \mu_1 < \dots < \mu_s \leq n$ is the $m|n$ -uple with x_1, \dots, x_r occupying the positions i_1, \dots, i_r and ξ_1, \dots, ξ_s occupying the positions μ_1, \dots, μ_s and all the other positions are occupied by zero.

Now define the subfunctors v_I of Gr as follows. The $v_I(A)$ are the morphism $\alpha : A^{m|n} \rightarrow L$, i.e. the elements in $\text{Gr}(A)$, such that $\alpha \circ \phi_I$ is surjective. Notice that if $\alpha \circ \phi : A^{r|s} \rightarrow A^{m|n} \rightarrow L$ is surjective, since $A^{r|s}$ and L are projective modules of the same rank, $\alpha \circ \phi$ is an isomorphism hence L is free. This is crucial for the representability of v_I and ultimately of Gr .

The functor v_I is in fact representable: $v_I \cong M_{(m|n) \times (m-r|n-s)}$, that is v_I is isomorphic to the functor of points of $(m|n) \times (m-r|n-s)$ matrices. In fact $v_I(A)$ consists of morphisms $\alpha : A^{m|n} \rightarrow L \cong A^{r|s}$, where we specify the images of $r|s$ elements in the canonical basis of $A^{m|n}$. We leave to the reader the easy checks involved.

Then one can also show that the v_I are open affine subfunctors of Gr and they cover $|\text{Gr}|$. This is done in [4] Sec. 10.4. It is useful to remark that $|\text{Gr}|$ indeed consists of pairs of subspaces (W_1, W_2) of \mathbb{C}^m and \mathbb{C}^n of dimension r and s respectively.

Finally to show that Gr is local, we identify $\text{Gr}(A)$ with locally free coherent sheaves of rank $r|s$ (see [4] Sec. 10.4).

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