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A conformal representation for linear Weingarten surfaces in the de Sitter space

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Abstract

In this paper we study a wide family of spacelike surfaces in $\mathbb{S}^3_1$ which includes, for instance, constant mean curvature 1 surfaces and flat surfaces: those whose mean and Gauss–Kronecker curvatures verify the lineal relationship $2a(H-1)+b(K-1)=0$ for $a, b \in \mathbb{R}$, $a+b \neq 0$. We show that these surfaces can be parametrized with holomorphic data and we use it to classify the complete surfaces from this family with non-negative Gaussian curvature.
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1. Introduction

The global study of constant mean curvature and constant Gaussian curvature surfaces in space forms has been of special relevance in submanifolds geometry, especially those ones whose structure equations are integrable in terms of holomorphic data. This is owing to the fact that, in this case, there are a great deal of global results from complex analysis which can be applied to such study. Some representative examples are the Enneper–Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ [17] and the McNertney–Kobayashi one for maximal surfaces in $\mathbb{L}^3$ [13].

In this paper we will deal with spacelike surfaces in the de Sitter space $\mathbb{S}^3_1$, which have been of increasing interest in recent years, motivated, in part, by the fact that they exhibit nice Bernstein-type properties. For instance, Ramanathan [18] proved that every compact spacelike surface in $\mathbb{S}^3_1$ with constant mean curvature is totally umbilical. This result was generalized to hypersurfaces of any dimension by Montiel [16]. In the same direction, the first author and Alías proved in [2] that the totally umbilical round spheres are the only complete spacelike hypersurfaces with bounded Gauss map and constant mean curvature.

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On the other hand, Li [14] showed that every compact spacelike surface in $\mathbb{S}^3_1$ with constant Gaussian curvature is totally umbilical (see also [3] for some generalizations to the case of hypersurfaces in $\mathbb{S}^n_1$ with a higher order mean curvature). More recently, the first author and Romero [5] have proved that the totally umbilical round spheres are the only compact spacelike surfaces in the de Sitter space such that the Gaussian curvature of the second fundamental form is constant.

As a natural generalization of Ramanathan and Li results, the first author and Gálvez [4] characterized the totally umbilical round spheres of $\mathbb{S}^3_1$ as the only compact linear Weingarten spacelike surfaces.

In this work we study spacelike surfaces in $\mathbb{S}^3_1$ whose mean curvature, H, and Gauss–Kronecker curvature, K, verify the relationship

$$2\,a\,(H-1)+b(K-1)=0,$$

with $a+b\neq 0$, that is, surfaces of elliptic type. We will refer to such surfaces as linear Weingarten surfaces of Bianchi type, in short BLW-surfaces. The reason for this terminology is that Bianchi [6] was the first to study such surfaces in the hyperbolic space $\mathbb{H}^3$.

We have organized the paper as follows. In Section 2 we introduce the notation and main concepts. Also, in Theorem 1 we find a special Riemannian metric σ on any BLW-surface and prove that its hyperbolic Gauss map is conformal for the conformal structure induced by σ. This fact will be the key, in Section 3, to obtaining a Weierstrass representation for such a surface (Theorem 2 and Corollary 1) in terms of meromorphic data which generalizes the one given by Aiyama and Akutagawa for $H=1$ [1] (see also [8]) and by Gálvez, Martínez and Milán for flat surfaces [9]. We also refer the reader to [10] where a representation for linear Weingarten surfaces of Bryant type in $\mathbb{H}^3$ is obtained.

By using this representation, in Section 4 we classify the complete BLW-surfaces with non-negative Gaussian curvature (Theorems 3 and 4).

2. Set-up

Let us denote by $\mathbb{L}^4$ the four-dimensional *Lorentz–Minkowski space* given as the vectorial space $\mathbb{R}^4$ with the Lorentzian metric $\langle,\rangle$ induced by the quadratic form $-x_0^2+x_1^2+x_2^2+x_3^2$, and consider the *de Sitter space* realized as the Lorentzian submanifold

$$\mathbb{S}^3_1=\{(x_0,x_1,x_2,x_3)\in\mathbb{L}^4 : -x_0^2+x_1^2+x_2^2+x_3^2=1\}.$$

As is well known, $\mathbb{S}^3_1$ inherits from $\mathbb{L}^4$ a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature 1.

We will also consider the *hyperbolic space*

$$\mathbb{H}^3=\{(x_0,x_1,x_2,x_3)\in\mathbb{L}^4 : -x_0^2+x_1^2+x_2^2+x_3^2=-1, x_0>0\}$$

and the *positive null cone* given by

$$\mathbb{N}^3_+=\{(x_0,x_1,x_2,x_3)\in\mathbb{L}^4 : -x_0^2+x_1^2+x_2^2+x_3^2=0, x_0>0\}.$$

A smooth immersion $\psi : S\longrightarrow\mathbb{S}^3_1$ of a two-dimensional connected manifold S is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on S, which, as usual, is also denoted by $\langle,\rangle$. The time orientation of $\mathbb{S}^3_1$ allows us to choose a timelike unit normal field η globally defined on S, tangent to $\mathbb{S}^3_1$, and hence we may assume that S is oriented by η.

Let us consider, associated with ψ, the map

$\phi : S\longrightarrow\mathbb{N}^3_+$ given by $\phi:=\psi+\eta=(\phi_0,\phi_1,\phi_2,\phi_3)$. Then the *hyperbolic Gauss map* of ψ is defined as the map

$$G=\frac{\phi_1+\mathrm{i}\phi_2}{\phi_0+\phi_3}\in\mathbb{C}\cup\{\infty\}$$

or equivalently

$$G\equiv\left(\frac{\phi_1}{\phi_0},\frac{\phi_2}{\phi_0},\frac{\phi_2}{\phi_0}\right)\in\mathbb{S}^2\subseteq\mathbb{R}^3$$

(see [7]). In addition, by considering the inclusion $\mathbb{R}^3 \longrightarrow \mathbb{L}^4$ given by $(x_1, x_2, x_3) \longmapsto (1, x_1, x_2, x_3)$ we can also identify

$$G \equiv \frac{1}{\phi_0}(\phi_0, \phi_1, \phi_2, \phi_3) \in \mathbb{N}^3_+.$$

Given a spacelike immersion $\psi : S \longrightarrow \mathbb{S}^3_1$, we will denote by $I = \langle \mathrm{d}\psi, \mathrm{d}\psi\rangle$, $II = \langle \mathrm{d}\psi, -\mathrm{d}\eta\rangle$ and $III = \langle \mathrm{d}\eta, \mathrm{d}\eta\rangle$ its first, second and third fundamental forms respectively.

Let A be a Riemannian metric on S and let us take a conformal parameter z for A. Given a 2-form $B = L\mathrm{d}z^2 + 2M|\mathrm{d}z|^2 + N\mathrm{d}\bar{z}^2$, we define $Q(B, A)$ as the 2-form $Q(B, A) = L\mathrm{d}z^2$. Observe that, in particular, $Q(II, I)$ is nothing but the Hopf differential of the immersion ψ.

In this paper we will deal with a wide family of linear Weingarten surfaces in $\mathbb{S}^3_1$, in particular those whose mean and Gauss–Kronecker curvatures, H and K respectively, satisfy a linear relationship of the type

$$2\,a\,(H - 1) + b\,(K - 1) = 0, \quad a, b \in \mathbb{R}, a + b \neq 0. \tag{1}$$

We will refer to these surfaces as *linear Weingarten surfaces of Bianchi type*, for short, BLW-surfaces.

In the following lemma we gather several technical results that will be essential for obtaining the conformal representation for the BLW-surfaces.

Theorem 1. *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a BLW-surface. Then we can choose $a, b \in \mathbb{R}$ such that $|a + b| = 1$, $2\,a\,(H - 1) + b\,(K - 1) = 0$, and $\sigma = a\,I + b\,II$ is a Riemannian metric on S. In addition, the hyperbolic Gauss map is conformal for σ. Moreover, either ψ is a totally umbilical surface contained in a degenerate hyperplane or $I_\phi = \langle \mathrm{d}\phi, \mathrm{d}\phi\rangle$ is a pseudometric which is conformal to σ.*

Proof. First, note that scaling a and b in (1), such relationship holds with $a + b = 1$.

In order to complete the proof, we will use some elemental facts from the theory of Codazzi pairs. In particular, the concepts and results that we will need can be found in [15].

Since (I, II) is a Codazzi pair and H and K verify the linear relationship (1), it follows from [15, Theorem 8] that σ is definite. Actually, it can also be seen by taking $\{e_1, e_2\}$ as a $\langle, \rangle$-orthonormal frame at a point $p \in S$ such that $\mathrm{d}\eta(e_i) = -k_i e_i$, $i = 1, 2$, where k_1, k_2 stand for the principal curvatures of ψ. Then

$$\begin{aligned}\sigma(e_1, e_1)\,\sigma(e_2, e_2) - \sigma(e_1, e_2)^2 &= (a + bk_1)\,(a + bk_2)\\ &= a^2 + b\,(2aH + bK) = (a + b)^2 = 1,\end{aligned}$$

and so we can assume that σ is definite.

Now, replacing a by $-a$ and b by $-b$ if necessary, we have that σ is positive definite and $|a + b| = 1$.

On the other hand, using the well-known relation $III = 2H\,II - K\,I$, one gets

$$I_\phi = \langle d(\psi + \eta), d(\psi + \eta)\rangle = -(K - 1)I + 2(H - 1)II. \tag{2}$$

Let us distinguish the two following cases:

- If $b = 0$, and so $H = 1$ on S, let us take a conformal parameter z for $\sigma = I$. Then, from [15, Theorem 8, (i)] $Q(II, I) = \langle \psi_z, -\eta_z\rangle \mathrm{d}z^2$ is a holomorphic 2-form for the conformal structure induced by σ. Observe that the set of singular points of $I_\phi = -(K - 1)I$ coincides with that of umbilical points of ψ with $K = 1$. Now, since the zeros of $Q(II, I)$ are precisely the umbilical points of S, either ψ is a totally umbilical surface contained in a degenerate hyperplane or I_ϕ is a pseudometric.
- If $b \neq 0$, let us take a conformal parameter z for σ. In this case [15, Theorem 8, (ii)] assures that $Q(I, \sigma) = \langle \psi_z, \psi_z\rangle \mathrm{d}z^2$ is holomorphic for the conformal structure induced by σ. Hence, since $0 = Q(\sigma, \sigma) = \langle a\psi_z - b\eta_z, \psi_z\rangle \mathrm{d}z^2$ then $Q(II, \sigma)$ is also holomorphic. Now, from (1) and (2) we have that $bI_\phi = 2(H - 1)\sigma$ and so its set of singular points coincides with that of umbilical points of ψ with $H = 1 = K$. The reasoning ends as in the above case.

Finally, let us see that G is conformal for the conformal structure given by σ. If we take z as a conformal parameter for σ and put $G = (1/\phi_0)(\phi_0, \phi_1, \phi_2, \phi_3)$, that is equivalent to seeing that $\langle G_z, G_z\rangle = 0$. In fact,

$$\langle G_z, G_z\rangle = \frac{1}{\phi_0^2}\langle(\psi + \eta)_z, (\psi + \eta)_z\rangle$$

which vanishes identically because the pseudometric I_ϕ is conformal to σ as we have seen above. $\square$

Remark 1. If S is compact and simply connected (and consequently $\psi(S)$ is a topological sphere), since Theorem 1 assures that $Q(I,\sigma)$ and $Q(II,\sigma)$ are holomorphic 2-forms, it follows that both $Q(I,\sigma)$ and $Q(II,\sigma)$ vanish identically on S and $\psi(S)$ must be a totally umbilical sphere.

Remark 2. From the proof of Theorem 1, it follows easily that ψ is totally umbilical if and only if I_ϕ vanishes identically.

3. Conformal representation of BLW-surfaces

Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a BLW-surface and let us take a conformal parameter z for the Riemannian metric σ, which is also conformal for the pseudometric I_ϕ from Theorem 1. If we put

$$G \equiv G_1 + \mathrm{i}G_2 = \frac{\phi_1 + \mathrm{i}\phi_2}{\phi_0 + \phi_3} \in \mathbb{C} \cup \{\infty\},$$

then ϕ can be written as

$$\phi = \frac{\rho}{2}(1 + |G|^2, G + \overline{G}, -\mathrm{i}(G - \overline{G}), 1 - |G|^2) \in \mathbb{N}^3_+ \tag{3}$$

where $\rho = \phi_0 + \phi_3$.

In the following theorem we obtain a conformal representation for the BLW-surface ψ in terms of the data G and ρ.

Theorem 2. *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a non-totally umbilical BLW-surface satisfying (1), with normal $\eta : S \longrightarrow \mathbb{H}^3$ and hyperbolic Gauss map G. Let us take $\phi := \psi + \eta = (\phi_0, \phi_1, \phi_2, \phi_3)$ and $\rho = \phi_0 + \phi_3$. Given a local conformal parameter z for the metric σ on S, ψ and η can be recovered as*

$$\begin{aligned}
\psi_0 &= -\frac{1}{\rho}(1 - (\rho/2)^2(1 + |G|^2)) - \frac{2}{\rho^2}\mathcal{R}\left(\frac{G\rho_z}{G_z}\right) - \frac{|\rho_z|^2(1 + |G|^2)}{|G_z|^2\rho^3}\\
\psi_1 + \mathrm{i}\psi_2 &= \frac{\rho G}{2} - 2\frac{\rho_{\bar z}}{\rho^2\overline{G_z}} - 2\frac{|\rho_z|^2 G}{|G_z|^2\rho^3}\\
\psi_3 &= -\frac{1}{\rho}(-1 - (\rho/2)^2(1 - |G|^2)) + \frac{2}{\rho^2}\mathcal{R}\left(\frac{G\rho_z}{G_z}\right) - \frac{|\rho_z|^2(1 - |G|^2)}{|G_z|^2\rho^3}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\eta_0 &= \frac{1}{\rho}(1 + (\rho/2)^2(1 + |G|^2)) + \frac{2}{\rho^2}\mathcal{R}\left(\frac{G\rho_z}{G_z}\right) + \frac{|\rho_z|^2(1 + |G|^2)}{|G_z|^2\rho^3}\\
\eta_1 + \mathrm{i}\eta_2 &= \frac{\rho G}{2} + 2\frac{\rho_{\bar z}}{\rho^2\overline{G_z}} + 2\frac{|\rho_z|^2 G}{|G_z|^2\rho^3}\\
\eta_3 &= \frac{1}{\rho}(-1 + (\rho/2)^2(1 - |G|^2)) - \frac{2}{\rho^2}\mathcal{R}\left(\frac{G\rho_z}{G_z}\right) + \frac{|\rho_z|^2(1 - |G|^2)}{|G_z|^2\rho^3},
\end{aligned} \tag{5}$$

where $\mathcal{R}(w)$ stands for the real part of $w \in \mathbb{C}$. Moreover, the pseudometric $I_\phi = \langle \mathrm{d}\phi, \mathrm{d}\phi\rangle$ on S has constant curvature

$$K_\phi = -\frac{a}{a+b}. \tag{6}$$

Conversely, let S be a simply connected Riemann surface, $G : S \longrightarrow \mathbb{S}^2$ a meromorphic map, $K_\phi \in \mathbb{R}$ and ρ a solution of the Liouville-type equation

$$(\ln\rho)_{z\bar z} = K_\phi \frac{|G_z|^2}{4}\rho^2. \tag{7}$$

Then the immersion given by (4) is a BLW-surface such that

$$2(-K_\phi)(H - 1) + (1 + K_\phi)(K - 1) = 0 \tag{8}$$

with normal (5) and whose hyperbolic Gauss map coincides with G. Moreover, the conformal structure of S as a Riemann surface coincides with the one induced by σ.

Proof. First, let us recover ψ and η from G and ρ. To do that, let us put ϕ as in (3). Since $\langle\psi,\psi\rangle = 1$, $\langle\psi,\eta\rangle = 0$ and $\langle\phi_z,\psi\rangle = 0 = \langle\phi_{\bar z},\psi\rangle$, the coordinates ψ_1, ψ_2 and ψ_3 can be obtained from

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ (\phi_1)_z & (\phi_2)_z & (\phi_3)_z \\ (\phi_1)_{\bar z} & (\phi_2)_{\bar z} & (\phi_3)_{\bar z} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 1+\phi_0\psi_0 \\ (\phi_0)_z\psi_0 \\ (\phi_0)_{\bar z}\psi_0 \end{pmatrix}$$

where z is a conformal parameter for σ. Once you have these three coordinates, ψ_0 can be calculated from $\langle\psi,\psi\rangle = 1$. Thus, (4) holds. Now, (5) follows from (4) and (3).

On the other hand, it is a standard calculation to check that

$$I_\phi = 2\langle\phi_z,\phi_{\bar z}\rangle|\mathrm{d}z|^2 = \rho^2|G_z|^2|\mathrm{d}z|^2.$$

Hence, the curvature of I_ϕ is given by

$$\rho^2|G_z|^2K_\phi = -4(\log\rho)_{z\bar z}. \tag{9}$$

Now, putting $\lambda = (\log\rho)_z$ and $\omega = \rho G_z$, and using (9), one gets from (4) and (5) that the first and second fundamental forms of ψ are given by

$$\begin{aligned} I &= -(1+K_\phi)\omega\left(\frac{\lambda}{\omega}\right)_z \mathrm{d}z^2 + 2\left(\frac{(1+K_\phi)^2}{8}|\omega|^2 + 2\left|\left(\frac{\lambda}{\omega}\right)_z\right|^2\right)|\mathrm{d}z|^2 - (1+K_\phi)\overline{\omega}\left(\frac{\overline{\lambda}}{\overline{\omega}}\right)_{\bar z}\mathrm{d}\bar z^2 \\ II &= -K_\phi\omega\left(\frac{\lambda}{\omega}\right)_z \mathrm{d}z^2 + 2\left(-\frac{1+K_\phi}{4}|\omega|^2 + \langle\psi_z,\psi_{\bar z}\rangle\right)|\mathrm{d}z|^2 - K_\phi\overline{\omega}\left(\frac{\overline{\lambda}}{\overline{\omega}}\right)_{\bar z}\mathrm{d}\bar z^2 \\ &= -K_\phi\omega\left(\frac{\lambda}{\omega}\right)_z \mathrm{d}z^2 + 2\left(\frac{K_\phi^2-1}{8}|\omega|^2 + 2\left|\left(\frac{\lambda}{\omega}\right)_z\right|^2\right)|\mathrm{d}z|^2 - K_\phi\overline{\omega}\left(\frac{\overline{\lambda}}{\overline{\omega}}\right)_{\bar z}\mathrm{d}\bar z^2 \end{aligned}$$

respectively, and so its Gauss–Kronecker and mean curvatures become

$$K = 1 + \frac{4K_\phi|\omega|^2}{16|(\lambda/\omega)_z|^2 - (1+K_\phi)^2|\omega|^2}, \qquad H = 1 + \frac{2(1+K_\phi)|\omega|^2}{16|(\lambda/\omega)_z|^2 - (1+K_\phi)^2|\omega|^2}. \tag{10}$$

Thus, from (9),

$$2K_\phi(H-1) = (K_\phi+1)(K-1),$$

and using that $2\,a\,(H-1) + b\,(K-1) = 0$, (6) finally holds.

The converse is a straightforward computation. □

Remark 3. It is well known that the solutions of the Liouville-type equation (7) can be expressed in terms of holomorphic functions (see, for instance, [11]). In fact, every solution of (7) can be written as

$$\rho^2 = \frac{4|h_z|^2}{|G_z|^2(1+K_\phi|h|^2)^2},$$

where h is a meromorphic function on S, holomorphic if $K_\phi \le 0$ and $1 + K_\phi|h|^2 > 0$. Thus, our representation is, actually, a conformal one.

Corollary 1. *Let S be a non-compact simply connected surface and $\psi : S \longrightarrow \mathbb{S}^3_1$ a BLW-immersion satisfying (1). Then there exists a pair (h,α), where h is a meromorphic function and α a holomorphic 1-form on S, such that its first and second fundamental forms are given by*

$$I = -(1+K_\phi)\alpha\mathrm{d}h + \left(\frac{(1+K_\phi)^2|\mathrm{d}h|^2}{(1+K_\phi|h|^2)^2} + (1+K_\phi|h|^2)^2|\alpha|^2\right) - (1+K_\phi)\bar\alpha\mathrm{d}\bar h \tag{11}$$

$$II = -K_\phi \alpha \mathrm{d}h + \left(\frac{(K_\phi^2 - 1)|\mathrm{d}h|^2}{(1 + K_\phi |h|^2)^2} + (1 + K_\phi |h|^2)^2 |\alpha|^2 \right) - K_\phi \bar{\alpha} \mathrm{d}\bar{h} \tag{12}$$

respectively, where $K_\phi = -\frac{a}{a+b}$. *In addition the metric* σ *becomes*

$$\sigma = (a+b) \left((1 + K_\phi |h|^2)^2 |\alpha|^2 - \frac{(1 + K_\phi)^2 |\mathrm{d}h|^2}{(1 + K_\phi |h|^2)^2} \right) \tag{13}$$

and the Gauss–Kronecker and mean curvatures of ψ *are given by*

$$K = 1 - \frac{4K_\phi |\mathrm{d}h|^2}{(1 + K_\phi)^2 |\mathrm{d}h|^2 - (1 + K_\phi |h|^2)^4 |\alpha|^2}, \qquad H = 1 + \frac{2(1 + K_\phi)|\mathrm{d}h|^2}{(1 + K_\phi |h|^2)^4 |\alpha|^2 - (1 + K_\phi)^2 |\mathrm{d}h|^2}. \tag{14}$$

Conversely, given a simply connected Riemann surface S, $K_\phi \in \mathbb{R}$ *and a pair* (h, α) *as above such that* (13) *is a positive definite metric, then there exists* $\psi : S \longrightarrow \mathbb{S}_1^3$ *a BLW-immersion, unique up to isometries of* $\mathbb{S}_1^3$, *satisfying* (8) *with* I, II *and* σ *given by* (11), (12) *and* (13) *respectively.*

Proof. We will suppose that ψ is not totally umbilical. Otherwise the result follows easily by taking h as a constant.

As S is simply connected and non-compact, we can choose a global isothermal parameter z on S and, from Theorem 1, $I_\phi = \langle d(\psi + \eta), d(\psi + \eta) \rangle$ is a pseudometric with constant curvature K_ϕ. Hence (see Remark 3), there exists a meromorphic function h on S (holomorphic if $K_\phi \leq 0$) such that

$$I_\phi = \rho^2 |\mathrm{d}G|^2 = \frac{4|\mathrm{d}h|^2}{(1 + K_\phi |h|^2)^2} \tag{15}$$

and

$$1 + K_\phi |h|^2 > 0. \tag{16}$$

Thus, it is clear that if h has a pole of order m at a point p, then $\mathrm{d}h$ has a pole of order $m+1$ at p and $\mathrm{d}G$ has a zero of order $m-1$ there. Analogously, if h has a zero of order k, both $\mathrm{d}h$ and $\mathrm{d}G$ have a zero of order $k-1$. In particular, the function $f = \mathrm{d}h/\mathrm{d}G$ is meromorphic on S (holomorphic if $K_\phi \leq 0$) without zeros on S.

On the other hand, from (9), we have

$$\rho_{\bar{z}} \left(\frac{\lambda}{\omega} \right)_z + \rho \left(\frac{\lambda}{\omega} \right)_{z\bar{z}} = 0$$

which means that $\rho(\lambda/\omega)_z \mathrm{d}z$ is a meromorphic 1-form on S. Hence, if we define the meromorphic 1-form

$$\alpha = \frac{\rho}{f} \left(\frac{\lambda}{\omega} \right)_z \mathrm{d}z$$

we can write I, II and σ as (11), (12) and (13) respectively. Regarding (14), it follows easily from (10).

Finally note that $(1 + K_\phi |h|^2)^2 |\alpha|^2$ is finite on S, because otherwise σ would not be a positive definite metric (see (13)) since

$$\frac{(1 + K_\phi)^2 |\mathrm{d}h|^2}{(1 + K_\phi |h|^2)^2}$$

has no poles, as we have reasoned in the paragraph after formula (16). Therefore we can assure that α is holomorphic.

The converse is, again, a standard computation. □

4. Complete BLW-surfaces

First, note that non-negative Gaussian curvature is equivalent to $K \leq 1$. Let us take, then, a complete immersion $\psi : S \longrightarrow \mathbb{S}_1^3$ with $K \leq 1$.

Observe that if the immersion is flat (that is, $K = 1$) then $a = 0$, whereas $a > 0$ if it is non-flat. In fact, from (13) and (14) one gets

$$K - 1 = -4a\frac{|\mathrm{d}h^2|}{(1 + K_\phi|h|^2)^2\sigma}$$

and the above assertion follows bearing in mind that σ is positive definite and $K \leq 1$.

We will distinguish these two cases. We will assume from now on that S is simply connected, by taking its simply connected cover if necessary.

Theorem 3. *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a complete flat BLW-surface satisfying (1). Then either $\psi(S)$ is a totally umbilical surface contained in a degenerate hyperplane or $\psi(S)$ is a hyperbolic cylinder.*

Proof. Since $a = 0$, it follows that $K_\phi = 0$ and, from (13), we have

$$\begin{aligned} &|\mathrm{d}h|^2 \leq |\alpha|^2 \quad \text{if } b = 1,\\ &|\mathrm{d}h|^2 \geq |\alpha|^2 \quad \text{if } b = -1. \end{aligned}$$

On the other hand, from (11),

$$\frac{1}{2}I \leq |\mathrm{d}h|^2 + |\alpha|^2$$

that is, in both cases ($b = 1$ and $b = -1$), we have a complete flat metric on S ($|\alpha|^2$ if $b = 1$ and $|\mathrm{d}h|^2$ if $b = -1$) conformal to σ, which assures that S is conformally equivalent to the complex plane.

When $b = 1$, it follows that $|\mathrm{d}h/\alpha| \leq 1$ and therefore $\mathrm{d}h/\alpha$ is constant. In particular, if $\mathrm{d}h$ vanishes identically, then from (11) and (12) $\psi(S)$ is a totally umbilical surface contained in a degenerate hyperplane. Otherwise, $\mathrm{d}h = c_1\alpha$ for a non-zero complex constant c_1, $|c_1| \leq 1$. Under this assumption, we can take locally a parameter ζ such that $\mathrm{d}h = \mathrm{d}\zeta$ and so $\alpha(\zeta) = c_1\mathrm{d}\zeta$. Then, (15) becomes

$$\rho^2 = \frac{4}{|G_\zeta|^2}, \tag{17}$$

whence

$$\frac{\rho_\zeta}{\rho} = -\frac{G_{\zeta\zeta}}{2G_\zeta} \tag{18}$$

and

$$\rho^2 G_\zeta = \frac{4}{\overline{G_\zeta}}. \tag{19}$$

Now, from (18) and (19) we get

$$\frac{\rho_\zeta}{\rho^2 G_\zeta} = -\frac{\rho G_{\zeta\zeta}\overline{G_\zeta}}{8G_\zeta}$$

and so, using again (18),

$$\left(\frac{\rho_\zeta}{\rho^2 G_\zeta}\right)_\zeta = -\frac{\rho\overline{G_\zeta}}{8}\left(\left(\frac{G_{\zeta\zeta}}{G_\zeta}\right)_\zeta - \frac{1}{2}\left(\frac{G_{\zeta\zeta}}{G_\zeta}\right)^2\right) = -\frac{\rho\overline{G_\zeta}}{8}\{G,\zeta\}$$

where $\{G,\zeta\} := \left(\left(\frac{G_{\zeta\zeta}}{G_\zeta}\right)_\zeta - \frac{1}{2}\left(\frac{G_{\zeta\zeta}}{G_\zeta}\right)^2\right)$ is the Schwarzian derivative of G (see [12, Chapter 10]).

Then, bearing in mind the notation used in Corollary 1, $f = 1/|G_\zeta|^2$ and

$$\alpha = \frac{\rho}{f}\left(\frac{\rho_\zeta}{\rho^2 G_\zeta}\right)_\zeta = -\frac{1}{2}\{G,\zeta\}$$

where we have also used (17). Consequently, $\{G, \zeta\} = -2c_1 d\zeta$ and therefore $G(\zeta) = \tanh(\sqrt{2c_1}\,\zeta)$ (see [12, Theorem 10.1.1]). Thus, from (17), $\rho = (\sqrt{2/|c_1|})|\cosh(\sqrt{2c_1}\zeta)|^2$ and it can be easily checked that the surface given by (4) for the data $K_\phi = 0$ and G, ρ as above is a hyperbolic cylinder.

Analogously, when $b = -1$, α/dh is constant. Thus, if $\alpha \equiv 0$, then $H \equiv -1$ and $\psi(S)$ is a totally umbilical surface contained in a degenerate hyperplane. Otherwise, $dh = c_2\alpha$ for a non-zero complex constant c_2 and, reasoning as above, $\psi(S)$ is a hyperbolic cylinder. □

Regarding the non-flat case, we have the following:

Theorem 4. *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a complete non-flat BLW-surface satisfying (1) with non-negative Gaussian curvature. Then either $\psi(S)$ is a totally umbilical sphere or $\psi(S)$ is a totally umbilical surface contained in a degenerate hyperplane.*

Proof. As we have seen in Theorem 1, we can assume that $|a + b| = 1$. So, we distinguish the following cases:

- If $a + b = -1$, then $K_\phi = a > 0$. From (13) and (15) we have that

$$\sigma = -\left(\frac{(1 + K_\phi)^2|dh|^2}{(1 + K_\phi|h|^2)^2} + (1 + K_\phi|h|^2)^2|\alpha|^2\right) + \frac{b^2}{2} I_\phi$$

and so

$$\frac{1}{2} I \leq \left(\frac{(1 + K_\phi)^2|dh|^2}{(1 + K_\phi|h|^2)^2} + (1 + K_\phi|h|^2)^2|\alpha|^2\right) \leq \frac{b^2}{2} I_\phi.$$

Consequently, I_ϕ is a complete metric with positive constant Gaussian curvature, which allows us to assure that S is compact from the Bonnet–Myers Theorem. Then, as we have seen in Remark 1, $\psi(S)$ is a totally umbilical sphere.

- If $a + b = 1$, from (13) we have that

$$\frac{1}{2} I \leq 2(1 + K_\phi|h|^2)^2|\alpha|^2 \leq 2|\alpha|^2$$

because $K_\phi = -a < 0$.

Thus, $|\alpha|^2$ is a complete flat metric on S conformal to σ, and so S is conformally equivalent to $\mathbb{C}$. Now, from (16), h is a bounded holomorphic function on $\mathbb{C}$ and therefore h is constant. Hence, $H = K = 1$ and $\psi(S)$ is a totally umbilical surface contained in a degenerate hyperplane. □

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