Journal of Geometry and Physics 60 (2010) 1634–1642

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Lower bounds for the eigenvalues of the Dirac operator on Spin^c manifolds

Roger Nakad

Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P. 239, 54506 Vandœuvre-Lès-Nancy Cedex, France

ARTICLE INFO

Article history:
Received 29 January 2010
Received in revised form 14 May 2010
Accepted 3 June 2010
Available online 12 June 2010

Keywords:
Spin^c structures
Dirac operator
Eigenvalues
Energy–momentum tensor
Perturbed Yamabe operator
Conformal geometry

ABSTRACT

In this paper, we extend the Hijazi inequality, involving the energy–momentum tensor, to the eigenvalues of the Dirac operator on Spin^c manifolds without boundary. The limiting case is then studied and an example is given.

1. Introduction

For a compact Riemannian spin manifold (M^n, g) of dimension $n \geqslant 2$, Friedrich [1] showed that any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)} \inf_M S_g, \tag{1}$$

where S_g denotes the scalar curvature of M. The limiting case of (1) is characterized by the existence of a special spinor called a real Killing spinor. This is a section ψ of the spinor bundle satisfying for every $X \in \Gamma(TM)$,

$$\nabla_X \psi = -\frac{\lambda_1}{n} X \cdot \psi,$$

where $X \cdot \psi$ denotes the Clifford multiplication and ∇ is the spinorial Levi-Civita connection [2]. On the complement set of zeros of any spinor field ϕ, we define ℓ^ϕ, the field of symmetric endomorphisms associated with the field of quadratic forms, denoted by T^ϕ, called the energy–momentum tensor, which is given, for any vector field X, by

$$T^\phi(X) = g(\ell^\phi(X), X) = \mathrm{Re}\left\langle X \cdot \nabla_X \phi, \frac{\phi}{|\phi|^2}\right\rangle.$$

The associated symmetric bilinear form is then given for every $X, Y \in \Gamma(TM)$ by

$$g(\ell^\phi(X), Y) = \frac{1}{2}\mathrm{Re}\left\langle X \cdot \nabla_Y \phi + Y \cdot \nabla_X \phi, \frac{\phi}{|\phi|^2}\right\rangle.$$

E-mail address: nakad@iecn.u-nancy.fr.

0393-0440/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2010.06.002

Note that if the spinor field ϕ is an eigenspinor, Bär showed that the zero set is contained in a countable union of $(n-2)$-dimensional submanifolds and has locally finite $(n-2)$-dimensional Hausdorff density [3]. In 1995, Hijazi [4] modified the connection ∇ in the direction of the endomorphism ℓ^ψ where ψ is an eigenspinor associated with an eigenvalue λ of the Dirac operator and established that

$$\lambda^2 \geqslant \inf_M \left(\frac{1}{4}S_g + |\ell^\psi|^2\right). \tag{2}$$

The limiting case of (2) is characterized by the existence of a spinor field ψ satisfying for all $X \in \Gamma(TM)$,

$$\nabla_X \psi = -\ell^\psi(X) \cdot \psi. \tag{3}$$

The trace of ℓ^ψ being equal to λ, Inequality (2) improves Inequality (1) since by the Cauchy–Schwarz inequality, $|\ell^\psi|^2 \geqslant \frac{(tr(\ell^\psi))^2}{n}$, where tr denotes the trace of ℓ^ψ. Ginoux and Habib showed in [5] that the Heisenberg manifold is a limiting manifold for (2) but equality in (1) cannot occur.

Using the conformal covariance of the Dirac operator, Hijazi [6] showed that, on a compact Riemannian spin manifold (M^n, g) of dimension $n \geqslant 3$, any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geqslant \frac{n}{4(n-1)}\mu_1, \tag{4}$$

where μ_1 is the first eigenvalue of the Yamabe operator given by

$$L := 4\frac{n-1}{n-2} \triangle_g + S_g,$$

and $\triangle_g$ is the Laplacian acting on functions. For dimension 2, Bär [7] proved that any eigenvalue of the Dirac operator on M satisfies

$$\lambda^2 \geqslant \frac{2\pi\chi(M)}{Area(M, g)}, \tag{5}$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M. The limiting case of (4) and (5) is also characterized by the existence of a real Killing spinor. In terms of the energy–momentum tensor, Hijazi [4] proved that, on such manifolds any eigenvalue of the Dirac operator satisfies the following:

$$\lambda^2 \geqslant \begin{cases} \frac{1}{4}\mu_1 + \inf\limits_M |\ell^\psi|^2 & \text{if } n \geqslant 3, \\ \frac{\pi\chi(M)}{Area(M, g)} + \inf\limits_M |\ell^\psi|^2 & \text{if } n = 2. \end{cases} \tag{6}$$

Again, the trace of ℓ^ψ being equal to λ, Inequality (6) improves Inequalities (4) and (5). The limiting case of (6) is characterized by the existence of a spinor field $\overline{\varphi}$ satisfying for all $X \in \Gamma(TM)$,

$$\overline{\nabla}_X \overline{\varphi} = -\ell^{\overline{\varphi}}(X) \overline{\cdot} \overline{\varphi}, \tag{7}$$

where $\overline{\varphi} = e^{-\frac{n-1}{2}u}\overline{\psi}$, the spinor field ψ is an eigenspinor associated with the first eigenvalue of the Dirac operator and $\overline{\psi}$ is the image of ψ under the isometry between the spinor bundles of (M^n, g) and $(M^n, \overline{g} = e^{2u}g)$. Suppose that on a spin manifold M, there exists a spinor field ϕ such that for all $X \in \Gamma(TM)$,

$$\nabla_X \phi = -E(X) \cdot \phi, \tag{8}$$

where E is a symmetric 2-tensor defined on TM. It is easy to see that E must be equal to ℓ^ϕ. For the two-dimensional case, Friedrich [8] proved that the existence of a pair (ϕ, E) satisfying (8) is equivalent to the existence of a local immersion of M into the euclidean space $\mathbb{R}^3$ with Weingarten tensor equal to E. In [9], Morel showed that if M^n is a hypersurface of a manifold N carrying a parallel spinor, then the energy–momentum tensor (associated with the restriction of the parallel spinor) appears, up to a constant, as the second fundamental form of the hypersurface. Habib [10] studied Eq. (8) for an endomorphism E, not necessarily symmetric. He showed that the symmetric part of E is ℓ^ϕ and the skew-symmetric part of E is q^ϕ defined on the complement set of zeros of ϕ by

$$g(q^\phi(X), Y) = \frac{1}{2}\mathrm{Re}\left\langle Y \cdot \nabla_X \phi - X \cdot \nabla_Y \phi, \frac{\phi}{|\phi|^2}\right\rangle,$$

for all $X, Y \in \Gamma(TM)$. Then he modified the connection in the direction of $\ell^\psi + q^\psi$ where ψ is an eigenspinor associated with an eigenvalue λ and obtained that

$$\lambda^2 \geqslant \inf_M \left(\frac{1}{4}S_g + |\ell^\psi|^2 + |q^\psi|^2\right). \tag{9}$$

The Heisenberg group and the solvable group are examples of limiting manifolds [10]. For a better understanding of the tensor q^ϕ, he studied Riemannian flows and proved that if the normal bundle carries a parallel spinor, the tensor q^ϕ plays the role of the O'Neill tensor of the flow. Here we prove the corresponding inequalities for Spin^c manifolds:

Theorem 1.1. *Let (M^n, g) be a compact Riemannian* Spin^c *manifold of dimension $n \geqslant 2$, and denote by $i\Omega$ the curvature form of the connection A on the $\mathbb{S}^1$-principal fibre bundle $(\mathbb{S}^1M, \pi, M)$. Then any eigenvalue of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \inf_M\Big(\frac{1}{4}S_g - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2 + |q^\psi|^2\Big), \tag{10}$$

where $c_n = 2[\frac{n}{2}]^{\frac{1}{2}}$ and $|\Omega|_g$ is the norm of Ω with respect to g.

In this paper, we only consider the deformation of the connection in the direction of the symmetric endomorphism ℓ^ϕ and hence under the same conditions as for Theorem 1.1, one gets

$$\lambda^2 \geqslant \inf_M\Big(\frac{1}{4}S_g - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2\Big). \tag{11}$$

In 1999, Moroianu and Herzlich [11] proved that on Spin^c manifolds of dimension $n \geqslant 3$, any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geqslant \lambda_1^2 := \frac{n}{4(n-1)}\mu_1, \tag{12}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator defined by

$$L^\Omega = L - c_n|\Omega|_g.$$

The limiting case of (12) is characterized by the existence of a real Killing spinor ψ satisfying $\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi$. In terms of the energy–momentum tensor we prove:

Theorem 1.2. *Under the same conditions as for Theorem 1.1, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \begin{cases} \frac{1}{4}\mu_1 + \inf\limits_M |\ell^\psi|^2 & \text{if } n \geqslant 3,\\ \frac{\pi\chi(M)}{Area(M,g)} - \frac{1}{2}\frac{\int_M |\Omega|_g v_g}{Area(M,g)} + \inf\limits_M |\ell^\psi|^2 & \text{if } n = 2,\end{cases} \tag{13}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator.

Using the Cauchy–Schwarz inequality in dimension $n \geqslant 3$, we have that Inequality (13) implies Inequality (12). As a corollary of Theorem 1.2, we compare the lower bound to a conformal invariant (the Yamabe number) and to a topological invariant, in the case of four-dimensional manifolds whose associated line bundle has self-dual curvature (see Corollaries 4.1 and 4.2). Finally, we study the limiting case of (11) and (13), and we give an example.

Even though the number $\inf_M |\ell^\psi|^2$ is not a nice geometric invariant, it appears naturally in some situations. For example, for hypersurfaces of certain limiting Spin^c manifolds it is easy to see, with the help of the Spin^c Gauss formula, that it is precisely the second fundamental form. Also, when deforming the Riemannian metric in the direction of the energy–momentum tensor, the eigenvalues of the Dirac operator on a Spin^c manifold are then critical (see [12]). The author would like to thank Oussama Hijazi for his support and encouragements.

2. Spin^c geometry and the Dirac operator

In this section, we briefly introduce basic notions concerning Spin^c manifolds and the Dirac operator. Details can be found in [13,2,14].

Let (M^n, g) be a compact connected oriented Riemannian manifold of dimension $n \geqslant 2$ without boundary. Furthermore, let SOM be the SO_n-principal bundle over M of positively oriented orthonormal frames. A Spin^c structure of M is a Spin^c_n-principal bundle (Spin^cM, π, M) and a $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1M, \pi, M)$ together with a double covering given by $\theta : \text{Spin}^cM \longrightarrow SOM \times_M \mathbb{S}^1M$ such that

$$\theta(ua) = \theta(u)\xi(a),$$

for every $u \in \text{Spin}^cM$ and $a \in \text{Spin}^c_n$, where ξ is the twofold covering of Spin^c_n over $SO_n \times \mathbb{S}^1$. A Riemannian manifold that admits a Spin^c structure is called a Riemannian Spin^c manifold.

Let $\Sigma^cM := \text{Spin}^cM \times_{\rho_n} \Sigma_n$ be the associated spinor bundle where $\Sigma_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$ and $\rho_n : \text{Spin}^c_n \longrightarrow \text{End}(\Sigma_n)$ the complex spinor representation. A section of Σ^cM will be called a spinor and the set of all spinors will be denoted by $\Gamma(\Sigma^cM)$. The spinor bundle Σ^cM is equipped with a natural Hermitian scalar product, denoted by $\langle .,.\rangle$ and satisfies

$$\langle X\cdot\psi, \varphi\rangle = -\langle\psi, X\cdot\varphi\rangle \quad \text{for every } X \in \Gamma(TM) \text{ and } \psi, \varphi \in \Gamma(\Sigma^cM),$$

where $X\cdot\psi$ denotes the Clifford multiplication of X and ψ. With this Hermitian scalar product we define an L^2-scalar product

$$(\psi,\phi)=\int_M\langle\psi,\phi\rangle v_g,$$

for any spinors ψ and ϕ. Additionally, given a connection 1-form A on $\mathbb{S}^1M$, $A:T(\mathbb{S}^1M)\longrightarrow i\mathbb{R}$ and the connection 1-form ω^M on SOM for the Levi-Civita connection ∇^M, induce a connection on the principal bundle $SOM\times_M\mathbb{S}^1M$, and hence a covariant derivative ∇ on $\Gamma(\Sigma^cM)$ [13], given by

$$\nabla_{e_i}\psi=\left[b, e_i(\sigma)+\frac{1}{4}\sum_{j=1}^{n}e_j\cdot\nabla^M_{e_i}e_j\cdot\sigma+\frac{1}{2}A(s_*(e_i))\sigma\right],\tag{14}$$

where $\psi=[b,\sigma]$ is a locally defined spinor field, $(e_1,\dots,e_n)$ is a local oriented orthonormal tangent frame and $s:U\longrightarrow\mathbb{S}^1M$ is a local section of $\mathbb{S}^1M$.

The curvature of A is an imaginary valued 2-form denoted by $F_A=\mathrm{d}A$, i.e., $F_A=i\Omega$, where Ω is a real valued 2-form on $\mathbb{S}^1M$. We know that Ω can be viewed as a real valued 2-form on M [13]. In this case $i\Omega$ is the curvature form of the associated line bundle L. It is the complex line bundle associated with the $\mathbb{S}^1$-principal bundle via the standard representation of the unit circle. The spinorial curvature $\mathcal{R}$ associated with the connection ∇, is given by

$$\mathcal{R}_{X,Y}=\frac{1}{4}\sum_{i,j=1}^{n}g\big(R_{X,Y}e_i,e_j\big)\,e_i\cdot e_j\cdot+\frac{i}{2}\Omega(X,Y).$$

In the Spin^c case, the Ricci identity translates to

$$\sum_j e_j\cdot\mathcal{R}_{e_j,X}\psi=\frac{1}{2}\mathrm{Ric}(X)\cdot\psi-\frac{i}{2}(X\lrcorner\Omega)\cdot\psi,\tag{15}$$

where $\lrcorner$ denotes the interior product. For every spinor ψ, the Dirac operator is locally defined by

$$D\psi=\sum_{i=1}^{n}e_i\cdot\nabla_{e_i}\psi.$$

It is an elliptic, self-adjoint operator with respect to the L^2-scalar product and verifies the Schrödinger–Lichnerowicz formula

$$D^2=\nabla^*\nabla+\frac{1}{4}S_g\ \mathrm{Id}_{\Gamma(\Sigma^cM)}+\frac{i}{2}\Omega\cdot,$$

where $\Omega\cdot$ is the extension of the Clifford multiplication to differential forms given by $(e_i^*\wedge e_j^*)\cdot\psi=e_i\cdot e_j\cdot\psi$.

3. Eigenvalue estimates on Spin^c manifolds

In this section, we prove the lower bound (10). This proof is based on the following lemma given by Moroianu and Herzlich in [11]:

Lemma 3.1 ([11]). *Let (M^n,g) be a Spin^c manifold. For any spinor $\psi\in\Gamma(\Sigma^cM)$ and a real 2-form Ω, we have*

$$\langle i\Omega\cdot\psi,\psi\rangle\geqslant-\frac{c_n}{2}|\Omega|_g|\psi|^2,\tag{16}$$

where $|\Omega|_g$ is the norm of Ω with respect to g given by $|\Omega|_g^2=\sum_{i<j}(\Omega_{ij})^2$, in any orthonormal local frame. Moreover, if equality holds in (16), then

$$\Omega\cdot\psi=i\frac{c_n}{2}|\Omega|_g\psi.\tag{17}$$

Proof of Theorem 1.1. Let E (resp. Q) be a symmetric (resp. skew-symmetric) 2-tensor defined on TM. For any spinor field ϕ, the modified connection

$$\widetilde{\nabla}_X\phi:=\nabla_X\phi+E(X)\cdot\phi+Q(X)\cdot\phi$$

satisfies $|\widetilde{\nabla}\phi|^2=|\nabla\phi|^2-|E|^2|\phi|^2-|Q|^2|\phi|^2$. After integration over M, the Schrödinger–Lichnerowicz formula gives

$$\int_M|\widetilde{\nabla}\phi|^2v_g=\int_M|D\phi|^2v_g-\int_M\frac{1}{4}S_g|\phi|^2v_g-\int_M(|E|^2+|Q|^2)|\phi|^2v_g-\int_M\left\langle\frac{i}{2}\Omega\cdot\phi,\phi\right\rangle v_g.$$

Let ψ be an eigenspinor corresponding to the eigenvalue λ of D. For $E = \ell^\psi$, $Q = q^\psi$ and by Lemma 3.1, it follows that

$$\lambda^2 \int_M |\psi|^2 v_g \geqslant \frac{1}{4}\int_M S_g|\psi|^2 v_g + \int_M (|\ell^\psi|^2 + |q^\psi|^2)|\psi|^2 v_g + \int_M \left\langle \frac{i}{2}\Omega\cdot\psi, \psi\right\rangle v_g$$
$$\geqslant \int_M \left(\frac{1}{4}S_g - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2 + |q^\psi|^2\right)|\psi|^2 v_g.$$

Finally,

$$\lambda^2 \geqslant \inf_M \left(\frac{1}{4}S_g - \frac{c_n}{4}|\Omega|_g + |\ell^\psi|^2 + |q^\psi|^2\right). \quad \square$$

4. Conformal geometry and eigenvalue estimates

Before proving Theorem 1.2, we give some basic facts on conformal Spin^c geometry. The conformal class of g is the set of metrics $\overline{g} = e^{2u}g$ for a real function u on M. At a given point x of M, we consider a g-orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xM. The corresponding $\overline{g}$-orthonormal basis is denoted by $\{\overline{e}_1 = e^{-u}e_1, \ldots, \overline{e}_n = e^{-u}e_n\}$. This correspondence extends to the Spin^c level to give an isometry between the corresponding spinor bundles. We put a "$^{-}$" above every object which is naturally associated with the metric $\overline{g}$, except for the scalar curvature where S_g (resp. S_u or S_h) denotes the scalar curvature associated with the metric g (resp. $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$). Then, for any spinor fields ψ and φ, one has

$$\langle\overline{\psi}, \overline{\varphi}\rangle = \langle\psi, \varphi\rangle,$$

where $\langle ., .\rangle$ denotes the natural Hermitian scalar products on $\Gamma(\Sigma^c M)$, and on $\Gamma(\Sigma^c\,\overline{M})$. The corresponding Dirac operators satisfy

$$\overline{D}\,(e^{-\frac{(n-1)}{2}u}\,\overline{\psi}) = e^{-\frac{(n+1)}{2}u}\,\overline{D\psi}.$$

The norms of any real 2-form Ω with respect to g and $\overline{g}$ are related by

$$|\Omega|_{\overline{g}} = e^{-2u}|\Omega|_g.$$

Hijazi [4] showed that on a spin manifold the energy–momentum tensor verifies

$$|\ell^{\overline{\varphi}}|^2 = e^{-2u}\,|\ell^\varphi|^2 = e^{-2u}\,|\ell^\psi|^2,$$

where $\varphi = e^{-\frac{(n-1)}{2}u}\psi$. We extend the result to a Spin^c manifold and get the same relation.

Lemma 4.1. *Under the same conditions as for Theorem 1.1, any eigenvalue λ of the Dirac operator to which is attached an eigenspinor ψ satisfies*

$$\lambda^2 \geqslant \frac{1}{4}\sup_u \inf_M (S_u e^{2u} - c_n|\Omega|_g) + \inf_M |\ell^\psi|^2.$$

Proof. For any spinor field ϕ and for any symmetric 2-tensor E defined on TM, the modified connection introduced in [4]:

$$\nabla^E_X\phi = \nabla_X\phi + E(X)\cdot\phi,$$

verifies $|\nabla^E\phi|^2 = |\nabla\phi|^2 - |E|^2|\phi|^2$. Using the Schrödinger–Lichnerowicz formula on M, applied to the spinor field $\overline{\phi}$ with respect to the metric $\overline{g}$, yields

$$\int_M |\overline{\nabla}^E\overline{\phi}|^2 v_{\overline{g}} = \int_M |\overline{D\phi}|^2 v_{\overline{g}} - \int_M \frac{1}{4}S_u|\overline{\phi}|^2 v_{\overline{g}} - \int_M |E|^2|\overline{\phi}|^2 v_{\overline{g}} - \int_M \left\langle \frac{i}{2}\Omega\cdot\overline{\phi}, \overline{\phi}\right\rangle v_{\overline{g}}. \tag{18}$$

For the spinor $\varphi = e^{-\frac{(n-1)}{2}u}\,\psi$ with $D\psi = \lambda\psi$, one gets $\overline{D}\,\overline{\varphi} = \lambda e^{-u}\,\overline{\varphi}$, and hence by Lemma 3.1 and for $E = \ell^{\overline{\varphi}}$,

$$\int_M \left[\lambda^2 - \left(\frac{1}{4}S_u e^{2u} + |\ell^\psi|^2 - \frac{c_n}{4}|\Omega|_g\right)\right] e^{-2u}|\overline{\varphi}|^2 v_{\overline{g}} \geqslant 0. \quad \square \tag{19}$$

Lemma 4.2. *Let (M^n, g) be a compact Riemannian* Spin^c *manifold of dimension $n \geqslant 2$ and S_g (resp. S_u or S_h) the scalar curvature associated with the metric g (resp. $\overline{g} = e^{2u}g = h^{\frac{4}{n-2}}g$). The 2-form $i\Omega$ denotes the curvature form on the $\mathbb{S}^1$-principal bundle associated with the* Spin^c *structure. We have*

$$\sup_u \inf_M (S_u e^{2u} - c_n|\Omega|_g) = \begin{cases} \mu_1 & \text{if } n \geqslant 3, \\ \dfrac{4\pi\chi(M) - 2\int_M |\Omega| v_g}{Area(M, g)} & \text{if } n = 2, \end{cases} \tag{20}$$

where μ_1 is the first eigenvalue of the perturbed Yamabe operator L^Ω.

Proof. For $n \geqslant 3$, let $\mathbf{h} > 0$ be an eigenfunction of L^{Ω} associated with the eigenvalue μ_1 such that $\int_M \mathbf{h}^2 v_g = 1$. For a conformal metric $\overline{g} = e^{2u} g = h^{\frac{4}{n-2}} g$, we have

$$S_h h^{\frac{4}{n-2}} - c_n |\Omega|_g = S_u e^{2u} - c_n |\Omega|_g = h^{-1} L^{\Omega} h.$$

So $\mu_1 = \mathbf{h}^{-1} L^{\Omega} \mathbf{h} = S_{\mathbf{h}} \mathbf{h}^{\frac{4}{n-2}} - c_n |\Omega|_g$. For any positive function H, we write $fH = \mathbf{h}$, where f is a positive function, and referring to [15] we get

$$\mu_1 = \int (H^{-1} L H) f^2 H^2 \, v_g - c_n \int_M |\Omega|_g f^2 H^2 \, v_g + \int_M H^2 |df|^2 \, v_g.$$

Finally,

$$\mu_1 \geqslant \inf_M (H^{-1} L^{\Omega} H) = \inf_M (S_v e^{2v} - c_n |\Omega|_g),$$

where $e^{2v} = H^{\frac{4}{n-2}}$; then $\mu_1 = \sup_u \inf_M (S_u e^{2u} - c_n |\Omega|_g)$. For $n = 2$ and for every u we have $S_u e^{2u} = S_g + 2 \triangle_g u$. The Stokes and Gauß–Bonnet theorems yield

$$\inf_M (S_u e^{2u} - 2|\Omega|_g) \leqslant \frac{\int_M \left(S_u e^{2u} - 2|\Omega|_g \right) v_g}{Area(M, g)} = \frac{4\pi \chi(M) - 2 \int_M |\Omega|_g v_g}{Area(M, g)}.$$

Let u_0 be a solution of the following equation [16]:

$$2 \triangle_g u = \frac{\int_M (S_g - 2|\Omega|_g) v_g}{Area(M, g)} - S_g + 2|\Omega|_g. \tag{21}$$

Hence,

$$S_{u_0} e^{2u_0} - 2|\Omega|_g = 2 \triangle_g u_0 + S_g - 2|\Omega|_g = \frac{4\pi \chi(M) - 2 \int_M |\Omega|_g v_g}{Area(M, g)}. \quad \square$$

Proof of Theorem 1.2. Combining Lemmas 4.2 and 4.1, Theorem 1.2 follows. $\square$

Remark 4.1. Inequality (11) improves Inequality (12), which itself implies the Friedrich Spin^c inequality given by

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \inf_M (S_g - c_n |\Omega|_g). \tag{22}$$

Equality holds in (22) if and only if equality holds in (12), i.e., if and only if the eigenspinor ψ associated with the first eigenvalue of D is a real Killing spinor and $\Omega \cdot \psi = i \frac{c_n}{2} |\Omega|_g \psi$.

Corollary 4.1. *Any eigenvalue of the Dirac operator on a compact Riemannian* Spin^c *manifold of dimension $n \geqslant 3$ satisfies*

$$\lambda^2 \geqslant \frac{1}{4} \mathrm{vol}(M, g)^{-\frac{2}{n}} \left(Y(M, [g]) - c_n \|\Omega\|_{\frac{n}{2}} \right) + \inf_M |\ell^{\psi}|^2,$$

where $Y(M, [g])$ is the Yamabe number given by

$$Y(M, [g]) = \inf_{\eta \neq 0} \frac{\int_M 4 \frac{n-1}{n-2} |d\eta|^2 + S_g \eta^2}{\left(\int_M |\eta|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}}.$$

Proof. Using the Hölder inequality, it follows that

$$\mu_1 = \inf_{\eta \neq 0} \frac{\int_M 4 \frac{n-1}{n-2} |d\eta|^2 + (S_g - c_n |\Omega|_g) \eta^2}{\int_M \eta^2} \geqslant \inf_{\eta \neq 0} \frac{\int_M 4 \frac{n-1}{n-2} |d\eta|^2 + (S_g - c_n |\Omega|) \eta^2}{\left(\int_M |\eta|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \mathrm{vol}(M, g)^{\frac{2}{n}}}.$$

Using the Hölder inequality again, we deduce

$$\mu_1 \, \mathrm{vol}(M, g)^{\frac{2}{n}} \geqslant \inf_{\eta \neq 0} \frac{\int_M 4 \frac{n-1}{n-2} |d\eta|^2 + S \eta^2}{\left(\int_M |\eta|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}} - c_n \left(\int_M |\Omega|^{\frac{n}{2}} \right)^{\frac{2}{n}} = Y(M, [g]) - c_n \|\Omega\|_{\frac{n}{2}}.$$

Finally, replacing in (13), we get the result. $\square$

Corollary 4.2. *On a compact four-dimensional* Spin^c *manifold with self-dual curvature form* $i\Omega$*, any eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geqslant \frac{1}{4}\mathrm{vol}(M,g)^{-\frac{1}{2}}\Big(Y(M,[g]) - 4\pi\sqrt{2}\sqrt{c_1(L)^2}\Big) + \inf_M |\ell^\psi|^2,$$

where $c_1(L)$ *is the Chern number of the line bundle* L *associated with the* Spin^c *structure.*

Proof. It follows directly from Corollary 4.1 and the fact that if $n = 4$ and Ω is self-dual, then $\int_M |\Omega|_g^2 v_g = 4\pi^2 c_1(L)^2$ (see [13]). □

5. The equality case

In this section, we study the limiting case of (11) and (13). An example is then given.

Proposition 5.1. *Under the same conditions as for Theorem 1.1,*

$$\text{Equality in (11) holds} \iff \begin{cases} \nabla_X\psi = -\ell^\psi(X)\cdot\psi, \\ \Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi, \end{cases} \tag{23}$$

for any $X \in \Gamma(TM)$ *and where* ψ *is an eigenspinor associated with the first eigenvalue of the Dirac operator.*

Proof. If equality in (11) is achieved, the two conditions follow directly. Now, suppose that $\nabla_X\psi = -\ell^\psi(X)\cdot\psi$ and $\Omega\cdot\psi = i\frac{c_n}{2}|\Omega|_g\psi$. The condition $\nabla_X\psi = -\ell^\psi(X)\cdot\psi$ implies that $|\psi|^2$ is constant. Denoting by $\mathcal{R}$ the curvature tensor on the Spin^c bundle associated with the connection ∇, one easily gets the following relation:

$$\mathcal{R}_{X,Y}\,\psi + d\ell^\psi(X,Y)\cdot\psi + [\ell^\psi(X),\ell^\psi(Y)]\cdot\psi = 0,$$

where $d\ell^\psi$ is a 2-form with values in $\Gamma(TM)$ given by

$$d\ell^\psi(X,Y) = (\nabla_X\ell^\psi)Y - (\nabla_Y\ell^\psi)X.$$

Taking $Y = e_j$ and performing its Clifford multiplication by e_j yields by the Ricci identity (15) on a Spin^c manifold

$$-\frac{1}{2}\mathrm{Ric}(X)\cdot\psi + \frac{i}{2}(X\lrcorner\Omega)\cdot\psi + \sum_j e_j\cdot d\ell^\psi(X,e_j)\cdot\psi + \sum_j e_j\cdot[\ell^\psi(X),\ell^\psi(e_j)]\cdot\psi = 0. \tag{24}$$

We then decompose the last two terms in (24) using that $X\cdot\alpha = X\wedge\alpha - X\lrcorner\alpha$ for any form α; it follows that

$$\sum_j e_j\cdot d\ell^\psi(X,e_j)\cdot\psi = \sum_j[e_j\wedge d\ell^\psi(X,e_j)]\cdot\psi - [X(\mathrm{tr}\,\ell^\psi) + \mathrm{div}\,\ell^\psi(X)]\psi.$$

$$\sum_j e_j\cdot[\ell^\psi(X),\ell^\psi(e_j)]\cdot\psi = 2\,(\mathrm{tr}\,\ell^\psi)\,\ell^\psi(X)\cdot\psi - 2\sum_j g(X,\ell^\psi(e_j))\,\ell^\psi(e_j)\cdot\psi.$$

Taking the scalar product of (24) with ψ, and separating real and imaginary parts, yields for every vector field X the relation

$$\Big(X(\mathrm{tr}\,\ell^\psi) + \mathrm{div}\,\ell^\psi(X)\Big)|\psi|^2 = \frac{i}{2}\langle(X\lrcorner\Omega)\cdot\psi,\psi\rangle. \tag{25}$$

But since Equality (17) holds we compute

$$\begin{aligned}\langle(X\lrcorner\Omega)\cdot\psi,\psi\rangle &= \langle(X\wedge\Omega)\cdot\psi,\psi\rangle - \langle X\cdot\Omega\cdot\psi,\psi\rangle \\ &= \langle(X\wedge\Omega)\cdot\psi,\psi\rangle - i\Big[\frac{n}{2}\Big]^{\frac{1}{2}}|\Omega|_g\langle X\cdot\psi,\psi\rangle.\end{aligned}$$

After separating the real and imaginary parts, $\langle(X\lrcorner\Omega)\cdot\psi,\psi\rangle$ must vanish. Using this and $\sum_{j=1}^n e_j\cdot(e_j\lrcorner\Omega) = 2\Omega$, Clifford multiplication of (24) with e_k, and for $X = e_k$, gives

$$-\frac{1}{2}S_g\psi - i\Omega\cdot\psi = \sum_{k,j} e_j\cdot(e_k\wedge d\ell^\psi(e_j,e_k))\cdot\psi - 2(\mathrm{tr}\,\ell^\psi)^2\psi + 2|\ell^\psi|^2\psi.$$

An easy computation implies that $\sum_{k,j} e_j\cdot(e_k\wedge d\ell^\psi(e_j,e_k))\cdot\psi = 0$; hence

$$-\frac{1}{2}S_g + \Big[\frac{n}{2}\Big]^{\frac{1}{2}}|\Omega|_g = -2(\mathrm{tr}\,\ell^\psi)^2 + 2|\ell^\psi|^2, \tag{26}$$

which implies equality in (11). □

Proposition 5.2. *On a compact Riemannian* Spin^c *manifold* (M^n, g) *of dimension* $n \geqslant 3$, *assume that the first eigenvalue* λ_1 *of the Dirac operator to which is attached an eigenspinor* ψ *satisfies the equality case in (13). Then,* $|\ell^\psi|$ *is constant and if* $\mathbf{h} > 0$ *denotes an eigenfunction of the Yamabe operator corresponding to* μ_1, *then for any vector field X*

$$g(X, \ell^\psi(d\mathbf{h}) - \lambda_1 d\mathbf{h}) = g(\lambda_1 X - \ell^\psi(X), d\mathbf{h}) = 0. \tag{27}$$

Proof. If $n \geqslant 3$ and equality holds in (13), we consider the positive function $\mathbf{v} > 0$ defined by $e^{2\mathbf{v}} = \mathbf{h}^{\frac{4}{n-2}}$ where $\mathbf{h}$ is an eigenfunction of the Yamabe operator corresponding to μ_1. Inequality (19) with $u = \mathbf{v}$ gives $|\ell^\psi|$ is constant, $\overline{\nabla}_X\overline{\varphi} = -\ell^{\overline{\varphi}}(X)\,\overline{\cdot}\overline{\varphi}$ and $\Omega\,\overline{\cdot}\,\overline{\varphi} = i\frac{c_n}{2}|\Omega|_{\overline{g}}\overline{\varphi}$. By Proposition 5.1, Equality (26) and (25) can be considered for the conformal metric $\overline{g} = e^{2\mathbf{v}}g = \mathbf{h}^{\frac{4}{n-2}}g$ to get

$$(\mathrm{tr}\,\ell^{\overline{\varphi}})^2 := f^2 = \frac{1}{4}S_{\mathbf{v}} - \frac{c_n}{4}|\Omega|_{\overline{g}} + |\ell^{\overline{\varphi}}|^2,$$
$$\mathrm{grad} f = -\mathrm{div}\,\ell^{\overline{\varphi}}.$$

It is straightforward to see that these two equalities give (27). □

Example. If the lower bound (22) is achieved, automatically equality holds in (11). Here we will give an example where equality holds in (11) but not in (22).

Let $(M^3, g) = (S^3, can)$ be endowed with its unique spin structure and consider a real Killing spinor ψ with Killing constant $\frac{1}{2}$. As the norm of ψ is constant, we may suppose that $|\psi| = 1$. Let ξ be the Killing vector field on M defined by

$$ig(\xi, X) = \langle X \cdot \psi, \psi \rangle.$$

In [11], it is shown that:

1. $id\xi(X, Y) = -\langle X \wedge Y \cdot \psi, \psi \rangle$ for any $X, Y \in \Gamma(TM)$.
2. $d|\xi|^2 = -2d\xi(\xi, .) = -2g(\nabla_\xi \xi, .) \simeq -2\nabla_\xi \xi = 0$.
3. $\xi \cdot \psi = i\psi$ and $|\xi| = 1$.
4. $\xi \cdot \psi = -e_1 \cdot e_2 \cdot \psi$, where $\{\xi/|\xi|, e_1, e_2\}$ is an oriented local orthonormal frame.

Let h be a real constant such that $h > 1$. We define the metric g^h on M by

$$\begin{cases} g^h(\xi, X) = g(\xi, X) \text{ pour tout } X \in \Gamma(TM), \\ g^h(X, Y) = h^{-2}g(X, Y) \text{ pour } X, Y \perp \xi. \end{cases}$$

Using the following isomorphism:

$$(TM, g) \longrightarrow (TM, g^h)$$
$$Z \longrightarrow Z^h = \begin{cases} Z & \text{si } Z = \xi, \\ hZ & \text{si } Z \perp \xi, \end{cases}$$

if $u = \{\xi, e_1, e_2\}$ is a positive local g-orthonormal frame defined in a neighborhood U of x, then $u^h = \{\xi^h = \xi, e_1^h = he_1, e_2^h = he_2\}$ is a positive local g^h-orthonormal frame defined in a neighborhood U of x.

There exists an isomorphism of vector bundles (see [11]) given by

$$\Sigma_g M \longrightarrow \Sigma_{g^h} M$$
$$\psi = [\tilde{u}, \phi] \longrightarrow \psi^h = [\tilde{u}^h, \phi],$$

satisfying

$$\langle \psi_1, \psi_2 \rangle_{\Sigma_g M} = \langle \psi_1^h, \psi_2^h \rangle_{\Sigma_{g^h} M} \quad \text{and} \quad (X \cdot \psi)^h = X^h \cdot \psi^h \text{ for any } X \in \Gamma(TM).$$

The covariant derivative of the spinor $\psi^h = [\tilde{u}^h, \phi]$ is given by (see [11])

$$\nabla^h_{X^h}\psi^h = \frac{h^2}{2}X^h \cdot \psi^h + i((1 - h^2)\xi)(X^h)\psi^h.$$

Let $\alpha = (1 - h^2)\xi$ be a 1-form on M. We may view $i\alpha$ as a connection 1-form on the trivial $\mathbb{S}^1$ bundle. Let $L = M \times \mathbb{C}$ be the induced trivial line bundle over M. We denote by σ the global section of L and by ∇^0 the covariant derivative on L induced by the above connection. It satisfies

$$\nabla^0_X \sigma = i\alpha(X)\sigma, \quad \text{for any } X \in \Gamma(TM).$$

We consider, on the twisted bundle $\Sigma_{g^h}M \otimes L$, the connection $\overline{\nabla} = \nabla^h \otimes \nabla^0$ and we calculate

$$\overline{\nabla}_{e_1^h}(\psi^h \otimes \sigma) = \frac{h^2}{2} e_1^h \cdot (\psi^h \otimes \sigma),$$
$$\overline{\nabla}_{e_2^h}(\psi^h \otimes \sigma) = \frac{h^2}{2} e_2^h \cdot (\psi^h \otimes \sigma),$$
$$\overline{\nabla}_{\xi}(\psi^h \otimes \sigma) = \left(\frac{-3h^2}{2} + 2\right) \xi \cdot (\psi^h \otimes \sigma).$$

The spinor $\psi^h \otimes \sigma$ is a section of $\Sigma_{g^h}M \otimes L$, which is, of course, the spinor bundle associated with the Spin^c structure with auxiliary line bundle L^2. It is easy to see that $\psi^h \otimes \sigma$ is an eigenspinor associated with the eigenvalue $\frac{h^2}{2} - 2$, and it is clear that $\psi^h \otimes \sigma$ is not a real Killing spinor since $h \neq 1$, so (M, g^h) is not a limiting manifold for the Friedrich Spin^c inequality. But it is a limiting manifold for the lower bound (11); in fact we will prove that (23) holds.

The complex 2-form $id\alpha$ is the curvature form associated with the connection ∇^0 on L. We have

$$d\alpha \cdot (\psi^h \otimes \sigma) = (1 - h^2) d\xi \cdot (\psi^h \otimes \sigma) = i(h^2 - 1)h^2 \psi^h \otimes \sigma.$$

The norm of $d\alpha$ with respect to the metric g^h is given by

$$|d\alpha|^2_{g^h} = (1 - h^2)^2 |d\xi|^2_{g^h} = (1 - h^2)^2 (d\xi(e_1^h, e_2^h))^2 = h^4 (1 - h^2)^2.$$

Since $h > 1$, $|d\alpha|_{g^h} = h^2(h^2 - 1)$, then the second equation of (23) is verified. Furthermore, it is easy to check that

$$T^{\psi^h \otimes \sigma}(e_1^h) = T^{\psi^h \otimes \sigma}(e_2^h) = g^h(\ell^{\psi^h \otimes \sigma}(e_1^h), e_1^h) = g^h(\ell^{\psi^h \otimes \sigma}(e_2^h), e_2^h) = -\frac{h^2}{2},$$
$$g^h(\ell^{\psi^h \otimes \sigma}(e_1^h), \xi) = g^h(\ell^{\psi^h \otimes \sigma}(e_2^h), \xi) = g^h(\ell^{\psi^h \otimes \sigma}(e_1^h), e_2^h) = 0,$$
$$T^{\psi^h \otimes \sigma}(\xi) = g^h(\ell^{\psi^h \otimes \sigma}(\xi), \xi) = \frac{3h^2}{2} - 2.$$

Finally, it is straightforward to verify that the first equation of (23) holds:

$$-\ell^{\psi^h \otimes \sigma}(e_1^h) \cdot (\psi^h \otimes \sigma) = \frac{h^2}{2} e_1^h \cdot (\psi^h \otimes \sigma) = \overline{\nabla}_{e_1^h}(\psi^h \otimes \sigma),$$
$$-\ell^{\psi^h \otimes \sigma}(e_2^h) \cdot (\psi^h \otimes \sigma) = \frac{h^2}{2} e_2^h \cdot (\psi^h \otimes \sigma) = \overline{\nabla}_{e_2^h}(\psi^h \otimes \sigma),$$
$$-\ell^{\psi^h \otimes \sigma}(\xi) \cdot (\psi^h \otimes \sigma) = \left(\frac{-3h^2}{2} + 2\right) \xi \cdot (\psi^h \otimes \sigma) = \overline{\nabla}_{\xi}(\psi^h \otimes \sigma).$$

References

[1] T. Friedrich, Der erste Eigenwert des Dirac-operators einer kompakten Riemannschen Mannigfaltigkeit nichtnagativer Skalarkrümmung, Math. Nach. 97 (1980) 117–146.
[2] H.B. Lawson, M.L. Michelson, Spin Geometry, Princeton University Press, Princeton, New Jersey, 1989.
[3] C. Bär, Zero sets of solutions to semilinear elliptic systems of first order, Invent. Math. 138 (1) (1999) 183–202.
[4] O. Hijazi, Lower bounds for the eigenvalues of the Dirac operator, J. Geom. Phys. 16 (1995) 27–38.
[5] N. Ginoux, G. Habib, A spectral estimate for the Dirac operator on Riemannian flows, preprint (2009).
[6] O. Hijazi, A conformal Lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors, Comm. Math. Phys. 104 (1986) 151–162.
[7] C. Bär, Lower eigenvalue estimates for Dirac operators, Math. Ann. 239 (1992) 39–46.
[8] T. Friedrich, On the spinor representation of surfaces in Euclidean 3-spaces, J. Geom. Phys. 28 (1998) 143–157.
[9] B. Morel, Tenseur d'impulsion-énergie et géométrie spinorielle extrinsèque, Ph.D. Thesis, Institut Élie Cartan, 2002.
[10] G. Habib, Energy–momentum tensor on foliations, J. Geom. Phys. 57 (2007) 2234–2248.
[11] M. Herzlich, A. Moroianu, Generalized Killing spinors and conformal eigenvalue estimates for *Spin^c* manifold, Ann. Global Anal. Geom. 17 (1999) 341–370.
[12] R. Nakad, The energy–momentum tensor on Spin^c manifolds, Preprint, Institut Elie Cartan Nancy, hal-00492141, 2010.
[13] T. Friedrich, Dirac operator's in Riemannian Geometry, in: Graduate Studies in Mathematics, vol. 25, American Mathematical Society.
[14] A. Moroianu, Parallel and Killing spinors on *Spin^c* manifolds, Comm. Math. Phys. 187 (1997) 417–428.
[15] O. Hijazi, Première valeur propre de l'opérateur de Dirac et nombre de Yamabe, C. R. Acad. Sci. Paris, Série I 313 (1991) 865–868.
[16] T. Aubin, Nonlinear Analysis on Manifolds, Monge–Ampère Equations, Springer-Verlag, 1982.