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The local zeta function for symmetric spaces of non-compact type

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ABSTRACT

The Mellin transform of the heat kernel on a non-compact symmetric space X gives rise to a zeta function $\zeta(s; x, b)$ that was studied when the rank of X was 1. In this case the special values of the zeta function and of its derivative at $s = 0$, for example, are relevant for the quantum field effective potential in space-times modelled on X, or especially on a compact locally symmetric quotient $\Gamma \setminus X$, where Γ is a discrete group of isometries of X. Also the special value of $\zeta(s; x, b)$ at $s = -\frac{1}{2}$ determines the Casimir energy of such a space-time.

In this paper we extend the study of $\zeta(s; x, b)$ to any symmetric space X of arbitrary real rank. One of our main results is Theorem 2.1, where we show that for general X and for $x \neq \bar{1}$, $\zeta(s; x, b)$ admits a continuation to an entire function. On the other hand, we show that under a mild condition, for $x = \bar{1}$, $\zeta(s; \bar{1}, b)$ has a meromorphic continuation to $\mathbb{C}$ with at most simple poles, all lying in the set of half-integers.

In case G is complex, we give a very explicit form of the meromorphic continuation and we compute special values of the zeta function and of its derivative at $s = 0$ and at $s = -\frac{1}{2}$, which give a local contribution to the Casimir energy of X. To illustrate the difficulties present in the general case, we work out explicitly the meromorphic continuation for two infinite families of higher rank groups.

1. Introduction

Let $X = G/K$ be a Riemannian symmetric space of non-compact type where G is a connected non-compact real semisimple Lie group with finite center, and K is a maximal compact subgroup of G. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ denote the Lie algebras of G, K, respectively, and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$ with Cartan involution θ. Thus, $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are, respectively, the ± 1-eigenspaces of θ, and the Killing form $(\ ,\ )$ of $\mathfrak{g}_0$ is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$. Furthermore $\theta : x+y \to x-y$, for $(x, y) \in \mathfrak{k}_0 \times \mathfrak{p}_0$, is an automorphism of $\mathfrak{g}_0$. Choose an Iwasawa decomposition $G = KA_pN$ of G where $A_p = \exp \mathfrak{a}_p$ for a maximal abelian subspace $\mathfrak{a}_p$ of $\mathfrak{p}_0$. Let Σ denote the set of *restricted* real roots of $(\mathfrak{g}_0, \mathfrak{a}_p)$. That is, $\alpha \in \Sigma \Leftrightarrow \alpha \in \mathfrak{a}_p^*$ (the dual space of $\mathfrak{a}_p$), $\alpha \neq 0$, and the corresponding root space $\mathfrak{g}_{0_\alpha} := \{Y \in \mathfrak{g}_0 \mid [H, Y] = \alpha(H)Y$ for all $H \in \mathfrak{a}_p\}$ is non-zero. We have $N = \exp \mathfrak{n}_0$ where $\mathfrak{n}_0 = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ for a choice of positive root system $\Sigma^+ \subset \Sigma$. For $\alpha \in \Sigma$, we set $m_\alpha := \dim \mathfrak{g}_{0_\alpha}$ and $\rho := \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\alpha$. The equation $(\lambda, \mu) := \left(H_\lambda, H_\mu\right)$ for $\lambda, \mu \in \mathfrak{a}_p^*$, where $H_\lambda \in \mathfrak{a}_p$ is the unique element such that $\lambda(H) = (H, H_\lambda)$ for all $H \in \mathfrak{a}_p$, defines an inner product $(\ ,\ )$ on $\mathfrak{a}_p^*$ (and thus a norm $|\cdot|$ on $\mathfrak{a}_p^*$) that extends $\mathbb{C}$-bilinearly to a form $(\ ,\ )$ on the complexification $\Lambda := \left(\mathfrak{a}_p^*\right)^{\mathbb{C}}$ of $\mathfrak{a}_p^*$. We regard Λ as the space of $\mathbb{R}$-linear, $\mathbb{C}$-valued maps on $\mathfrak{a}_p$; $\mathbb{R}$ and $\mathbb{C}$ are the real and complex fields.

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For $\lambda \in \Lambda$, the corresponding Harish-Chandra spherical function $\phi_\lambda(x)$ on G is given by

$$\phi_\lambda(x) = \int_K \mathrm{e}^{(\mathrm{i}\lambda-\rho)(H(xk))}\mathrm{d}k \tag{1.1}$$

where $\mathrm{d}k$ is the normalized Haar measure on K with total mass one, and $H : G \to \mathfrak{a}_p$ is the Iwasawa projection defined by $x \in K \exp H(x) N$ for all $x \in G$. The Weyl group of $(\mathfrak{g}_0, \mathfrak{a}_p)$ is given by $W = M'/M$ where M and M' are the centralizer and normalizer respectively, of $\mathfrak{a}_p$ in K. For further details regarding the preceding definitions and results, the reader can consult [1–3]. Using $\phi_\lambda(x)$ and the Harish-Chandra c-function $c(\lambda)$ on Λ, the heat kernel on G is given, for each $t > 0$, by convolution with the function:

$$h_t(x) = \frac{1}{|W|}\int_{\mathfrak{a}_p^*} \mathrm{e}^{-t(|\lambda|^2+|\rho|^2)}\phi_\lambda(x)|c(\lambda)|^{-2}\mathrm{d}\lambda, \tag{1.2}$$

(see Gangolli [4]); h_t is K-bi-invariant, and as a function on X it satisfies the heat equation $\Delta h_t = \frac{\partial h_t}{\partial t}$, where Δ is the Laplace–Beltrami operator on X.

The zeta function that we attach to X and will be of interest to us is given by the Laplace–Mellin transform

$$\zeta(s; x, b) := \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{e}^{-tb^2} h_t(x) t^{s-1}\mathrm{d}t \tag{1.3}$$

for $x \in X$ fixed, and for a fixed parameter $b \in \mathbb{R}$. We shall see that $\zeta(s; x, b)$ is a well-defined holomorphic function of s for $\mathrm{Re}\, s > d/2$, $d := \dim X$. The goal of this paper is to study its meromorphic continuation to $\mathbb{C}$ in the case when G is semisimple, locally without compact factors and of arbitrary real rank.

Among other reasons, there are motivations from physics for considering $\zeta(s; x, b)$, in particular when $x = \overline{1} := 1K$ is the origin in X and 1 is the identity element of G. For example, if $G = SO_1(4, 1)$ and $K = SO(4)$ (so that X is the real hyperbolic 4-space) then the meromorphic continuation of $\zeta(s; \overline{1}, b)$ (for an appropriate choice of b) is used to compute the one-loop effective potential $V^{(1)}$ of a scalar field in anti de Sitter space. Indeed Camporesi in [5] computed that

$$V^{(1)} = -\frac{1}{2}\zeta'(0; \overline{1}, b) - \frac{1}{2}\zeta(0; \overline{1}, b)\left[\frac{1}{s} + \log \mu^2\right], \tag{1.4}$$

where μ is a fixed renormalization parameter (see also [6–8]). In this example, X is a rank one symmetric space.

Let now $\Gamma \subset G$ be a co-compact torsion free discrete subgroup of G and let $X_\Gamma = \Gamma \setminus X$ be the associated locally symmetric space. For all rank one X, the meromorphic continuation of $\zeta(s; \overline{1}, b)$ was carried out in [9,10]. Also for the locally symmetric spaces $X_\Gamma = \Gamma \setminus X$, the explicit meromorphic continuation of the spectral zeta function, which in turn involves the meromorphic continuation of $\zeta(s; \overline{1}, b)$, was carried out in [10] by an application of the Selberg trace formula. In fact, apart from contributions to the meromorphic continuation of certain functions indexed by hyperbolic elements $\gamma \in \Gamma \setminus \{1\}$, the contribution of $\zeta(s; \overline{1}, b)$ is by way of the identity element $1 \in \Gamma$, which is why we shall refer to $\zeta(s; x, b)$ as a *local* zeta function.

It is also of interest in physics to compute the quantity $E_\Gamma := \frac{1}{2}\zeta_{X_\Gamma}(-\frac{1}{2})$, which comes up in the expression of the *Casimir energy* of the locally symmetric space X_Γ and involves the special value $\zeta(-\frac{1}{2}; \overline{1}, 0)$; E_Γ was computed in some special rank one cases in [11–13] and in the general rank one case in [14], and is given by the zeta regularization of the formal (divergent) expression $\frac{1}{2}\sum_{\lambda_j\neq 0}\lambda_j^{1/2}$ that arises in the canonical quantization of a scalar field on space–times modelled on X_Γ. Here, $\{\lambda_j : 1 \le j \le \infty\}$ is the spectrum of $-\Delta_\Gamma$, the Laplace–Beltrami operator of X_Γ.

Regarding mathematical applications, one can find explicit formulas for *all* Minakshisundaram–Pleijel short-time asymptotic heat kernel coefficients, using the meromorphic structure of zeta [15]; see also [16,17] and compare with [18,19].

The physical applications that we have mentioned involve results that have been worked out in the rank one case only. A study of the local zeta function $\zeta(s; x, b)$ for higher rank spaces is still open, and is the focus of the present paper.

One of our main results is Theorem 2.1, which shows in particular that for general X, for all $x \neq \overline{1}$, $\zeta(s; x, b)$ has an analytic continuation to an entire function of s. In case G is complex, we give a very explicit form of the meromorphic continuation of $\zeta(s; x, b)$ (for all $x \in X$) in Theorem 3.1. We also compute special values of the zeta function and of its derivative at $s = 0$ (with Eq. (1.4) in mind) and at $s = -\frac{1}{2}$. However, the special value at $s = -\frac{1}{2}$ gives only a *local* contribution to the Casimir energy.

To illustrate the difficulties present in the general case, we work out the meromorphic continuation for two higher rank families in the last section. In future work we intend to compute the full energy E_Γ by way of an appropriate version of the trace formula.

2. Meromorphic continuation

The main goal of this section is to prove one of our main results, Theorem 2.1, on the meromorphic continuation of the zeta function. As a first step, we will check the convergence of $\zeta(s; x, b)$ (c.f. definition (1.5)) for s such that $\sigma = \mathrm{Re}\,(s) > \frac{d}{2}$ where $d = \dim X$.

We use the estimate

$$|c(\lambda)|^{-2} \leq C(1+|\lambda|)^n$$

on $\mathfrak{a}_p^*$, where $n = \dim N$ and C is some positive constant. Also for $\lambda \in \mathfrak{a}_p^*$, $\phi_\lambda(x)$ is positive definite on G and therefore $|\phi_\lambda(x)| \leq \phi_\lambda(1) = 1$. Also, let $d = \dim X$, $l = \dim\ \mathfrak{a}_p$, the real rank of X, and let $S_1(0) := \{\lambda \in \mathfrak{a}_p^* \mid |\lambda| = 1\}$ denote the unit sphere in $\mathfrak{a}_p^*$, with surface element $\mathrm{d}S$.

We have the following bound,

$$\begin{aligned}
\int_{\mathfrak{a}_p^*}\int_0^\infty \left| \mathrm{e}^{-tb^2}\mathrm{e}^{-t(|\lambda|^2+|\rho|^2)}\phi_\lambda(x)|c(\lambda)|^{-2}t^{s-1}\right| \mathrm{d}t\mathrm{d}\lambda &= \int_{\mathfrak{a}_p^*}\int_0^\infty \mathrm{e}^{-(|\lambda|^2+|\rho|^2+b^2)t}t^{\sigma-1}\,|\phi_\lambda(x)||c(\lambda)|^{-2}\mathrm{d}t\mathrm{d}\lambda \\
&= \Gamma(\sigma)\int_{\mathfrak{a}_p^*}\frac{|\phi_\lambda(x)||c(\lambda)|^{-2}}{(|\lambda|^2+|\rho|^2+b^2)^\sigma}\mathrm{d}\lambda \\
&\leq C\,\Gamma(\sigma)\int_{\mathfrak{a}_p^*}\frac{(1+|\lambda|)^n}{(|\lambda|^2+|\rho|^2+b^2)^\sigma}\mathrm{d}\lambda \\
&= C\Gamma(\sigma)\int_{S_1(0)}\int_0^\infty \frac{(1+r)^n}{(r^2+|\rho|^2+b^2)^\sigma}r^{l-1}\mathrm{d}r\mathrm{d}S < \infty
\end{aligned}$$

valid for $2\sigma - n - l + 1 > 1$, i.e. for $\sigma > \frac{n+l}{2} = \frac{d}{2}$.

Thus, by Fubini's theorem and definitions (1.2) and (1.3),

$$\begin{aligned}
\zeta(s;x,b) &= \frac{1}{|W|\Gamma(s)}\int_0^\infty \mathrm{e}^{-tb^2}\int_{\mathfrak{a}_p^*}\mathrm{e}^{-t(|\lambda|^2+|\rho|^2)}\phi_\lambda(x)|c(\lambda)|^{-2}\mathrm{d}\lambda t^{s-1}\mathrm{d}t \\
&= \frac{1}{|W|\Gamma(s)}\int_{\mathfrak{a}_p^*}\int_0^\infty \mathrm{e}^{-(|\lambda|^2+|\rho|^2+b^2)t}t^{s-1}\mathrm{d}t\phi_\lambda(x)|c(\lambda)|^{-2}\mathrm{d}\lambda. \qquad (2.1)
\end{aligned}$$

That is, we obtain:

$$\zeta(s;x,b) = \frac{1}{|W|}\int_{\mathfrak{a}_p^*}\frac{\phi_\lambda(x)|c(\lambda)|^{-2}}{(|\lambda|^2+|\rho|^2+b^2)^s}\mathrm{d}\lambda, \qquad (2.2)$$

for any $x \in G/K$, $b \in \mathbb{R}$ and s with Re $(s) > \frac{d}{2}$.

Since $G = KA_pK$ and h_t is K-bi-invariant we can write for every $x \in X$, $x = (k\exp H)K$ with $(k, H) \in K \times \mathfrak{a}_p$ so that

$$\zeta(s;x,b) := \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{e}^{-tb^2}h_t(\exp H)t^{s-1}\mathrm{d}t. \qquad (2.3)$$

Furthermore, if

$$\mathfrak{a}_p{}^+ := \{H \in \mathfrak{a}_p \mid \alpha(H) > 0\ \forall\alpha \in \Sigma^+\},$$

then by the Cartan decomposition $G = K\exp\overline{\mathfrak{a}_p{}^+}K$ and one can assume that $H \in \overline{\mathfrak{a}_p{}^+}$, with H uniquely determined by x.

We now state one of the main results in this paper.

Theorem 2.1. *Let X be a symmetric space of the noncompact type and arbitrary real rank.*

(i) *If $x \in X$, $x \neq \overline{1}$, the origin in $X = G/K$, the function $\zeta(s;x,b)$ can be analytically continued to an* entire *function of s.*
(ii) *If X is such that the asymptotic expansion (2.9) holds, then $\zeta(s;\overline{1},b)$ extends meromorphically to $\mathbb{C}$ for all X. Furthermore, its poles are simple and are located at s of the form $\frac{d}{2} - r$, with $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the residues can be expressed explicitly in terms of the coefficients of the asymptotic expansion a_k in (2.9).*

Remark 2.2. Actually, we believe that the requirement in (ii) should be true always as it was shown in [16] for any symmetric space of classical type, with an explicit determination of the coefficients a_k in all cases. Thus, only the symmetric spaces associated to exceptional groups need to be investigated. However, we do believe that the methods in [16] apply in these cases as well to yield an expansion as in (2.9). After the proof of the theorem, we will give a simple argument that allows one to obtain the meromorphic continuation by way of the special structure of the Plancherel measure for a class of higher rank symmetric spaces which includes the case when $G = SU(p, l+1-p)$ with $2 \leq p \leq \frac{l}{2}$.

Proof. We base the proof on a very useful global estimate for the heat kernel due to Anker and Ostellari (see [20,21]; also see [22–25]) that gives for $t > 0$ and $H \in \mathfrak{a}_p$:

$$h_t(\exp H) \leq Ct^{-\frac{d}{2}}\left[\prod_{\alpha\in\Sigma_0^+}(1+\alpha(H))(1+\alpha(H)+t)^{\frac{m_\alpha+m_{2\alpha}}{2}-1}\right]\cdot \mathrm{e}^{-\rho(H)}\mathrm{e}^{-|\rho|^2t-|H|^2/4t}. \qquad (2.4)$$

Here, $C > 0$ is some constant and $\Sigma_0^+ := \{\alpha \in \Sigma^+ \mid \alpha/2 \notin \Sigma^+\}$ is the set of positive, *indivisible* roots. This estimate, together with a lower bound estimate had been conjectured by Anker.

For $t > 0$ and $H \in \overline{\mathfrak{a}_p{}^+}$, one has $1 + \alpha(H) < 1 + \alpha(H) + t \leq (1 + \alpha(H))(1 + t)$, hence,

$$\begin{aligned}\prod_{\alpha\in\Sigma_0^+}(1+\alpha(H)+t)^{(m_\alpha+m_{2\alpha})/2} &\leq \prod_{\alpha\in\Sigma_0^+}(1+\alpha(H))^{(m_\alpha+m_{2\alpha})/2}(1+t)^{(m_\alpha+m_{2\alpha})/2}\\ &\leq \left(\prod_{\alpha\in\Sigma_0^+}(1+\alpha(H))^{(m_\alpha+m_{2\alpha})/2}\right)(1+t)^{\frac{d-l}{2}}\end{aligned}$$

since $\sum_{\alpha\in\Sigma_0^+}(m_\alpha + m_{2\alpha}) = \dim \mathfrak{n} = d - l$. This estimate, together with (2.4) implies

$$h_t(\exp H) \leq \gamma(H)t^{-\frac{d}{2}}\mathrm{e}^{-|\rho|^2t-|H|^2/4t}(1+t)^{\frac{d-l}{2}} \tag{2.5}$$

where

$$\gamma(H) := C\mathrm{e}^{-\rho(H)}\prod_{\alpha\in\Sigma_0^+}(1+\alpha(H))^{(m_\alpha+m_{2\alpha})/2}, \tag{2.6}$$

for $t > 0$, $H \in \overline{a^+}$. In particular, for $t \geq 1$, $(1+t)^{\frac{d-l}{2}} \leq (2t)^{\frac{d-l}{2}} \Rightarrow$

$$\begin{aligned}\left|\mathrm{e}^{-tb^2}h_t(\exp H)t^{s-1}\right| &\leq \gamma(H)2^{\frac{d-l}{2}}t^{-\frac{l}{2}+\sigma-1}\mathrm{e}^{-(|\rho|^2+b^2)t-\frac{|H|^2}{4t}}\\ &\leq \gamma(H)2^{\frac{d-l}{2}}t^{-\frac{l}{2}+\sigma-1}\mathrm{e}^{-|\rho|^2t}\end{aligned} \tag{2.7}$$

for $\sigma = \mathrm{Re}\,(s)$, a weaker estimate, but enough to show that the integral

$$\int_1^\infty \mathrm{e}^{-tb^2}h_t(\exp H)t^{s-1}\mathrm{d}t$$

converges uniformly on $\mathrm{Re}\,(s) \leq A$ for each $A \in \mathbb{R}$. In particular, it converges uniformly on compact subsets of $\mathbb{C}$, and hence yields an entire function of s (c.f. [26, Ch. XII, Lemma 1.1, p. 308]).

Similarly, $\int_0^1 \mathrm{e}^{-tb^2}h_t(\exp H)t^{s-1}\mathrm{d}t = \int_1^\infty \mathrm{e}^{\frac{-b^2}{t}}h_{1/t}(\exp H)t^{-s-1}\mathrm{d}t$, and for $t > 0$, inequality (2.5) gives

$$h_{1/t}(\exp H) < \gamma(H)t^{\frac{d}{2}}\mathrm{e}^{\frac{-|\rho|^2}{t}-\frac{|H|^2t}{4}}\left(1+\frac{1}{t}\right)^{\frac{d-l}{2}}.$$

In particular, for $t \geq 1$, $1 + \frac{1}{t} \leq 2$, thus

$$\begin{aligned}\left|\mathrm{e}^{-b^2/t}h_{1/t}(\exp H)t^{-s-1}\right| &\leq \gamma(H)\,2^{\frac{d-l}{2}}\,t^{\frac{d}{2}-\sigma-1}\mathrm{e}^{\frac{-|\rho|^2}{t}}\mathrm{e}^{-\frac{|H|^2t}{4}}\\ &\leq \gamma(H)2^{\frac{d-l}{2}}t^{\frac{d}{2}-\sigma-1}\mathrm{e}^{-\frac{|H|^2t}{4}},\end{aligned}$$

which shows that if $H \neq 0$ then, as before, the integral

$$\int_1^\infty \mathrm{e}^{-b^2/t}h_{1/t}(\exp H)t^{-s-1}\mathrm{d}t$$

converges uniformly on $\mathrm{Re}\,(s) \geq -A$, for every $A \in \mathbb{R}$, and thus it defines an entire function of s.

From these considerations and (2.3) we see that $\zeta(s; x, b)$ is an entire function of s provided that $H \neq 0$, which is the case if $x \neq \overline{1}$. This completes the proof of part (i) in Theorem 2.1.

Note that if $H = 0$, the preceding estimate gives

$$\left|\mathrm{e}^{-b^2/t}h_{1/t}(\exp H)t^{-s-1}\right| \leq \gamma(H)2^{\frac{d-l}{2}}t^{\frac{d}{2}-\sigma-1}\mathrm{e}^{-|H|^2t/4} = \gamma(0)2^{\frac{d-l}{2}}t^{\frac{d}{2}-\sigma-1},$$

where for $\sigma \geq \frac{d}{2} + \epsilon$, $\epsilon > 0$, $\int_1^\infty t^{-\frac{d}{2}-\sigma-1}\mathrm{d}t < \infty$, which shows (as is consistent with our earlier remarks) that $\zeta(s; \overline{1}, b)$ is only holomorphic, a priori, on $\mathrm{Re}\,(s) > \frac{d}{2}$.

We now consider assertion (ii). Although $X = G/K$ is non-compact, we employ some ideas that are standard in the study of spectral zeta functions for compact manifolds.

The initial step is to assign to X an appropriate *theta function* $\theta_X(t)$. Namely, for $t > 0$ we take

$$\theta_X(t) := \int_{\mathfrak{a}_p^*}\mathrm{e}^{-t|\lambda|^2}\,|c(\lambda)|^{-2}\,\mathrm{d}\lambda. \tag{2.8}$$

A short time asymptotic expansion

$$\theta_X(t) \overset{t\to 0^+}{\sim} (4\pi t)^{-d/2} \sum_{k=0}^{\infty} a_k t^k \tag{2.9}$$

is assumed. In [16], it was shown that (2.9) is valid for all G of classical type and the coefficients were computed. More precisely, (2.9) means that given any non-negative integer N one has that

$$\lim_{t\to 0^+} \left[(4\pi t)^{d/2} \theta_X(t) - \sum_{k=0}^{N} a_k t^k \right] t^{-N} = 0. \tag{2.10}$$

Eq. (2.10) is equivalent to the existence, for each N, of a positive constant C_N such that

$$\left| \theta_X(t) - (4\pi)^{-d/2} \sum_{k=0}^{N} a_k t^{k-d/2} \right| \le C_N t^{N+1-d/2} \tag{2.11}$$

for any $0 < t \le 1$.

For $\mathrm{Re}\,(s) > d/2$, define $F(s)$ by

$$F(s) := |W| \Gamma(s) \zeta(s; \overline{1}, b). \tag{2.12}$$

If we set $\delta = b^2 + |\rho|^2 > 0$, for convenience, we can use Eq. (2.2) and definition (2.8) to write

$$\begin{aligned} F(s) &= \int_0^\infty e^{-\delta t} \theta_X(t) t^{s-1} dt \\ &= \int_0^1 e^{-\delta t} \left[\theta_X(t) - (4\pi)^{-d/2} \sum_{k=0}^{N} a_k t^{k-d/2} \right] t^{s-1} dt \\ &\quad + \int_0^1 e^{-\delta t} (4\pi)^{-d/2} \left[\sum_{k=0}^{N} a_k t^{k-d/2} \right] t^{s-1} dt + \int_1^\infty e^{-\delta t} \theta_X(t) t^{s-1} dt. \end{aligned} \tag{2.13}$$

The second term here is

$$(4\pi)^{-d/2} \sum_{k=0}^{N} a_k \left[\int_0^\infty e^{-\delta t} t^{k-d/2+s-1} dt - \int_1^\infty e^{-\delta t} t^{k-d/2+s-1} dt \right] = (4\pi)^{-d/2} \sum_{k=0}^{N} \frac{a_k \Gamma\left(k - \frac{d}{2} + s\right)}{\delta^{k-d/2+s}} + F_N(s) \tag{2.14}$$

(since $\mathrm{Re}\,(s) > d/2 - k$ for $0 \le k \le N$), where

$$F_N(s) := -(4\pi)^{-d/2} \sum_{k=0}^{N} a_k \int_1^\infty e^{-\delta t} t^{k-d/2+s-1} dt, \tag{2.15}$$

is a finite sum of entire functions.

Let

$$I_N(s) := \int_0^1 e^{-\delta t} \left[\theta_X(t) - (4\pi)^{-d/2} \sum_{k=0}^{N} a_k t^{k-d/2} \right] t^{s-1} dt \tag{2.16}$$

denote the first integral in (2.13), which we claim is uniformly convergent on $\mathrm{Re}\,(s) \ge \frac{d}{2} - (N+1) + \epsilon$ for every $\epsilon > 0$. Namely, for $\mathrm{Re}\,(s) \ge d/2 - (N+1) + \epsilon$ and $0 < t \le 1$, by (2.11) we have

$$\begin{aligned} \left| e^{-\delta t} \left[\theta_X(t) - (4\pi)^{-d/2} \sum_{k=0}^{N} a_k t^{k-d/2} \right] t^{s-1} \right| &\le e^{-\delta t} C_N t^{N+1-d/2+\mathrm{Re}\,(s)-1} \\ &\le e^{-\delta t} C_N t^{\epsilon-1}. \end{aligned}$$

It follows that $I_N(s)$ is holomorphic on $\mathrm{Re}\,(s) > \frac{d}{2} - (N+1)$.

Now for $H \in \overline{\mathfrak{a}^+}$ we have noted that $E_H(s) := \int_1^\infty e^{-tb^2} h_t(\exp H) t^{s-1} dt$ is an entire function of s. In particular, for $H = 0$, we see by definitions (1.2) and (2.8) that

$$\begin{aligned} E_0(s) &= \int_1^\infty e^{-tb^2} \frac{1}{|W|} \int_{\mathfrak{a}_p^*} e^{-t(|\lambda|^2+|\rho|^2)} |c(\lambda)|^{-2} t^{s-1} d\lambda \\ &= \frac{1}{|W|} I(s), \end{aligned} \tag{2.17}$$

where

$$I(s) := \int_1^\infty e^{-\delta t}\theta_X(t)t^{s-1}dt, \tag{2.18}$$

the third integral in (2.13), which shows that $I(s)$ is an entire function of s.

We have therefore established that

$$\zeta(s;\bar{1},b) = \frac{I_N(s)}{|W|\Gamma(s)} + \frac{(4\pi)^{-d/2}}{|W|}\sum_{k=0}^{N}\frac{a_k\Gamma(k-d/2+s)}{\delta^{k-d/2+s}\Gamma(s)} + \frac{F_N(s)}{|W|\Gamma(s)} + \frac{I(s)}{|W|\Gamma(s)} \tag{2.19}$$

for $\mathrm{Re}\,(s) > d/2$, where $F_N(s)$ and $I(s)$ are entire and $I_N(s)$ is holomorphic on the domain $\mathrm{Re}\,(s) > \frac{d}{2} - (N+1)$. Since $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is arbitrary, Eq. (2.19) proves the meromorphic continuation asserted in (ii) of the theorem.

Concerning the location of poles, there are two possibilities. Suppose, first that $d = 2m$ is even. Choose $N > \frac{d}{2} = m$ and write $\sum_{n=0}^{N} = \sum_{k=0}^{m} + \sum_{k=m+1}^{N}$ for the sum in (2.19). For $k \geq m+1$, $\Gamma(k-m+s)/\Gamma(s) = \prod_{j=0}^{k-m-1}(s+j)$ is a polynomial of degree $k-m$, therefore

$$\sum_{k=m+1}^{N}\frac{a_k\Gamma(k-d/2-s)}{\delta^{k-d/2+s}\Gamma(s)} \tag{2.20}$$

is a polynomial in s, hence an entire function.

On the other hand

$$\sum_{k=0}^{m}\frac{a_k\,\Gamma(k-d/2+s)}{\delta^{k-d/2+s}\,\Gamma(s)} = \sum_{k=0}^{m}\frac{a_{m-k}}{\delta^{s-k}(s-1)(s-2)\cdots(s-k)}. \tag{2.21}$$

From (2.19)–(2.21) we see that if $d = 2m$ is *even*, then $\zeta(s;\bar{1},b)$ has finitely many poles, all simple, possibly at the points $s = 1, 2, \ldots, m = \frac{d}{2}$.

If d is odd, the poles of $\zeta(s;\bar{1},b)$ (all of which, again, are simple) lie at $s = \frac{d}{2} - l$, $l \in \mathbb{N}_0$ due to the summands containing the factor $\Gamma(k-d/2+s)$ in (2.19). It is clear from (2.19) that the residues of $\zeta(s;\bar{1},b)$ are computable in terms of the coefficients a_k in the asymptotic expansion (2.9). $\square$

Remark 2.3. We note that if the rank of X is 1, then one can use the last equation in (2.2) and the formulas for the Harish-Chandra Plancherel density $|c(\lambda)|^{-2}$ to meromorphically continue $\zeta(s;\bar{1},b)$ to $\mathbb{C}$. This task is carried out in [9]; also see [14,10].

We have mentioned that condition (ii) of Theorem 2.1 holds for symmetric spaces of non-compact type, where G is a classical group as proved in [16]. We conjecture it holds for all G. Here we will show that this condition is valid, in a rather simple way, for a certain class of groups G of higher rank that extends the class of rank one groups.

Indeed, we will assume that the Plancherel measure is of a special kind, namely of the form

$$c(\lambda)^{-2} = c_0 p(\lambda)\prod_{i=1}^{l}\tanh(\pi x_i) \quad \left(\text{or } \prod_i \coth(\pi x_i)\right), \tag{2.22}$$

where $\lambda = \sum_{i=1}^{l} x_i\epsilon_i$ and $\epsilon_1, \ldots, \epsilon_l$ is an orthonormal basis of $\mathfrak{a}_\mathfrak{p}^*$, c_0 is a constant and $p(\lambda)$ is a polynomial function. This includes the following choices of G:

$$SU(p, l+1-p)\left(2 \leq p \leq \frac{l}{2}\right), \qquad Sl(l+1,\mathbb{H}), \qquad SO^*(4p), \qquad SO^*(4p+2), \tag{2.23}$$

together with any G of split $\mathbb{R}$-rank one.

For these groups, to obtain the asymptotic expansion of (2.9) one can proceed as in the case of rank one groups, since the integral over $\mathfrak{a}_\mathfrak{p}^*$ in (2.8) essentially splits as a product of integrals of the type appearing for rank one groups. Hence the asymptotic expansion can be obtained in this case by the methods in [17].

3. The zeta function for complex *G*

As a first illustrative example, in this section we determine the structure of $\zeta(s;x,b)$ in the special case when G is complex semisimple Lie group. The main point here is that in this case we can use Gangolli's simple formula in [4] for the heat kernel $h_t(x)$. Namely

$$h_t(\exp H) = p(H)t^{-d/2}e^{-|\rho|^2 t}e^{-\frac{|H|^2}{4t}}, \tag{3.1}$$

for $H \in \mathfrak{a}_p$, where

$$p(H) := \pi^{-l/2} 2^{-d} \prod_{\alpha \in \Sigma^+} \frac{\alpha(H)}{(\rho, \alpha) \sinh \alpha(H)}. \tag{3.2}$$

We have, by [27, p. 340],

$$\int_0^\infty e^{-\gamma t - \frac{\beta}{t}} t^{\nu-1} dt = 2 \left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \tag{3.3}$$

for $\beta, \gamma > 0$, where $K_\nu(r)$ is Macdonald's modified Bessel function. From Eq. (1.3) it follows, for any $x = (k \exp H)K \in X$, with $k \in K, H \in \mathfrak{a}_p, H \neq 0$, that

$$\begin{aligned} \zeta(s; x, b) &= \frac{p(H)}{\Gamma(s)} \int_0^\infty e^{-[|\rho|^2 + b^2]t - \frac{|H|^2}{4t}} t^{s - \frac{d}{2} - 1} dt \\ &= \frac{2p(H)}{\Gamma(s)} \left(\frac{|H|^2}{4(|\rho|^2 + b^2)}\right)^{s/2 - d/4} K_{s - \frac{d}{2}} \left(|H| \sqrt{|\rho|^2 + b^2}\right). \end{aligned} \tag{3.4}$$

Here for any positive number r, $K_\nu(r)$ is an entire function of ν. On the other hand if $H = 0$, then

$$\begin{aligned} \zeta(s; \bar{1}, b) &= \frac{p(0)}{\Gamma(s)} \int_0^\infty e^{-[|\rho|^2 + b^2]t} t^{s - \frac{d}{2} - 1} dt \\ &= \frac{p(0)}{(|\rho|^2 + b^2)^{s - \frac{d}{2}}} \frac{\Gamma\left(s - \frac{d}{2}\right)}{\Gamma(s)}, \end{aligned}$$

for $\mathrm{Re}\,(s) > \frac{d}{2}$, by [27, p. 317].

In particular if $d = 2m$ is even, then

$$\zeta(s; \bar{1}, b) = \frac{p(0)}{(|\rho|^2 + b^2)^{s - \frac{d}{2}} \prod_{j=1}^{m} (s - j)} \tag{3.5}$$

for $\mathrm{Re}\,(s) > \frac{d}{2}$. Summing up, the following result is obtained directly (without appealing to Theorem 2.1):

Theorem 3.1. *Suppose G is complex semisimple. If $x \in X = G/K$ and $x \neq \bar{1}$, then the zeta function $\zeta(s; x, b)$ is an entire function of s given by the explicit formula:*

$$\zeta(s; x, b) = \frac{2p(H)}{\Gamma(s)} \left(\frac{|H|^2}{4(|\rho|^2 + b^2)}\right)^{s/2 - d/4} K_{s - \frac{d}{2}} \left(|H| \sqrt{|\rho|^2 + b^2}\right). \tag{3.6}$$

On the other hand, for $\mathrm{Re}\,(s) > \frac{d}{2}$, $x = \bar{1}$ *we have:*

$$\zeta(s; \bar{1}, b) = \frac{p(0)}{(|\rho|^2 + b^2)^{s - \frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} \tag{3.7}$$

where $p(H)$ is as in (3.2). This gives a meromorphic function with possibly simple poles located at $s = \frac{d}{2} - k$, for $k \in \mathbb{N}_0$, the set of non-negative integers.

Furthermore,

$$\mathrm{Res}_{s = \frac{d}{2} - k} \zeta(s; \bar{1}, b) = \frac{(-1)^k p(0) (|\rho|^2 + b^2)^k}{k! \Gamma\left(\frac{d}{2} - k\right)}. \tag{3.8}$$

In particular, if d is even, *then $\zeta(s; \bar{1}, b)$ is given by formula (3.5), which shows it has* finitely *many poles, all of them simple and located at $s = \frac{d}{2} - k$, for $0 \leq k \leq \frac{d}{2} - 1$.*

Examples of G as in Theorem 2.1 are given in Table 1 where the corresponding symmetric space X is irreducible of type IV [1].

It is useful to note also that for all complex G, $|\rho|^2$ in the various formulas is given by

$$|\rho|^2 = \frac{d}{12}, \tag{3.9}$$

a consequence of the so-called "strange formula" of Freudenthal–deVries [28] (compare [2, p. 487], for example).

Table 1

G	K	d	l = rank of X
$SL(n,\mathbb{C})$	$SU(n)$	n^2-1	$n-1$
$SO(n,\mathbb{C})$	$SO(n)$	$n(n-1)/2$	$[n/2]$
$Sp(n,\mathbb{C})$	$Sp(n)$	$n(2n+1)$	n
$G_2^{\mathbb{C}}$	G_2	14	2
$F_4^{\mathbb{C}}$	F_4	52	4
$E_6^{\mathbb{C}}$	E_6	78	6
$E_7^{\mathbb{C}}$	E_7	133	7
$E_8^{\mathbb{C}}$	E_8	248	8

Formulas (3.5)–(3.7) allow to compute special values of $\zeta(s;x,b)$. Given remarks in the introduction on the physical applications, we have a particular interest in the values $\zeta(0;\overline{1},b)$, $\zeta'(0;\overline{1},b)$ that come up in the computation of one-loop effective potential (see Eq. (1.4)) and in the value $\frac{1}{2}\zeta(-\frac{1}{2};\overline{1},0)$ which is the local Casimir energy. We continue to assume here that G is a complex Lie group. We note from (3.2) that

$$p(0)=\frac{\pi^{-l/2}2^{-d}}{\prod\limits_{\alpha\in\Sigma^+}(\rho,\alpha)}. \tag{3.10}$$

Proposition 3.2. *If d is odd, then $\zeta(-k;\overline{1},b)=0$ for any $k\in\mathbb{N}_0$. Furthermore, $\zeta(-\frac{1}{2};\overline{1},b)=\infty$. If $d=2m$ is even then*

$$\zeta(-k;\overline{1},b)=\frac{(-1)^m p(0)(|\rho|^2+b^2)^{m+k}}{\prod\limits_{j=1}^{m}(k+j)}. \tag{3.11}$$

for $k\in\mathbb{N}_0$ and

$$\zeta\left(-\frac{1}{2};\overline{1},b\right)=\frac{(-1)^m 2^m p(0)(|\rho|^2+b^2)^{m+\frac{1}{2}}}{\prod\limits_{j=1}^{m}(2j+1)}. \tag{3.12}$$

Proof. These statements are easy to check. Indeed if $d=2m+1$ is odd, then for $k\in\mathbb{N}_0$, $\Gamma\left(-k-\frac{d}{2}\right)/\Gamma(-k)=0$, hence, by formula (3.7), $\zeta(-k;\overline{1},b)=0$ and $\zeta\left(-\frac{1}{2};\overline{1},b\right)=\infty$.

If $d=2m$ is even, then (3.11) and (3.12) follow from formula (3.5). $\square$

Proposition 3.3. *If $d=2m+1$ is odd, then*

$$\zeta'(0;\overline{1},b)=\frac{(-1)^{m+1}2^{m+1}\sqrt{\pi}\,p(0)(|\rho|^2+b^2)^{m+\frac{1}{2}}}{\prod\limits_{j=1}^{m}(2j+1)}. \tag{3.13}$$

If $d=2m$ is even, then

$$\zeta'(0;\overline{1},b)=\frac{(-1)^{m+1}}{m!}p(0)(|\rho|^2+b^2)^m\left(\log(|\rho|^2+b^2)-H_m\right), \tag{3.14}$$

where $H_m:=\sum_{j=1}^{m}\frac{1}{j}$ is the mth harmonic number and $p(0)$ is as in (3.10).

Proof. Eq. (3.13) follows from formula (3.7) in conjunction with the fact that $\Gamma(-m-\frac{1}{2})=(-1)^{m+1}2^{m+1}\sqrt{\pi}/\prod_{j=1}^{m}(2j+1)$.

Eq. (3.14) follows from formula (3.5), together with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[(s-1)(s-2)\cdots(s-m)\right]\Big|_{s=0}=(-1)^{m+1}m!H_m. \quad\square$$

Proposition 3.4. *If $x\neq\overline{1}$, then $\zeta(-k;x,b)=0$ for $k\in\mathbb{N}_0$. Furthermore, if $p(0)$ is as in (3.10), then*

$$\zeta'(0;x,b)=2p(0)\left(\frac{|H|^2}{4(|\rho|^2+b^2)}\right)^{-\frac{d}{4}}K_{\frac{d}{2}}\left(|H|\sqrt{|\rho|^2+b^2}\right). \tag{3.15}$$

Proof. Proposition 3.4 follows from formula (3.6), and the fact that $K_{-\nu}(t) = K_\nu(t)$. In particular if $d = 2m+1$ is odd, the formula

$$K_{\frac{d}{2}}(t) = K_{m+\frac{1}{2}}(t) = \sqrt{\frac{\pi}{2t}}\mathrm{e}^{-t}\sum_{k=0}^{m}\frac{(m+k)!}{k\,!(m-k)!\,(2t)^k} \tag{3.16}$$

in [27, p. 967] for $t > 0$, can be used to further explain Eq. (3.15). If $d = 2m$ is even, a more complicated expression for $K_{\frac{d}{2}}(t)$ is given in [27, p. 961]. □

4. Some further examples

In this section, for the reader's benefit, we carry out the explicit meromorphic continuation for a couple of higher rank examples. We shall see that for these families, the expressions of the zeta functions become much more involved. We will make use of some results in [16].

Consider first the space $X = SU^*(2(l+1))/Sp(l+1)$, of rank l and dimension $d = l(2(l+1)+1)$; see for example [1, p. 354, Table II]. $G = SU^*(2(l+1)) = SL(l+1,\mathbb{H})$, where $\mathbb{H}$ denotes the algebra of quaternions. Here,

$$|c(\lambda)|^{-2} = c_0{}^{-2}\prod_{1\le i<j\le l+1}\left(\frac{\lambda_i-\lambda_j}{2}\right)^2\left[1+\left(\frac{\lambda_i-\lambda_j}{2}\right)^2\right], \tag{4.1}$$

for a suitable constant $c_0 = c_0(l) > 0$, and a suitable basis $\{e_i\}_{i=1}^{l+1}$

$$\mathfrak{a}_p^* = \left\{\lambda = \sum_{i=1}^{l+1}\lambda_i e_i \,\middle|\, \sum_{i=1}^{l+1}\lambda_i = 0\right\}; \tag{4.2}$$

(see Section 4.2.6 of [16], for example). We can write

$$|c(\lambda)|^{-2} = \sum_I a_I\lambda^I := \sum_{r_1,\ldots,r_{l+1}} a_{r_1,\ldots,r_{l+1}}\lambda_1^{r_1}\lambda_2^{r_2}\cdots\lambda_{l+1}^{r_{l+1}} \tag{4.3}$$

for $I = (r_1,\ldots,r_{l+1})$, where the r_i are nonnegative integers and, for each I, $|I| = \sum_{i=1}^{l+1} r_i \le 2l(l+1)$, while there is some I with $|I| = 2l(l+1)$. The norms $|e_i|$ have a common value, say v.

In Theorem 5.5.9 of [16], Godoy derives the formula

$$\int_{\mathfrak{a}_p^*}\mathrm{e}^{-t|\lambda|^2}|c(\lambda)|^{-2}\mathrm{d}\lambda = v^l\sum_{I:r_i\ \mathrm{even}}\frac{a_I}{(v\sqrt{t})^{|I|+l}}\prod_{i=1}^{l+1}\Gamma\left(\frac{r_i}{2}+\frac{1}{2}\right) \tag{4.4}$$

for any $t > 0$. The first equation in (2.1) therefore gives

$$\begin{aligned}\zeta(s;\bar{1},b) &= \frac{1}{|W|\Gamma(s)}\sum_{r_i\in 2\mathbb{N}_0}\frac{a_I}{v^{|I|}}\prod_{i=1}^{l+1}\Gamma\left(\frac{r_i}{2}+\frac{1}{2}\right)\int_0^\infty \mathrm{e}^{-t(|\rho|^2+b^2)}t^{s-\left(\frac{|I|+l}{2}\right)-1}\mathrm{d}t\\ &= \frac{1}{|W|}\sum_{r_i\in 2\mathbb{N}_0}\frac{a_I}{v^{|I|}}\prod_{i=1}^{l+1}\frac{\Gamma\left(\frac{r_i}{2}+\frac{1}{2}\right)\Gamma\left(s-\left(\frac{|I|+l}{2}\right)\right)}{(|\rho|^2+b^2)^{s-(|I|+l)/2}\,\Gamma(s)}\end{aligned} \tag{4.5}$$

for s such that Re $(s) > (|I|+l)/2$ for each I. But since, for each I, $|I| \le 2l(l+1)$ with $|I| = 2l(l+1)$ for some I, then, for each I, $(|I|+l)/2 \le [2l(l+1)+l]/2 = d/2$, hence the formula holds for Re $(s) > \frac{d}{2}$, and it thus provides the meromorphic continuation of $\zeta(s;\bar{1},b)$ to $\mathbb{C}$.

Suppose for example that d is even, that is, l is even. In (4.5) write $l = 2m$, $r_i = 2m_i$, and $J = J_I = (m_1,\ldots,m_{l+1})$, where the m_i are nonnegative integers. Then

$$\frac{\Gamma\left(s-\left(\frac{|I|+l}{2}\right)\right)}{\Gamma(s)} = \frac{\Gamma(s-(m+|J_I|))}{\Gamma(s)} = \prod_{i=1}^{m+|J_I|}(s-i)^{-1},$$

thus

$$\zeta(s;\bar{1},b) = \frac{1}{|W|}\sum_{m_i\in\mathbb{N}_0}\frac{a_{2J}}{v^{2|J|}}\frac{\prod_{i=1}^{l+1}\Gamma\left(m_i+\frac{1}{2}\right)}{(|\rho|^2+b^2)^{s-(m+|J|)}\prod_{i=1}^{m+|J|}(s-i)}$$

for Re $(s) > \frac{d}{2}$, where some $|J| = l(l+1)$ (since some $|I| = 2l(l+1)$) hence, for some J, one has $m + |J| = \frac{l}{2} + l(l+1) = \frac{d}{2}$. Thus, when d is *even*, we see from (4.6) that $\zeta(s; \overline{1}, b)$ has only finitely many poles, all of them simple, located at the points $s = 1, 2, \ldots, \frac{d}{2}$.

As a final example we consider the symmetric space $X = SU(p+1, p+1)/S(U(p+1) \times U(p+1))$, which has rank $l = p+1$ and dimension $d = 2(p+1)^2$. We look first at $X = SU(2,2)/S(U(2) \times U(2))$, which has rank $l = 2$ and dimension $d = 8$, since the general case can be handled in a similar way. For $(x, y) \in \mathbb{R}^2$, let

$$M_{x,y} := \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{bmatrix} \tag{4.6}$$

and choose

$$\mathfrak{a}_p := \{M_{x,y} \mid (x, y) \in \mathbb{R}^2\} \simeq \mathbb{R}^2. \tag{4.7}$$

If $\alpha, \beta \in \mathfrak{a}_p^*$ are defined by $\alpha(M_{x,y}) = x$, $\beta(M_{x,y}) = y$, then the set of restricted roots of $(\mathfrak{su}(2,2), \mathfrak{a}_p)$ is $\Sigma = \{\pm\alpha \pm \beta, \pm 2\alpha, \pm 2\beta\}$ and $\Sigma^+ = \{\alpha \pm \beta, 2\alpha, 2\beta\}$ is a choice of positive roots in Σ. The roots $\pm 2\alpha, \pm 2\beta$ have multiplicity 1, whereas those of the form $\pm\alpha \pm \beta$ have multiplicity 2.

For $N = (n_1, n_2)$, $n_1, n_2 \in \mathbb{N}_0$ and $a, \delta > 0$, we set

$$I_N(s; \delta, a) := \int_{\mathbb{R}^2} \frac{x_1^{2n_1+1} x_2^{2n_2+1} \tanh(ax_1) \tanh(ax_2)}{(x_1^2 + x_2^2 + \delta)^s} \mathrm{d}x_1 \mathrm{d}x_2 \tag{4.8}$$

for Re $(s) > |N| + 2 = n_1 + n_2 + 2$.

Theorem 4.1. *If $N! = (n_1!)(n_2!)$, $|x|^2 := x_1^2 + x_2^2$, we have, for* Re $(s) > |N| + 2$*:*

$$I_N(s; \delta, a) = \sum_{\substack{0 \le j_1 \le n_1 \\ 0 \le j_2 \le n_2}} \frac{N! \left(\frac{a}{2}\right)^2}{(n_1 - j_1)!(n_2 - j_2)! \prod_{j=1}^{j_1+j_2+2} (s - j)} \cdot \int_{\mathbb{R}^2} \frac{x_1^{2(n_1-j_1)} x_2^{2(n_2-j_2)} \mathrm{sech}^2(ax_1)\, \mathrm{sech}^2(ax_2)}{(|x|^2 + \delta)^{s-(j_1+j_2+2)}} \mathrm{d}x. \tag{4.9}$$

Proof. Theorem 4.1 is a consequence of the formula

$$\int_{\mathbb{R}} \frac{x^{2n+1} \tanh(ax)}{(x^2 + \delta)^s} \mathrm{d}x = \frac{n! a}{2} \sum_{j=0}^{n} \frac{K_{n-j}(s - j - 1; \delta, a)}{(n-j)! \prod_{i=1}^{j+1} (s - i)} \tag{4.10}$$

for Re $(s) > n + 1$, where we have set, for $m \in \mathbb{N}_0$ and for any $s \in \mathbb{C}$,

$$K_m(s; \delta, a) = \int_{\mathbb{R}} \frac{x^{2m} \mathrm{sech}^2(ax)}{(x^2 + \delta)^s} \mathrm{d}x. \quad \square \tag{4.11}$$

Formula (4.10), which in turn follows by induction on m and by integration by parts, is used in [14] to meromorphically continue $\zeta(s; \overline{1}, b)$ in the rank one case, since the $K_m(s; \delta, a)$ are entire functions of s. Similarly, the integrals over $\mathbb{R}^2$ in (4.8) are entire functions of s, hence Theorem 4.1 yields the meromorphic continuation of $I_N(s; a, \delta)$ to $\mathbb{C}$. In particular we see that all poles of $I_N(s; a, \delta)$ (finite in number) are simple and located at the points $s = 1, 2, \ldots, |N| + 2$.

Now for a suitable constant $c_0 > 0$ depending on the normalization of various measures

$$|c(\lambda)|^{-2} = c_0^{-2} p(\lambda) \prod_{i=1}^{p+1} \pi \tanh\left(\frac{\pi \lambda_i}{2}\right), \tag{4.12}$$

where $p(\lambda) = \prod_{1 \le i < j \le p+1} \pi \left(\frac{\lambda_i - \lambda_j}{2}\right)^2 \pi \left(\frac{\lambda_i + \lambda_j}{2}\right)^2 \prod_{i=1}^{p+1} \lambda_i$, for $G = SU(p+1, p+1)$. Compare Section 4.2.6 of [16], for example. We can thus write

$$p(\lambda) = \sum_N a_N \lambda^N = \sum_{\substack{N = (n_1, \ldots, n_{p+1}) \\ 0 \le n_i \in \mathbb{Z}}} a_{n_1, \ldots, n_{p+1}} \lambda_1^{2n_1+1} \cdots \lambda_{p+1}^{2n_{p+1}+1}, \tag{4.13}$$

where $|N| = p(p+1)$. By the last equation of (2.2), the choice $p = 1$ in (4.13) and definition (4.8), we have for $\operatorname{Re}(s) > 4$

$$\begin{aligned}\zeta(s;\bar{1},b) &= \frac{\pi^2 c_0^{-2}}{|W|} \sum_{\substack{N=(n_1,n_2)\\|N|=2}} a_N \int_{\mathbb{R}^2} \frac{x_1^{2n_1+1}x_2^{2n_2+1}\tanh\left(\frac{\pi x_1}{2}\right)\tanh\left(\frac{\pi x_2}{2}\right)}{(x_1^2+x_2^2+|\rho|^2+b^2)^s}\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{\pi^2 c_0^{-2}}{|W|} \sum_{\substack{N=(n_1,n_2)\\|N|=2}} a_N I_N\left(s;|\rho|^2+b^2,\frac{\pi}{2}\right).\end{aligned}$$

Here $|W| = 8$. Since each $I_N(s;\delta,a)$ extends meromorphically to $\mathbb{C}$, by Theorem 4.1, this formula shows that (for $X = SU(2,2)/S(U(2)\times U(2))$) $\zeta(s;\bar{1},b)$ extends meromorphically to $\mathbb{C}$, by way of Theorem 4.1. All of its poles are simple and are located at the points $s = 1,2,3,4 = d/2$.

One can formulate a generalization of Theorem 4.1 valid for the space $X = SU(p+1,p+1)/S(U(p+1)\times U(p+1))$, which coupled with Eqs. (4.12) and (4.13) allows to give the meromorphic continuation of $\zeta(s;\bar{1},b)$ for this space. Namely, let $N = (n_1,\ldots,n_{p+1}), J = (j_1,\ldots,j_{p+1})$, again for integers n_i, j_i, and write $0 \le J \le N$ if $0 \le j_i \le n_i$ for each i. For $N! := \prod_{i=1}^{p+1}(n_i)!$ and $(N-J)! = \prod_{i=1}^{p+1}(n_i-j_i)!$, the obvious generalization of definition (4.8) is

$$I_N(s;\delta,a) := \int_{\mathbb{R}^{p+1}} \frac{\prod_{i=1}^{p+1} x_i^{2n_i+1}\tanh ax_i}{\left(\sum_{i=1}^{p+1} x_i^2+\delta\right)^s}\mathrm{d}x_i, \tag{4.14}$$

for $\operatorname{Re}(s) > |N|+p+1$, $a,\delta > 0$, while that of Theorem 4.1 is

Theorem 4.2. *For* $\operatorname{Re}(s) > |N|+p+1$,

$$I_N(s;\delta,a) = \sum_{J,N:0\le J\le N} \frac{N!\left(\frac{a}{2}\right)^{p+1}}{(N-J)!\prod_{j=1}^{|J|+p+1}(s-j)} \cdot \int_{\mathbb{R}^{p+1}} \frac{\prod_{i=1}^{p+1} x_i^{2(n_i-j_i)}\operatorname{sech}^2(ax_i)}{\left(\sum_{i=1}^{p+1}x_i^2+\delta\right)^{s-(|J|+p+1)}}\mathrm{d}x_i \tag{4.15}$$

which is proved by induction on p.

The meromorphic continuation of $\zeta(s;\bar{1},b)$ for the symmetric space $X = SU(p+1,p+1)/S(U(p+1)\times U(p+1))$ can be carried out as before. Again start with the final equation in (2.2) and apply formulas (4.12) and (4.13), and definition (4.14): For $\operatorname{Re}(s) > (p+1)^2$,

$$\begin{aligned}\zeta(s;\bar{1},b) &= \frac{\pi^{p+1}c_0^{-2}}{|W|} \sum_{\substack{N=(n_1,\ldots,n_{p+1})\\|N|=p(p+1)}} a_N \int_{\mathbb{R}^{p+1}} \frac{\prod_{i=1}^{p+1} x_i^{2n_i+1}\tanh\left(\frac{\pi x_i}{2}\right)}{\sum_{i=1}^{p+1}(x_i^2+|\rho|^2+b^2)^s}\mathrm{d}x_i \\ &= \frac{\pi^{p+1}c_0^{-2}}{|W|} \sum_{N:|N|=p(p+1)} a_N I_N\left(s;|\rho|^2+b^2,\frac{\pi}{2}\right).\end{aligned} \tag{4.16}$$

Now, we can apply (4.15) to each $I_N(s;|\rho|^2+b^2,\frac{\pi}{2})$ in (4.16), provided that $\operatorname{Re}(s) >$ each $|N|+p+1 = (p+1)^2$, which is the assumption in place. Thus, by (4.15) and formula (4.16), we see that $\zeta(s;\bar{1},b)$ extends meromorphically to $\mathbb{C}$, with only finitely many poles (all of them simple), located at the points $s = 1,2,\ldots,(p+1)^2 = d/2$.

In general, $\mathfrak{sl}(r+s,\mathbb{C})$ is the complexification of the Lie algebra $\mathfrak{su}(r,s)$. Thus the Killing form of $\mathfrak{su}(r,s)$ is given by $(X,Y) = 2(r+s)\operatorname{tr}XY$. In particular, for $\mathfrak{g}_0 = \mathfrak{su}(p+1,p+1)$ it is given by $(X,Y) = 4(p+1)\operatorname{tr}XY$, for $X,Y \in \mathfrak{g}_0$, and $|\rho|^2 = (H_\rho,H_\rho)_{\mathfrak{sl}(2(p+1),\mathbb{C})} = \frac{1}{24}\dim SL(2(p+1),\mathbb{C})/SU(2(p+1))$, by Freudenthal–deVries formula (see (3.9)). Thus by Table 1,

$$|\rho|^2 = (4p^2+8p+3)/24 \tag{4.17}$$

for $X = SU(p+1,p+1)/S(U(p+1)\times U(p+1))$. In particular (for $p=1$), $|\rho|^2 = 5/8$ for $X = SU(2,2)/S(U(2)\times U(2))$. This value also follows directly from (4.6), given that $(X,Y) = 8\operatorname{tr}XY$ is the Killing form of $\mathfrak{su}(2,2)$.

The constant c_0 in formula (4.16) is given by

$$c_0 = \prod_{\alpha\in\Sigma_0^+} \frac{\Gamma\left(\frac{1}{2}\left(\frac{m_\alpha}{2}+1+(\rho,\alpha_0)\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{m_\alpha}{2}+m_{2\alpha}+(\rho,\alpha_0)\right)\right)}{2^{-(\rho,\alpha_0)}\Gamma((\rho,\alpha_0))}, \tag{4.18}$$

where $\alpha_0 = \alpha/(\alpha, \alpha)$ for $\alpha \in \Sigma$. As in [2, p. 109], for example, c_0 accounts for the normalization condition $c(-i\rho) = 1$ of Harish-Chandra's c-function on Λ. Recall from the introduction that Λ is the complexification of the dual space $\mathfrak{a}_p^*$ and Σ_0^+ is the set of indivisible roots in Σ^+. In the present case $\Sigma_0^+ = \Sigma^+ = \{\alpha_i \pm \alpha_j \mid 1 \le i < j \le p+1\} \cup \{2\alpha_i \mid 1 \le i \le p+1\}$, with $m_\alpha = 2$ for $\alpha = \alpha_i \pm \alpha_j$, $i < j$, and $m_\alpha = 1$ for $\alpha = 2\alpha_i$. Here, in a similar way to (4.6), an element of $\mathfrak{a}_p$ has the form $H = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}$, where D is a $(p+1) \times (p+1)$ diagonal matrix with diagonal entry $(x_1, \ldots, x_{p+1}) \in \mathbb{R}^{p+1}$, and the roots α_i are such that $\alpha_i(H) = x_i$. Also the Weyl group is given by $W \simeq (\mathbb{Z}/2\mathbb{Z})^{p+1} \times S_{p+1}$, where S_{p+1} is the symmetric group on $p+1$ letters. Therefore $|W| = 2^{p+1}(p+1)!$ in formula (4.16).

By formulas (4.15) and (4.16), the local Casimir energy $\frac{1}{2}\zeta\left(-\frac{1}{2}; \overline{1}, 0\right)$ assumes the form

$$\begin{aligned}\frac{1}{2}\zeta\left(-\frac{1}{2}; \overline{1}, 0\right) &= \frac{\pi^{p+1}c_0^{-2}}{2|W|} \sum_{N:|N|=p(p+1)} a_N I_N \left(-\frac{1}{2}; |\rho|^2, \frac{\pi}{2}\right) \\ &= \frac{\pi^{p+1}c_0^{-2}}{2|W|} \left(\frac{\pi}{4}\right)^{p+1} \sum_{\substack{J,N:0\le J\le N \\ |N|=p(p+1)}} \frac{\binom{N}{J} J! a_N}{(-1)^{|J|+p+1} \prod_{j=1}^{|J|+p+1} \frac{2j+1}{2}} \cdot \int_{\mathbb{R}^{p+1}} \frac{\prod_{i=1}^{p+1} x_i^{2(n_i-j_i)} \operatorname{sech}\left(\frac{\pi x_i}{2}\right)^2}{\left(\sum_{i=1}^{p+1} x_i^2 + |\rho|^2\right)^{-\frac{1}{2}-(|J|+p+1)}} \mathrm{d}x_i\end{aligned}$$

for $|\rho|^2$ given by Eq. (4.17), which shows that in some cases one obtains, unfortunately, less tractable formulas for the energy, in contrast to the very simple, explicit formula obtained in Eq. (3.12), for example.

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