



$U(1) \times U(1)$ Chern–Simons vortices in line bundles over a compact Riemann surface

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ABSTRACT

The relativistic $U(1) \times U(1)$ Chern–Simons model with two Higgs particles and two gauge fields has been studied in \mathbb{R}^2 and on a flat torus. We formulate a model for line bundles over a compact Riemann surface, and prove an existence theorem for the induced system of self-dual Chern–Simons vortex equations.

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1. Introduction

Models governed by Chern–Simons-type dynamics have been developed to explain certain phenomena in various fields of physics. For example, in particle physics, Chern–Simons terms allow both electrically and magnetically charged vortex-like solitons known as dyons; in condensed matter physics, Chern–Simons terms are necessary parts of various anyon models describing many-fermion systems such as electron pairing in high-temperature superconductors and the integral and fractional quantum Hall effect.

Parity-breaking is a property of the simplest models with only one Chern–Simons field. However, when there is an even number of Chern–Simons fields and their coupling constants are appropriately chosen, the parity invariance can be restored. One of the simplest models of this type is the $U(1) \times U(1)$ model of two Higgs fields, each of which is coupled to one of the two Chern–Simons fields, as proposed by Kim et al. [1]. This model is suitable for the anionic mechanism of superconductivity since experiments with high-temperature superconductors support parity invariance.

The Chern–Simons–Higgs energy functional on \mathbb{R}^2 [2,1,3] is given by

$$E(A, B, \phi, \psi) = \int_{\mathbb{R}^2} \left(\frac{\kappa^2}{4} \frac{|F_B|^2}{q_1^2 |\phi|^2} + \frac{\kappa^2}{4} \frac{|F_A|^2}{q_2^2 |\psi|^2} + |d_A \phi|^2 + |d_B \psi|^2 + V(\phi, \psi) \right) dx,$$

where $\kappa > 0$ is a coupling constant, ϕ and ψ are two scalar fields (complex-valued functions in \mathbb{R}^2) representing two Higgs particles of charges q_1 and q_2 , A and B are two associated gauge fields of real-valued one-forms in \mathbb{R}^2 , $F_A = dA$ and $F_B = dB$ are the field strength, $d_A \phi = d\phi - iq_1 A\phi$ and $d_B \psi = d\psi - iq_2 B\psi$ are the covariant derivatives, and $V(\phi, \psi)$ is the potential

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energy density, defined as

$$V(\phi, \psi) = \frac{q_1^2 q_2^2}{\kappa^2} (|\phi|^2 (c_2^2 - |\psi|^2)^2 + |\psi|^2 (c_1^2 - |\phi|^2)^2).$$

Here the positive vacuum states $\langle \phi \rangle = c_1 > 0$ and $\langle \psi \rangle = c_2 > 0$ lead to spontaneous broken symmetries.

A gauge transformation (g_1, g_2) , a $U(1) \times U(1)$ -valued function in \mathbb{R}^2 , acts on gauge fields and scalar fields as $g_1.A = A - ig_1^{-1}dg_1$ ($g_2.B = B - ig_2^{-1}dg_2$) and $g_1.\phi = g_1\phi$ ($g_2.\psi = g_2\psi$). Hence, $d_{g_1.A}(g_1.\phi) = g_1d_A\phi$ ($d_{g_2.B}(g_2.\psi) = g_2d_B\psi$) and $F_{g_1.A} = F_A$ ($F_{g_2.B} = F_B$), from which it follows that the Chern–Simons–Higgs energy functional is gauge-invariant.

With \mathbb{R}^2 and \mathbb{C} identified in the usual way, we have $d_A\phi = \partial_A\phi + \bar{\partial}_A\phi$ ($d_B\psi = \partial_B\psi + \bar{\partial}_B\psi$), with the covariant derivative separated into its holomorphic and anti-holomorphic parts. Applying integration by parts, the energy functional can be written as

$$E(A, B, \phi, \psi) = \int_{\mathbb{R}^2} \left(2|\bar{\partial}_A\phi|^2 + 2|\bar{\partial}_B\psi|^2 + \left| \frac{\kappa}{2} \frac{*F_A}{q_2|\psi|} - \frac{q_1q_2}{\kappa} |\psi| (c_1^2 - |\phi|^2) \right|^2 + \left| \frac{\kappa}{2} \frac{*F_B}{q_1|\phi|} - \frac{q_1q_2}{\kappa} |\phi| (c_2^2 - |\psi|^2) \right|^2 \right) dx + \int_{\mathbb{R}^2} (c_1^2 q_1 F_A + c_2^2 q_2 F_B)$$

and

$$E(A, B, \phi, \psi) = \int_{\mathbb{R}^2} \left(2|\partial_A\phi|^2 + 2|\partial_B\psi|^2 + \left| \frac{\kappa}{2} \frac{*F_A}{q_2|\psi|} + \frac{q_1q_2}{\kappa} |\psi| (c_1^2 - |\phi|^2) \right|^2 + \left| \frac{\kappa}{2} \frac{*F_B}{q_1|\phi|} + \frac{q_1q_2}{\kappa} |\phi| (c_2^2 - |\psi|^2) \right|^2 \right) dx - \int_{\mathbb{R}^2} (c_1^2 q_1 F_A + c_2^2 q_2 F_B).$$

Below we use the first expression together with $\int_{\mathbb{R}^2} (c_1^2 q_1 F_A + c_2^2 q_2 F_B) \geq 0$. We thus have

$$E \geq \int_{\mathbb{R}^2} (c_1^2 q_1 F_A + c_2^2 q_2 F_B)$$

and the minimum energy is attained if and only if (A, B, ϕ, ψ) satisfies the system of self-dual Chern–Simons vortex equations

$$\begin{cases} \bar{\partial}_A\phi = 0, \\ \bar{\partial}_B\psi = 0, \\ *F_A = \frac{2q_1q_2}{\kappa^2} |\psi|^2 (c_1^2 - |\phi|^2), \\ *F_B = \frac{2q_1^2q_2}{\kappa^2} |\phi|^2 (c_2^2 - |\psi|^2). \end{cases}$$

Furthermore, as $|\phi| \rightarrow c_1$ and $|\psi| \rightarrow c_2$ at infinity, we can show that $\frac{1}{2\pi} \int_{\mathbb{R}^2} F_A$ and $\frac{1}{2\pi} \int_{\mathbb{R}^2} F_B$ are nonnegative integers, called the vortex numbers. The existence theorem for these equations was proved by Lin et al. [3]. A similar result for periodic domains was proved by Lin and Prajapat [4].

Note that the system has as limiting cases both the familiar self-dual Ginzburg–Landau equations [5–7] and the well-studied single-Higgs-particle self-dual Chern–Simons vortex equations [8–11].

We study the following more general situation. We define the $U(1) \times U(1)$ Chern–Simons energy functional for holomorphic line bundles over a compact Riemann surface, from which we induce the self-dual Chern–Simons vortex equations over the Riemann surface involving two Higgs particles and two gauge fields, using the Kähler identities to replace integration by parts in the \mathbb{R}^2 case. Our aim is to prove the existence of the equations.

The remainder of the paper is organized as follows. In Section 2 we introduce the self-dual $U(1) \times U(1)$ Chern–Simons vortex equations over a compact Riemann surface, deduce a property of $U(1) \times U(1)$ Chern–Simons vortices, and state our main result. In Section 3 we prove the main theorem.

2. $U(1) \times U(1)$ Chern–Simons vortex equations

In this section we introduce the self-dual Chern–Simons vortex equations for holomorphic line bundles over a compact Riemann surface involving two Higgs fields and two gauge fields. As in the \mathbb{R}^2 case, they appear as equations satisfied by minima of the $U(1) \times U(1)$ Chern–Simons energy functional for the line bundles.

Let S be a compact Riemann surface with a Hermitian metric and the associated Kähler form ω . Let L_1 and L_2 be C^∞ line bundles over S . We fix a Hermitian metric h_1 (h_2) on L_1 (L_2). We denote by \mathcal{A} (\mathcal{B}) the space of unitary connections on (L_1, h_1) ((L_2, h_2)), and by $\Omega^0(L_1)$ ($\Omega^0(L_2)$) the space of smooth sections of L_1 (L_2).

Definition 1. The $U(1) \times U(1)$ Chern–Simons energy functional $E : \mathcal{A} \times \mathcal{B} \times \Omega^0(L_1) \times \Omega^0(L_2) \rightarrow \mathbb{R}$ is defined as

$$E(A, B, \phi, \psi) = \frac{\kappa^2}{4} \left\| \frac{F_B}{q_1 |\phi|_{h_1}} \right\|^2 + \frac{\kappa^2}{4} \left\| \frac{F_A}{q_2 |\psi|_{h_2}} \right\|^2 + \|\mathrm{d}_A \phi\|^2 + \|\mathrm{d}_B \psi\|^2 \\ + \frac{q_1^2 q_2^2}{\kappa^2} \left\| |\phi|_{h_1} (c_2^2 - |\psi|_{h_2}^2) \right\|^2 + \frac{q_1^2 q_2^2}{\kappa^2} \left\| |\psi|_{h_2} (c_1^2 - |\phi|_{h_1}^2) \right\|^2,$$

where $\|\cdot\|$ denotes the L^2 norm, $F_A \in \Omega_S^2$ ($F_B \in \Omega_S^2$) is the curvature of the connection A (B), $\mathrm{d}_A \phi \in \Omega^1(L_1)$ ($\mathrm{d}_B \psi \in \Omega^1(L_2)$) is the covariant derivative of ϕ (ψ), $|\phi|_{h_1}$ ($|\psi|_{h_2}$) is the norm of ϕ (ψ) with respect to h_1 (h_2), and $c_1, c_2 > 0$ are constants.

Let \mathcal{G}_1 (\mathcal{G}_2) be the group of unitary transformations of (L_1, h_1) ((L_2, h_2)). The functional \mathcal{E} is invariant under the standard action of the group $\mathcal{G}_1 \times \mathcal{G}_2$ of gauge transformations.

Proposition 1. If $(A, B, \phi, \psi) \in \mathcal{A} \times \mathcal{B} \times \Omega^0(L_1) \times \Omega^0(L_2)$, then

$$E(A, B, \phi, \psi) = 2 \|\bar{\partial}_A \phi\|^2 + 2 \|\bar{\partial}_B \psi\|^2 + \left\| \frac{\kappa}{2} \frac{i \Lambda F_A}{q_2 |\psi|_{h_2}} - \frac{q_1 q_2}{\kappa} |\psi|_{h_2} (c_1^2 - |\phi|_{h_1}^2) \right\|^2 \\ + \left\| \frac{\kappa}{2} \frac{i \Lambda F_B}{q_1 |\phi|_{h_1}} - \frac{q_1 q_2}{\kappa} |\phi|_{h_1} (c_2^2 - |\psi|_{h_2}^2) \right\|^2 + 2\pi c_1^2 q_1 \deg L_1 + 2\pi c_2^2 q_2 \deg L_2,$$

where $\bar{\partial}_A$ ($\bar{\partial}_B$) is the anti-holomorphic part of the covariant derivative d_A (d_B), $\Lambda F_A \in \Omega_S^0$ ($\Lambda F_B \in \Omega_S^0$) is the contraction of F_A (F_B) with the Kähler form, and $\deg L_1$ ($\deg L_2$) is the degree of L_1 (L_2), that is, its first Chern class.

Proof. We can expand

$$\left\| \frac{\kappa}{2} \frac{i \Lambda F_A}{q_2 |\psi|_{h_2}} - \frac{q_1 q_2}{\kappa} |\psi|_{h_2} (c_1^2 - |\phi|_{h_1}^2) \right\|^2 = \frac{\kappa^2}{4} \left\| \frac{\Lambda F_A}{q_2 |\psi|_{h_2}} \right\|^2 + \frac{q_1^2 q_2^2}{\kappa^2} \left\| |\psi|_{h_2} (c_1^2 - |\phi|_{h_1}^2) \right\|^2 \\ + \langle i q_1 \Lambda F_A, |\phi|_{h_1}^2 I_1 \rangle - \langle i q_1 \Lambda F_A, c_1^2 I_1 \rangle$$

and similarly

$$\left\| \frac{\kappa}{2} \frac{i \Lambda F_B}{q_1 |\phi|_{h_1}} - \frac{q_1 q_2}{\kappa} |\phi|_{h_1} (c_2^2 - |\psi|_{h_2}^2) \right\|^2 = \frac{\kappa^2}{4} \left\| \frac{\Lambda F_B}{q_1 |\phi|_{h_1}} \right\|^2 + \frac{q_1^2 q_2^2}{\kappa^2} \left\| |\phi|_{h_1} (c_2^2 - |\psi|_{h_2}^2) \right\|^2 \\ + \langle i q_2 \Lambda F_B, |\psi|_{h_2}^2 I_2 \rangle - \langle i q_2 \Lambda F_B, c_2^2 I_2 \rangle,$$

where $I_1 \in \Omega^0(\mathrm{End} L_1) \cong \Omega^0(L_1 \otimes L_1^*)$ ($I_2 \in \Omega^0(\mathrm{End} L_2) \cong \Omega^0(L_2 \otimes L_2^*)$) is the identity section. The result follows from the identities

$$\langle i q_1 \Lambda F_A, |\phi|_{h_1}^2 \rangle = -\|\bar{\partial}_A \phi\|^2 + \|\partial_A \phi\|^2, \quad \langle i q_2 \Lambda F_B, |\psi|_{h_2}^2 \rangle = -\|\bar{\partial}_B \psi\|^2 + \|\partial_B \psi\|^2, \\ \|\Lambda F_A\|^2 = \|F_A\|^2, \quad \|\Lambda F_B\|^2 = \|F_B\|^2, \\ \int_S i \Lambda F_A \omega = 2\pi \deg L_1 \quad \text{and} \quad \int_S i \Lambda F_B \omega = 2\pi \deg L_2,$$

and their substitution into the above expansions. \square

We conclude that the functional \mathcal{E} is bounded below by $2\pi c_1^2 q_1 \deg L_1 + 2\pi c_2^2 q_2 \deg L_2$. This lower bound is attained if and only if $(A, B, \phi, \psi) \in \mathcal{A} \times \mathcal{B} \times \Omega^0(L_1) \times \Omega^0(L_2)$ satisfies

$$\begin{cases} \bar{\partial}_A \phi = 0, \\ \bar{\partial}_B \psi = 0, \\ \Lambda F_A = -\frac{2i q_1 q_2^2}{\kappa^2} |\psi|_{h_2}^2 (c_1^2 - |\phi|_{h_1}^2), \\ \Lambda F_B = -\frac{2i q_1^2 q_2}{\kappa^2} |\phi|_{h_1}^2 (c_2^2 - |\psi|_{h_2}^2). \end{cases} \quad (1)$$

These equations are our self-dual Chern–Simons vortex equations involving two Higgs particles and two gauge fields. The first (second) equation simply states that ϕ (ψ) is holomorphic with respect to the holomorphic structure on L_1 (L_2) induced by $A \in \mathcal{A}$ ($B \in \mathcal{B}$).

Using the change of variables $q_1 A \mapsto A$, $q_2 B \mapsto B$, $\phi \mapsto c_1 \phi$, and $\psi \mapsto c_2 \psi$, and the suppressed parameter $\lambda = 4c_1^2 c_2^2 q_1^2 q_2^2 / \kappa^2$, we can simplify (1) as

$$\begin{cases} \bar{\partial}_A \phi = 0, \\ \bar{\partial}_B \psi = 0, \\ \Delta F_A = -\frac{i\lambda}{2} |\psi|_{h_2}^2 (1 - |\phi|_{h_1}^2), \\ \Delta F_B = -\frac{i\lambda}{2} |\phi|_{h_1}^2 (1 - |\psi|_{h_2}^2). \end{cases} \quad (2)$$

We note an important property of $U(1) \times U(1)$ Chern–Simons vortices.

Proposition 2. Let L_1 (L_2) be a line bundle with a Hermitian metric h_1 (h_2). If $(A, B, \phi, \psi) \in \mathcal{A} \times \mathcal{B} \times \Omega^0(L_1) \times \Omega^0(L_2)$ are $U(1) \times U(1)$ Chern–Simons vortices, a solution of the system (2), then

$$|\phi|_{h_1}^2 < 1 \quad \text{and} \quad |\psi|_{h_2}^2 < 1. \quad (3)$$

Proof. Since A (B) is the metric connection compatible with $\bar{\partial}$ and h_1 (h_2), $\bar{\partial}\phi = 0$ ($\bar{\partial}\psi = 0$) and $F_A = A'' \wedge A' + A' \wedge A''$ ($F_B = B'' \wedge B' + B' \wedge B''$), we have

$$\begin{aligned} \bar{\partial}\partial|\phi|_{h_1}^2 &= (A'' \wedge A'\phi, \phi)_{h_1} - (A'\phi, A'\phi)_{h_1} \\ &= (F_A \phi, \phi)_{h_1} - (A'\phi, A'\phi)_{h_1} \end{aligned}$$

and

$$\begin{aligned} \bar{\partial}\partial|\psi|_{h_2}^2 &= (B'' \wedge B'\psi, \psi)_{h_2} - (B'\psi, B'\psi)_{h_2} \\ &= (F_B \psi, \psi)_{h_2} - (B'\psi, B'\psi)_{h_2}, \end{aligned}$$

where A' (B') and A'' (B'') are the $(1, 0)$ and $(0, 1)$ parts of A (B), respectively. Using the Kähler identities, we obtain

$$\begin{aligned} \Delta|\phi|_{h_1}^2 &= 2i\Lambda\bar{\partial}\partial|\phi|_{h_1}^2 \\ &= (2i\Lambda F_A \phi, \phi)_{h_1} - 2i\Lambda(A'\phi, A'\phi)_{h_1} \end{aligned}$$

and

$$\begin{aligned} \Delta|\psi|_{h_2}^2 &= 2i\Lambda\bar{\partial}\partial|\psi|_{h_2}^2 \\ &= (2i\Lambda F_B \psi, \psi)_{h_2} - 2i\Lambda(B'\psi, B'\psi)_{h_2}, \end{aligned}$$

where Δ is the Hodge Laplacian. By system (2), it follows that

$$\Delta|\phi|_{h_1}^2 = \lambda|\psi|_{h_2}^2 |\phi|_{h_1}^2 (1 - |\phi|_{h_1}^2) - 2i\Lambda(A'\phi, A'\phi)_{h_1}$$

and

$$\Delta|\psi|_{h_2}^2 = \lambda|\phi|_{h_1}^2 |\psi|_{h_2}^2 (1 - |\psi|_{h_2}^2) - 2i\Lambda(B'\psi, B'\psi)_{h_2}.$$

This can be rewritten as

$$(-\Delta - \lambda|\psi|_{h_2}^2 |\phi|_{h_1}^2) (1 - |\phi|_{h_1}^2) = -2i\Lambda(A'\phi, A'\phi)_{h_1} \quad (4)$$

and

$$(-\Delta - \lambda|\phi|_{h_1}^2 |\psi|_{h_2}^2) (1 - |\psi|_{h_2}^2) = -2i\Lambda(B'\psi, B'\psi)_{h_2}. \quad (5)$$

Since the right-hand side of (4) and (5) is less than or equal to zero, we can apply the strong maximum principle to the elliptic operators $(-\Delta - \lambda|\psi|_{h_2}^2 |\phi|_{h_1}^2)$ and $(-\Delta - \lambda|\phi|_{h_1}^2 |\psi|_{h_2}^2)$. This yields the desired result (3). \square

By virtue of Proposition 2, we can integrate the last two equations of (2) over S to find an obstruction to its solution:

$$\begin{aligned} 2\pi \deg L_1 &= \frac{\lambda}{2} \int_S |\psi|_{h_2}^2 (1 - |\phi|_{h_1}^2) < \frac{\lambda \text{vol}(S)}{2}, \\ 2\pi \deg L_2 &= \frac{\lambda}{2} \int_S |\phi|_{h_1}^2 (1 - |\psi|_{h_2}^2) < \frac{\lambda \text{vol}(S)}{2}. \end{aligned}$$

Thus, we see that a necessary condition for the existence of solutions is that

$$\lambda > \frac{4\pi}{\text{vol}(S)} \max\{\deg L_1, \deg L_2\}. \quad (6)$$

We show below that any sufficiently large λ satisfying this obstruction is also sufficient. More precisely, we have the following theorem.

Theorem 3. *Let S be a compact Riemann surface equipped with a Hermitian metric. Let L_1 (L_2) be a C^∞ line bundle of degree $d_1 > 0$ ($d_2 > 0$) with a fixed Hermitian metric h_1 (h_2). Let $D_1 = \sum_{i=1}^{d_1} s_i$ ($D_2 = \sum_{i=1}^{d_2} t_i$) be an effective divisor of degree d_1 (d_2) (possibly with multiplicities). Then there exists a smooth solution (A, B, ϕ, ψ) of system (2) if λ satisfying (6) is sufficiently large. Moreover, this solution is such that $(L_1, \bar{\partial}_A) = [D_1]$ ($(L_2, \bar{\partial}_B) = [D_2]$), the holomorphic line bundle determined by D_1 (D_2), and the set of zeros of ϕ (ψ) is the divisor D_1 (D_2).*

3. The elliptic system

In this section we reduce the $U(1) \times U(1)$ Chern–Simons vortex equations, the last two equations of (2), to a system of non-linear elliptic equations.

Let L_1 (L_2) be a holomorphic line bundle over S . The holomorphic structure on L_1 (L_2) is given by

$$\bar{\partial}_{L_1} : \Omega^0(L_1) \longrightarrow \Omega^{0,1}(L_1) \quad (\bar{\partial}_{L_2} : \Omega^0(L_2) \longrightarrow \Omega^{0,1}(L_2)).$$

Let $\phi \in \Omega^0(L_1)$ ($\psi \in \Omega^0(L_2)$) be a holomorphic section of L_1 (L_2). Let h_1 (h_2) be a fixed Hermitian metric on L_1 (L_2). Note that any other Hermitian metric on L_1 (L_2) is related to h_1 (h_2) by a positive self-adjoint bundle endomorphism. Since L_1 (L_2) is a line bundle, the two metrics are related by a positive real-valued function $\rho \in C^\infty(S)$ ($\sigma \in C^\infty(S)$). Given a Hermitian metric k_1 (k_2), we can thus write $k_1 = h_1 e^{2u}$ ($k_2 = h_2 e^{2v}$), where $u \in C^\infty(S)$ ($v \in C^\infty(S)$). The factor 2 is for later convenience.

Lemma 4. *If $k_1 = h_1 e^{2u}$ and $k_2 = h_2 e^{2v}$, then the $U(1) \times U(1)$ self-dual Chern–Simons vortex equations are, as equations for u and v ,*

$$\begin{cases} \Delta u + \frac{\lambda}{2} |\psi|_{h_2}^2 e^{2v} (|\phi|_{h_1}^2 e^{2u} - 1) + i\Lambda F_{h_1} = 0, \\ \Delta v + \frac{\lambda}{2} |\phi|_{h_1}^2 e^{2u} (|\psi|_{h_2}^2 e^{2v} - 1) + i\Lambda F_{h_2} = 0, \end{cases} \quad (7)$$

where F_{h_1} (F_{h_2}) is the curvature of the metric connection A_{h_1} (A_{h_2}) compatible with $\bar{\partial}_{L_1}$ ($\bar{\partial}_{L_2}$) and h_1 (h_2).

Proof. If $k_1 = h_1 \rho$ ($k_2 = h_2 \sigma$), then we have

$$i\Lambda F_{k_1} = i\Lambda F_{h_1} + i\Lambda \bar{\partial}_{L_1} (\rho^{-1} A'_{h_1} \rho) \quad (i\Lambda F_{k_2} = i\Lambda F_{h_2} + i\Lambda \bar{\partial}_{L_2} (\sigma^{-1} B'_{h_2} \sigma))$$

by straightforward calculation. Here F_{k_1} (F_{k_2}) is the curvature of the metric connection A_{k_1} (B_{k_2}) compatible with $\bar{\partial}_{L_1}$ ($\bar{\partial}_{L_2}$) and k_1 (k_2), and A'_{h_1} (B'_{h_2}) is the $(1, 0)$ part of the connection induced by A_{h_1} (B_{h_2}) on $\text{End } L_1$ ($\text{End } L_2$). Since L_1 (L_2) is a line bundle, $\text{End } L_1$ ($\text{End } L_2$) is a trivial bundle and $A_{h_1} = d$ ($B_{h_2} = d$) on $\text{End } L_1$ ($\text{End } L_2$). Hence, if $\rho = e^{2u}$ ($\sigma = e^{2v}$), then

$$i\Lambda F_{k_1} = i\Lambda F_{h_1} + 2i\Lambda \bar{\partial} \partial u \quad (i\Lambda F_{k_2} = i\Lambda F_{h_2} + 2i\Lambda \bar{\partial} \partial v). \quad (8)$$

The Kähler identities give

$$2i\Lambda \bar{\partial} \partial u = 2\partial^* \partial u = 2\Delta_{\partial} u = \Delta u \quad (9)$$

and

$$2i\Lambda \bar{\partial} \partial v = \Delta v, \quad (10)$$

where $\Delta := d^* d + d d^*$. For example, for a function f on Euclidean n -space, $\Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

However, as $k_1 = h_1 e^{2u}$ ($k_2 = h_2 e^{2v}$),

$$|\phi|_{k_1}^2 = |\phi|_{h_1}^2 e^{2u} \quad (|\psi|_{k_2}^2 = |\psi|_{h_2}^2 e^{2v}). \quad (11)$$

The result follows from the identities (8)–(11).

Proof of Theorem 3. Let $u_0 \in C^\infty(S)$ ($v_0 \in C^\infty(S)$) be a solution to the equation

$$-\Delta u = i\Lambda F_{h_1} - \frac{2\pi \deg L_1}{\text{vol}(S)} \quad \left(-\Delta v = i\Lambda F_{h_2} - \frac{2\pi \deg L_2}{\text{vol}(S)} \right),$$

in accordance with the Hodge theorem [12, Section 0.6] and [13, Theorem 4.7]. Inserting $u = u_0 + \frac{U}{2}$ and $v = v_0 + \frac{V}{2}$ into (7), we obtain

$$\begin{cases} -\Delta U = \lambda (|\psi|_{h_2}^2 e^{2v_0}) e^V ((|\phi|_{h_1}^2 e^{2u_0}) e^U - 1) + \frac{4\pi \deg L_1}{\text{vol}(S)}, \\ -\Delta V = \lambda (|\phi|_{h_1}^2 e^{2u_0}) e^U ((|\psi|_{h_2}^2 e^{2v_0}) e^V - 1) + \frac{4\pi \deg L_2}{\text{vol}(S)}. \end{cases} \quad (12)$$

It is now sufficient to show that there exists a solution of (12) if λ satisfying (6) is sufficiently large.

Set

$$e^{U_0} = |\phi|_{h_1}^2 e^{2u_0} \quad \text{and} \quad e^{V_0} = |\psi|_{h_2}^2 e^{2v_0}.$$

Then $U_0 \in C^\infty(S \setminus \{s_1, \dots, s_{d'_1}\})$ ($V_0 \in C^\infty(S \setminus \{t_1, \dots, t_{d'_2}\})$), and for sufficiently small $\varepsilon > 0$,

$$U_0 \sim 2\ell_j \ln |x - s_j| \quad \text{in } B(s_j, \varepsilon) \quad (V_0 \sim 2m_j \ln |x - t_j| \text{ in } B(t_j, \varepsilon))$$

for $j = 1, \dots, d'_1$ ($j = 1, \dots, d'_2$), where $s_1, \dots, s_{d'_1}$ ($t_1, \dots, t_{d'_2}$) are the distinct zeros of ϕ (ψ), and each ℓ_j (m_j) is the multiplicity of s_j (t_j). We can now reduce (12) to

$$\begin{cases} -\Delta U = \lambda e^{V_0+V} (e^{U_0+U} - 1) + \frac{4\pi \deg L_1}{\text{vol}(S)}, \\ -\Delta V = \lambda e^{U_0+U} (e^{V_0+V} - 1) + \frac{4\pi \deg L_2}{\text{vol}(S)}. \end{cases} \quad (13)$$

To find a solution to (13), we follow the construction of Lin and Prajapat [4], which in turn is modeled on that of Caffarelli and Yang [8].

We choose $\varepsilon > 0$ sufficiently small so that the balls $B(s_j, 2\varepsilon)$ ($j = 1, \dots, d'_1$) and $B(t_j, 2\varepsilon)$ ($j = 1, \dots, d'_2$) are mutually disjoint. Let f_ε (g_ε) be a smooth function such that

$$\begin{cases} 0 \leq f_\varepsilon \leq 1 & (0 \leq g_\varepsilon \leq 1), \\ f_\varepsilon \equiv 1 & \text{in } B(s_j, \varepsilon), j = 1, \dots, d'_1 \quad (g_\varepsilon \equiv 1 \text{ in } B(t_j, \varepsilon), j = 1, \dots, d'_2), \\ f_\varepsilon \equiv 0 & \text{in } S \setminus B(s_j, 2\varepsilon), j = 1, \dots, d'_1 \quad (g_\varepsilon \equiv 0 \text{ in } S \setminus B(t_j, 2\varepsilon), j = 1, \dots, d'_2). \end{cases}$$

Then we have

$$C_1(\varepsilon) := \frac{1}{\text{vol}(S)} \int_S \frac{8\pi \deg L_1}{\text{vol}(S)} f_\varepsilon \leq \frac{8\pi (\deg L_1)^2}{\text{vol}(S)^2} \max_{1 \leq j \leq d'_1} \{\text{vol}(B(s_j, 2\varepsilon))\} \quad (14)$$

and

$$C_2(\varepsilon) := \frac{1}{\text{vol}(S)} \int_S \frac{8\pi \deg L_2}{\text{vol}(S)} g_\varepsilon \leq \frac{8\pi (\deg L_2)^2}{\text{vol}(S)^2} \max_{1 \leq j \leq d'_2} \{\text{vol}(B(t_j, 2\varepsilon))\}. \quad (15)$$

We define $\tilde{f}_\varepsilon = \frac{8\pi \deg L_1}{\text{vol}(S)} f_\varepsilon - C_1(\varepsilon)$ and $\tilde{g}_\varepsilon = \frac{8\pi \deg L_2}{\text{vol}(S)} g_\varepsilon - C_2(\varepsilon)$. Since $\int_S \tilde{f}_\varepsilon = 0$ and $\int_S \tilde{g}_\varepsilon = 0$, the equations

$$\Delta W = \tilde{f}_\varepsilon \quad (16)$$

and

$$\Delta Z = \tilde{g}_\varepsilon \quad (17)$$

have, by the Hodge theorem, a unique solution up to an additive constant. In addition, by (14) and (15) we have, for $\varepsilon > 0$ sufficiently small,

$$\tilde{f}_\varepsilon \geq \frac{4\pi \deg L_1}{\text{vol}(S)} \left(2 - \frac{2 \deg L_1}{\text{vol}(S)} \max_{1 \leq j \leq d'_1} \{\text{vol}(B(s_j, 2\varepsilon))\} \right) > \frac{4\pi \deg L_1}{\text{vol}(S)} \quad (18)$$

in $B(s_j, \varepsilon)$, $j = 1, \dots, d'_1$, and

$$\tilde{g}_\varepsilon \geq \frac{4\pi \deg L_2}{\text{vol}(S)} \left(2 - \frac{2 \deg L_2}{\text{vol}(S)} \max_{1 \leq j \leq d'_2} \{\text{vol}(B(t_j, 2\varepsilon))\} \right) > \frac{4\pi \deg L_2}{\text{vol}(S)} \quad (19)$$

in $B(t_j, \varepsilon)$, $j = 1, \dots, d'_2$. We fix ε so that the inequalities (18) and (19) are valid. Now we choose a solution W_0 (Z_0) of (16) [(17)] to fulfill

$$e^{U_0+W_0} < 1 \quad (e^{V_0+Z_0} < 1).$$

We then have, for any $\lambda > 0$, the inequalities

$$\begin{aligned}\Delta W_0 &= \tilde{f}_\varepsilon > \frac{4\pi \deg L_1}{\text{vol}(S)} \\ &\geq \lambda e^{V_0+Z_0} (e^{U_0+W_0} - 1) + \frac{4\pi \deg L_1}{\text{vol}(S)}\end{aligned}\quad (20)$$

in $B(s_j, \varepsilon)$, $j = 1, \dots, d'_1$, and

$$\begin{aligned}\Delta Z_0 &= \tilde{g}_\varepsilon > \frac{4\pi \deg L_2}{\text{vol}(S)} \\ &\geq \lambda e^{U_0+W_0} (e^{V_0+Z_0} - 1) + \frac{4\pi \deg L_2}{\text{vol}(S)}\end{aligned}\quad (21)$$

in $B(t_j, \varepsilon)$, $j = 1, \dots, d'_2$. Furthermore, we can choose $\lambda_0 > 0$ sufficiently large so that for all $\lambda > \lambda_0$, the inequalities (20) and (21) hold over all of \tilde{S} . Therefore, (W_0, Z_0) is a strict subsolution of (13) for all $\lambda > \lambda_0$.

We now define $(\tilde{U}_0, \tilde{V}_0) = (-U_0, -V_0)$ and consider the iteration scheme

$$\begin{cases} (-\Delta - K)\tilde{U}_n = \lambda e^{V_0+\tilde{V}_{n-1}} (e^{U_0+\tilde{U}_{n-1}} - 1) - K\tilde{U}_{n-1} + \frac{4\pi \deg L_1}{\text{vol}(S)}, \\ (-\Delta - K)\tilde{V}_n = \lambda e^{U_0+\tilde{U}_{n-1}} (e^{V_0+\tilde{V}_{n-1}} - 1) - K\tilde{V}_{n-1} + \frac{4\pi \deg L_2}{\text{vol}(S)} \end{cases}$$

for $n = 1, 2, 3, \dots$, where $K > 0$ is a constant. Then $\{\tilde{U}_n\}$ and $\{\tilde{V}_n\}$ are monotone sequences such that

$$\begin{cases} -U_0 > \tilde{U}_1 > \tilde{U}_2 > \dots > \tilde{U}_n > \dots > W_0, \\ -V_0 > \tilde{V}_1 > \tilde{V}_2 > \dots > \tilde{V}_n > \dots > Z_0 \end{cases}$$

and $(\tilde{U}_n, \tilde{V}_n) \rightarrow (U_\lambda, V_\lambda)$ in $H^1(S) \times H^1(S)$ as in Lin and Prajapat [4]. It follows that (U_λ, V_λ) is a solution to (13) for sufficiently large λ satisfying (6). By virtue of the standard elliptic regularity theory, this completes the proof of Theorem 3. \square

Remark. The functions $(U_\lambda$ and $V_\lambda)$ are maximal in the sense that if (U, V) is any other solution of (13), then $U < U_\lambda$ and $V < V_\lambda$.

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