Journal of Geometry and Physics 84 (2014) 30–42

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

W-algebras and the equivalence of bihamiltonian, Drinfeld–Sokolov and Dirac reductions

Yassir Ibrahim Dinar [*,1]

Department of Mathematics and Statistics, Faculty of Science, Sultan Qaboos University, Oman
The Abdus Salam International Centre for Theoretical Physics (ICTP), Italy

ARTICLE INFO

Article history:
Received 7 June 2011
Received in revised form 1 April 2014
Accepted 12 June 2014
Available online 19 June 2014

MSC:
primary 37K10
secondary 35D45

Keywords:
W-algebras
Bihamiltonian reduction
Drinfeld–Sokolov reduction
Dirac reduction
Slodowy slice
Transverse Poisson structure

ABSTRACT

We prove that the classical W-algebra associated to a nilpotent orbit in a simple Lie-algebra can be constructed by preforming bihamiltonian, Drinfeld–Sokolov or Dirac reductions. We conclude that the classical W-algebra depends only on the nilpotent orbit but not on the choice of a good grading or an isotropic subspace. In addition, using this result we prove again that the transverse Poisson structure to a nilpotent orbit is polynomial and we better clarify the relation between classical and finite W-algebras.

1. Introduction

A classical W-algebra is a local Poisson bracket on a loop space $\mathcal{L}(M)$ of a manifold M where in some local coordinates $(u^1, \dots, u^n)$, $u^1(x)$ is a Virasoro density and $u^i(x),\ i > 1$ are primary fields of conformal weights η_i [1], i.e. they satisfy the identities

$$\begin{aligned}\{u^1(x), u^1(y)\} &= \epsilon\delta'''(x-y) + 2u^1(x)\delta'(x-y) + u^1_x\delta(x-y),\\ \{u^1(x), u^i(y)\} &= (\eta_i+1)u^i(x)\delta'(x-y) + \eta_i u^i_x\delta(x-y).\end{aligned} \tag{1.1}$$

Classical W-algebras have a significant role in conformal field theory as their quantization give W-algebras, i.e. polynomial extensions of a particular central extension of the Lie algebra of vector fields on the circle [2]. They are also associated to integrable hierarchies of partial differential equations of KdV type [3]. However, we are interested in classical W-algebras because, possibly after a Poisson reduction, we can construct algebraic Frobenius manifolds from the dispersionless limit [4–6].

A wide literature is devoted to construct examples of classical W-algebras within the theory of integrable systems (see [2] for some details). One of the most general and uniform construction was obtained by Feher et al. in [7], where the authors introduced a generalization of Drinfeld–Sokolov reduction that can be performed for any nilpotent element in simple Lie

* Correspondence to: Department of Mathematics and Statistics, Faculty of Science, Sultan Qaboos University, Oman.
E-mail addresses: dinar@ictp.it, yassird@gmail.com.

1 On leave from: Faculty of Mathematical Sciences, University of Khartoum, Sudan.

http://dx.doi.org/10.1016/j.geomphys.2014.06.003
0393-0440/

algebras named *canonical Drinfeld–Sokolov reduction*. This reduction is performed on a standard Lie–Poisson bracket on loop algebra using Dynkin grading, Slodowy slice and a choice of a maximal isotropic subspace (more details are explained below). From the construction it is clear that nilpotent elements belonging to the same nilpotent orbit give equal classical W-algebras. By a nilpotent orbit we mean the conjugacy class of nilpotent elements under the adjoint group action. In [7], the authors also argued that canonical Drinfeld–Sokolov reduction is equivalent to Dirac reduction of the Lie–Poisson bracket on Slodowy slice.

Moreover, several attempts have been made to construct classical W-algebras by performing a bihamiltonian reduction on Lie–Poisson brackets using the theory of nilpotent orbits. This was obtained by Casati and Pedroni [8] to regular nilpotent orbits in simple Lie algebras via proving the equivalence between the bihamiltonian and standard Drinfeld–Sokolov reductions. We refer also to [9,2] for the construction of classical W-algebras associated to regular nilpotent orbits in Lie algebras of type A_n.

Furthermore, in [4] we obtained a generalization of the bihamiltonian reduction. This generalization enabled us to perform bihamiltonian reduction for any nilpotent orbit in simple Lie algebras. It makes use of the Dynkin grading and the minimal isotropic subspace. In the case of regular nilpotent orbits, our approach made possible to verify directly that the bihamiltonian reduction leads to classical W-algebras. For arbitrary nilpotent orbit we proved that the bihamiltonian reduction is equivalent to a generalization of Drinfeld–Sokolov reduction [4, Section 4]. Thus to show that the bihamiltonian reduction leads to classical W-algebras it is sufficient to prove the equivalence between different types of Drinfeld–Sokolov reductions.

Actually in this work we find further results. We prove that the bihamiltonian, Dirac and Drinfeld–Sokolov reductions are all equivalent. For a given nilpotent element, we prove that the associated classical W-algebra does not depend on the choice of a good grading or an isotropic subspace. As a consequence we prove again that the transverse Poisson structure to a nilpotent orbit is polynomial and we better clarify the relation between classical and finite W-algebras.

2. Poisson geometry and reductions

In this section we fix some notations and terminologies. We review our work in [4] and we add some minor results.

A Poisson manifold M is a manifold endowed with a Poisson bracket $\{.,.\}$, i.e. a bilinear skewsymmetric form on the space of smooth functions satisfying the Leibnitz rule and the Jacobi identity. Let M be a Poisson manifold with a Poisson bracket $\{.,.\}$. Then the corresponding Poisson tensor P is a linear map $P : T^*M \to TM$ defined by requiring that

$$\{F, G\} = \langle dF|P\, dG\rangle$$

for any smooth functions F and G on M. A smooth function F on M is called a Casimir function, if it satisfies

$$P(dF) = \{., F\} = 0.$$

A bihamiltonian manifold M is a manifold endowed with two Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$ such that

$$\{.,.\}_\lambda := \{.,.\}_2 + \lambda\{.,.\}_1$$

is a Poisson bracket for any constant λ. The Jacobi identity for $\{.,.\}_\lambda$ gives the following equation

$$\{\{F,G\}_1,H\}_2 + \{\{G,H\}_1,F\}_2 + \{\{H,F\}_1,G\}_2 + \{\{F,G\}_2,H\}_1 + \{\{G,H\}_2,F\}_1 + \{\{H,F\}_2,G\}_1 = 0 \tag{2.1}$$

for any smooth functions F, G and H on M. It follows from this equation that the set of all Casimir functions of $\{.,.\}_1$ are closed with respect to $\{.,.\}_2$.

Let M be a bihamiltonian manifold with Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$. Let P_1 and P_2 denote the corresponding Poisson tensors, respectively. We assume there is a set

$$\Xi = \{K_1, K_2, \dots, K_n\} \tag{2.2}$$

of independent Casimirs of $\{.,.\}_1$ which are closed with respect to $\{.,.\}_2$. For the standard bihamiltonian reduction [10] we assume Ξ to be a complete set of independent Casimirs of $\{.,.\}_1$. Let us fix a level set S of Ξ and let $i_s : S \to M$ be the canonical immersion. Then we consider the integrable distribution D on M generated by the Hamiltonian vector fields

$$X_{K_i} = P_2(dK_i), \quad i = 1, \dots, n. \tag{2.3}$$

Let E denote the distribution induced on S by D. We assume the foliation of E on S is regular, so that $N = S/E$ is a smooth manifold and $\pi : S \to N$ is a submersion. Then applying Marsden–Ratiu reduction theorem [11], we get the following result.

Proposition 2.1 ([4]). *The space N has a natural bihamiltonian structure $\{.,.\}_2^N$, $\{.,.\}_1^N$ defined as follows. For any functions f, g on N we have*

$$\{f,g\}_2^N \circ \pi = \{F,G\}_2 \circ i_s \tag{2.4}$$

$$\{f,g\}_1^N \circ \pi = \{F,G\}_1 \circ i_s,$$

where F and G are functions on M which extend f and g, respectively, and are constant on D.

2.1. Poisson tensor procedure

In this section we give a procedure to obtain the reduced bihamiltonian structure, it was introduced for the standard bihamiltonian reduction in [8]. We assume that there is a submanifold $Q \subset S$ transverse to E, i.e.

$$T_qS = E_q \oplus T_qQ, \quad \text{for all } q \in Q. \tag{2.5}$$

Then we have an isomorphism

$$\Psi : Q \to N$$

sending a point to the foliation of E containing that point. The composition $\Psi^{-1} \circ \pi$ is an inverse of the inclusion map $i_Q : Q \to S$. Hence, the bihamiltonian structure on N can be defined on Q as follows. For any functions f, g on Q we have

$$\begin{aligned}\{f, g\}_2^Q &= \{F, G\}_2 \\ \{f, g\}_1^Q &= \{F, G\}_1,\end{aligned} \tag{2.6}$$

where F, G are functions on M extending f, g and constant along D. Let P_λ^Q denote the Poisson tensor of $\{.,.\}_\lambda^Q := \{f, g\}_2^Q + \lambda\{f, g\}_1^Q$.

Lemma 2.2 *([4]). Let $q \in Q$ and $w \in T_q^*Q$. Then there exists $v \in T_q^*M$ such that:*
(1) v is an extension of w, i.e. $(v, \dot q) = (w, \dot q)$ for any $\dot q \in T_qQ$.
(2) $P_\lambda(v) \in T_qQ$.

Moreover, the Poisson tensor $P_\lambda^Q(w)$ is given by

$$P_\lambda^Q w = P_\lambda v \tag{2.7}$$

for any extension v satisfying conditions (1) and (2).

The previous lemma leads to a procedure to calculate the reduced Poisson bracket. We refer to it simply by *Poisson tensor procedure*.

2.2. Bihamiltonian and Dirac reductions

We show that under further hypothesis the bihamiltonian reduction is equivalent to Dirac reduction.

Corollary 2.3. *In the notations of Lemma 2.2, an extension v of w is unique if and only if P_λ^Q is the Dirac reduction of P_λ to Q.*

Proof. We apply Poisson tensor procedure. Let us choose local coordinates $(q^1, \dots, q^n)$ on M such that Q is defined by the equations $q^\alpha = 0$ for $\alpha = m+1, \dots, n$. We introduce three types of indices differing by their ranges to simplify the formulas below; capital letters $I, J, K, \dots = 1, \dots, n$, small letters $i, j, k, \dots = 1, \dots, m$ which label the coordinates on the submanifold Q and Greek letters $\alpha, \beta, \delta, \dots = m+1, \dots, n$. In these notations a covector $w \in T^*Q$ will have the form

$$w = a_i\, dq^i \tag{2.8}$$

and an extension of this covector to $v \in T^*M$ satisfying Lemma 2.2 is given by

$$v = a_I\, dq^I, \tag{2.9}$$

where the coefficients a_α's are obtained from requiring that

$$P_\lambda(v) = P_\lambda{}^{IJ} a_J \frac{\partial}{\partial q^I} \in TQ. \tag{2.10}$$

This means that the coefficients of $\frac{\partial}{\partial q^\beta}$ equal 0 and we get a system of linear equations

$$-P_\lambda{}^{\alpha i} a_i = P_\lambda{}^{\alpha\beta} a_\beta. \tag{2.11}$$

Then the uniqueness of the extension v is equivalent to the fact that the minor matrix $P_\lambda{}^{\alpha\beta}$ is invertible. Let $(P_\lambda)_{\alpha\beta}$ denote its inverse, then

$$a_\beta = -(P_\lambda)_{\beta\alpha} P_\lambda{}^{\alpha i} a_i. \tag{2.12}$$

Substituting this into the formula of $P_\lambda(v)$, we get

$$P_\lambda(v) = \left(P_\lambda{}^{ij} a_j + P_\lambda{}^{i\beta} a_\beta\right)\frac{\partial}{\partial q^i} = \left(P_\lambda{}^{ij} - P_\lambda{}^{i\beta}(P_\lambda)_{\beta\alpha}P_\lambda{}^{\alpha j}\right)a_j \frac{\partial}{\partial q^i}. \tag{2.13}$$

Using the identity $P_\lambda^Q(w) = P_\lambda(v)$, we end with Dirac formula for the reduced Poisson tensor

$$({P_\lambda}^Q)^{ij} = {P_\lambda}^{ij} - {P_\lambda}^{i\beta}(P_\lambda)_{\beta\alpha}{P_\lambda}^{\alpha j} \tag{2.14}$$

meaning that P_λ^Q is the Dirac reduction of P_λ to Q. □

We observe that the bihamiltonian reduction guarantees that P_λ^Q is linear in λ and hence we have a bihamiltonian structure on Q. This fact is not obvious when we use Dirac reduction because Dirac formula used to evaluate the reduced Poisson tensor depends on the inverse of a matrix.

2.3. Local Poisson brackets and Dirac reduction

Let M be a manifold. The loop space $\mathfrak{L}(M)$ of M is the space of smooth maps from the circle to M. A local Poisson bracket $\{.,.\}$ on $\mathfrak{L}(M)$ is a Poisson bracket on the space of local functional on $\mathfrak{L}(M)$. If we choose local coordinates $(u^1, \ldots, u^n)$, then $\{.,.\}$ is a finite summation of the form

$$\{u^i(x), u^j(y)\} = \sum_{k=-1}^{\infty} \{u^i(x), u^j(y)\}^{[k]}, \tag{2.15}$$

$$\{u^i(x), u^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A^{i,j}_{k,s}(u(x))\delta^{(k-s+1)}(x-y),$$

where $A^{i,j}_{k,s}(u(x))$ are homogeneous polynomials in $\partial_x^m u^r(x)$ of degree s when we assign $\partial_x^m u^r(x)$ degree m and $\delta(x-y)$ is the Dirac delta function defined by

$$\int_{S^1} f(y)\delta(x-y)dy = f(x).$$

In particular, the first terms can be written as follows:

$$\{u^i(x), u^j(y)\}^{[-1]} = F^{ij}(u(x))\delta(x-y) \tag{2.16}$$

$$\{u^i(x), u^j(y)\}^{[0]} = F_0^{ij}(u(x))\delta'(x-y) + \Gamma_k^{ij}(u(x))u_x^k\delta(x-y)$$

where $F_0^{ij}(u)$, $F^{ij}(u)$ and $\Gamma_k^{ij}(u)$ are smooth functions on M. It follows from the definition that $F^{ij}(u)$ defines a Poisson bracket on M.

Assume we have a local Poisson bracket on the loop space $\mathfrak{L}(M)$ of a manifold M. Let $N \subset M$ be a submanifold of dimension m. Then under some assumptions the Poisson bracket can be reduced to N using Dirac reduction. For this end we assume N is defined by the equations $u^\alpha = 0$ for $\alpha = m+1, \ldots, n$. We introduce three types of indexes; capital letters $I, J, K, \ldots = 1, \ldots, n$, small letters $i, j, k, \ldots = 1, \ldots, m$ which parameterize the submanifold N and Greek letters $\alpha, \beta, \gamma, \delta, \ldots = m+1, \ldots, n$.

We write the Poisson bracket on $\mathfrak{L}(M)$ in the form

$$\{u^I(x), u^J(y)\} = \mathbb{F}^{IJ}(u(x))\delta(x-y)$$

where $\mathbb{F}^{IJ}(u)$ is a matrix differential operator

$$\mathbb{F}^{IJ}(u) = \sum_{k\geq -1}\sum_{s=0}^{k+1} A^{I,J}_{k,s}(u(x))\frac{d^{k-s+1}}{dx^{k-s+1}}. \tag{2.17}$$

Proposition 2.4. *Assume the minor matrix $\mathbb{F}^{\alpha\beta}(u)$ restricted to $\mathfrak{L}(N)$ has an inverse $\mathbb{S}_{\alpha\beta}(u)$ which is a matrix differential operator of finite order, i.e. a finite sum*

$$\mathbb{S}_{\alpha\beta}(u) = \sum_{k=-1}^{\infty}\sum_{s=0}^{k+1} B^{\alpha,\beta}_{k,s}(u(x))\frac{d^{k-s+1}}{dx^{k-s+1}}. \tag{2.18}$$

Then Dirac reduction of $\{.,.\}$ to $\mathfrak{L}(N)$ is well defined and gives a local Poisson structure. The reduced Poisson structure is given by

$$\{u^i(x), u^j(y)\}^N = \widetilde{\mathbb{F}}^{ij}(u)\delta(x-y)$$

where

$$\widetilde{\mathbb{F}}^{ij}(u) = \mathbb{F}^{ij}(u) - \mathbb{F}^{i\alpha}(u)\mathbb{S}_{\alpha\beta}(u)\mathbb{F}^{\beta j}(u). \tag{2.19}$$

Proof. Let $\mathcal{F}$ be a Hamiltonian functional on $\mathfrak{L}(M)$. Then the Hamiltonian flows have the equations

$$u_t^I = \mathbb{F}^{IJ}\frac{\delta\mathcal{F}}{\delta u^J} \tag{2.20}$$

where $\frac{\delta\mathcal{F}}{\delta u^J}$ is the variational derivative of $\mathcal{F}$ with respect to $u^J(x)$. Following the spirit of [12] (see also [13, Chapter 3]), the Dirac procedure for the reduction of (2.20) to $\mathfrak{L}(N)$ has the form

$$\begin{aligned} u_t^i &= \mathbb{F}^{iJ}\frac{\delta\mathcal{F}}{\delta u^J} + \int \{u^i(x), u^\beta(y)\} C_\beta(y)dy \\ &= \mathbb{F}^{ij}\frac{\delta\mathcal{F}}{\delta u^j} + \mathbb{F}^{i\beta}\left(\frac{\delta\mathcal{F}}{\delta u^\beta} + C_\beta(x)\right), \end{aligned} \tag{2.21}$$

where the Lagrange multiplier $C_\beta(y)$ is found from the system of linear equations

$$\begin{aligned} 0 = u_t^\alpha &= \mathbb{F}^{\alpha J}\frac{\delta\mathcal{F}}{\delta u^J} + \int \{u^\alpha(x), u^\beta(y)\} C_\beta(y)dy \\ &= \mathbb{F}^{\alpha j}\frac{\delta\mathcal{F}}{\delta u^j} + \mathbb{F}^{\alpha\beta}\left(\frac{\delta\mathcal{F}}{\delta u^\beta} + C_\beta(x)\right). \end{aligned} \tag{2.22}$$

Applying the inverse operator $\mathbb{S}_{\alpha\beta}$, we get

$$\frac{\delta\mathcal{F}}{\delta u^\beta} + C_\beta(x) = -\mathbb{S}_{\beta\alpha}\mathbb{F}^{\alpha j}\frac{\delta\mathcal{F}}{\delta u^j}. \tag{2.23}$$

Substituting in (2.21),

$$u_t^i = (\mathbb{F}^{ij} - \mathbb{F}^{i\beta}\mathbb{S}_{\beta\alpha}\mathbb{F}^{\alpha j})\frac{\delta\mathcal{F}}{\delta u^j}. \tag{2.24}$$

Hence, the operator $\widetilde{\mathbb{F}}^{ij} = \mathbb{F}^{ij} - \mathbb{F}^{i\beta}\mathbb{S}_{\beta\alpha}\mathbb{F}^{\alpha j}$ defines the Poisson bracket of the Dirac reduction of $\{.,.\}$ to $\mathfrak{L}(N)$. □

We show the existence of the inverse operator $\mathbb{S}_{\beta\alpha}$ under certain condition.

Proposition 2.5 (*[4]*). *In the notations of Eq. (2.16), if the minor matrix $F^{\alpha\beta}$ is nondegenerate on N, then the operator $\mathbb{F}^{IJ}$ has an inverse. Moreover, if $F_{\alpha\beta}$ is the inverse matrix of $F^{\alpha\beta}$ and we write the leading terms of the reduced Poisson bracket on $\mathfrak{L}(N)$ in the form*

$$\{u^i(x), u^j(y)\}_N^{[-1]} = \widetilde{F}^{ij}(u(x))\delta(x-y), \tag{2.25}$$

$$\{u^i(x), u^j(y)\}_N^{[0]} = \widetilde{F}_0^{ij}(u(x))\delta'(x-y) + \widetilde{\Gamma}_k^{ij}(u(x))u_x^k\delta(x-y) \tag{2.26}$$

then

$$\widetilde{F}^{ij} = F^{ij} - F^{i\beta}F_{\beta\alpha}F^{\alpha j}, \tag{2.27}$$

$$\widetilde{F}_0^{ij} = F_0^{ij} - F_0^{i\beta}F_{\beta\alpha}F^{\alpha j} + F^{i\beta}F_{\beta\alpha}F_0^{\alpha\varphi}F_{\varphi\gamma}F^{\gamma j} - F^{i\beta}F_{\beta\alpha}F_0^{\alpha j} \tag{2.28}$$

and

$$\widetilde{\Gamma}_k^{ij}u_x^k = \left(\Gamma_k^{ij} - \Gamma_k^{i\beta}F_{\beta\alpha}F^{\alpha j} + F^{i\lambda}F_{\lambda\alpha}\Gamma_k^{\alpha\beta}F_{\beta\varphi}F^{\varphi j} - F^{i\beta}F_{\beta\alpha}\Gamma_k^{\alpha j}\right)u_x^k - \left(F_0^{i\beta} - F^{i\lambda}F_{\lambda\alpha}F_0^{\alpha\beta}\right)\partial_x(F_{\beta\varphi}F^{\varphi j}). \tag{2.29}$$

The other terms of the reduced Poisson structure can be found by solving certain recursive equations.

Corollary 2.6. *The Poisson bracket defined on N via the matrix $\widetilde{F}^{ij}(u)$ equals the Dirac reduction of the Poisson bracket defined on M via the matrix $F^{ij}(u)$.*

3. Constructing classical *W*-algebra

We review some facts about nilpotent elements in simple Lie algebras. A good reference for the material in this section is the book by Collingwood and McGovern [14].

We fix a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a nilpotent element $f \in \mathfrak{g}$. A good grading for f is a $\mathbb{Z}$-grading

$$\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i; \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \tag{3.1}$$

where

(1) $f \in \mathfrak{g}_{-2}$, and
(2) The map

$$\mathrm{ad} f : \mathfrak{g}_j \to \mathfrak{g}_{j-2}; \qquad a \mapsto \mathrm{ad}f(a) = [f, a]$$

is injective for $j \geq 1$ and surjective for $j \leq 1$.

All good gradings for nilpotent elements are classified in [15].

We fix a good grading Γ for f. Then we choose, by using Jacobson–Morozov theorem, a semisimple element h and a nilpotent element $e \in \mathfrak{g}$ such that the set $\mathcal{A} = \{e, h, f\}$ forms an sl_2-triple, i.e.

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{3.2}$$

We can assume without loss of generality that $\mathcal{A}$ is compatible with Γ in the sense that $h \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_2$ [15].

We observe that if Γ' denotes the grading on $\mathfrak{g}$ defined by

$$\widehat{\mathfrak{g}_i} := \{x \in \mathfrak{g} : \operatorname{ad} h(x) = ix\},$$

then it follows from representation theory of sl_2 algebras that Γ' is a good grading for f. Such a good grading obtained from a sl_2-triple is called Dynkin grading. We can map this grading canonically to a weighted Dynkin diagram of $\mathfrak{g}$. It is known that two nilpotent elements are conjugate under the adjoint group action if and only if they have the same weighted Dynkin diagram. Hence, we conclude that the construction of classical W-algebras, by the methods we will introduce in next sections, depends only on the nilpotent orbit of f.

Let $\langle .|. \rangle$ denote the Killing form on $\mathfrak{g}$. Then there is a natural symplectic bilinear form on $\mathfrak{g}_{-1}$ defined by

$$(.,.) : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C}, \qquad (x, y) \mapsto \langle e|[x, y]\rangle. \tag{3.3}$$

We use this symplectic structure to fix an isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{-1}$. Let $\mathfrak{l}'$ denote the symplectic complement of $\mathfrak{l}$ and introduce the following nilpotent subalgebras

$$\mathfrak{m} := \mathfrak{l} \oplus \bigoplus_{i \le -2} \mathfrak{g}_i; \qquad \mathfrak{n} := \mathfrak{l}' \oplus \bigoplus_{i \le -2} \mathfrak{g}_i. \tag{3.4}$$

Let $\mathfrak{g}_f$ denote the subspace $\ker(\operatorname{ad} f)$ and $\mathfrak{b}$ denote the orthogonal complement of $\mathfrak{n}$ under $\langle .|. \rangle$. Then from the properties of the good grading we get [16]

$$\dim \mathfrak{g}_f = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{-1} \quad \text{and} \quad \mathfrak{g}_f \subset \oplus_{i \le 0} \mathfrak{g}_i \subset \mathfrak{b}. \tag{3.5}$$

Lemma 3.1. *The space $\mathfrak{b}$ has the following form*

$$\mathfrak{b} = [\mathfrak{m}, e] \oplus \mathfrak{g}_f. \tag{3.6}$$

Proof. We get from the properties of good grading that

$$0 = \langle [\mathfrak{m}, \mathfrak{n}]|e\rangle = -\langle \mathfrak{n}|[\mathfrak{m}, e]\rangle$$

which implies that $[\mathfrak{m}, e] \subset \mathfrak{b}$. We observe that the properties of $\operatorname{ad} f$ has its counterpart on $\operatorname{ad} e$. In particular $\operatorname{ad} e : \mathfrak{g}_i \to \mathfrak{g}_{i+2}$ is injective for $i < 0$. Hence, $\dim[\mathfrak{m}, e] = \dim \mathfrak{m}$. Also, from representation theory of sl_2-triples we get

$$[\mathfrak{m}, e] \cap \mathfrak{g}_f = 0. \tag{3.7}$$

Computing the dimension of $\mathfrak{b}$ we find that

$$\begin{aligned} \dim \mathfrak{b} &= \dim \oplus_{i \le 0} \mathfrak{g}_i + \dim \mathfrak{g}_1 - \dim \mathfrak{l}' = \dim \oplus_{i \le 0} \mathfrak{g}_i + \dim \mathfrak{l} \\ &= \dim \mathfrak{m} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{-1} = \dim[\mathfrak{m}, e] + \dim \mathfrak{g}_f. \end{aligned} \tag{3.8}$$

Hence, from (3.7) and (3.8) we get the direct sum (3.6). □

3.1. Standard Lie–Poisson structures on loop algebra

We define a bihamiltonian structure on the loop algebra $\mathfrak{L}(\mathfrak{g})$ as follows. We extend the Killing form on $\mathfrak{g}$ to $\mathfrak{L}(\mathfrak{g})$ by setting

$$(u|v) = \int_{S^1} \langle u(x)|v(x)\rangle dx, \quad u, v \in \mathfrak{L}(M). \tag{3.9}$$

We use $(.|.)$ to identify $\mathfrak{L}(\mathfrak{g})$ with $\mathfrak{L}(\mathfrak{g})^*$. We define the gradient $\delta\mathcal{F}(q)$ for a functional $\mathcal{F}$ on $\mathfrak{L}(\mathfrak{g})$ to be the unique element in $\mathfrak{L}(\mathfrak{g})$ satisfying

$$\frac{d}{d\theta}\mathcal{F}(q + \theta\dot{s})\,|_{\theta=0} = \int_{S^1} \langle \delta\mathcal{F}|\dot{s}\rangle dx \quad \text{for all } \dot{s} \in \mathfrak{L}(\mathfrak{g}). \tag{3.10}$$

Then we choose an element $a \in \mathfrak{g}$ which centralizes the subalgebra $\mathfrak{n}$, i.e.

$$\mathfrak{n} \subset \ker \operatorname{ad} a. \tag{3.11}$$

Such an element always exists. For example, we can take a to be a homogeneous element of the minimal grading. Finally, we introduce a bihamiltonian structure $\{.,.\}_2$ and $\{.,.\}_1$ on $\mathfrak{L}(\mathfrak{g})$, respectively, by means of Poisson tensors

$$\begin{aligned} &P_2(q(x))(v) = [\partial_x + q(x), v(x)] \\ &P_1(q(x))(v) = [a, v(x)], \end{aligned} \tag{3.12}$$

for every $q \in \mathfrak{L}(\mathfrak{g})$ and $v \in T_q^*\mathfrak{L}(\mathfrak{g}) \cong \mathfrak{L}(\mathfrak{g})$. It is a well known fact that they define a bihamiltonian structure on $\mathfrak{L}(\mathfrak{g})$ [17].

We mention that $\{.,.\}_2$ can be interpreted as the restriction to $\mathfrak{L}(\mathfrak{g})$ of the Lie–Poisson bracket on the untwisted affine Kac–Moody algebra associated to $\mathfrak{g}$. In particular, the leading term $\{.,.\}_2^{[-1]}$ defines the Lie–Poisson bracket on $\mathfrak{g}$.

3.2. Generalized Drinfeld–Sokolov reduction

We introduce a generalization of Drinfeld–Sokolov reduction by applying Marsden–Weinstein reduction theorem [11].

Let us define a gauge action of the adjoint group $\mathcal{N}$ of $\mathfrak{L}(\mathfrak{n})$ by

$$q(x) \to \exp(-\mathrm{ad}s(x))[\partial_x + q(x)] - \partial_x \tag{3.13}$$

where $s(x) \in \mathfrak{L}(\mathfrak{n})$ and $q(x) \in \mathfrak{L}(\mathfrak{g})$.

Proposition 3.2 (*[4]*)**.** *The action of $\mathcal{N}$ on $\mathfrak{L}(\mathfrak{g})$ under the Poisson tensor*

$$P_\lambda := P_2 + \lambda P_1$$

is Hamiltonian for all λ. It admits a momentum map J to be the projection

$$J : \mathfrak{L}(\mathfrak{g}) \to \mathfrak{L}(\mathfrak{n}^+),$$

where $\mathfrak{n}^+$ is the embedding of n^* in $\mathfrak{g}$ under the Killing form. Moreover, J is Ad^*-equivariant.

We choose e as a regular value of J. Since $\mathfrak{b}$ is the orthogonal complement to $\mathfrak{n}$, the level set $J^{-1}(e)$ is given by

$$S := \mathfrak{L}(\mathfrak{b}) + e. \tag{3.14}$$

Proposition 3.3. *The isotropy group of e is the adjoint group $\mathcal{M}$ of $\mathfrak{L}(\mathfrak{m})$.*

Proof. The isotropy group of e is the subgroup of $\mathcal{N}$ generated by the set

$$G_e = \{s \in \mathfrak{L}(\mathfrak{n}) : (\exp(\mathrm{ad}\, s)n, e) = (n, e),\ \forall\, n \in \mathfrak{L}(\mathfrak{n})\}.$$

Let $s \in G_e$. Then from the grading properties we have

$$(\exp(\mathrm{ad}\, s)n, e) = (n, e), \quad \forall\, n \in \mathfrak{L}(\mathfrak{n}) \Leftrightarrow ([s, e], \mathfrak{L}(\mathfrak{n})) = 0.$$

The last equality is satisfied if and only if the projection s_l of s to $\mathfrak{L}(\mathfrak{l}')$ satisfies $([s_l, e], \mathfrak{L}(\mathfrak{l}')) = 0$. From the definition this means that $s_l \in \mathfrak{L}(\mathfrak{l})$ and therefore $G_e = \mathfrak{L}(\mathfrak{m})$. $\square$

Proposition 3.3 implies, using Marsden–Weinstein reduction theorem [11], that the space $S/\mathcal{M}$ is a manifold and inherits a Poisson tensor P'_λ from P_λ.

3.3. Generalized bihamiltonian reduction

We perform a bihamiltonian reduction by considering the set Ξ of Casimirs of $\{.,.\}_1$ whose gradient belongs to $\mathfrak{L}(\mathfrak{n})$. For example, for any element $b \in \mathfrak{n}$ we have that

$$\mathcal{F}_b(q(x)) := (b|q(x))$$

belongs to Ξ. Since $\mathfrak{n}$ is a Lie subalgebra, it is easy to verify that Ξ is closed under $\{.,.\}_2$. We take as a level surface the affine subspace

$$S := \mathfrak{L}(\mathfrak{b}) + e. \tag{3.15}$$

Then the distribution D equals $P_2(\mathfrak{L}(\mathfrak{n}))$. Let E be the restriction of D to S, i.e. $E = P_2(\mathfrak{L}(\mathfrak{n})) \cap \mathfrak{L}(\mathfrak{b})$.

Proposition 3.4. *The distribution E is given by*

$$E = P_2(\mathfrak{L}(\mathfrak{m})). \tag{3.16}$$

Moreover, the foliation of E on S is given by the orbits of the adjoint group $\mathcal{M}$ of $\mathfrak{L}(\mathfrak{m})$ acting on S by

$$s(x) + e \to \exp(-\mathrm{ad}\, m(x))[\partial_x + s(x) + e] - \partial_x, \tag{3.17}$$

where $m(x) \in \mathfrak{L}(\mathfrak{m})$ and $s(x) \in \mathfrak{L}(\mathfrak{b})$.

Proof. By definition, E consists of all elements $v \in \mathfrak{L}(\mathfrak{n})$ such that

$$\langle v_x + [q, v] + [e, v]|w\rangle = 0 \tag{3.18}$$

for every $q \in \mathfrak{L}(\mathfrak{b})$ and $w \in \mathfrak{L}(\mathfrak{n})$. We note that this equation is satisfied if $v \in \mathfrak{L}(\oplus_{i\leq -2}\, \mathfrak{g}_i)$. Hence, it is sufficient to assume that $v \in \mathfrak{L}(\mathfrak{l}')$. But then v satisfies the above equation iff

$$\langle [e, v]|\mathfrak{L}(\mathfrak{l}')\rangle = 0.$$

This implies that v belongs to the symplectic complement $\mathfrak{L}(\mathfrak{l})$ of $\mathfrak{L}(\mathfrak{l}')$. Thus

$$E = P_2(\mathfrak{l}) \oplus P_2\left(\bigoplus_{i\leq -2} \mathfrak{g}_i\right) = P_2(\mathfrak{L}(\mathfrak{m})). \tag{3.19}$$

In Proposition 3.6, we prove that the action (3.17) is free, which implies that E is its infinitesimal generator. □

From this proposition it follows that the space $N = S/\mathcal{M}$ is well defined as it is the orbit space of the action (3.17). Hence, we get a bihamiltonian structure P_1^N and P_2^N on N from P_1 and P_2, respectively. At this point we already proved the equivalence between Drinfeld–Sokolov and bihamiltonian reductions.

Theorem 3.5. *The generalized Drinfeld–Sokolov reduction coincides with the generalized bihamiltonian reduction.*

Proof. This follows directly from Propositions 3.3 and 3.4 as in both reductions the reduced space is $N = S/\mathcal{M}$, where $S = \mathfrak{L}(\mathfrak{b}) + e$ and $\mathcal{M}$ is the adjoint group of $\mathfrak{L}(\mathfrak{m})$. □

Following the work [18,19], we study the manifold N by introducing a transverse subspace to the orbits in S. Slodowy slice is a natural choice of such transverse subspace since it is coherent with the theory of nilpotent elements. It is defined as the affine loop subspace

$$Q := e + \mathfrak{L}(\mathfrak{g}_f) \subset S. \tag{3.20}$$

Proposition 3.6. *The manifold Q is transverse to E on S. Hence, for any element $s(x) + e \in S$ there is a unique element $m(x) \in \mathfrak{L}(\mathfrak{m})$ such that*

$$q(x) + e = \exp(-\mathrm{ad}\, m(x))[\partial_x + s(x) + e] - \partial_x \tag{3.21}$$

belongs to Q. The entries of $q(x)$ give a system of generators for the ring R of differential polynomials on S invariant under the action (3.17).

Proof. We must prove that for any $q \in \mathfrak{L}(\mathfrak{g}_f)$ and $\dot{s} \in \mathfrak{L}(\mathfrak{b})$ there are a unique $v \in \mathfrak{L}(\mathfrak{m})$ and a unique $\dot{w} \in \mathfrak{L}(\mathfrak{g}_f)$ such that

$$\dot{s} = P_2(e + q)(v) + \dot{w}. \tag{3.22}$$

We write this equation using the good grading of $\mathfrak{g}$. For $t \in \mathfrak{L}(\mathfrak{g})$, let t_i denote its projection to $\mathfrak{L}(\mathfrak{g}_i)$. Then we can rewrite (3.22) as

$$[e, v_{i-2}] + \dot{w}_i = \dot{s}_i - v_i' - \sum_k [q_k, v_{i-k}]. \tag{3.23}$$

This gives a linear system of equations which can be solved recursively because the map ad e is injective for $i < 0$ and we have

$$\mathfrak{L}(\mathfrak{g}_f) \oplus [e, \mathfrak{L}(\mathfrak{m})] = \mathfrak{L}(\mathfrak{b}) \tag{3.24}$$

from Lemma 3.1. The second part of the proposition can be proved similarly. □

Now we explain what we call *Drinfeld–Sokolov method* for calculating the reduced bihamiltonian structure. We write the coordinates of Q as differential polynomials in the coordinates of S by means of Eq. (3.21) and then apply the Leibnitz rule. If $s^i(x)$ denote the coordinates on S, then the Leibnitz rule for $u, v \in R$ have the following form

$$\{u(x), v(y)\}_\lambda = \frac{\partial u(x)}{\partial(\partial^m s^i)} \partial_x^m \left(\frac{\partial v(y)}{\partial(\partial^n s^j)} \partial_y^n \left(\{s^i(x), s^j(y)\}_\lambda\right)\right). \tag{3.25}$$

3.3.1. Fractional KdV

We demonstrate Drinfeld–Sokolov method when $\mathfrak{g}$ is the Lie algebra sl_3 and f is a minimal nilpotent element. We explain the different choices of good gradings, isotropic subspaces and first Poisson brackets. To this end, let us denote $e_{i,j}$ the fundamental 3×3 matrix, i.e. $(e_{i,j})_{s,t} := \delta_{i,s}\delta_{j,t}$. We consider the sl_2-triple $\mathcal{A} = \{e, h, f\}$, where $e = e_{1,3}$, $h = e_{1,1} - e_{3,3}$ and $f = e_{3,1}$. There are three good gradings compatible with $\mathcal{A}$. The following matrices summarize the degrees assigned to $e_{i,j}$ by these gradings. The grading Γ_1 is Dynkin grading.

$$\Gamma_1 := \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}, \qquad \Gamma_2 := \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}, \qquad \Gamma_3 := \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}. \tag{3.26}$$

Let us list some possible choices for the element a which can be used to define the first Poisson tensor P_1 on $\mathfrak{L}(\mathfrak{g})$ (3.12). First, we can take $a = e_{3,1}$ since it has the minimal degree in all good gradings. We can also choose $a = e_{3,2}$ (resp. $a = e_{2,1}$) since it has the minimal degree in the grading Γ_2 (resp. Γ_3). Moreover, we can set $a = e_{2,1} + e_{3,2}$ (resp. $a = e_{2,1} - e_{3,2}$) when we consider the grading Γ_1 and fix the isotropic subspace $\mathfrak{l} = \mathbb{C}(e_{2,1} + e_{3,2})$ (resp. $\mathfrak{l} = \mathbb{C}(e_{2,1} - e_{3,2})$).

Under any choice of a good grading or isotropic subspace, the transverse subspace Q is the same. We fix for Q the following coordinates. Here we use lower indices for convenience.

$$q(x) = \begin{pmatrix} q_4(x) & 0 & 1 \\ q_3(x) & -2q_4(x) & 0 \\ q_1(x) & q_2(x) & q_4(x) \end{pmatrix}. \tag{3.27}$$

Let us consider the grading Γ_1. We fix the isotropic subspace $\mathbb{C}(e_{2,1}+e_{3,2})$ and define P_1 by taking $a = e_{2,1}+e_{3,2}$. Then the subspace S takes the form

$$s(x) = \begin{pmatrix} s_4(x)+s_5(x) & s_6(x) & 1 \\ s_3(x) & -2s_4(x) & -s_6(x) \\ s_1(x) & s_2(x) & s_4(x)-s_5(x) \end{pmatrix}. \tag{3.28}$$

Eq. (3.21) leads to the following system of generators for the invariant ring R

$$\begin{aligned} q_1(x) &= s_1(x) - \frac{3}{4}s_6^4(x) + 3s_4(x)s_6^2(x) - s_2(x)s_6(x) + s_3(x)s_6(x) + s_5^2(x) - s_5'(x); \\ q_2(x) &= s_2(x) + s_6(x)^3 - 3s_4(x)s_6(x) + s_5(x)s_6(x) - s_6'(x); \\ q_3(x) &= s_3(x) - s_6^3(x) + 3s_4(x)s_6(x) + s_5(x)s_6(x) - s_6'(x); \\ q_4(x) &= s_4(x) - \frac{1}{2}s_6^2(x). \end{aligned}$$

Calculating the reduced Poisson brackets by using the Leibnitz rule (3.25), the nonzero brackets of $\{.,.\}_1^Q$ are

$$\begin{aligned} \{q_1(x), q_2(y)\}_1^Q &= \frac{3}{2}\,\delta'(x-y) - 3\,q_4(x)\delta(x-y); \\ \{q_1(x), q_3(y)\}_1^Q &= \frac{3}{2}\,\delta'(x-y) + 3q_4(x)\,\delta(x-y); \\ \{q_2(x), q_4(y)\}_1^Q &= -\frac{1}{2}\,\delta(x-y); \\ \{q_3(x), q_4(y)\}_1^Q &= \frac{1}{2}\,\delta(x-y), \end{aligned} \tag{3.29}$$

while the nonzero ones of $\{.,.\}_2^Q$ are

$$\begin{aligned} \{q_1(x), q_1(y)\}_2^Q &= -\frac{1}{2}\delta'''(x-y) + 2\,q_1(x)\,\delta'(x-y) + \partial_x q_1 \delta(x-y); \\ \{q_1(x), q_2(y)\}_2^Q &= \frac{3}{2}\,q_2(x)\,\delta'(x-y) + \frac{1}{2}\,\left(-6\,q_2(x)\,q_4(x) + q_2'(x)\right)\delta(x-y); \\ \{q_1(x), q_3(y)\}_2^Q &= \frac{3}{2}\,q_3(x)\,\delta'(x-y) + \frac{1}{2}\,\left(6\,q_3(x)\,q_4(x) + q_3'(x)\right)\delta(x-y); \\ \{q_2(x), q_3(y)\}_2^Q &= -\delta''(x-y) + \left(q_1(x) - 9q_4(x)^2 - 3q_4'(x)\right)\delta(x-y) - 6\,q_4(x)\delta'(x-y); \\ \{q_2(x), q_4(y)\}_2^Q &= -\frac{1}{2}q_2(x)\delta(x-y); \\ \{q_3(x), q_4(y)\}_2^Q &= \frac{1}{2}\,q_3(x)\delta(x-y); \\ \{q_4(x), q_4(y)\}_2^Q &= \frac{1}{6}\delta'(x-y). \end{aligned} \tag{3.30}$$

If we consider the grading Γ_3 and we define P_1 by taking $a = e_{2,1}$, then the space S will take the form

$$s(x) = \begin{pmatrix} s_4(x)+s_5(x) & 0 & 1 \\ s_3(x) & -2s_4(x) & s_6(x) \\ s_1(x) & s_2(x) & s_4(x)-s_5(x) \end{pmatrix} \tag{3.31}$$

and the system of generators will change to

$$\begin{aligned} q_1(x) &= s_1(x) + s_5(x)^2 + s_2(x)s_6(x) - s_5'(x); \\ q_2(x) &= s_2(x); \\ q_3(x) &= s_3(x) - 3s_4(x)s_6(x) - s_5(x)s_6(x) + s_6'(x); \\ q_4(x) &= s_4(x). \end{aligned}$$

Calculating $\{.,.\}_2^Q$ using this system of generators, we get again the brackets (3.30). This suggests that the reduced second Poisson bracket is independent of the choice of good grading and isotropic subspace. We prove this result in the next section.

We mention here that the Poisson bracket (3.30) is known in the literature as *fractional KdV algebra* and the Poisson bracket (3.29) is used in [3,20] to construct an integrable hierarchy.

3.4. Poisson tensor procedure and Dirac reduction

Let us apply Poisson tensor procedure to construct P_λ^Q.

Proposition 3.7. *Let $z \in Q$ and $w \in T_z^*Q$. Then an extension $v \in T_z^*\mathfrak{L}(\mathfrak{g})$ of w satisfying the hypothesis of Lemma 2.2 is unique. The reduced Poisson tensor in this case is given by*

$$P_\lambda^Q(w) = P_\lambda(v). \tag{3.32}$$

Proof. We identify $T_z^*Q \simeq \mathfrak{L}(\mathfrak{g}_f)^*$ with $\mathfrak{L}(\mathfrak{g}_e)$ using the Killing form. Let $w \in T_z^*Q$. Then a vector $v \in \mathfrak{L}(\mathfrak{g})$ extends w if $(w, s) = (v, s)$ for all $s \in \mathfrak{L}(g_f)$. Using the direct sum $\mathfrak{g} = [\mathfrak{g}, f] \oplus \mathfrak{g}_e$, we find that a vector $v \in \mathfrak{L}(\mathfrak{g})$ extends w if and only if the projection v_e of v to $\mathfrak{L}(\mathfrak{g}_e)$ equals w. Let us rewrite the condition $P_\lambda(v) \in T_zQ$ of Lemma 2.2 under the grading Γ. Here for $s \in \mathfrak{L}(\mathfrak{g})$, we denote s_i its projection to $\mathfrak{L}(\mathfrak{g}_i)$. For $i \geq 0$, we get a recursive linear system of equations on the coordinates of v_i

$$[v_i, e] = v'_{i+2} + \lambda[a, v]_{i+2} + \sum_{k\leq 0}[q_k, v_{i+2-k}] \tag{3.33}$$

which can be solved uniquely since ad e restricted to $\mathfrak{g}_i$ is surjective and the projection of v_i to kernel ad e equals $(v_e)_i$. For $i \leq -1$, we have $\mathfrak{g}_{i+2} = (\mathfrak{g}_f)_{i+2} \oplus [\mathfrak{g}_i, e]$ and we get a recursive linear system of equations on the coordinates of v_i by setting the projection of

$$[e, v_i] + v'_{i+2} + \lambda[a, v]_{i+2} + \sum_{k\leq 0}[q_k, v_{i+2-k}] \tag{3.34}$$

to $[\mathfrak{g}_i, e]$ equals 0, which can be solved uniquely as the map ad e restricted to $\mathfrak{g}_i$ is injective. □

Now we are in a position to prove the following theorem.

Theorem 3.8. *The reduced second Poisson bracket $\{.,.\}_2^Q$ on Q is independent of the choice of a good grading and an isotropic subspace.*

Proof. We observe that the calculation of P_λ^Q in Proposition 3.7 can be done by using any other choice of good grading. This implies that this calculation depends only on the properties of sl_2-triples $\{e, h, f\}$. The Poisson bracket $\{.,.\}_2^Q$ is obtained by setting $\lambda = 0$ in the recursive equations (3.33) and (3.34). This ends the proof. □

We obtain the following theorem by applying Corollary 2.3.

Theorem 3.9. *The Poisson bracket $\{.,.\}_\lambda^Q$ equals the Dirac reduction of $\{.,.\}_\lambda$ to Q. It can be calculated by using Dirac formulas given in Proposition 2.5.*

In [7], the authors proved the following.

Theorem 3.10. *When Γ is the Dynkin grading and $\mathfrak{l}$ is a Lagrangian subspace, the Poisson bracket $\{.,.\}_2^Q$ is a classical W-algebra.*

Combining this result with Theorem 3.8 we get the following.

Theorem 3.11. *The classical W-algebra associated to a nilpotent orbit is independent of the choice of a good grading and an isotropic subspace and it can be calculated equally by using Drinfeld–Sokolov method, Poisson tensor procedure or Dirac formula.*

Let us explain in some detail, how we apply Dirac reduction to find $\{.,.\}_\lambda^Q$. We fix a homogeneous basis $\xi_1, \ldots, \xi_n$ for $\mathfrak{g}$ with $\xi_1, \ldots, \xi_m$ a basis for $\mathfrak{g}_f$. Let $\xi^1, \ldots, \xi^n \in \mathfrak{g}$ be the dual basis satisfying

$$\langle \xi_i | \xi^j \rangle = \delta_i^j.$$

Note that if $\xi_i \in \mathfrak{g}_j$ then $\xi^i \in \mathfrak{g}_{-j}$ and $\xi^1, \ldots, \xi^m$ are a basis for $\mathfrak{g}_e$. We calculate in this basis the structure constants and the matrix of the Killing form

$$[\xi^i, \xi^j] := c_k^{ij}\xi^k, \qquad \langle [\xi^i, \xi^j] | a \rangle = c_a^{ij}, \qquad g^{ij} = \langle \xi^i | \xi^j \rangle. \tag{3.35}$$

Let us consider the following coordinates on $\mathfrak{L}(\mathfrak{g})$

$$q^i(z) := \langle z - e | \xi^i \rangle, \quad i = 1, \ldots, n. \tag{3.36}$$

Then matrix differential operator

$$\mathbb{F}_\lambda^{ij} = -g^{ij}\partial_x - \sum_k c_k^{ij} q^k(x) - \lambda c_a^{ij} \tag{3.37}$$

defines the Poisson brackets

$$\{q^i(x), q^j(y)\}_\lambda = \mathbb{F}_\lambda^{ij}\delta(x-y). \tag{3.38}$$

From the construction, Slodowy slice Q is defined by $q^\alpha = 0$ for $\alpha = m+1, \dots, n$. Then we can directly apply Dirac formulas given in Proposition 2.5 to find the reduction of $\{.,.\}_\lambda$ to Q.

Example 3.12 (*The KdV Bihamiltonian Structure*)**.** Let $\mathfrak{g}$ be the Lie algebra sl_2 with its standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.39}$$

For a point $q \in \mathfrak{L}(\mathfrak{g})$ we use the notations

$$q(x) = q_e(x)e + \frac{1}{2}q_h(x)h + q_f(x)f \tag{3.40}$$

and we define P_1 by setting $a = f$. Then the matrix differential operator on $Q := e + q_f(x)f$ is given by

$$\mathbb{F}_\lambda^{\alpha,\beta} = \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 2\partial_x & 0 \\ \partial_x & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2(q_f(x)+\lambda) & 0 \\ -2(q_f(x)+\lambda) & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}. \tag{3.41}$$

Here, we order the coordinates as $\big(q_f(x), q_h(x), q_e(x)\big)$. The minor matrix operator $\mathbb{F}_\lambda^{\alpha\beta}$, $\alpha, \beta := 2, 3$ has the following inverse

$$\mathbb{S} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\partial_x \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{3.42}$$

Then apply Dirac formula to get

$$P_\lambda^Q = -\frac{1}{2}\partial_x^3 + 2(q_f + \lambda)\partial_x + q_f \tag{3.43}$$

which gives the bihamiltonian structure associated to the KdV equation

$$\{q_f(x), q_f(y)\}_2^Q = -\frac{1}{2}\delta'''(x-y) + 2(\, q_f(x) + \lambda)\,\delta'(x-y) + \partial_x q_f \delta(x-y). \tag{3.44}$$

4. Conclusions and remarks

4.1. Transverse Poisson structure

Let us consider the leading terms $\{.,.\}_2^{[-1]}$ and $\{.,.\}_1^{[-1]}$ of the bihamiltonian structure $\{.,.\}_2$ and $\{.,.\}_1$ on $\mathfrak{L}(\mathfrak{g})$. In the notations introduced after Theorem 3.11, we have

$$\{q^i, q^j\}_2^{[-1]} = -\sum_k c_k^{ij} q^k, \tag{4.1}$$

$$\{q^i, q^j\}_1^{[-1]} = -c_a^{ij}.$$

In the same manner as in Proposition 3.2, we can prove that the restriction of the action (3.13) to the adjoint group of $\mathfrak{n}$ on $\mathfrak{g}$ is Hamiltonian and admits a momentum map. Taking e as a regular value, we obtain a bihamiltonian structure $\{.,.\}_1^{Q[-1]}$, $\{.,.\}_2^{Q[-1]}$ on Slodowy slice $\widetilde{Q} = e + \mathfrak{g}_f$. From Corollary 2.6, this bihamiltonian structure is the leading term of the bihamiltonian structure $\{.,.\}_\lambda^Q$ on Q.

The Poisson structure $\{.,.\}_2^{Q[-1]}$ is known in the literature as the transverse Poisson structure (TPS) to the adjoint orbit of e. It was originally defined as the Dirac reduction of $\{.,.\}_2^{[-1]}$ to $\widetilde{Q}$ (see [21] and the references within). There were many papers devoted to prove that the TPS is a polynomial structure. This was not a trivial problem as the method used to calculate the TPS was Dirac formulas and it depends on the inverse of a polynomial matrix. In this paper we proved that, in addition to Dirac formulas, the TPS can be calculated by using Poisson tensor procedure and Drinfeld–Sokolov method. Both lead to a simpler proof for the polynomiality of the TPS as the former uses the linear recursive equations obtained in Proposition 3.7 and the latter uses the Leibnitz rule (3.25) on differential polynomials.

4.2. Classical and finite W-algebras

We mention that Slodowy slice $\widetilde{Q}$ is associated to the theory of finite W-algebras initiated by Premet [22]. More precisely, let $\chi \in \mathfrak{g}^*$ be given by

$$\chi(x) = \langle e|x\rangle$$

and consider the one dimensional character $\mathbb{C}_\chi$ on $\mathfrak{m}$ given by the restriction of χ. Let $U(\mathfrak{g})$ and $U(\mathfrak{m})$ be the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{m}$, respectively, and define the associative algebra

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi .$$

Then the finite W-algebra is a noncommutative algebra defined as

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}. \tag{4.2}$$

In [23], Gan and Ginzburg proved that W_χ is a quantization of TPS and it is independent of the choice of isotropic subspace, while Brundan and Goodwin [24] proved that W_χ is independent of the choice of a good grading (see [16] and the references within for more details). In this work we proved a similar argument for classical W-algebras. We hope this will contribute in clarifying more the relation between classical and finite W-algebras.

4.3. Integrable hierarchies of KdV type

Let $\{.,.\}_2^Q$ be a classical W-algebra associated to a nilpotent element e. In this paper we gave a procedure to obtain a Poisson bracket $\{.,.\}_1^Q$ such that it forms with $\{.,.\}_2^Q$ a bihamiltonian structure. This Poisson bracket is a reduction of a Poisson bracket defined on $\mathfrak{L}(\mathfrak{g})$ by means of an element a satisfying the following sufficient condition (see Eq. (3.12)): There exists a good grading Γ for e and an isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{-1}$ such that

$$\mathfrak{n} := \mathfrak{l}' \oplus \bigoplus_{i\leq -2} \mathfrak{g}_i \subset \ker \mathrm{ad}\, a \tag{4.3}$$

where $\mathfrak{l}'$ is the symplectic complement of $\mathfrak{l}$. Examples above suggest that this may be a necessary condition as well. Classifying such elements a may help in studying integrable hierarchies associated to classical W-algebras. In particular, if a is such that $a+e$ is regular semisimple then one can obtain an integrable hierarchy by using Zakharov–Shabat scheme, i.e. analyzing the spectrum of the matrix differential operator

$$P_\lambda = \partial_x + q(x) + e + \lambda a, \quad q(x) \in \mathfrak{L}(\mathfrak{b}).$$

This includes the generalized Drinfeld–Sokolov hierarchy developed in [18,25,3,26]. We mention here that in the case of the subregular nilpotent element in the Lie algebra of type C_3 there exist an element $a \in \mathfrak{g}$ such that $e+a$ is regular semisimple. Unfortunately, the sufficient condition (4.3) is not satisfied. In other words, the bihamiltonian structure defined by using this element a cannot be reduced to bihamiltonian structure on Slodowy slice by the methods introduced in this paper.

4.4. General remark

It is well known that, under certain assumptions, from a local bihamiltonian structure on $\mathfrak{L}(M)$, where M is a smooth manifold, one can construct a Frobenius structure on M. Our main motivation in studying local bihamiltonian structures related to classical W-algebras is the classification and construction of algebraic Frobenius manifolds [4–6]. The classification of Frobenius manifolds is the first step to classify local bihamiltonian structures using the concept of central invariants [27]. In the case of a regular nilpotent element in a simply laced Lie algebra the bihamiltonian structure obtained from applying standard Drinfeld–Sokolov reduction [18] gives a polynomial Frobenius manifolds and the central invariants are all equal to $\frac{1}{24}$.

In a subsequent publication we will consider further examples of Frobenius manifolds and investigate the central invariants on bihamiltonian manifolds that are produced by applying the reduction methods introduced in this paper.

Acknowledgments

The author thanks B. Dubrovin for useful discussions and N. Pagnon for providing Ref. [21]. Part of this work was inspired by the "Summer School and Conference in Geometric Representation Theory and Extended Affine Lie Algebras" at the University of Ottawa, organized by the Fields Institute. This work is partially supported by the European Science Foundation Programme (grant no. MISGAM-2116) "Methods of Integrable Systems, Geometry, Applied Mathematics" (MISGAM). The author also likes to thank anonymous reviewers who gave corrections and valuable comments that has helped to improve the quality of the manuscript.

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