



Integrability of S -deformable surfaces: Conservation laws, Hamiltonian structures and more



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ABSTRACT

We present infinitely many nonlocal conservation laws, a pair of compatible local Hamiltonian structures and a recursion operator for the equations describing surfaces in three-dimensional space that admit nontrivial deformations which preserve both principal directions and principal curvatures (or, equivalently, the shape operator).

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1. Introduction

The class of surfaces in three-dimensional space that admit nontrivial deformations which simultaneously preserve principal directions and principal curvatures (or, equivalently, the shape operator, also known as the Weingarten operator) has [1] a long and distinguished history: it was studied already by Finikoff and Gambier [2,3] more than 80 years ago but the investigation of preservation of principal directions and principal curvatures dates back to Bonnet [4,5]. For the sake of brevity we shall refer to the surfaces from the class in question as to the S -deformable surfaces.

Ferapontov [1] has established integrability of the corresponding Gauss–Codazzi equations (1) by presenting the associated Lax pair with a nonremovable spectral parameter; cf. also [6] and references therein for the general study of integrability of the Gauss–Codazzi equations.

A natural next step in the study of the equations in question, which we rewrite in the form (2), is to explore their geometric structures naturally related to integrability: symmetries, conservation laws, Hamiltonian structures and recursion operators.

In what follows we implement this program. Namely, after recalling the explicit form of the equations under study and of their Lax pair in Section 2 we proceed to construct an infinite sequence of nontrivial nonlocal conservation laws for (2) in Section 3, and a recursion operator along with a pair of compatible local Hamiltonian structures in Sections 4 and 5.

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2. Preliminaries

Consider the system [1] of the Gauss–Codazzi equations describing the S -deformable surfaces

$$\begin{aligned}\partial_1 H_2 &= \beta_{12} H_1, & \partial_2 H_1 &= \beta_{21} H_2, \\ \partial_1 \beta_{12} + \partial_2 \beta_{21} &= 0, \\ \eta_1 \partial_1 \beta_{12} + \eta_2 \partial_2 \beta_{21} + \frac{1}{2} \eta'_1 \beta_{12} + \frac{1}{2} \eta'_2 \beta_{21} + H_1 H_2 &= 0,\end{aligned}\tag{1}$$

where $\eta_1 = \eta_1(x)$ and $\eta_2 = \eta_2(y)$ are arbitrary smooth functions.

Upon expressing β_{ij} via H_k we rewrite this system in the form

$$\begin{aligned}u_{yy} &= \frac{\frac{1}{2} \eta'_1 v^2 v_x + \frac{1}{2} \eta'_2 u v u_y + (\eta_1 - \eta_2) u u_y v_y + u^2 v^3}{u v (\eta_1 - \eta_2)}, \\ v_{xx} &= \frac{\frac{1}{2} \eta'_1 u v v_x + \frac{1}{2} \eta'_2 u^2 u_y + (\eta_2 - \eta_1) v v_x u_x + u^3 v^2}{u v (\eta_2 - \eta_1)},\end{aligned}\tag{2}$$

where $u = H_1$, $v = H_2$. Its (complex) $\mathfrak{sl}_2$ -valued zero-curvature representation reads [1]

$$\begin{aligned}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x &= \frac{1}{2\sqrt{\lambda + \eta_1}} \begin{pmatrix} i\sqrt{\lambda + \eta_2} \cdot \frac{u_y}{v} & u \\ -u & -i\sqrt{\lambda + \eta_2} \cdot \frac{u_y}{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_y &= \frac{i}{2\sqrt{\lambda + \eta_2}} \begin{pmatrix} -\sqrt{\lambda + \eta_1} \cdot \frac{v_x}{u} & v \\ v & \sqrt{\lambda + \eta_1} \cdot \frac{v_x}{u} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},\end{aligned}\tag{3}$$

where $\lambda \in \mathbb{R}$.

For any zero-curvature representation of the form

$$\begin{aligned}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}_x &= \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \\ \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}_y &= \begin{pmatrix} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}\end{aligned}$$

we can (cf. e.g. [7] and references therein) consider the associated Riccati covering

$$\begin{aligned}w_x &= A_2^1 + (A_1^1 - A_2^2)w - A_1^2 w^2, \\ w_y &= B_2^1 + (B_1^1 - B_2^2)w - B_1^2 w^2\end{aligned}$$

where $w = \psi^1/\psi^2$; see e.g. [8] and references therein for more details on (differential) coverings.

In particular, for (3) the Riccati covering is

$$\begin{aligned}w_x &= i\sqrt{\frac{\lambda + \eta_2}{\lambda + \eta_1}} \cdot \frac{u_y}{v} w + \frac{u}{2\sqrt{\lambda + \eta_1}} (1 + w^2), \\ w_y &= -i\sqrt{\frac{\lambda + \eta_1}{\lambda + \eta_2}} \cdot \frac{v_x}{u} w + \frac{iv}{2\sqrt{\lambda + \eta_2}} (1 - w^2).\end{aligned}\tag{4}$$

By changing the parametrization $\lambda = 1/\mu^2$, $\mu > 0$, we transform (4) to

$$\begin{aligned}w_x &= i\sqrt{\frac{1 + \eta_2 \mu^2}{1 + \eta_1 \mu^2}} \cdot \frac{u_y}{v} w + \frac{u\mu}{2\sqrt{1 + \eta_1 \mu^2}} (1 + w^2), \\ w_y &= -i\sqrt{\frac{1 + \eta_1 \mu^2}{1 + \eta_2 \mu^2}} \cdot \frac{v_x}{u} w + \frac{iv\mu}{2\sqrt{1 + \eta_2 \mu^2}} (1 - w^2).\end{aligned}\tag{5}$$

3. Infinite hierarchy of nonlocal conservation laws

Consider the formal Taylor expansions

$$\begin{aligned} w &= w_0 + w_1\mu + w_2\mu^2 + w_3\mu^3 \dots, \\ w^2 &= w_0^2 + 2w_0w_1\mu + (2w_0w_2 + w_1^2)\mu^2 + 2(w_0w_3 + w_1w_2)\mu^3 + \dots, \\ \frac{1}{\sqrt{1 + \eta_1\mu^2}} &= \alpha_0^1 + \alpha_2^1\mu^2 + \dots + \alpha_{2k}^1\mu^{2k} + \dots, \\ \frac{1}{\sqrt{1 + \eta_2\mu^2}} &= \alpha_0^2 + \alpha_2^2\mu^2 + \dots + \alpha_{2k}^2\mu^{2k} + \dots, \\ \sqrt{\frac{1 + \eta_1\mu^2}{1 + \eta_2\mu^2}} &= \alpha_0^{12} + \alpha_2^{12}\mu^2 + \dots + \alpha_{2k}^{12}\mu^{2k} + \dots, \\ \sqrt{\frac{1 + \eta_2\mu^2}{1 + \eta_1\mu^2}} &= \alpha_0^{21} + \alpha_2^{21}\mu^2 + \dots + \alpha_{2k}^{21}\mu^{2k} + \dots, \end{aligned}$$

where

$$\alpha_{2k}^j = \left(-\frac{\eta_j}{2}\right)^k (2k-1)!!, \quad j = 1, 2,$$

and

$$\alpha_{2k}^{jl} = \sum_{a+b=2k} \frac{\varkappa^+(\varkappa^+ - 1) \dots (\varkappa^+ - a + 1)}{a!} \cdot \frac{\varkappa^-(\varkappa^- - 1) \dots (\varkappa^- - b + 1)}{b!} \eta_j^a \eta_l^b,$$

where $\varkappa^\pm = \pm 1/2$ and $j = 1, l = 2$ or $j = 2, l = 1$.

Substituting these expansions into (5) and equating coefficients at powers of μ , we obtain the following infinite tower of 1-dimensional coverings:

$$\begin{aligned} w_{0,x} &= \frac{iu_y}{v} \alpha_0^{21} w_0, \\ w_{1,x} &= \frac{iu_y}{v} \alpha_0^{21} w_1 + \frac{u}{2} \alpha_0^1 (1 + w_0^2), \\ w_{2,x} &= \frac{iu_y}{v} (\alpha_2^{21} w_0 + \alpha_0^{21} w_2) + u \alpha_0^1 w_0 w_1, \\ w_{3,x} &= \frac{iu_y}{v} (\alpha_2^{21} w_1 + \alpha_0^{21} w_3) + \frac{u}{2} (\alpha_2^1 (1 + w_0^2) + \alpha_0^1 (2w_0 w_2 + w_1^2)), \\ &\dots \\ w_{2k,x} &= \frac{iu_y}{v} (\alpha_{2k}^{21} w_0 + \alpha_{2k-2}^{21} w_2 + \dots + \alpha_0^{21} w_{2k}) \\ &\quad + u (\alpha_{2k-2}^1 w_0 w_1 + \alpha_{2k-4}^1 (w_0 w_3 + w_1 w_2) + \dots + \alpha_0^1 (w_0 w_{2k-1} + \dots + w_{k-1} w_k)), \\ w_{2k+1,x} &= \frac{iu_y}{v} (\alpha_{2k}^{21} w_1 + \alpha_{2k-2}^{21} w_3 + \dots + \alpha_0^{21} w_{2k+1}) \\ &\quad + \frac{u}{2} (\alpha_{2k}^1 (1 + w_0^2) + \alpha_{2k-2}^1 (2w_0 w_2 + w_1^2) + \dots + \alpha_0^1 (2w_0 w_{2k} + \dots + 2w_{k-1} w_{k+1} + w_k^2)), \\ &\dots \end{aligned}$$

and

$$\begin{aligned} w_{0,y} &= -\frac{iv_x}{u} \alpha_0^{12} w_0, \\ w_{1,y} &= -\frac{iv_x}{u} \alpha_0^{12} w_1 - \frac{iv}{2} \alpha_0^2 (w_0^2 - 1), \\ w_{2,y} &= -\frac{iv_x}{u} (\alpha_2^{12} w_0 + \alpha_0^{12} w_2) - iv \alpha_0^2 w_0 w_1, \\ w_{3,y} &= -\frac{iv_x}{u} (\alpha_2^{12} w_1 + \alpha_0^{12} w_3) - \frac{iv}{2} (\alpha_2^2 (w_0^2 - 1) + \alpha_0^2 (2w_0 w_2 + w_1^2)), \end{aligned}$$

$$\begin{aligned}
w_{2k,y} &= -\frac{iv_x}{u}(\alpha_{2k}^{12}w_0 + \alpha_{2k-2}^{12}w_2 + \cdots + \alpha_0^{12}w_{2k}) \\
&\quad - iv\left(\alpha_{2k-2}^2w_0w_1 + \alpha_{2k-4}^2(w_0w_3 + w_1w_2) + \cdots + \alpha_0^2(w_0w_{2k-1} + \cdots + w_{k-1}w_k)\right), \\
w_{2k+1,y} &= -\frac{iv_x}{u}(\alpha_{2k}^{12}w_1 + \alpha_{2k-2}^{12}w_3 + \cdots + \alpha_0^{12}w_{2k+1}) \\
&\quad - \frac{iv}{2}\left(\alpha_{2k}^2(w_0^2 - 1) + \alpha_{2k-2}^2(2w_0w_2 + w_1^2) + \cdots + \alpha_0^2(2w_0w_{2k} + \cdots + 2w_{k-1}w_{k+1} + w_k^2)\right), \\
&\quad \dots
\end{aligned}$$

Apply the following gauge transformation in the above covering:

$$w_0 = e^{\theta_0}, \quad w_k = \theta_k e^{\theta_0}, \quad k > 0.$$

Then the latter transforms to

$$\begin{aligned}
\theta_{0,x} &= \frac{iu_y}{v}\alpha_0^{21}, \\
\theta_{1,x} &= u\alpha_0^1 \cosh \theta_0, \\
\theta_{2,x} &= \frac{iu_y}{v}\alpha_2^{21} + u\alpha_0^1\theta_1 e^{\theta_0}, \\
\theta_{3,x} &= \frac{iu_y}{v}\alpha_2^{21}\theta_1 + u\left(\alpha_2^1 \cosh \theta_0 + \alpha_0^1\left(\theta_2 + \frac{1}{2}\theta_1^2\right)e^{\theta_0}\right), \\
&\quad \dots \\
\theta_{2k,x} &= \frac{iu_y}{v}X_{2k}^{21} + uX_{2k}^1, \\
\theta_{2k+1,x} &= \frac{iu_y}{v}X_{2k+1}^{21} + uX_{2k+1}^1, \\
&\quad \dots
\end{aligned}$$

where

$$\begin{aligned}
X_{2k}^{21} &= \alpha_{2k}^{21} + \alpha_{2k-2}^{21}\theta_2 + \cdots + \alpha_2^{21}\theta_{2k-2}, \\
2k^1 &= \left(\alpha_{2k-2}^1\theta_1 + \alpha_{2k-4}^1(\theta_3 + \theta_1\theta_2) + \cdots + \alpha_0^1(\theta_{2k-1} + \theta_1\theta_{2k-2} + \cdots + \theta_{k-1}\theta_k)\right)e^{\theta_0}, \\
X_{2k+1}^{21} &= \alpha_{2k}^{21}\theta_1 + \alpha_{2k-2}^{21}\theta_3 + \cdots + \alpha_2^{21}\theta_{2k-1}, \\
X_{2k+1}^1 &= \alpha_{2k}^1 \cosh \theta_0 + \left(\alpha_{2k-2}^1\left(\theta_2 + \frac{1}{2}\theta_1^2\right) + \alpha_{2k-4}^1\left(\theta_4 + \theta_1\theta_3 + \frac{1}{2}\theta_2^2\right) \right. \\
&\quad \left. + \cdots + \alpha_0^1\left(\theta_{2k} + \theta_1\theta_{2k-1} + \cdots + \theta_{k-1}\theta_{k+1} + \frac{1}{2}\theta_k^2\right)\right)e^{\theta_0},
\end{aligned}$$

and

$$\begin{aligned}
\theta_{0,y} &= -\frac{iv_x}{u}\alpha_0^{12}, \\
\theta_{1,y} &= -iv\alpha_0^2 \sinh \theta_0, \\
\theta_{2,y} &= -\frac{iv_x}{u}\alpha_2^{12} - iv\alpha_0^2\theta_1 e^{\theta_0}, \\
\theta_{3,y} &= -\frac{iv_x}{u}\alpha_2^{12}\theta_1 - iv\left(\alpha_2^2 \sinh \theta_0 + \alpha_0^2\left(\theta_2 + \frac{1}{2}\theta_1^2\right)e^{\theta_0}\right), \\
&\quad \dots \\
\theta_{2k,y} &= -\frac{iv_x}{u}Y_{2k}^{12} - ivY_{2k}^2, \\
\theta_{2k+1,y} &= -\frac{iv_x}{u}Y_{2k+1}^{12} - ivY_{2k+1}^2,
\end{aligned}$$

where

$$\begin{aligned}
Y_{2k}^{12} &= \alpha_{2k}^{12} + \alpha_{2k-2}^{12}\theta_2 + \cdots + \alpha_2^{12}\theta_{2k-2}, \\
Y_{2k}^2 &= \left(\alpha_{2k-2}^2\theta_1 + \alpha_{2k-4}^2(\theta_3 + \theta_1\theta_2) + \cdots + \alpha_0^2(\theta_{2k-1} + \theta_1\theta_{2k-2} + \cdots + \theta_{k-1}\theta_k)\right)e^{\theta_0},
\end{aligned}$$

$$Y_{2k+1}^{12} = \alpha_{2k}^{12}\theta_1 + \alpha_{2k-2}^{12}\theta_3 + \cdots + \alpha_2^{12}\theta_{2k-1},$$

$$Y_{2k+1}^2 = \alpha_{2k}^2 \sinh \theta_0 + \left(\alpha_{2k-2}^2 \left(\theta_2 + \frac{1}{2}\theta_1^2 \right) + \left(\theta_4 + \theta_1\theta_3 + \frac{1}{2}\theta_2^2 \right) \right. \\ \left. + \cdots + \alpha_0^2 \left(\theta_{2k} + \theta_1\theta_{2k-1} + \cdots + \theta_{k-1}\theta_{k+1} + \frac{1}{2}\theta_k^2 \right) \right) e^{\theta_0}.$$

Remark. The above complex covering can be reduced to the real form if we set $\theta_k = p_k + iq_k$, $k = 0, 1, \dots$, and consider real and imaginary parts of the defining equations separately. In the particular case $p_0 = 0$ we obtain a real covering which is employed below (see Section 4) to construct a recursion operator for symmetries of (2).

Thus, we have obtained an infinite tower of Abelian coverings

$$\mathcal{E} \longleftarrow \mathcal{E}_0 \longleftarrow \cdots \longleftarrow \mathcal{E}_{2k} \longleftarrow \mathcal{E}_{2k+1} \longleftarrow \cdots,$$

where $\mathcal{E}$ is the initial system (2) and $\mathcal{E}_s$ is obtained by extending $\mathcal{E}$ with the nonlocal variables $\theta_0, \dots, \theta_s$. It only remains to prove that for any $s \in \mathbb{N}$ the conservation law

$$\omega_s = \left(\frac{i u_y}{v} X_s^{21} + u X_s^1 \right) dx - \left(\frac{i v_x}{u} Y_s^{12} + i v Y_s^2 \right) dy \quad (6)$$

is nontrivial on $\mathcal{E}_{s-1}$.

Denote by $D_x^{[l]}, D_y^{[l]}$ the total derivatives on $\mathcal{E}_l$, $l \geq 1$.

Proposition 1. *The only solutions of the system*

$$D_x^{[l]}(f) = 0, \quad D_y^{[l]}(f) = 0, \quad f \in C^\infty(\mathcal{E}_l), \quad (7)$$

are constants.

Proof of Proposition 1 (Part I). The total derivatives on $\mathcal{E}_l$ have the form

$$D_x^{[l]} = D_x + \sum_{j=0}^l \left(\frac{i u_y}{v} X_s^{21} + u X_s^1 \right) \frac{\partial}{\partial \theta_s}, \quad D_y^{[l]} = D_y - \sum_{j=0}^l \left(\frac{i v_x}{u} Y_s^{12} + i v Y_s^2 \right) \frac{\partial}{\partial \theta_s},$$

where D_x and D_y are the total derivatives on $\mathcal{E}$. Obviously, a function $f \in C^\infty(\mathcal{E}_l)$ is a solution to (7) if and only if

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial u} = 0,$$

and

$$\sum_{s=0}^l X_s^{12} \frac{\partial f}{\partial \theta_s} = 0, \quad \sum_{s=0}^l Y_s^{21} \frac{\partial f}{\partial \theta_s} = 0, \quad \sum_{s=1}^l X_s^1 \frac{\partial f}{\partial \theta_s} = 0, \quad \sum_{s=1}^l Y_s^2 \frac{\partial f}{\partial \theta_s} = 0. \quad (8)$$

We shall proceed with the proof after establishing Lemma 1. $\square$

Consider the vector fields

$$TX_l = \sum_{s=1}^l X_s^1 \frac{\partial}{\partial \theta_s}, \quad TY_l = \sum_{s=1}^l Y_s^2 \frac{\partial}{\partial \theta_s}$$

on $\mathcal{E}_l$ and let

$$Z_l = [[\dots [TX_l, \underbrace{TY_l}_{l-1 \text{ times}}, \dots], TY_l]]$$

for $l \geq 2$.

Lemma 1. *For any $l \geq 2$ we have*

$$Z_l = \begin{cases} \alpha_0^1 (\alpha_0^2)^{l-1} \frac{\partial}{\partial \theta_l} & \text{for even } l, \\ \alpha_0^1 (\alpha_0^2)^{l-1} \cosh(\theta_0) \frac{\partial}{\partial \theta_l} & \text{for odd } l. \end{cases}$$

Proof of Lemma 1. Consider the formal series

$$Q = \frac{1}{2} \left(1 + \sum_{j=1}^{\infty} \theta_j \lambda^j \right)^2$$

and denote by $Q_{[j]}$ its coefficient at λ^j .

Assign to any monomial $p = \varphi(\theta_0) \theta_{i_1} \dots \theta_{i_r}$ the weight

$$|p| = i_1 + \dots + i_r.$$

Then the quantities $Q_{[j]}$ are homogeneous and $|Q_{[j]}| = j$. In addition, one has

$$\frac{\partial Q_{[j]}}{\partial \theta_i} = \begin{cases} 1, & \text{if } i = j, \\ \theta_{j-i}, & \text{if } i < j, \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

for any $i \geq 1$.

Consider the vector fields

$$TX = \sum_{s=1}^{\infty} X_s^1 \frac{\partial}{\partial \theta_s}, \quad TY = \sum_{s=1}^{\infty} Y_s^2 \frac{\partial}{\partial \theta_s}$$

on the space $\mathcal{E}_{\infty} = \liminf_{l \rightarrow \infty} \mathcal{E}_l$. Then obviously, $TX|_{\mathcal{E}_l} = TX_l$, $TY|_{\mathcal{E}_l} = TY_l$ and $[TX, TY]|_{\mathcal{E}_l} = [TX_l, TY_l]$. These fields can be written as follows:

$$\begin{aligned} TX &= \alpha_0^1 \left(\cosh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \left((Q_{[1]} + o(0)) \frac{\partial}{\partial \theta_2} + (Q_{[2]} + o(1)) \frac{\partial}{\partial \theta_3} + \dots + (Q_{[l-1]} + o(l-2)) \frac{\partial}{\partial \theta_l} + \dots \right) \right), \\ TY &= \alpha_0^2 \left(\sinh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \left((Q_{[1]} + o(0)) \frac{\partial}{\partial \theta_2} + (Q_{[2]} + o(1)) \frac{\partial}{\partial \theta_3} + \dots + (Q_{[l-1]} + o(l-2)) \frac{\partial}{\partial \theta_l} + \dots \right) \right), \end{aligned}$$

where $o(j)$ denotes terms of weight $\leq j$. To simplify notation, we shall use the short form

$$\begin{aligned} TX &= \alpha_0^1 \left(\cosh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \left(Q_{[1]} \frac{\partial}{\partial \theta_2} + Q_{[2]} \frac{\partial}{\partial \theta_3} + \dots + Q_{[l-1]} \frac{\partial}{\partial \theta_l} + \dots \right) \right) + o, \\ TY &= \alpha_0^2 \left(\sinh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \left(Q_{[1]} \frac{\partial}{\partial \theta_2} + Q_{[2]} \frac{\partial}{\partial \theta_3} + \dots + Q_{[l-1]} \frac{\partial}{\partial \theta_l} + \dots \right) \right) + o. \end{aligned}$$

One readily checks that

$$[A + o, B + o] = [A, B] + o$$

in all subsequent computations.

We shall now prove, by induction on l , that the fields

$$Z_l = [\underbrace{[TX, TY], \dots, [TX, TY]}_{l-1 \text{ times}}]$$

are of the form

$$\alpha_0^1 (\alpha_0^2)^{l-1} \left(\frac{\partial}{\partial \theta_l} + \theta_1 \frac{\partial}{\partial \theta_{l+1}} + \dots + \theta_j \frac{\partial}{\partial \theta_{l+j}} + \dots \right) + o$$

if l is even and

$$\alpha_0^1 (\alpha_0^2)^{l-1} \left(\cosh(\theta_0) \frac{\partial}{\partial \theta_l} + e^{\theta_0} \left(Q_{[1]} \frac{\partial}{\partial \theta_{l+1}} + \dots + Q_{[j]} \frac{\partial}{\partial \theta_{l+j}} + \dots \right) \right) + o$$

for odd l . The claim of Lemma 1 readily follows from these formulas for Z_l (please note, however, that the Z_l in the statement of Lemma 1 are not identical to those defined above).

Let $l = 2$. Then by virtue of (9) we have

$$\begin{aligned} Z_2 &= [TX, TY] \\ &= \left[\cosh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \sum_{j=1}^{\infty} Q_{[j+1]} \frac{\partial}{\partial \theta_j}, \sinh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \sum_{j=1}^{\infty} Q_{[j+1]} \frac{\partial}{\partial \theta_j} \right] + o \end{aligned}$$

$$\begin{aligned}
&= \alpha_0^1 \alpha_0^2 e^{\theta_0} (\cosh(\theta_0) - \sinh(\theta_0)) \left(\frac{\partial}{\partial \theta_2} + \sum_{j=1}^{\infty} \theta_j \frac{\partial}{\partial \theta_{j+2}} \right) + o \\
&= \alpha_0^1 \alpha_0^2 \left(\frac{\partial}{\partial \theta_2} + \sum_{j=1}^{\infty} \theta_j \frac{\partial}{\partial \theta_{j+2}} \right) + o.
\end{aligned}$$

For $l = 3$ one has

$$\begin{aligned}
Z_3 &= [Z_2, TY] \\
&= \left[\alpha_0^1 \alpha_0^2 \left(\frac{\partial}{\partial \theta_2} + \sum_{j=1}^{\infty} \theta_j \frac{\partial}{\partial \theta_{j+2}} \right), \alpha_0^2 \left(\sinh(\theta_0) \frac{\partial}{\partial \theta_1} + e^{\theta_0} \sum_{j=1}^{\infty} Q_{[j+1]} \frac{\partial}{\partial \theta_j} \right) \right] + o \\
&= \alpha_0^1 (\alpha_0^2)^2 e^{\theta_0} \left(\frac{\partial}{\partial \theta_3} + \sum_{j=1}^{\infty} \theta_j \frac{\partial}{\partial \theta_{j+3}} + \sum_{s=1}^{\infty} \theta_s \left(\frac{\partial}{\partial \theta_{s+3}} + \sum_{j=1}^{\infty} \theta_j \frac{\partial}{\partial \theta_{j+s+3}} \right) \right) \\
&\quad - \alpha_0^1 (\alpha_0^2)^2 \left(\sinh(\theta_0) \frac{\partial}{\partial \theta_3} + e^{\theta_0} \sum_{j=1}^{\infty} Q_{[j]} \frac{\partial}{\partial \theta_{j+3}} \right) + o \\
&= \alpha_0^1 (\alpha_0^2)^2 e^{\theta_0} \left(\frac{\partial}{\partial \theta_3} + 2 \sum_{j=1}^{\infty} Q_{[j]} \frac{\partial}{\partial \theta_{j+3}} \right) - \alpha_0^1 (\alpha_0^2)^2 \left(\sinh(\theta_0) \frac{\partial}{\partial \theta_3} + e^{\theta_0} \sum_{j=1}^{\infty} Q_{[j]} \frac{\partial}{\partial \theta_{j+3}} \right) + o \\
&= \alpha_0^1 (\alpha_0^2)^2 \left(\cosh(\theta_0) \frac{\partial}{\partial \theta_3} + e^{\theta_0} \left(Q_{[1]} \frac{\partial}{\partial \theta_{j+3}} \right) \right) + o.
\end{aligned}$$

The induction step also uses property (9) of the quantities $Q_{[j]}$ and is accomplished by the computations quite similar to those given above. $\square$

Let us complete the proof of Proposition 1.

Proof of Proposition 1 (Part II). We prove by induction on l that system (7) possesses constant solutions only.

For $l = 1$, the equations are

$$\begin{aligned}
D_x(f) + \alpha_0^{21} \frac{i u_y}{v} \frac{\partial f}{\partial \theta_0} + \alpha_0^1 \cosh(\theta_0) u \frac{\partial f}{\partial \theta_1} &= 0, \\
D_y(f) - \alpha_0^{12} \frac{i v_x}{u} \frac{\partial f}{\partial \theta_0} - \alpha_0^2 i \sinh(\theta_0) v \frac{\partial f}{\partial \theta_1} &= 0,
\end{aligned}$$

where f is a function on $\mathcal{E}_1$, i.e., it may depend on jet variables as well as on θ_0 and θ_1 . Analyzing the coefficients of the fields D_x and D_y , one immediately sees that f can depend on x , y , θ_0 , and θ_1 only and from the above equations it readily follows that $f = \text{const}$.

Assume now that the result is valid for some $l > 1$ and let $f \in C^\infty(\mathcal{E}_{l+1})$ be such that $D_x^{[l+1]}(f) = D_y^{[l+1]}(f) = 0$. Then, by (8),

$$TX_{l+1}(f) = 0, \quad TY_{l+1}(f) = 0.$$

Consequently, $Z_{l+1}(f) = 0$. Using Lemma 1, we see that

$$\frac{\partial f}{\partial \theta_{l+1}} = 0.$$

Thus, f is a function on $\mathcal{E}_l$ and it is constant by the induction hypothesis. $\square$

Corollary 1. The conservation law ω_s given by (6) is nontrivial on $\mathcal{E}_s$.

Proof. Indeed, if $\omega_s = d_h f$, where $d_h = dx \wedge D_x^{[s]} + dy \wedge D_y^{[s]}$ is the horizontal de Rham differential on $\mathcal{E}_s$ then $D_x^{[s+1]}(f - \theta_{s+1}) = 0$ and $D_y^{[s+1]}(f - \theta_{s+1}) = 0$, which contradicts Proposition 1. $\square$

Thus we have constructed an infinite hierarchy of nonlocal conservation laws ω_s , $s \in \mathbb{N}$, for (2).

4. Recursion operator and Hamiltonian structures

It is readily checked that (2) possesses an Abelian covering $\mathcal{S}$ with the fiber coordinates s_i defined by the formulas

$$\begin{aligned}(s_0)_x &= \frac{u_y}{v}, & (s_0)_y &= -\frac{v_x}{u}, \\ (s_1)_x &= \cos(s_0)u, & (s_1)_y &= v \sin(s_0), \\ (s_2)_x &= -\sin(s_0)u, & (s_2)_y &= v \cos(s_0).\end{aligned}$$

Note that the conservation laws associated with s_i , $i = 1, 2$, are potential conservation laws in the terminology of [9] and that we have $\theta_0 = is_0$ and $\theta_1 = \alpha_0^1 s_1$ modulo the addition of arbitrary constants.

The following result is readily verified by straightforward computation.

Proposition 2. Suppose that U, V and S_i are fiber coordinates of the tangent covering $V\mathcal{S}$.

Then the tangent covering over (2) admits a Bäcklund auto-transformation (i.e., a recursion operator for (2)) of the form

$$\begin{aligned}\tilde{U} &= \eta_1 U + \left(\frac{1}{2} \eta'_1 \cos(s_0) + \frac{\sin(s_0)(\eta_2 - \eta_1)}{v} u_y \right) S_1 + \left(-\frac{1}{2} \eta'_1 \sin(s_0) + \frac{\cos(s_0)(\eta_2 - \eta_1)}{v} u_y \right) S_2, \\ \tilde{V} &= \eta_2 V + \left(\frac{1}{2} \eta'_2 \sin(s_0) - \frac{\cos(s_0)(\eta_2 - \eta_1)}{u} v_x \right) S_1 + \left(\frac{1}{2} \eta'_2 \cos(s_0) + \frac{\sin(s_0)(\eta_2 - \eta_1)}{u} v_x \right) S_2.\end{aligned}\quad (10)$$

Eqs. (10) define a recursion operator for (15) in the following fashion (see e.g. [10–13] and references therein for details).

Suppose $(U, V)^T$ is a symmetry shadow for (15) in a covering $\mathcal{C}$ over $\mathcal{S}$ (here and below the superscript T indicates the transposed matrix). Then we have a (possibly trivial) covering $\mathcal{C}'$ over $\mathcal{C}$ arising from substituting our U and V into the equations defining S_i . Under these assumptions (10) defines a new symmetry shadow $(\tilde{U}, \tilde{V})^T$ for (15) in $\mathcal{C}'$, i.e., we have a recursion operator for (15).

Note that (10) was found using the method from [11]; cf. e.g. [10, 14–16, 13] for some other related techniques.

Starting with a simple seed symmetry like $(0, 0)^T$ yields, through the repeated application of (10), an infinite hierarchy of shadows of nonlocal symmetries for (2). It is an interesting open problem to find the minimal covering in which it is possible to lift all these shadows to full-fledged nonlocal symmetries of (2) and to find the commutation relations among these nonlocal symmetries, cf. e.g. [17].

Moreover, one can readily establish the following result:

Proposition 3. System (2) admits (in a generalized sense of [18, 14, 8]) a pair of compatible local Hamiltonian structures $\mathcal{P}_i$ of the form

$$\mathcal{P}_i = f_{i,3} D_x D_y + D_y \circ f_{i,2} + D_x \circ f_{i,1} + f_{i,0}, \quad i = 1, 2,$$

where $f_{i,j}$ are 2×2 matrices of the form

$$\begin{aligned}f_{1,3} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & f_{2,3} &= \begin{pmatrix} \eta_1 & 0 \\ 0 & -\eta_2 \end{pmatrix} \\ f_{1,2} &= \begin{pmatrix} 0 & 0 \\ \frac{v_x}{u} + \frac{v\eta'_1}{2u\Delta} & -\frac{u_x}{u} - \frac{\eta'_1}{2\Delta} \end{pmatrix}, \\ f_{2,2} &= \begin{pmatrix} \frac{\eta'_1}{2} & 0 \\ \frac{\eta_1 v_x}{u} + \frac{v\eta'_1 \eta_2}{2u\Delta} & -\frac{\eta_2 u_x}{u} - \frac{\eta'_1 \eta_2}{2\Delta} \end{pmatrix}, \\ f_{1,1} &= \begin{pmatrix} \frac{v_y}{v} - \frac{\eta'_2}{2\Delta} & -\frac{u_y}{v} + \frac{\eta'_2 u}{2v\Delta} \\ 0 & 0 \end{pmatrix}, \\ f_{2,1} &= \begin{pmatrix} \frac{\eta_1 v_y}{v} - \frac{\eta_1 \eta'_2}{2\Delta} & -\frac{\eta_2 u_y}{v} + \frac{\eta_1 \eta'_2 u}{2v\Delta} \\ 0 & -\frac{\eta'_2}{2} \end{pmatrix} \\ f_{1,0} &= \begin{pmatrix} \frac{\eta'_1 u_y}{2u\Delta} & \frac{u_{xy}}{v} - \frac{u_x u_y}{vu} - \frac{v_x u_y}{v^2} - \frac{\eta'_1 u_y}{2v\Delta} \\ -\frac{v_{xy}}{u} + \frac{v_x v_y}{vu} + \frac{v_x u_y}{u^2} - \frac{\eta'_2 v_x}{2u\Delta} & \frac{\eta'_2 v_x}{2v\Delta} \end{pmatrix}\end{aligned}$$

$$f_{2,0} = \begin{pmatrix} -\frac{\eta'_1 v_y}{2v} + \frac{\eta'_1 \eta_2 u_y}{2u\Delta} + \frac{\eta'_1 \eta'_2}{4\Delta} & (f_{2,0})_{12} \\ (f_{2,0})_{21} & \frac{\eta'_2 u_x}{2u} + \frac{\eta_1 \eta'_2 v_x}{2v\Delta} + \frac{\eta'_1 \eta'_2}{4\Delta} \end{pmatrix},$$

$$(f_{2,0})_{12} = \frac{\eta_2 u_{xy}}{v} - \frac{\eta_2 u_x u_y}{vu} + \frac{\eta_2 v_x u_y}{v^2} - \frac{\eta'_1 \eta_2 u_y}{2v\Delta} - \frac{\eta'_1 \eta'_2 u}{4v\Delta},$$

$$(f_{2,0})_{21} = -\frac{\eta_1 v_{xy}}{u} + \frac{\eta_1 v_x v_y}{vu} + \frac{\eta_1 v_x u_y}{u^2} - \frac{\eta_1 \eta'_2 v_x}{2u\Delta} - \frac{\eta'_1 \eta'_2 v}{4u\Delta},$$

and $\Delta = \eta_2 - \eta_1$.

Proof. Our system (2) is readily shown to be contact equivalent to an evolutionary one (cf. Section 5). Consequently, system (2) is normal and its cotangent covering (the ℓ^* -covering) is well-defined, cf. [14], and the Schouten brackets of variational bivectors can be computed in any representation of the system under study. Let us employ to this end an evolutionary representation.

Namely, let $\mathcal{E}$ denote now an evolutionary representation for our system. Let u^i_σ be the jet variables on $\mathcal{E}$ and p^j_σ be the associated variables in the fiber of the cotangent covering of $\mathcal{E}$. Then to any differential operator $\Delta = \|\sum_\sigma a^{ij}_\sigma D_\sigma\|$ in total derivatives acting from the space of symmetries of $\mathcal{E}$ to that of cosymmetries of $\mathcal{E}$ there corresponds the function

$$W_\Delta = \sum_{i,j,\sigma} a^{ij}_\sigma p^j_\sigma p^i \quad (11)$$

on the space of the cotangent covering, see [19].

The operator Δ is skew-adjoint (and to the function W_Δ we can associate a variational bivector, cf. [19]) if and only if [19]

$$\sum_j \frac{\delta W_\Delta}{\delta p^j} p^j = -2W_\Delta, \quad (12)$$

while a variational bivector is a Poisson structure (and the corresponding operator is Hamiltonian) if and only if [19]

$$\delta \sum \frac{\delta W}{\delta u^i} \frac{\delta W}{\delta p^j} = 0, \quad (13)$$

where δ is the Euler operator.

Finally, two Poisson structures are compatible if and only if the associated bivectors satisfy [19]

$$\delta \sum_j \left(\frac{\delta W_1}{\delta u^i} \frac{\delta W_2}{\delta p^j} + \frac{\delta W_2}{\delta u^i} \frac{\delta W_1}{\delta p^j} \right) = 0. \quad (14)$$

Identities (12)–(14), although quite complicated, are easily checked for the quantities W_i associated to the counterparts of $\mathcal{P}_i$ for $\mathcal{E}$ by an appropriate computer algebra software. $\square$

The ratio of the Hamiltonian structures $\mathcal{R} = \mathcal{P}_2 \circ \mathcal{P}_1^{-1}$ is nothing but the recursion operator (10) written in the pseudodifferential form, cf. e.g. [20,10,7,11,21] and references therein. The compatibility of $\mathcal{P}_1$ and $\mathcal{P}_2$ implies that $\mathcal{R}$ is hereditary, cf. e.g. [20,22] and references therein for details.

5. Compatible Hamiltonian structures in evolutionary representation

In order to make contact with the standard theory of Hamiltonian structures, see e.g. [20] and references therein, we should rewrite (2) in evolutionary form and find the counterparts of $\mathcal{P}_i$ for this new form.

First of all, in order to simplify further computations note [23] that in general position we can without loss of generality assume that $\eta_i = x^i$.

To see this, introduce new independent variables $\tilde{x}^i = \eta_i(x^i)$, and let h_i stand for the inverse functions, i.e., $x^i = h_i(\tilde{x}^i)$, and new dependent variables $\tilde{H}_i$ such that $H_i = \eta'_i \tilde{H}_i$ (no sum over i). Next, for $i \neq j$ put $\tilde{\beta}_{ij} = \tilde{\partial}_i \tilde{H}_j / \tilde{H}_i$, where $\tilde{\partial}_i = \partial / \partial \tilde{x}^i$.

We find that $\tilde{H}_i = \tilde{H}_i(\tilde{x}^1, \tilde{x}^2)$ satisfy

$$\begin{aligned} \tilde{\partial}_1 \tilde{H}_2 &= \tilde{\beta}_{12} \tilde{H}_1, & \tilde{\partial}_2 \tilde{H}_1 &= \tilde{\beta}_{21} \tilde{H}_2, \\ \tilde{\partial}_1 \tilde{\beta}_{12} + \tilde{\partial}_2 \tilde{\beta}_{21} &= 0, \\ \tilde{x}^1 \tilde{\partial}_1 \tilde{\beta}_{12} + \tilde{x}^2 \tilde{\partial}_2 \tilde{\beta}_{21} + \frac{1}{2} \tilde{\beta}_{12} + \frac{1}{2} \tilde{\beta}_{21} + \tilde{H}_1 \tilde{H}_2 &= 0. \end{aligned}$$

The above system is, modulo the tildes, nothing but a special case of (1) with $\eta_i = x^i$. Thus, in what follows we shall assume without loss of generality that $\eta_1 = x^1$ and $\eta_2 = x^2$.

With this assumption in mind, we shall work below with the system

$$\begin{aligned} u_{yy} &= \frac{\frac{1}{2}v^2v_x + \frac{1}{2}uvu_y + (x-y)uu_yv_y + u^2v^3}{uv(x-y)}, \\ v_{xx} &= \frac{\frac{1}{2}uvv_x + \frac{1}{2}u^2u_y + (y-x)v v_xu_x + u^3v^2}{uv(y-x)} \end{aligned} \quad (15)$$

instead of (2).

In order to study the Hamiltonian structures admitted by (15), perform the following change of variables: put $z = (x + y)/2$ and $t = (x - y)/2$. Then (15) can be rewritten in an evolutionary form:

$$\begin{aligned} u_t &= p, \\ v_t &= q, \\ p_t &= -u_{zz} + 2p_z + \frac{u_zv_z}{v} - \frac{(v + 2zq)u_z}{2vz} - \frac{(2zup - v^2)v_z}{2uvz} + \frac{2u^2v^3 + uvp + 2zupq + v^2q}{2uvz}, \\ q_t &= -v_{zz} - 2q_z + \frac{u_zv_z}{u} + \frac{(u^2 + 2zvq)u_z}{2uvz} - \frac{(u - 2zp)v_z}{2zu} - \frac{2u^3v^2 + u^2p + uvq - 2zvpq}{2uvz}. \end{aligned} \quad (16)$$

Introduce 4×4 matrices g_i of the form

$$\begin{aligned} g_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2(z+t) & 0 \\ 0 & 0 & 0 & 2(t-z) \end{pmatrix}, \\ g_2 &= \begin{pmatrix} 0 & 0 & 2(z+t) & 0 \\ 0 & 0 & 0 & 2(z-t) \\ -2(z+t) & 0 & 3 & (g_2)_{34} \\ 0 & 2(t-z) & -(g_2)_{34} & -3 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} (g_2)_{34} &= -\frac{(z-t)u_z}{v} - \frac{(z+t)v_z}{u} - \frac{(z+t)q}{u} + \frac{(z-t)p}{v}, \\ g_1 &= \begin{pmatrix} -2(z+t) & 0 & 1 & (g_1)_{14} \\ 0 & 2(z-t) & -(g_1)_{14} & 1 \\ -3 & -(g_1)_{14} & (g_1)_{33} & (g_1)_{34} \\ (g_1)_{14} & -3 & (g_1)_{43} & (g_1)_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} (g_1)_{14} &= \frac{(z-t)u_z}{v} - \frac{(z+t)v_z}{u} - \frac{(z+t)q}{u} - \frac{(z-t)p}{v}, \\ (g_1)_{33} &= 2(z+t) \left(-\frac{u_zv_z}{uv} - \frac{qu_z}{uv} + \frac{pv_z}{uv} + 2\frac{v^2}{z} + \frac{pq}{uv} \right), \\ (g_1)_{34} &= \frac{(z-t)u_{zz}}{v} - 3\frac{(z+t)v_{zz}}{u} + \frac{(z(3v^2 - u^2) + t(u^2 + 3v^2))u_zv_z}{u^2v^2} \\ &\quad + 3\frac{(z+t)(qu_z - uq_z)}{u^2} - \frac{((t-z)up + 2v^2)v_z}{v^2u} + \frac{(t-z)p_z}{v} - 2\frac{u^2v + q}{u}, \\ (g_1)_{43} &= 3\frac{(z-t)u_{zz}}{v} - \frac{(z+t)v_{zz}}{u} + \frac{(3(t-z)u^2 + (t+z)v^2)u_zv_z}{u^2v^2} \\ &\quad + \frac{(2u^2 + (z+t)vq)u_z}{u^2v} + 3\frac{(z-t)(pv_z - vp_z)}{v^2} - \frac{(z+t)q_z}{u} - 2\frac{uv^2 + p}{v}, \\ (g_1)_{44} &= \frac{(t-z)}{(t+z)}(g_1)_{33} + 4\frac{(u^2 + v^2)(z-t)}{z}, \end{aligned}$$

and

$$g_0 = \begin{pmatrix} -1 & (g_0)_{12} & (g_0)_{13} & (g_0)_{14} \\ -(g_0)_{12} & 1 & (g_0)_{23} & (g_0)_{24} \\ -(g_0)_{13} & (g_0)_{32} & (g_0)_{33} & (g_0)_{34} \\ (g_0)_{41} & -(g_0)_{24} & (g_0)_{43} & (g_0)_{44} \end{pmatrix},$$

where

$$\begin{aligned}
 (g_0)_{12} &= \frac{(z-t)u_z}{v} + \frac{(z+t)v_z}{u} + \frac{(t-z)up + (t+z)vq}{uv}, \\
 (g_0)_{13} &= -\frac{(z-t)u_z^2}{v^2} - \frac{(z+t)u_z v_z}{uv} - \frac{(2(t-z)up + (z+t)vq)u_z}{uv^2} \\
 &\quad + \frac{(z+t)pv_z}{uv} + \frac{(z+t)uv^4 + z(t-z)up^2 + z(z+t)vpq}{v^2uz}, \\
 (g_0)_{14} &= -2\frac{(z+t)v_{zz}}{u} + 2\frac{(z+t)u_z v_z}{u^2} + 2\frac{(z+t)qu_z}{u^2} - \frac{v_z}{u} - 2\frac{(z+t)q_z}{u} - \frac{zu^2v + tu^2v + zq}{zu}, \\
 (g_0)_{23} &= -2\frac{(z-t)u_{zz}}{v} + 2\frac{(z-t)u_z v_z}{v^2} - \frac{u_z}{v} - 2\frac{(z-t)pv_z}{v^2} + 2\frac{(z-t)p_z}{v} + \frac{(z-t)uv^2 + zp}{vz}, \\
 (g_0)_{24} &= -\frac{(z-t)u_z v_z}{uv} - \frac{(z+t)v_z^2}{u^2} - \frac{(z-t)qu_z}{uv} - \frac{((t-z)up + 2(z+t)vq)v_z}{u^2v} \\
 &\quad - \frac{(z-t)u^4v + z(t-z)upq + z(z+t)vq^2}{zu^2v}, \\
 (g_0)_{32} &= \frac{(z-t)u_{zz}}{v} + \frac{(z+t)v_{zz}}{u} - \frac{(zu^2 - tu^2 + zv^2 + tv^2)u_z v_z}{u^2v^2} - \frac{(z+t)qu_z}{u^2} \\
 &\quad + \frac{((z-t)up + v^2)v_z}{uv^2} - \frac{(z-t)p_z}{v} + \frac{(z+t)q_z}{u} + \frac{(t-z)u^2v + zq}{zu}, \\
 (g_0)_{33} &= -\frac{(z+t)v_z u_{zz}}{uv} - \frac{(z+t)(q + v_z)u_{zz}}{uv} + \frac{(z+t)pv_{zz}}{uv} + \frac{(z+t)u_z^2 v_z}{u^2v} \\
 &\quad + \frac{(z+t)u_z v_z^2}{uv^2} + \frac{(z+t)qu_z^2}{u^2v} + \frac{((z+t)(uq - vp) - uv)u_z v_z}{u^2v^2} \\
 &\quad - \frac{(z+t)u_z q_z}{uv} - \frac{(z+t)pv_z^2}{uv^2} + \frac{(z+t)v_z p_z}{uv} - \frac{q(u + (t+z)p)u_z}{u^2v} \\
 &\quad + \frac{(4(z+t)uv^3 + zvp - z(t+z)pq)v_z}{zu^2v} + \frac{(z+t)(qp_z + pq_z)}{uv} + \frac{z^2pq - 2tuv^3}{z^2uv}, \\
 (g_0)_{34} &= -2\frac{(z+t)v_{zzz}}{u} + 2\frac{(z+t)(q + v_z)u_{zz}}{u^2} - 3\frac{v_{zz}}{u} + 4\frac{(z+t)u_z v_{zz}}{u^2} \\
 &\quad - 2\frac{(z+t)q_{zz}}{u} - \frac{((t-z)u^2 + 4(t+z)v^2)u_z^2(v_z + q)}{u^3v^2} + \frac{(z+t)u_z v_z^2}{u^2v} \\
 &\quad + \frac{(2(t-z)up + 3v^2 + 2(t+z)vq)u_z v_z}{u^2v^2} + 4\frac{(z+t)u_z q_z}{u^2} - \frac{(z+t)pv_z^2}{u^2v} \\
 &\quad + \frac{((z+t)u^4v - (z+t)u^2v^3 + 2z(t-z)upq + 3zv^2q + z(z+t)vq^2)u_z}{u^2v^2z} \\
 &\quad - \frac{((t+z)u^3v^2 + (t+z)uv^4 + z(t-z)up^2 + 2z(z+t)vpq)v_z}{u^2v^2z} - 3\frac{q_z}{u} \\
 &\quad - \frac{z(z-t)u^4vp - 2tu^3v^3 + z(z+t)uv^4q + z^2(t-z)up^2q + z^2(z+t)vpq^2}{v^2z^2u^2}, \\
 (g_0)_{41} &= (g_0)_{32} + \frac{u_z - p}{v} - \frac{v_z + q}{u} + 2uv, \\
 (g_0)_{43} &= 2\frac{(z-t)u_{zzz}}{v} - 4\frac{(z-t)v_z u_{zz}}{v^2} + 3\frac{u_{zz}}{v} + 2\frac{(z-t)pv_{zz}}{v^2} - 2\frac{(z-t)u_z v_{zz}}{v^2} \\
 &\quad - 2\frac{(z-t)p_{zz}}{v} - \frac{(z-t)u_z^2 v_z}{uv^2} - \frac{(4(t-z)u^2 + (t+z)v^2)u_z v_z^2}{u^2v^3} - \frac{(z-t)qu_z^2}{uv^2} \\
 &\quad - 2\frac{(z+t)qu_z v_z}{u^2v} + 2\frac{(z-t)pu_z v_z}{uv^2} - 3\frac{u_z v_z}{v^2} + \frac{(4(t-z)u^2 + (t+z)v^2)pv_z^2}{u^2v^3} \\
 &\quad + 4\frac{(z-t)v_z p_z}{v^2} - \frac{(z-t)(u^2 + v^2)u_z}{vz} - \frac{(z+t)q^2 u_z}{u^2v} + 2\frac{(z-t)pqu_z}{uv^2} \\
 &\quad + \frac{((t-z)u^2 + (t+z)v^2)v_z}{zu} - \frac{(z-t)p^2 v_z}{uv^2} + 2\frac{(z+t)pqv_z}{u^2v} + 3\frac{pv_z}{v^2}
 \end{aligned}$$

$$\begin{aligned}
& -3\frac{p_z}{v} + \frac{(z-t)u^2p}{vz} - 2\frac{utv}{z^2} + \frac{(z+t)v^2q}{zu} - \frac{(z-t)p^2q}{uv^2} + \frac{(z+t)pq^2}{u^2v}, \\
(g_0)_{44} = & \frac{(z-t)v_z u_{zz}}{uv} + \frac{(z-t)qu_{zz}}{uv} - \frac{(z-t)pv_{zz}}{uv} + \frac{(z-t)u_z v_{zz}}{uv} \\
& - \frac{(z-t)u_z^2 v_z}{u^2 v} - \frac{(z-t)u_z v_z^2}{uv^2} - \frac{(z-t)qu_z^2}{u^2 v} - \frac{(z-t)qu_z v_z}{uv^2} + \frac{u_z v_z}{uv} \\
& + \frac{(z-t)pu_z v_z}{u^2 v} + \frac{(z-t)u_z q_z}{uv} + \frac{(z-t)pv_z^2}{uv^2} - \frac{(z-t)v_z p_z}{uv} \\
& + \frac{u_z q}{uv} + \frac{(z-t)pqu_z}{u^2 v} + 4\frac{(z-t)uu_z}{z} - \frac{pv_z}{uv} + \frac{(z-t)pqv_z}{uv^2} \\
& - \frac{(z-t)qp_z}{uv} - \frac{(z-t)pq_z}{uv} + 2\frac{tu^2}{z^2} - \frac{pq}{uv}.
\end{aligned}$$

It is readily checked that the operator $\mathfrak{P} = \sum_{i=0}^3 g_i D_z^i$ is a local Hamiltonian operator and the flow (16) preserves $\mathfrak{P}$. Moreover, as $\mathfrak{P}$ is linear in t , the operators $\mathfrak{P}$ and $\partial\mathfrak{P}/\partial t$ form a Hamiltonian pair and are both preserved by the flow (16), cf. e.g. [24,25] and references therein. Clearly, $\mathfrak{P}$ is the counterpart of $\mathcal{P}_2$ while $\partial\mathfrak{P}/\partial t$ is the counterpart of $\mathcal{P}_1$ for (16).

As $\mathfrak{P}$ explicitly depends on t while the right-hand side of (16) does not, Lemma 3.8 from [20] implies that there exists no functional $\mathcal{H}$ such that (16) would be Hamiltonian with respect to $\mathfrak{P}$ with the Hamiltonian $\mathcal{H}$. Hence (16) cannot be written in a bihamiltonian form with respect to the Hamiltonian pair under study.

On the other hand, (16) is Hamiltonian with respect to $\partial\mathfrak{P}/\partial t$, that is, we can write (16) in the form

$$\mathbf{u}_t = \frac{\partial\mathfrak{P}}{\partial t} \left(\frac{\delta\mathcal{H}}{\delta\mathbf{u}} \right),$$

where $\mathbf{u} = (u, v, p, q)^T$, T indicates the transposed matrix, and $\mathcal{H} = \int h \, dz$,

$$h = \frac{zu_z^2}{v^2} + \frac{zv_z^2}{u^2} - 2\frac{zpu_z}{v^2} + 2\frac{zqv_z}{u^2} - (u-v)(v+u) + \frac{(q^2v^2 + u^2p^2)z}{u^2v^2}.$$

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