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Applying the index theorem to non-smooth operators

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ABSTRACT

We give a simple way to extend index-theoretical statements from partial differential operators with smooth coefficients to operators with coefficients of finite Sobolev order.
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In the theory of partial differential equations and in particular in index theory it is most common to consider only partial differential operators P with smooth coefficients, or, more generally, smooth pseudo-differential operators, which still map smooth sections to smooth sections. If P is moreover elliptic, this allows defining a parametrix of P, that is, an inverse up to infinitely smoothing operators. However, sometimes the assumption of smooth coefficients is not justified, e.g. in the theory of geodesic curves on some natural spaces of mappings. Here, to ensure some minimal amount of computability, one usually restricts the considerations to spaces of mappings of finite Sobolev or Hölder regularity instead of smooth mappings. Consequently, the induced metrics are also of finite regularity only, and so are the differential operators defined in terms of them, which appear in the geodesic equations. Often, as in [1], the proof of local existence and uniqueness for the geodesic equations requires an application of the index theorem to such an operator with Sobolev coefficients seen as a Fredholm map between Sobolev spaces $H^{k+p}(\pi) \to H^k(\pi)$ of sections of a vector bundle $\pi : E \to M$ over a compact manifold M. One could hope that the notion of abstract elliptic operators due to Atiyah [2] and elaborated on topological manifolds by Teleman [3] embark the case of partial differential operators with Sobolev coefficients. Amazingly enough, this does not seem to be the case, as one of the assumptions of the theorems in those articles is that multiplication with every $f \in C^0(M)$ is a bounded operator on the Hilbert spaces used, which is not true for any Sobolev Hilbert space except $L^2(M)$. Here we want to present a solution to the above problem, assuming coefficients of regularity increasing linearly with the order, a condition satisfied by the Laplacian. After the prepublication of the first version of the article, the author learned about the techniques presented in the third chapter of [4]. It would be interesting to compare the results of this note to the results about special first and second order pseudodifferential elliptic operators obtained there. The author wants to thank Ulrich Bunke for an interesting discussion on the topic of this note and Martin Bauer, Martins Bruveris, Peter Michor and an anonymous referee for many useful suggestions.

Conventions: We impose the convention $0 \in \mathbb{N}$, and for $T \in \mathbb{N}$ we define $\mathbb{N}_T := \mathbb{N} \cap [1, T]$. Throughout the article, M denotes an arbitrary smooth compact n-dimensional manifold. Contrary e.g. to [5], no smoothness assumption is made on

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the coefficients of a differential operator (as functions on the corresponding jet bundles) — they are a priori only assumed to be measurable.

Let M be a smooth compact manifold of dimension n, let $k, N \in \mathbb{N}$, and let π be an H^k vector bundle of rank N over M, i.e. there is a bundle atlas (U_t, D_t), $t \in \mathbb{N}_T$, with H^k trivialization changes. If g is an H^k vector bundle scalar product on π, ω is a volume form on M and ∇ is a metric connection on π with Christoffel symbols of regularity H^{k-1} in the bundle charts, then (π, g, ∇) is called a **metric H^k vector bundle with connection**. In this case we define in the usual way the norm $W^{l,q}(g, \nabla)$ (and the scalar product $H^l(g, \nabla)$ associated to $W^{l,2}(g, \nabla)$) on the Sobolev space $W^{l,q}(\pi)$ of $W^{l,q}$ sections of π for any $l \leq k$. A differential operator P of order $s \leq k$ on π is called k-**safe** iff for every U_t and every multiindex $i = (i_1, \ldots, i_r)$ there are $(N \times N)$-matrices $P_{i,t}$ with

$$(\mathrm{jet}^s D_t) \circ P \circ (D_t)^{-1} = \sum_{r=0}^{s} \sum_{1 \leq \ldots i_1 \leq i_2 \leq \ldots \leq i_r \leq n} P_{i,t}\, \partial_{i_1} \ldots \partial_{i_r},$$

and such that every matrix $P_{i,t}$ is of regularity $H^{p(i,k)}$ where

$$p(i,k) := \max\left\{k - s, |i| - s + \left\lfloor \frac{n}{2} \right\rfloor + 1\right\},$$

where, for $i \in \mathbb{N}$, $\lfloor i \rfloor$ is the Gauss bracket of i. The first thing to note is that $p(i,k) > \frac{n}{2}$ for all i with $|i| = s$, i.e., the leading coefficients are continuous, which means that ellipticity can be defined as usual. Obviously, the k-safe operators of degree s form a linear subspace $S^s_k(\pi)$ of the space of operators of degree s, which we equip with the scalar product $(\cdot, \cdot)$ defined by

$$(P, Q) := \sum_{t \in \mathbb{N}_T} \sum_{|i| \leq s} H^{a(i,k)}(\mathrm{pr}_2^* h, \nabla_0)(P_{i,t}, Q_{i,t})$$

where h is any scalar product on $\mathbb{R}^{N^2}$ (any two choices yield topologically equivalent scalar products), ∇_0 is the trivial connection. There are more invariant ways to define a scalar product of this kind, but the above definition will fully suffice for our purposes. We will often write S^s_k instead of $S^s_k(\pi)$. By definition, we have $S^s_{k+1} \subset S^s_k$. Obviously, if π_1 is a metric $H^{k(1)}$-vector bundle with connection and π_2 is a metric $H^{k(2)}$-vector bundle with connection then $\pi_1 \oplus \pi_2$ and $\pi_1 \otimes \pi_2$ with the corresponding natural metrics and connections are metric H^k-vector bundles with connections, for $k := \min\{k_1, k_2\}$. To make an analogous statement on the dual vector bundle, recall that on a compact manifold standard estimates (see e.g. [6], VI.3 (p. 386)) give $W^{k,p} \cdot W^{l,q} \subset W^{m,p}$ for all $p, q \in \mathbb{R}$ with $1 < p \leq q \leq \infty$ and for all $k, l, m \in \mathbb{N}$ with $k + l > m + n/q$ and $k, l \geq m$. Specializing to Sobolev Hilbert spaces ($p = q = 2$), this means $H^k \cdot H^l \subset H^{S(k,l)}$ for

$$S(k,l) := \min\{k, l, k + l - \lfloor n/2 \rfloor - 1\}, \tag{1}$$

as long as $k + l - \lfloor n/2 \rfloor - 1 \geq 0$.

Theorem 1. *Let P be a differential operator of order s on a vector bundle $\pi : E \to M$, let $l > n/2$.*

1. *$P^{(l)} := P|_{H^l(\pi)} : H^l(\pi) \to H^{l-s}(\pi)$ is bounded if and only if P is l-safe, and $e^{(l)} : P \mapsto P^{(l)}$ is a smooth linear map from $S^s_l(\pi)$ to the space of bounded linear maps from $H^l(\pi)$ to $H^{l-s}(\pi)$.*
2. *If $k > n/2$ and if π is equipped with a vector bundle metric g of regularity H^k and M has a volume form ω of regularity H^{k-s}, then for $k - s > n/2$, taking the formal $H^0(g, \omega)$-adjoint $P'_{g,\omega}$ of a k-safe operator P of order s is a bounded map $S^s_k(\pi) \to S^s_{k-s}(\pi)$.*
3. *If $k > n/2$ and $s_1 + s_2 \leq k$, then composition of operators is a bounded bilinear map $S^{s_1}_k(\pi) \times S^{s_2}_k(\pi) \to S^{s_1+s_2}_k(\pi)$.*

Proof. The first assertion can be verified by simply counting orders. The necessity of the condition can be seen by the classical fact of sharpness of $S(u, v)$ in Eq. (1) (which can be verified in an open ball in $\mathbb{R}^n$ considering the functions $x \mapsto |x|^\alpha$, for different values of α) combined with appropriate partitions of unity.

Now the formal $H^0(g)$-adjoint $P'_{g,\omega}$ of an operator $P := \sum_{|j| \leq s} P_j \partial_j$ (by linearity w.l.o.g. applied to a section u supported in a single trivialization chart) is given as

$$(P'_{g,\omega})u(x) = \sum_{|j| < s} (-1)^{|j|} \partial_j (P^g_j(x) w(x) u(x)),$$

where P^g denotes the transpose of a matrix with respect to g and $\omega = w\, dx_1 \wedge \cdots \wedge dx_n$ in the local coordinates. Thus for the coefficients of $P'_{g,\omega}$ we get

$$(P'_{g,\omega})_j = \sum_{l \geq j} (-1)^{|j|} \binom{l}{j} \partial_{l-j}(P^g_l \cdot w),$$

where we used subtraction of multiindices. If P is k-safe, then P_l, $P^g_l \in H^{a(l,k)}$, and $P'_{g,\omega}$ is seen to be $(k - s)$-safe by counting orders. It is easy to see that the map is continuous.

For the third assertion, one can invoke the equivalence of the first assertion or simply calculate as follows: Let P, Q be two k-safe operators with coefficients P_j and Q_j, respectively. Then, for the coefficient of $A := P \circ Q$ belonging to the multiindex J we get similarly

$$A_J = \sum_{j_1+j_2\geq J\geq j_2} \binom{j_1}{j_1+j_2-J} P_{j_1}\partial_{j_1+j_2-J}Q_{j_2} \in H^{k-(s_1+s_2)+|J|} = H^{k-\deg A+|J|},$$

thus A is k-safe, and clearly the map is continuous. ■

Theorem 2. *Let P be an elliptic l-safe operator of order s with $l > n/2$.*

1. *If the formal adjoint of P is $(l-s)$-safe, then $P^{(m)}$ is Fredholm for all $m \leq l$, and its index equals the topological index.*
2. *If P is formally self-adjoint, then* $\mathrm{ind}(P^{(l)}) = 0$.

Proof. Theorem 1 (Item 2) and the fact that $S^s_{k+1} \subset S^s_k$ imply that we can w.l.o.g. focus on the case $m = l$. To show that $P^{(l)} : H^l \to H^{l-s}$ is Fredholm, the main tool is the following elliptic estimate for nonsmooth coefficients proven in the Appendix:

Lemma 1 (*Nonsmooth Elliptic Regularity*)**.** *Let $k > n/2$, and let P be an elliptic k-safe operator of order s, then there is $C > 0$ such that for every $q \in \mathbb{N}$ with $q \leq k$ and every $u \in H^{q+s-1}$ with $Pu \in H^q$ we have $u \in H^{q+s}$, and $\|u\|_{q+s} \leq C(\|Pu\|_q + \|u\|_{q+s-1})$.*

We also use the following quite classical closed range lemma whose proof we leave to the reader:

Lemma 2. *Let X, Y, Z be Banach spaces, let $P : X \to Y$, $K : X \to Z$ be bounded linear operators, K compact. If there is a constant $C > 0$ such that for each $x \in X$ we have*

$$\|x\|_X \leq C\big(\|Px\|_Y + \|Kx\|_Z\big),$$

then $P(X)$ is a closed subspace of Y, and $\ker(P)$ *is finite-dimensional.*

The lemma applied to P and $q = s - l$ and to the compact canonical inclusion of H^{q+s} in H^{q+s-1} yields that P has closed image and finite-dimensional kernel. For the corresponding statement on the cokernel, we use $\mathrm{coker}(P) := P(H^l)^\perp = \ker((P'_{g,\omega})^{(l-s)})$ for the adjoint $(P'_{g,\omega})^{(l-s)} : H^{l-s} \to H^{l-2s}$ and the fact that for an elliptic operator P its formal adjoint $P'_{g,\omega}$ is elliptic again, due to the relation $\sigma(P') = (-1)^s\sigma(P)^g$ between the principal symbols $\sigma(P)$ of P and $\sigma(P')$ of P'. Thus we can apply Lemma 1 and the closed range lemma to the formal adjoint P' to show the statement on the cokernel in H^{l-s}.

Using that C^∞ is dense in any Sobolev space, we approximate every local coefficient $P_{i,t}$ of P by sequences $P_{i,t}(n)$ of smooth coefficients converging in $H^{a(i,k)}$. The (by means of a smooth partition of unity) so defined differential operators $P(n)$ are elliptic for sufficiently large n, and they converge to P in the operator norm from H^l to H^{l-s} due to Theorem 1, first assertion. As $P^{(l)}$ is Fredholm, and as the index is locally constant in the open subset of Fredholm operators with the subspace topology, application of the classical index theorem for smooth coefficients completes the proof of the first assertion.

For the second assertion, use that Fredholmness implies the last equality in
$\dim \mathrm{coker} P^{(l)} = \dim \ker(P')^{(l-s)} = \dim \ker P^{(l-s)} = \dim \ker P^{(l)}$. ■

Let us consider examples of k-safe operators.

Theorem 3. *Let $(\pi : E \to M, \gamma, \nabla)$ be a metric H^k vector bundle with connection over M. Then $\nabla : \Gamma(\pi) \to \Gamma(\tau'M) \otimes \Gamma(\pi)$ is k-safe. If moreover the smooth tangent bundle $\tau M : TM \to M$ is equipped with a metric H^{k-1} metric for $k > \frac{n}{2} + 1$, then the Bochner Laplacian Δ^π on π is k-safe.*

Proof. The operator ∇ is k-safe by counting orders: The first coefficients are smooth (and constant) and the second are in H^k. Now, $\tau'M \otimes \pi$ is a metric H^{k-1}-vector bundle. Concretely, its connection $\tilde\nabla : \Gamma(\tau'M \otimes \pi) \to \Gamma((\tau'M)^2 \otimes \pi)$ maps $b \otimes f$ to $\sum a_i \otimes (\nabla_{a_i^g} b \otimes f + b \otimes \nabla_{a_i^g} f)$, where a_i is a frame for $\tau'M$ and a_i^g denotes the g-equivalent vector field. The regularities of the coefficients P_2, P_1, P_0 of $P := \tilde\nabla \circ \nabla$ are $p_2 = \infty$, $p_1 = k-1$, $p_0 = k-1$. As $tr^g = g^{-1}\circ$, which is an H^{k-1} endomorphism (by the cofactor formula for the inverse matrix and e.g. the analogue of Faà di Bruno's formula for higher (weak) derivatives of the reciprocal of a function, always keeping in mind that g is continuous and thus is uniformly positive by compactness), the corresponding regularities q_2, q_1, q_0 of $Q := \Delta^\pi := \mathrm{tr}^g(\tilde\nabla \circ \nabla)$ are $q_2 = k-1$, $q_1 = k-1$, $q_0 = k-1$, in particular $q_j \geq j - 2 + \lfloor n/2 \rfloor + 1$ and $q_j \geq k-2$, thus Δ^π is k-safe due to Theorem 1. ■

Theorem 4. *Let $(N, \overline{g})$ be a Riemannian manifold, $f \in H^k(M, N)$ an immersion. Then $(f^*\tau N, f^*\overline{g}, f^*\nabla)$ is a metric H^k vector bundle with connection, whereas $(\tau M, h := f^*\overline{g}, \nabla^h)$ is a metric H^{k-1} vector bundle with connection. Moreover, the second fundamental form of f is an H^{k-1} endomorphism.*

Proof. If we denote the bundle chart transitions of a bundle π by t_{ij}, the bundle chart transitions of the pull-back bundle $f^*\pi$ are $t_{ij} \circ f$. Let $S \in C^1(f^*\tau N)$ and let $\{x_i\}$ be local coordinates in a neighborhood U of $q = f(y) \in f(M)$ with associated

Christoffel symbols $\Gamma^\beta_{\gamma\alpha}$ and the corresponding decomposition $S(y') = \sum_\alpha S_\alpha(y') \cdot (\frac{\partial}{\partial x_\alpha} \circ f)$ for y' in the neighborhood $f^{-1}(U)$ of y. For $S \in H^l$, we have

$$(f^*\nabla)_X S = \left(\sum_\beta X(S_\beta) + \sum_{\gamma,\alpha,\beta} \Gamma^\beta_{\gamma\alpha} \cdot (d_y f \cdot X)_\gamma \cdot S_\alpha \right) \cdot \left(\frac{\partial}{\partial x_\beta} \circ f \right),$$

and as $\Gamma^\beta_{\gamma\alpha} \in C^\infty(U)$ and $d_y f \cdot X \in H^{k-1}$, the Christoffel symbols $\tilde{\Gamma}^\beta_{i\alpha} := \sum \Gamma^\beta_{\gamma\alpha} \cdot \frac{\partial f_\gamma}{X_i}$ in these charts are of regularity H^{k-1}, thus the bundle $f^*\tau N$ is a metric H^k vector bundle with connection. As for the second fundamental form, $T(f(N))$ is an H^k submanifold of TN, and the orthogonal projection p to its pullback in $f^*\tau N$ is an H^k endomorphism. Thus $S := p \circ f^*\nabla$ is a k-safe operator of order 1 and an endomorphism of regularity H^{k-1}. ■

We write $g := f^*\overline{g}$ on $f^*\tau N$, and for $m \in \mathbb{N}$ and $C \in \mathbb{R}$ we define $P_{C,m} := \Delta^g - C \cdot Q_m$ where C is a constant and Q_m is the mth invariant polynomial of the second fundamental form of f, e.g. $Q_0 := |\mathrm{tr}_g S^f|^2$, the mean curvature. Then Theorems 2–4 imply

Theorem 5. *Let $(N, \overline{g})$ be a Riemannian manifold, $k > n/2$ and $f \in H^k(M, N)$. If $l \in \mathbb{N}$, $2 \le l$, then $P_{C,m}|_{H^l}$ is a Fredholm operator from H^l to H^{l-2} for all $l \le k$ and its index is zero. Moreover, $P_{C,m}$ is a smooth map from $\mathrm{Imm}^k(M, N)$ to $H^l(f^*\tau N)' \otimes H^{l-2}(f^*\tau N)$ for all $2 \le l \le k$.* ■

In order to illustrate the subtleties of the use of the index theorem in the context of the geodesic equation on Hilbert manifolds, let us consider the remarkable article [1] of Bauer, Harms and Michor on spaces of mappings and shape spaces. As noted by the authors themselves, the proof of their statement on Page 415 about local well-posedness of the geodesic equation for a metric defined by an operator P is not entirely correct, as the assumptions 1–3 on $P : f \mapsto P_f$ on Page 414 concern exclusively smooth maps f, thus it is unclear whether each such P necessarily extends continuously from the space of smooth immersions between two manifolds M and N to $\mathrm{Imm}^{k+2q}(M, N)$ taking at one such immersion f a value P_f in the bounded linear operators from $T_f\mathrm{Imm}^{k+2q}(M, N) = H^{k+2q}(f^*\tau N)$ to $T_f\mathrm{Imm}^k(M, N) = H^k(f^*\tau N)$ (consider e.g. $P_f := (1 + Q_0) \cdot \Delta^{f^*\tau N}$). To make things easy, let us just include in the assumptions that *P extends continuously to each Sobolev manifold $\mathrm{Imm}^{k+2q}(M, N)$, and that this extension is a $(k + 2q)$-safe operator from each $T_f\mathrm{Imm}^{k+2q}(M, N)$ to $T_f\mathrm{Imm}^k(M, N)$*. As shown in Theorem 5, this assumption is satisfied by the principal example $\mathbf{1} + \Delta^q$ of that article. Let us call the local well-posedness of geodesics with this new assumption on P the *modified BHM Theorem*. In the continuation of the proof, the authors refer to Theorem 26.2 of Shubin's book [5] to show vanishing index (implying then bijectivity) of any P_f. But the theorem in this reference applies to operators with smooth coefficients only, using explicitly that the operator maps smooth sections to smooth sections. Thus the theorem in Shubin's book cannot be applied in the present situation. Rather, one should use Theorem 5 for $l = d = k + 2q$, completing the proof of the modified BHM Theorem. In a nutshell, the gap in the cited article can be closed by the additional assumption of safeness, which, due to Theorems 1 and 5, holds true in the example $P := \mathbf{1} + \Delta^q$, inducing in fact a C^∞ map

$$P : \mathrm{Imm}^{k+2q} \ni f \mapsto P_f \in L(T_f\mathrm{Imm}^{k+2q}, T_f\mathrm{Imm}^k).$$

Let us remark that the results obtained in this article could be extended in various ways:

1. Given sufficient regularity, using difference operators it is easy to see that even the stronger estimate $|u|_{l+s} < C(|Pu|_l + |u|_l)$ holds in Lemma 1.
2. Everything could be done for nonempty boundary and Dirichlet or Neumann boundary conditions.
3. One can generalize the definition of k-safeness to pseudodifferential operators such that the statements above still hold.
4. Using the treatment of multiplicativity properties of negative Sobolev spaces as e.g. in [7] one could extend the k-safeness condition correspondingly.
5. Finally, requiring appropriate decay conditions and using carefully modified Sobolev ring properties one could transfer the statement to the noncompact setting.

Appendix. Elliptic estimates for Sobolev coefficients

Now we want to the prove Lemma 1, which is to be performed in close analogy to the proof for the classical standard elliptic estimates in the smooth case as laid down for example in Taylor's book [8], Th. 5.11.1., one of the most foundational results of elliptic theory, and it is quite likely that the case of Sobolev coefficients has been considered already, yet the author could not find a version of the lemma in the literature.

First of all, pick a covering U of M by finitely many open chart neighborhoods $U_1, \dots, U_n$ with charts $\kappa_1, \dots, \kappa_n$. W.l.o.g. the chart neighborhoods $\kappa_i(U_i)$ are precompact domains in $\mathbb{R}^n$. Choose a smooth partition of unity ψ_i subordinate to U. Then the operator $Q_i := [P, \psi_i \cdot]$ is of order $s - 1$, and Theorem 1 implies that it is k-safe.

It is easy to show the estimate of the lemma for $p = 0$: Ellipticity (entailing uniform ellipticity as M is compact) implies that $\|\nabla^i u\|_{L^2} \le C\|Pu\|_{L^2}$ if $|i| = s$ (cf. [8]), and application of the interpolation estimates yields $\|u\|_{H^s} \le C\|Pu\|_{L^2} + \|u\|_{L^2}$ (cf. [8]).

For $p > 0$, as a first step we want to prove that we can *localize* the statement in the sense that it suffices to show it for $\mathrm{supp}(u) \in V$, where V is an open precompact set in $\mathbb{R}^n$. To see this, suppose we know the estimate for that case, then (the ψ_i being elements of a partition of unity and the C_n constants independent of u) we get

$$\begin{aligned}\|u\|_{q+s} &\leq C_1 \sum_i \|\psi_i u\|_{q+s} \leq C_2 \sum_i \big(\|P\psi_i u\|_q + \|\psi_i u\|_{q+s-1}\big)\\ &\leq C_3 \sum_i \big(\|\psi_i P u\|_q + \|Q_i u\|_q + \|\psi_i u\|_{q+s-1}\big) \leq C_4(\|Pu\|_q + \|u\|_{q+s-1}).\end{aligned}$$

Thus, indeed, the statement for compactly supported operators in $\mathbb{R}^n$ implies the statement on compact manifolds. Consequently, in the following we suppress the t-indices in the coefficients P_i.

Our ansatz to prove the statement on $V \subset \mathbb{R}^n$ is to freeze the coefficients, like in Taylor's proof. But evaluating the coefficients on a lattice, as it is done in [8], would not be a well-defined operation in our case, unless the coefficient P_i in question is continuous, that is to say, for $|i| = s$. Instead of evaluating we will consider means of coefficients, i.e. associate for $z \in \mathbb{Z}^n$ and the point εz on the lattice $\varepsilon\mathbb{Z}^n$ the value $\hat{P}_{i,\varepsilon}(\varepsilon z) = \fint_{B_\varepsilon(\varepsilon z)} P_i = \mathrm{vol}(B_\varepsilon(\varepsilon z))^{-1} \cdot \int_{B_\varepsilon(\varepsilon z)} P_i$, and we define $P_\varepsilon(y) := \sum_{|i|\leq s} \hat{P}_i(y)\partial_i$. Choose a partition of unity in $\mathbb{R}^n$ of the form $\{\phi_z = \phi_0(\cdot - z) | z \in \mathbb{Z}^n\}$, for a smooth compactly supported function ϕ_0 on $\mathbb{R}^n$. Such a partition can easily be constructed by pulling back a smooth partition of unity on the torus $\mathbb{T}^n$ subordinated to a trivializing atlas of the universal covering $\mathbb{R}^n \to \mathbb{T}^n$. Then define partitions of unity $\{\phi_{\varepsilon z,\varepsilon} | z \in \mathbb{Z}^n\}$ in $\mathbb{R}^n$ centered at the rescaled lattice $\varepsilon\mathbb{Z}^n$ by $\phi_{\varepsilon z,\varepsilon}(x) := \phi_0(\frac{x-\varepsilon z}{\varepsilon}) = \phi_0(\frac{x}{\varepsilon} - z) = \phi_z(x/\varepsilon)$. In the end we will use $\lim_{\varepsilon\to 0} \sum_{y\in\varepsilon\mathbb{Z}^n} \hat{P}_{i,\varepsilon}(y)\phi_{y,\varepsilon} = P_i$ in C^0 for $|i| = s$, as those coefficients are continuous. The constant coefficient case is then treated by standard Fourier methods. Now $Pu = f$ implies

$$\phi_{\varepsilon z,\varepsilon} P_\varepsilon(\varepsilon z)u = \phi_{\varepsilon z,\varepsilon} f - R_{z,\varepsilon} u$$

where

$$R_{z,\varepsilon}(x) := \phi_{\varepsilon z,\varepsilon} \sum_{|i|\leq s} \big(P_i(x) - \hat{P}_{i,\varepsilon}(\varepsilon z)\big)\partial_i. \tag{2}$$

Thus, with $Q_{z,\varepsilon} := [P_\varepsilon(\varepsilon z), \phi_{\varepsilon z,\varepsilon}]$, which is a smooth differential operator of order $s - 1$, we get

$$P_\varepsilon(\varepsilon z)(\phi_{\varepsilon z,\varepsilon} u) = \phi_{\varepsilon z,\varepsilon} f - R_{z,\varepsilon} u - Q_{z,\varepsilon} u. \tag{3}$$

Now, for a differential operator A on $\mathbb{R}^n$ let $\tilde{A} := \mathcal{F}^{-1} \circ A \circ \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform. Ellipticity of P means that $\widetilde{P_\varepsilon(\varepsilon z)}|_{\mathbb{R}^n\setminus B_K(0)}$ is bounded from zero for all $K > 0$. Therefore homogeneity of $\widetilde{P_{\varepsilon z,\varepsilon}}$ implies that we can pick $\kappa \in C^\infty(\mathbb{R}^n, \mathbb{R})$ of compact support and with $\kappa(B_K(0)) = \{1\}$ and define a pseudodifferential operator $E_{z,\varepsilon}$ by

$$\widetilde{E_{z,\varepsilon}} := (1 - \kappa)\widetilde{P_\varepsilon(\varepsilon z)}^{-1}.$$

Then there is a $C > 0$ independent of ε such that $|\widetilde{E_{z,\varepsilon}}(\xi)| \leq C(1 + \xi)^{-s}$, and thus , for all $\sigma \in \mathbb{N}$, $E_{z,\varepsilon} : H^\sigma \to H^{\sigma+s}$ is continuous with operator norm bounded uniformly in ε:

$$\exists K_\sigma > 0 \ \forall \varepsilon > 0 \ \forall v \in H^\sigma : \|v\|_{\sigma+s} \leq K \cdot \|v\|_\sigma.$$

Furthermore, we have

$$E_{z,\varepsilon} \circ P_\varepsilon(\varepsilon z) = \mathbf{1} + \rho_{z,\varepsilon} \tag{4}$$

for a pseudodifferential operator $\rho_{z,\varepsilon}$ with $\widetilde{\rho_{z,\varepsilon}} \in C_c^\infty$, thus $\rho_{z,\varepsilon}$ is infinitely smoothing, that is, maps any Sobolev space H^σ to C^∞, and its operator norms $H^\sigma \to H^\tau$ are bounded for all choices of σ and τ. Now we apply $E_{z,\varepsilon}$ to Eq. (3), insert Eq. (4) and take the sum over all $z \in \mathbb{Z}^n$, obtaining

$$u = \sum_{z\in\mathbb{Z}^n} E_{z,\varepsilon}(\phi_{\varepsilon z,\varepsilon} f) + E_{z,\varepsilon} R_{z,\varepsilon} u + E_{\varepsilon z,\varepsilon} Q_{z,\varepsilon} u + \rho_{z,\varepsilon}(\phi_{\varepsilon z,\varepsilon} u).$$

Now it is all about estimating the four terms on the right-hand side. The last one is unproblematic as it contains an infinitely smoothing operator. For the first one, using that ϕ is smooth and compactly supported and counting orders, we obtain

$$\left\| \sum_{z\in\mathbb{Z}^n} E_{z,\varepsilon}(\phi_{\varepsilon z,\varepsilon} f) \right\|_l \leq C(l, \varepsilon)\|f\|_{l-s}.$$

Taking into account that $\deg(Q_{z,\varepsilon}) < s - 1$, the two last terms can be estimated as

$$\left\| \sum_{z\in\mathbb{Z}^n} E_{z,\varepsilon} Q_{z,\varepsilon} u + \rho_{z,\varepsilon}(\phi_{\varepsilon z,\varepsilon} u) \right\|_l \leq C(l, \varepsilon)\|u\|_{l-1}.$$

Finally, looking at Eq. (2), using that the highest coefficients of P are continuous (as $k > n/2$) and thus uniformly continuous as M is compact, for all $l \leq k$ the second term can be estimated as

$$\begin{aligned}
\left\| \sum_{z\in\mathbb{Z}^n} E_{z,\varepsilon} R_{z,\varepsilon} u \right\|_l &\leq K_l \|R_{z,\varepsilon} u\|_{l-s} \\
&= K_l \cdot \left\| \sum_{z\in\mathbb{Z}^n} \Phi_{\varepsilon z,\varepsilon} \sum_{|i|\leq s} (P_i(x) - \hat{P}_{i,\varepsilon}(\varepsilon z))\partial_i u \right\|_{l-s} \\
&= K_l \cdot \left\| \sum_{z\in\mathbb{Z}^n} \sum_{|k|\leq l-s} \partial_k \left(\Phi_{\varepsilon z,\varepsilon} \sum_{|i|\leq s} (P_i(x) - \hat{P}_{i,\varepsilon}(\varepsilon z))\partial_i u \right) \right\|_0 \\
&\leq K_l \cdot \left\| \sum_{z\in\mathbb{Z}^n} \sum_{|k|=l-s,|i|=s} \Phi_{\varepsilon z,\varepsilon} \cdot (P_i(x) - \hat{P}_{i,\varepsilon}(\varepsilon z))\partial_i \partial_k u \right\|_0 + C(l,\varepsilon)\|u\|_{l-1} \\
&\leq C(l)A(\varepsilon)\|u\|_l + C(l,\varepsilon)\|u\|_{l-1},
\end{aligned}$$

with a constant $C(l)$ independent of ε and with $A(\varepsilon) = \|\sum_{|i|=s}\sum_{z\in\mathbb{Z}^n} \Phi_{\varepsilon z,\varepsilon} \cdot (P_i(x) - \hat{P}_{i,\varepsilon}(\varepsilon z))\|_\infty$. Now, as $|i| = s$ means considering the highest coefficients, which are continuous, we have $\lim_{\varepsilon\to 0} A(\varepsilon) = 0$. Thus we can choose $\varepsilon < (2C(l))^{-1}$ and incorporate the second term into the left-hand side yielding a factor $1/2$, by which the desired form of the estimate is obtained. ■

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