Journal of Geometry and Physics 151 (2020) 103622

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Spacetimes with different forms of energy–momentum tensor

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ARTICLE INFO

Article history:
Received 4 October 2019
Received in revised form 17 January 2020
Accepted 13 February 2020
Available online 19 February 2020

MSC:
53B30
53C50
53C80

Keywords:
Perfect fluid spacetime
Einstein's field equation
Energy–momentum tensor
Codazzi type tensor

ABSTRACT

The object of the present paper is to characterize spacetimes with different types of energy–momentum tensor. At first we consider spacetimes with pseudo symmetric energy–momentum tensor T. We obtain a necessary and sufficient condition for a spacetime with pseudo symmetric energy–momentum tensor to be a pseudo Ricci symmetric spacetime. Next we consider the spacetimes with Codazzi type of energy–momentum tensor and several interesting results are pointed out. Moreover, some results related to perfect fluid spacetimes with different forms of energy–momentum tensors have been obtained. We study spacetimes with quadratic Killing energy–momentum tensor T and show that a GRW spacetime with quadratic Killing energy–momentum tensor is an Einstein space. Finally, we have considered general relativistic spacetimes with semisymmetric energy–momentum tensor and obtained some important results.

1. Introduction

The semi-Riemannian geometry is an important branch of modern differential geometry. It has many applications (directly or indirectly) in the sciences, the medical sciences and engineering. A semi-Riemannian manifold of dimension n is a smooth n-dimensional differentiable manifold equipped with a pseudo-Riemannian metric of signature (p, q), where $n = p+q$. Lorentzian manifold is a special case of a semi-Riemannian manifold. The signature of the metric of a Lorentzian manifold is $(-, +, +, \ldots\ldots, +)$. In a Lorentzian manifold there exists three type of vectors named timelike, spacelike and null vector. In general, a Lorentzian manifold may not have a globally timelike vector field. A Lorentzian manifold has many applications in applied physics, especially in the theory of relativity and cosmology. To study the Lorentzian manifold, the causal character of the vector fields plays an important role and thus it becomes the advantageous choice for the researchers to study the theory of relativity and cosmology. By launching its study on Lorentzian manifolds the general theory of relativity opens the way to the study of global questions about it [2,6,12,16,17] and many others. If a Lorentzian manifold admits a globally timelike vector field, it is called time oriented Lorentzian manifold, physically known as spacetime. Thus a spacetime is a 4-dimensional time oriented Lorentzian manifold.

Modern differential geometry has become more and more important in theoretical physics which it has led to a greater simplicity in mathematics and a more fundamental understanding of physics.

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1 The third author is supported by Grant Project No. NRF-2018-R1D1A1B-05040381 from National Research Foundation of Korea.

https://doi.org/10.1016/j.geomphys.2020.103622
0393-0440/© 2020 Published by Elsevier B.V.

Einstein's field equation without cosmological constant is conferred by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{1.1}$$

where r is the scalar curvature of the manifold and $\kappa \neq 0$. Eq. (1.1) of Einstein imply that matter determines the geometry of spacetime and conversely that the motion of matter is determined by the metric tensor of the space which is not flat.

A Lorentzian manifold is said to be a perfect fluid spacetime if the Ricci tensor S is of the form:

$$S = ag + bA \otimes A,$$

where a, b are scalars and A is non-zero 1-form. The vector field ρ metrically equivalent to the 1-form A is a unit timelike vector, that is, $A(\rho) = -1$.

In perfect fluid spacetime, the energy–momentum tensor T of type $(0, 2)$ is of the form [26]:

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{1.2}$$

where σ and p are the energy density and the isotropic pressure respectively. The velocity vector field ρ metrically equivalent to the non-zero 1-form A is a timelike vector, that is, $g(\rho, \rho) = -1$. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [16].

A perfect fluid is called isentropic [16] if the equation of state satisfies the condition

$$p = p(\sigma),$$

where σ and p are the energy density and the isotropic pressure respectively.

Recently, De et al. [8]) studied spacetimes with semisymmetric energy–momentum tensor. Also Chaki et al. [5] studied perfect fluid spacetimes with covariant constant energy–momentum tensor. Moreover, in [18] Mallick, Suh and De studied spacetime with pseudo-projective curvature tensor. Also several authors have studied spacetimes in different way such as [7,8,11,14,15,24,31] and many others.

A non-flat Lorentzian manifold is called pseudo-Ricci symmetric [4] and denoted by $(PRS)_n$ if the Ricci tensor S of type (0,2) of the manifold is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \tag{1.3}$$

where ∇ denotes the Levi-Civita connection and G is a non-zero 1-form such that

$$g(X,\rho) = A(X), \tag{1.4}$$

for all vector fields X; ρ being the vector field corresponding to the associated 1-form A. If in (1.3) the 1-form $A = 0$, then the manifold reduces to Ricci symmetric manifold ($\nabla S = O$).

Let (M, g) be a spacetime with Levi-Civita connection ∇. A quadratic Killing tensor is a generalization of a Killing vector and is defined as a second order symmetric tensor K satisfying the condition

$$(\nabla_X K)(Y,W) + (\nabla_Y K)(W,X) + (\nabla_W K)(X,Y) = 0. \tag{1.5}$$

The conformal curvature tensor play an important role in differential geometry and also in general theory of relativity. The Weyl conformal curvature tensor C in a Lorentzian manifold (M^n, g) $(n > 3)$ is defined by [30]

$$\begin{aligned} C(X,Y)W = {} & R(X,Y)W - \frac{1}{n-2}[g(Y,W)QX - g(X,W)QY \\ & + S(Y,W)X - S(X,W)Y] \\ & + \frac{r}{(n-1)(n-2)}[g(Y,W)X - g(X,W)Y], \end{aligned} \tag{1.6}$$

where r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(QX, Y) = S(X, Y)$. If the dimension $n = 3$, then the conformal curvature tensor vanishes identically. A Lorentzian manifold of dimension $n(n > 3)$ is said to be conformally flat if the conformal curvature tensor C is identically zero.

It is known [9] that in a manifold of dimension $n > 3$,

$$\begin{aligned} (divC)(X,Y)Z = {} & \frac{n-3}{n-2}[\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ & + \frac{1}{2(n-1)}\{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}]. \end{aligned} \tag{1.7}$$

Motivated by the above studies in the present paper we characterize spacetimes with different types of energy–momentum tensor. After introduction in the different Sections we give some results and definitions and prove several results but few are:

Theorem 1.1. *In a spacetime with pseudo symmetric energy–momentum tensor, $\frac{5r}{6}$ is an eigenvalue corresponding to the eigenvector ρ and the associated 1-form A is closed.*

Theorem 1.2. *A necessary and sufficient condition for a spacetime with pseudo symmetric energy–momentum tensor to be a pseudo Ricci symmetric spacetime is that the scalar curvature vanishes.*

Theorem 1.3. *A perfect fluid spacetime with pseudo symmetric energy–momentum tensor is isentropic.*

Theorem 1.4. *A spacetime with pseudo symmetric and Codazzi type of energy–momentum tensor is Ricci flat.*

Theorem 1.5. *A general relativistic spacetime with Codazzi type of energy–momentum tensor is shear-free, irrotational and its energy density is constant over the spacelike hypersurface orthogonal to the 4-velocity vector.*

Theorem 1.6. *A spacetime has quadratic Killing energy–momentum tensor iff the Ricci tensor of the spacetime is quadratic Killing tensor.*

Theorem 1.7. *In a perfect fluid spacetime with quadratic Killing energy–momentum tensor, the integral curves of the flow vector are geodesics.*

Theorem 1.8. *A perfect fluid spacetime with semisymmetric energy–momentum tensor is limiting case of dark energy and limiting case of violating the strong energy conditions.*

Theorem 1.9. *A GRW spacetime with quadratic Killing energy–momentum tensor is an Einstein space.*

Theorem 1.10. *A general relativistic spacetime with semisymmetric energy–momentum tensor is Riemann compatible.*

2. Spacetimes with pseudo symmetric energy–momentum tensor

In this section we study spacetimes with pseudo symmetric energy–momentum tensor.
Energy–momentum tensor T is said to be pseudo symmetric if

$$(\nabla_X T)(Y,Z) = 2A(X)T(Y,Z) + A(Y)T(X,Z) + A(Z)T(Y,X), \tag{2.1}$$

where A is a non-zero 1-form such that

$$A(X) = g(X,\rho), \tag{2.2}$$

ρ is a unit timelike vector field.

Taking covariant derivative of (1.1) with respect to Z we get

$$(\nabla_Z S)(X,Y) - \frac{1}{2}dr(Z)g(X,Y) = \kappa(\nabla_Z T)(X,Y). \tag{2.3}$$

Combining equations (1.1), (2.1) and (2.3) we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) - \frac{1}{2}dr(Z)g(X,Y) =\ & 2A(Z)S(X,Y) + A(X)S(Y,Z)\\ & + A(Y)S(X,Z) - rA(Z)g(X,Y)\\ & - \frac{1}{2}rA(X)g(Y,Z) - \frac{1}{2}rA(Y)g(X,Z).\end{aligned} \tag{2.4}$$

Proof of Theorem 1.1. Since the energy–momentum tensor is pseudo symmetric, Eq. (2.1) holds.

Let $\tilde{T}$ be the (1, 1)-energy momentum tensor such that $g(\tilde{T}X,Y) = T(X,Y)$. Then Eq. (2.1) can be rewritten as

$$(\nabla_X\tilde{T})(Y) = 2A(X)\tilde{T}(Y) + A(Y)\tilde{T}(X) + T(X,Y)\rho,$$

which implies

$$g((\nabla_X\tilde{T})Y,Z) = 2A(X)g(\tilde{T}(Y),Z) + A(Y)g(\tilde{T}(X),Z) + T(X,Y)g(\rho,Z), \tag{2.5}$$

where $g(X,\rho) = A(X)$ for all X.

Now taking a frame field and putting $X = Z = e_i$ in (2.5), where $\{e_i\}$ forms an orthonormal basis of the tangent space at each point of the spacetime, we infer

$$(div\tilde{T})(Y) = 3T(Y,\rho) + A(Y)t, \tag{2.6}$$

where t = trace of $\tilde{T}$.

Remembering that $(div\tilde{T})(Y) = 0$, from (2.6) we get

$$3T(Y, \rho) = -A(Y)t. \tag{2.7}$$

Also from (1.1) we obtain

$$\kappa t = -r. \tag{2.8}$$

Eqs. (2.7) and (2.8) together give

$$S(Y, \rho) = \frac{5r}{6}A(Y). \tag{2.9}$$

Again taking frame field and putting $Y = Z = e_i$, from (2.1) we obtain

$$(\nabla_X T)(e_i, e_i) = 2A(X)t + 2T(X, \rho). \tag{2.10}$$

In virtue of Eqs. (2.7) and (2.10) we have

$$\nabla_X t = \frac{4}{3}A(X)t. \tag{2.11}$$

Using Eq. (2.8) in Eq. (2.11) we infer

$$dr(X) = \frac{4r}{3}A(X), \tag{2.12}$$

which implies that the 1-form A is closed.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let us assume that the spacetime has vanishing scalar curvature. Then from (2.4) it follows that

$$(\nabla_Z S)(X, Y) = 2A(Z)S(X, Y) + A(X)S(Y, Z) + A(Y)S(X, Z), \tag{2.13}$$

which implies that the spacetime is pseudo Ricci symmetric.

Conversely, suppose that the spacetime is pseudo Ricci symmetric. Then we have

$$(\nabla_Z S)(X, Y) = 2A(Z)S(X, Y) + A(X)S(Y, Z) + A(Y)S(X, Z), \tag{2.14}$$

Using (2.13)in (2.3), we obtain

$$\begin{aligned}(\nabla_Z T)(X, Y) = {} & 2A(Z)T(X, Y) + A(X)T(Y, Z) + A(Y)T(X, Z) \\ & + \frac{1}{\kappa}\{rA(Z)g(X, Y) + \frac{r}{2}A(X)g(Y, Z) + \frac{r}{2}A(Y)g(X, Z) \\ & - \frac{1}{2}dr(Z)g(X, Y)\}.\end{aligned} \tag{2.15}$$

In virtue of Eqs. (2.13) and (2.15), proof of Theorem 1.2 follows.

Proof of Theorem 1.3. Now we consider perfect fluid spacetime whose energy–momentum tensor is pseudo symmetric. Then Eq. (1.1) takes the form:

$$S(X, Y) - \frac{r}{2}g(X, Y) = \kappa\{(\sigma + p)A(X)A(Y) + pg(X, Y)\}. \tag{2.16}$$

Putting $Y = \rho$ in (2.16) and then using (2.9) we obtain

$$r + 3\kappa\sigma = 0, \tag{2.17}$$

since $A \neq 0$ and $A(\rho) = -1$.

Taking a frame field and contracting over X and Y, we have from (2.16) that

$$r = \kappa(\sigma - 3p). \tag{2.18}$$

Eqs. (2.17) and (2.18) together yield

$$p = \frac{4\sigma}{3}. \tag{2.19}$$

This is what the proof of Theorem 1.3.

3. Spacetimes with pseudo symmetric and Codazzi type of energy–momentum tensor

A $(1, 1)$-tensor field K on a Riemannian or a semi-Riemannian manifold (M, g) is said to be of Codazzi type [3] if it satisfies the condition

$$(\nabla_X K)Y = (\nabla_Y K)X,$$

where ∇ is the Riemannian connection on g and X, Y are arbitrary vector fields on M. A (0,2)-tensor is said to be of Codazzi type if the metrically associated $(1, 1)$-tensor is of Codazzi type [3].

Proof of Theorem 1.4. In this section we consider a type of spacetime whose energy–momentum tensor is pseudo symmetric as well as Codazzi type. Thus equations (2.1) and

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z) \tag{3.1}$$

both hold. Hence equations (2.1) and (3.1) together give

$$A(Y)T(X, Z) = A(X)T(Y, Z). \tag{3.2}$$

On the other hand from $(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z)$ we obtain

$$div\tilde{T}(Y) = \nabla_Y t.$$

Since $div\tilde{T} = 0$, it follows from the above equation that

$$\nabla_Y t = 0. \tag{3.3}$$

Now using Eq. (2.8) in Eq. (3.3) we get

$$dr(Y) = 0. \tag{3.4}$$

Again from Eqs. (2.12) and (3.4) we infer

$$r = 0. \tag{3.5}$$

Thus, Einstein's field equation (1.1) and Eq. (2.9) become

$$S(X, Y) = \kappa T(X, Y) \tag{3.6}$$

and

$$S(Y, \rho) = 0. \tag{3.7}$$

In this way putting $Y = \rho$ in (3.2) and using Eqs. (3.6) and (3.7) we obtain

$$S(X, Z) = 0,$$

which implies that the spacetime is Ricci flat.
This finishes the proof of Theorem 1.4.

Also from the definition of Wyel conformal curvature tensor we get

$$C = R.$$

From the above equation, we immediately get $R \cdot C = R \cdot R$. If $R \cdot R = 0$, then $R \cdot C = 0$, that is, the spacetimes under consideration are conformally semisymmetric. Eriksson and Senovilla [10] considered the semisymmetric spacetime and proved that it is of Petrov types D, N and O. Thus we have

Theorem 3.1. *Every conformally semisymmetric spacetime with pseudo symmetric and Codazzi type of energy–momentum tensor is of Petrov type D, N and O.*

4. Spacetimes with Codazzi type of energy–momentum tensor

In this section we consider a spacetime whose energy–momentum tensor T is of Codazzi type. Then

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z) \tag{4.1}$$

Proof of Theorem 1.5. Using Eqs. (2.3) and (4.1) we infer that

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) - \frac{1}{2}dr(X)g(Y, Z) + \frac{1}{2}dr(Y)g(X, Z) = 0 \tag{4.2}$$

Now taking a frame field and contracting (4.2) over X and Y, we get

$$dr(Z) = 0, \tag{4.3}$$

which means that the scalar curvature r of the spacetime is constant. Using (4.3) we conclude from (4.2) that

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0. \tag{4.4}$$

This means that the Ricci tensor of the spacetime is also Codazzi type and consequently, from Eq. (1.7) it follows that $divC = 0$, where C denotes the Weyl conformal curvature tensor. The Weyl conformal curvature tensor of a general perfect fluid spacetime M is divergence-free iff M is shear-free, irrotational and its energy density is constant over the spacelike hypersurface orthogonal to the 4-velocity vector [29]. It follows the proof of Theorem 1.5.

In their review of Yang's gravitational theory, Guilfoyle and Nolan [13] named "Yang Pure Space" a 4-dimensional Lorentzian manifold (M, g) whose metric tensor solves Yang's equations:

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0.$$

Thus we are in the position to state the following:

Theorem 4.1. *A spacetime with Codazzi type of energy–momentum tensor T is a Yang Pure Space.*

5. Spacetimes with quadratic Killing energy–momentum tensor

In this section we consider a spacetime whose energy–momentum tensor T is quadratic Killing. That is,

$$(\nabla_X T)(Y, Z) + (\nabla_Y T)(X, Z) + (\nabla_Z T)(X, Y) = 0. \tag{5.1}$$

Proof of Theorem 1.6. Differentiating (1.1) covariantly with respect to Z, we infer that

$$(\nabla_Z S)(X, Y) - \frac{1}{2} dr(Z) g(X, Y) = \kappa(\nabla_Z T)(X, Y). \tag{5.2}$$

Similarly, we obtain

$$(\nabla_X S)(Y, Z) - \frac{1}{2} dr(X) g(Y, Z) = \kappa(\nabla_X T)(Y, Z). \tag{5.3}$$

and

$$(\nabla_Y S)(X, Z) - \frac{1}{2} dr(Y) g(X, Z) = \kappa(\nabla_Y T)(X, Z). \tag{5.4}$$

Equations (5.1), (5.2), (5.3) and (5.4) together give

$$\begin{aligned} &(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) - \frac{1}{2} dr(X) g(Y, Z) \\ &- \frac{1}{2} dr(Y) g(X, Z) - \frac{1}{2} dr(Z) g(X, Y) = 0. \end{aligned} \tag{5.5}$$

Now taking a frame field and contracting over Y and Z, we get from (5.5) that

$$dr(X) = 0. \tag{5.6}$$

This means that r is a constant. Hence using (5.6) in (5.5) we conclude that the Ricci tensor of the spacetime is a quadratic Killing tensor. This completes the proof of Theorem 1.6.

Let us now consider perfect fluid spacetimes with quadratic Killing energy–momentum tensor T.

Then the Ricci tensor takes the form

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{5.7}$$

where α, β are reals.

Proof of Theorem 1.7. From (5.7) we have the following:

$$(\nabla_Z S)(X, Y) = \beta[(\nabla_Z A)XA(Y) + A(X)(\nabla_Z A)Y], \tag{5.8}$$

$$(\nabla_X S)(Y, Z) = \beta[(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z] \tag{5.9}$$

and

$$(\nabla_Y S)(X, Z) = \beta[(\nabla_Y A)XA(Z) + A(X)(\nabla_Y A)Z]. \tag{5.10}$$

Since $\beta \neq 0$, Eqs. (5.8), (5.9) and (5.10) together yield

$$(\nabla_X A)YA(Z) + A(Y)(\nabla_X A)Z + (\nabla_Y A)XA(Z) + A(X)(\nabla_Y A)Z + (\nabla_Z A)XA(Y) + A(X)(\nabla_Z A)Y = 0. \tag{5.11}$$

Putting $Z = \rho$ in (5.11) we get

$$(\nabla_X A)Y + (\nabla_Y A)X = (\nabla_\rho A)XA(Y) + A(X)(\nabla_\rho A)Y. \tag{5.12}$$

Putting $Y = \rho$ in the above equation we have

$$(\nabla_\rho A)(X) = 0,$$

which gives $\nabla_\rho \rho = 0$. That is, the integral curves of the flow vector field are geodesics.

6. Spacetimes with semisymmetric energy–momentum tensor

Now we consider relativistic perfect fluid spacetimes with semisymmetric energy–momentum tensor.
In a paper De and Velimirovic [8] obtained the following result:

Theorem 6.1. *A general relativistic spacetime with semisymmetric energy–momentum tensor is Ricci semisymmetric and vice-versa.*

A semi-Riemannian manifold is said to be Ricci semisymmetric if the Ricci tensor S of type (0,2) satisfies the condition

$$R(X, Y) \cdot S = 0, \tag{6.1}$$

for all X, Y$\in \chi(M)$, where $R(X, Y)$ acts as a derivation on the curvature tensor R.
This means that

$$S(R(X, Y)Z, U) + S(Z, R(X, Y)U) = 0. \tag{6.2}$$

Proof of Theorem 1.8. From Eqs. (5.7) and (6.2) it follows that

$$A(R(X, Y)Z)A(U) + A(Z)A(R(X, Y)U) = 0, \tag{6.3}$$

since $\beta \neq 0$.
Since $g(\rho, \rho) = -1$, putting $U = \rho$ in (6.3) we obtain

$$A(R(X, Y)Z) = 0,$$

which gives $A(QX) = 0$. Hence from Eq. (5.7) it follows that $\alpha = \beta$ which yields $\sigma + 3p = 0$. This means that the spacetime is limiting case of dark energy and limiting case of violating the strong energy conditions. This finishes the proof of Theorem 1.8.

We recall now the definition of a generalized Robertson–Walker spacetime [1,21,22,25,27,28]

Definition 1. An $n(n \geq 3)$-dimensional Lorentzian manifold is named generalized Robertson–Walker spacetime if the metric takes the local shape:

$$ds^2 = -(dt)^2 + q(t)^2 g^*_{\alpha\beta} dx^\alpha dx^\beta, \tag{6.4}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only$(\alpha, \beta, \gamma = 2, 3, \ldots, n)$ and q is a function of t only.

The generalized Robertson–Walker(GRW) spacetime is thus the warped product $(-I) \times q^2 M^*$ [1,21,22,25,27,28], where M^* is a $(n-1)$- dimensional semi-Riemannian manifold. If M^* is a 3-dimensional semi-Riemannian manifold of constant curvature, the spacetime is called Robertson–Walker spacetime.
Mantica and Molinari [21] proved that a Lorentzian manifold of dimension n is a GRW spacetime if and only if it admits a unit timelike torse-forming vector field.

Proof of Theorem 1.9. In Theorem 1.6 it is proved that if the energy–momentum tensor T is quadratic Killing, then the Ricci tensor S is also quadratic Killing tensor. That is, the Ricci tensor S satisfies the following condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0.$$

In [23], the authors proved that an n-dimensional GRW spacetime with the condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0$$

is an Einstein space. Thus the Killing tensor is trivial.
This completes the proof of Theorem 1.9.

The concept of compatibility for symmetric tensors was first introduced in [19] and investigated in [20]. A symmetric tensor b_{ij} is Riemann compatible if:

$$b_{am}R^m_{bcl} + b_{bm}R^m_{cal} + b_{cm}R^m_{abl} = 0.$$

The metric tensor is trivially Riemann compatible because of the first Bianchi identity.

Proof of Theorem 1.10. Changing cyclically X, Y, U in (6.2) and then adding we obtain

$$S(R(X,Y)Z,U) + S(R(Y,U)Z,X) + S(R(U,X)Z,Y) = 0.$$

Thus we can state that a general relativistic spacetime with semisymmetric energy–momentum tensor is Riemann compatible. This completes the proof of Theorem 1.10.

Acknowledgement

The authors are thankful to the referee for his/her valuable suggestions and comments in the improvement of the paper.

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