Journal of Geometry and Physics 62 (2012) 800–803

Contents lists available at SciVerse ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Spectral bundles and the DRY-Conjecture

Björn Andreas [a,*], Gottfried Curio [b]

[a] *Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany*

[b] *Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstr. 37, 80333 München, Germany*

ARTICLE INFO

Article history:
Received 17 January 2011
Received in revised form 14 December 2011
Accepted 27 December 2011
Available online 2 January 2012

Keywords:
Stable vector bundles
Calabi–Yau threefolds
Heterotic string theory

ABSTRACT

Supersymmetric heterotic string models, built from a Calabi–Yau threefold X endowed with a stable vector bundle V, usually start from a phenomenologically motivated choice of a bundle V_v in the visible sector, the spectral cover construction on an elliptically fibered X being a prominent example. The ensuing anomaly mismatch between $c_2(V_v)$ and $c_2(X)$, or rather the corresponding differential forms, is often 'solved', on the cohomological level, by including a fivebrane. This leads to the question whether the difference can be alternatively realized by a further stable bundle. The 'DRY'-conjecture of Douglas, Reinbacher and Yau in math.AG/0604597 gives a sufficient condition on cohomology classes on X to be realized as the Chern classes of a stable sheaf. In 1010.1644 [hep-th], we showed that infinitely many classes on X exist for which the conjecture is true. In this note, we give the sufficient condition for the mentioned fivebrane classes to be realized by a further stable bundle in the hidden sector. Using a result obtained in 1011.6246 [hep-th], we show that corresponding bundles exist, thereby confirming this version of the DRY-Conjecture.

1. Introduction

To get four-dimensional $N = 1$ supersymmetric models from the ten-dimensional $E_8 \times E_8$ heterotic string, one compactifies on a Calabi–Yau threefold X endowed with a polystable holomorphic vector bundle $V' = (V_v, V_h)$. V_v is usually a stable bundle with structure group considered to be embedded in the first (visible) E_8 (V_h plays the corresponding role for the hidden E_8).

Usually one specifies V_v to fulfill some phenomenological requirements, like the generation number $N_{gen} := |c_3(V)/2|$. This specifies discrete parameters of the bundle construction. Anomaly freedom of the construction is encoded in the integrability condition for the existence of a solution to the anomaly cancelation equation

$$c_2(X) = c_2(V_v) + W. \tag{1.1}$$

Here W has just the meaning to indicate a possible mismatch between the second Chern classes; it can be understood either as the cohomology class of (the compact part of the world-volume of) a fivebrane, or as second Chern class of a further stable bundle V_h in the hidden sector. In the first case, the class of W has to be effective for supersymmetry to be preserved. If one wants to solve without a fivebrane one has to make sure that a corresponding hidden bundle V_h with $c_2(V_h) = W := c_2(X) - c_2(V_v)$ exists.

In [1], it has been shown that when (1.1) can be satisfied with $W = 0$, then X and V_v can be deformed to a solution of the anomaly equation even already on the level of differential forms.

* Corresponding author.
E-mail addresses: andreas@math.hu-berlin.de (B. Andreas), gottfried.curio@physik.uni-muenchen.de (G. Curio).

0393-0440/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2011.12.012

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern class $c_2(V)$. Concerning this, a conjecture is put forward in [2] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with $c_1(V) = 0$). We will actually use a weaker version of the conjecture, considered already in [3]. We recall the following definition and conjecture.

Definition. Let X be a Calabi–Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbf{Z})$,
(i) c is called a *Chern class* if a stable $SU(\mathcal{N})$ vector bundle V on X exists with $c = c_2(V)$
(ii) c is called a *DRY class* if an ample class $H \in H^2(X, \mathbf{R})$ exists (and an integer $\mathcal{N}$) with

$$c = \mathcal{N}\left(H^2 + \frac{c_2(X)}{24}\right). \tag{1.2}$$

Weak DRY-conjecture. *On a Calabi–Yau threefold X of $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbf{Z})$ is a Chern class.*

Here it is understood that the integer $\mathcal{N}$ occurring in the two definitions is the same.

This note has two parts. In Section 2 we prove, on an elliptically fibered Calabi–Yau threefold X, the weak DRY conjecture in the generalized sense of constructing a *poly* stable bundle with prescribed second Chern class. In Section 3, we apply this to the problem mentioned above of constructing a hidden bundle such that the anomaly cancelation equation is fulfilled.

2. The weak DRY conjecture for the rank 4 case

We choose X smooth and elliptically fibered over a rational base surface B with section $\sigma : B \longrightarrow X$ (we will also denote by σ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbf{Z})$), a case particularly well studied in phenomenological applications. Typical examples for B are Hirzebruch or del Pezzo surfaces (or suitable blow-ups of these). As in [3], we consider bases B for which $c_1 := c_1(B)$ is ample (this excludes in particular the Hirzebruch surface $\mathbf{F_2}$). (The classes c_1^2 and $c_2 := c_2(B)$ will be considered as (integral) numbers.) We will consider actually B being $\mathbf{F_0}$ or $\mathbf{F_1}$ or one of the del Pezzo surfaces $\mathbf{dP_k}$ with $k = 1, \ldots, 8$.

On X, one has according to the general decomposition $H^4(X, \mathbf{Z}) \cong H^2(B, \mathbf{Z})\sigma \oplus H^4(B, \mathbf{Z})$

$$c_2(V) = \phi\sigma + \omega \tag{2.1}$$

where ω is understood as an integral number (pullbacks from B to X will be usually suppressed). Similarly one has $c_2(X) = 12c_1\sigma + 11c_1^2 + c_2$ or, with Noether's theorem, $12c_1\sigma + 10c_1^2 + 12$.

One gets the following theorem [3].

Theorem on DRY classes. *A class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$ is a DRY class if and only if the following condition is fulfilled (where b is some $b \in \mathbf{R}^{>0}$ and $\omega \in H^4(B, \mathbf{Z}) \cong \mathbf{Z}$):*

$\phi - \mathcal{N}(\frac{1}{2} + b)c_1$ is ample and $\frac{1}{\mathcal{N}}\omega > \omega_0(\phi; b) := r + \frac{c_1^2}{4}(b + \frac{q}{b})$.

Here one uses the following abbreviations (cf. [3])

$$r := \frac{1}{2\mathcal{N}}\phi c_1 + \frac{1}{6}c_1^2 + \frac{1}{2} \tag{2.2}$$

$$q := \frac{\left(\phi - \frac{\mathcal{N}}{2}c_1\right)^2}{\mathcal{N}^2 c_1^2}. \tag{2.3}$$

Note that under the hypothesis that $\phi - \frac{\mathcal{N}}{2}c_1 = A + b\mathcal{N}c_1$ with an ample class A on B (and with $c_1 \neq 0$ effective) one has $(\phi - \frac{\mathcal{N}}{2}c_1)^2 > b^2\mathcal{N}^2c_1^2$, thus one has $b < \sqrt{q}$.

To make the condition more explicit let us choose a concrete b: the choice $b = 1/2$, for example, gives the conditions

$$\phi - \mathcal{N}c_1 \quad \text{ample} \tag{2.4}$$

$$\frac{1}{\mathcal{N}}\omega > \frac{5}{12}c_1^2 + \frac{1}{2} + \frac{1}{2\mathcal{N}^2}\phi^2. \tag{2.5}$$

These are the conditions for the application of the weak DRY conjecture. By contrast recall that the conditions of the effectivity of the fivebrane class $W = c$ are

$$\phi \quad \text{effective} \tag{2.6}$$

$$\omega \geq 0. \tag{2.7}$$

In the case of our application, where c arises as $c_2(X) - c_2(V_v)$ for an $SU(n)$ spectral cover bundle V_v, one has for the effective fivebrane class $W = c$ the following [4]

$$\phi = 12c_1 - \eta_v \tag{2.8}$$

$$\omega = 10c_1^2 + 12 + \frac{n^3 - n}{24}c_1^2 - \left(\lambda_v^2 - \frac{1}{4}\right)\frac{n}{2}\eta_v(\eta_v - nc_1). \tag{2.9}$$

Here η_v is an effective class in B with $\eta_v - nc_1$ also effective and λ_v is a half-integer satisfying the following conditions: λ_v is strictly half-integral for n being odd; for n even an integral λ_v requires $\eta_v \equiv c_1 \pmod 2$ while a strictly half-integral λ_v requires c_1 even. (Often one assumes that $\eta_v - nc_1$ is not only effective but even ample in B.)

As the visible bundle is a spectral cover bundle, and so is stable with respect to the typical spectral Kähler class (the polarization) $H = \epsilon H_0 + H_B$ (where H_0 is an ample class on X, ϵ is chosen sufficiently small [5] and H_B is an ample class on B), we have to choose a hidden bundle V_h which is stable, or polystable, with respect to the same class. This suggests to take for V_h also an $SU(N)$ spectral bundle (which we take with ample spectral cover surface, thus implying that $\eta_h - Nc_1$ is ample). This will allow to generate the part $\phi\sigma$ of the class $W = c = \phi\sigma + \omega$ by $c_2(V_h)$. However the special form of the remaining fiber term of $c_2(V_h)$ for a spectral bundle V_h will usually not be able to represent a given ω part in c. Therefore one needs a second input to match also this part of c. Suitable to represent this part is a pull-back bundle π^*E (where E is a stable bundle on B) as it has $c_2(\pi^*E) = k$, and so consists just of the needed part; here the integer k is arbitrarily choosable as long $k \geq r(E) + 2$ where $r(E) := \mathrm{rank}(E)$ [6]. Therefore, in total, we will choose in the hidden sector a combination of a spectral $SU(N)$ bundle and a pull-back bundle π^*E of rank $r(E)$ such that $\mathcal{N} = N + r(E)$. It is possible to combine the advantages of a hidden spectral bundle with those of a hidden pullback bundle because, according to [7], the pullback bundle is stable with respect to the same polarization as the spectral bundle.

Thus combining the flexibilities of the (hidden) spectral bundle in the ϕ-term (in the $c = \phi\sigma + \omega$ decomposition of $c_2(V_h) + c_2(\pi^*E)$; we call this the "σ"-term of c_2) with that of the pullback bundle in the ω-term (which we call the "fiber"-term) one has the following theorem.

Theorem. *Let X and B be as above. Let $c = \phi\sigma + \omega$ be a class in $H^4(X, \mathbf{Z})$ and $\mathcal{N} = N + r$ (where $N, r \geq 2$) and let either $\mathcal{N} \geq 5$ (and choose N odd), or $\mathcal{N} = 4$ and $B = \mathbf{F_0}$.*

(a) *Sufficient for c to be a Chern class (cf. remark below) are the following conditions*

$$\phi - Nc_1 \quad \text{ample} \tag{2.10}$$

$$\omega \geq 4. \tag{2.11}$$

(b) *All DRY classes with $\mathcal{N} = 4$ fulfill the conditions of* (a).

Proof. (a) Let us check that all classes, as described in (2.10) and (2.11), are indeed realizable as $c_2(V_h) + c_2(\pi^*E)$. First notice that one has

$$c_2(V_h \oplus \pi^*E) = \eta_h\sigma - \frac{N^3 - N}{24}c_1^2 + \frac{N}{2}\left(\lambda_h^2 - \frac{1}{4}\right)\eta_h(\eta_h - Nc_1) + k. \tag{2.12}$$

For the σ term, one takes $\eta_h := \phi$. The instanton number k will be arbitrarily specifiable [6] if one has $k \geq rk(E) + 2$. For the special case $r(E) = 2$ one has, however, the stronger result that $k \geq 2$ is sufficient; cf. [8, Chapter. 10, Theorem 37]. Let us discuss the two cases N odd or N even separately.

For N odd, we take $\lambda_h = \frac{1}{2}$. The ω term takes the form $-\frac{N^3-N}{24}c_1^2 + k$. The first term is smallest in absolute value for $N = 3$; thus the sequence of realizable ω terms starts in the worst case with $-c_1^2 + r(E) + 2$. Now an arbitrary given $\mathcal{N}$ can be decomposed as $N + r(E)$ with N odd and $r(E)$ being 2 or 3. Therefore the mentioned sequence starts with $-c_1^2 + 5$ (in the worst case of $r(E)$) and so actually with 4 in the worst case for B.

Let us now discuss the other case where N is even. This case needs only to be discussed for $\mathcal{N} = 4$, that is, $N = r(E) = 2$. By our assumption in this case we have $B = \mathbf{F_0}$ and thus can take again $\lambda_h = \frac{1}{2}$. The ω term takes the form $-\frac{1}{4}c_1^2 + k$. Thus the sequence of realizable ω terms starts with $-\frac{1}{4}c_1^2 + 2$, that is, actually with $\omega \geq 2$.

(b) The conditions for a DRY class were

$$\phi - \mathcal{N}\left(\frac{1}{2} + b\right)c_1 \quad \text{ample} \tag{2.13}$$

$$\frac{1}{\mathcal{N}}\omega > r + \frac{c_1^2}{4}\left(b + \frac{q}{b}\right). \tag{2.14}$$

Thus one sees that with $N = 2 = r(E)$ all DRY classes are realized as second Chern classes of a polystable bundle $V_h \oplus \pi^*E$: for a DRY class one has that $\phi - \frac{\mathcal{N}}{2}c_1$ is ample, so the condition in (2.10) is satisfied (this works always as long as one has $\mathcal{N}/2 \geq N$, i.e. $r(E) \geq N$). Furthermore, the necessary condition $\frac{1}{\mathcal{N}}\omega \geq r + s$ for the fiber part of a DRY class (cf. [3], here $s = \frac{1}{2\mathcal{N}}\sqrt{c_1^2}\sqrt{(\phi - \frac{\mathcal{N}}{2}c_1)^2}$) implies that

$$\omega \geq \frac{1}{2}\phi c_1 + \frac{\mathcal{N}}{6}c_1^2 + \frac{\mathcal{N}}{2} + \frac{1}{2}\sqrt{c_1^2}\sqrt{\left(\phi - \frac{\mathcal{N}}{2}c_1\right)^2}. \tag{2.15}$$

We saw above in (a) that the ω bound for $\mathcal{N} = 4$ is actually $\omega \geq 2$. The third term in (2.15) is already 2; thus this gives for $\mathcal{N} = 4$ that ω bound for realizable classes is fulfilled for the fiber part of all DRY classes. □

Remark. Thus, taking part (a) and (b) together, we have shown the weak DRY conjecture for the rank 4 case, in the sense of realizing a DRY class by the c_2 of a corresponding rank 4 bundle; however, we have shown only a realization by a *polystable* bundle which is a sum $V_h \oplus \pi^*E$. (Starting from this bundle one can build, as a variant of this construction, an irreducible, stable bundle, constructed as a non-split extension.)

3. Application to the heterotic string

Finally, let us come back to the original problem which motivated our consideration of the DRY conjecture. Notice that, after our consideration of the question in which classes c are realizable as Chern classes, the focus of the original question has actually somewhat shifted: it will be enough to investigate, whether the fivebrane class W belongs to these "realizable" classes; for note that in any case the DRY class condition is only sufficient for a class to be realizable as Chern class.

This demand of "realizability" means, concerning the σ part, whether $\phi = 12c_1 - \eta_v$, which is effective by assumption, fulfills actually the stronger condition that $\phi - 2c_1$ is ample; concerning the fiber part, whether ω in (2.9), which by assumption is ≥ 0, is actually ≥ 2.

Let us conclude: usually one supplemented a phenomenological heterotic spectral bundle construction, which was constructed in the visible sector, with a fivebrane class W (to solve the anomaly cancelation equation); then the only condition one had to satisfy was that this class $W = c = \phi\sigma + \omega$ is effective (i.e. ϕ effective and $\omega \geq 0$). Here we have seen that, sharpening this demand just slightly (to $\phi - 2c_1$ ample and $\omega \geq 2$), one can actually turn on a polystable bundle $V_h \oplus \pi^*E$ in the hidden sector whose c_2 just realizes the class W (furthermore the total bundle including also V_v is polystable).

Corollary. *Let the spectral cover bundle V_v be given and assume that the anomaly cancelation equation is solved with a fivebrane of class $W = c_2(X) - c_2(V_v)$, that is, $W = \phi\sigma + \omega$ is effective, or equivalently*

$$\phi = 12c_1 - \eta_v \quad \textit{effective}, \tag{3.1}$$

$$\omega \geq 0. \tag{3.2}$$

We find that $W = c_2(V_h)$ with V_h a polystable bundle if the slightly stronger conditions hold

$$\phi - 2c_1 \quad \textit{ample}, \tag{3.3}$$

$$\omega \geq 2. \tag{3.4}$$

One can thus solve the anomaly for (almost) all the visible spectral bundles (for which one can solve it *with* fivebrane) also *without* any fivebrane (and instead just with a hidden bundle). Then one can actually solve the anomaly already on the level of differential forms [7].

References

[1] B. Andreas, M. Garcia-Fernandez, Solution of the strominger system via stable bundles on Calabi–Yau threefolds, http://arxiv/org/abs/1008.1018 [math.DG].
[2] M.R. Douglas, R. Reinbacher, S.-T. Yau, Branes, bundles and attractors: bogomolov and beyond, math.AG/0604597.
[3] B. Andreas, G. Curio, On possible chern classes of stable bundles on Calabi–Yau threefolds, http://arxiv/org/abs/1010.1644 [hep-th].
[4] R. Friedman, J. Morgan, E. Witten, Vector bundles and F-theory, Comm. Math. Phys. 187 (1997) 679. hep-th/9701162.
[5] R. Friedman, J. Morgan, E. Witten, Vector bundles over elliptic fibrations, http://arxiv/org/abs/alg-geom/9709029.
[6] I.V. Artamkin, Deforming torsion-free sheaves on an algebraic surface, Math. USSR. Izv. 36 (1991) 449.
[7] B. Andreas, M. Garcia-Fernandez, Heterotic non-Kähler geometries via polystable bundles on Calabi–Yau threefolds, http://arxiv/org/abs/1011.6246.
[8] R. Friedman, Algebraic Surfaces and Holomorphic Vector Bundles, Springer, 1998.