Journal of Geometry and Physics 98 (2015) 535–555

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Spinorial description of SU(3)- and $G_2$-manifolds

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ARTICLE INFO

*Article history:*
Received 22 August 2015
Accepted 30 August 2015
Available online 25 September 2015

*MSC:*
primary 53C10
secondary 53C25
53C27
53C29
53C80

*Keywords:*
Spinor
SU(3)-structure
$G_2$-structure
Intrinsic torsion
Characteristic connection
Killing spinor with torsion

ABSTRACT

We present a uniform description of SU(3)-structures in dimension 6 as well as $G_2$-structures in dimension 7 in terms of a characterising spinor and the spinorial field equations it satisfies. We apply the results to hypersurface theory to obtain new embedding theorems, and give a general recipe for building conical manifolds. The approach also enables one to subsume all variations of the notion of a Killing spinor.



## 1. Introduction

This paper is devoted to a systematic and uniform description of SU(3)-structures in dimension 6, as well as $G_2$-structures in dimension 7, using a spinorial formalism. Any SU(3)- or $G_2$-manifold can be understood as a Riemannian spin manifold of dimension 6 or 7, respectively, equipped with a real spinor field $\phi$ or $\bar{\phi}$ of length one. Let us denote by $\nabla$ the Levi-Civita connection and its lift to the spinor bundle. We prove that an SU(3)-manifold admits a 1-form $\eta$ and an endomorphism field $S$ such that the spinor $\phi$ solves, for any vector field $X$,

$$\nabla_X \phi = \eta(X) j(\phi) + S(X) \cdot \phi,$$

where $j$ is the Spin(6)-invariant complex structure on the spin representation space $\Delta = \mathbb{R}^8$ realising the isomorphism $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$. In a similar vein, there exists an endomorphism $\bar{S}$ such that the spinor $\bar{\phi}$ of a $G_2$-manifold satisfies the even simpler equation

$$\nabla_X \bar{\phi} = \bar{S}(X) \cdot \bar{\phi}.$$

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http://dx.doi.org/10.1016/j.geomphys.2015.08.023
0393-0440/

We identify the characterising entities $\eta$, $S$, and $\bar{S}$ with certain components of the intrinsic torsion and use them to describe the basic classes of SU(3)- and $G_2$-manifolds by means of a spinorial field equation. For example, it is known that nearly Kähler manifolds correspond to $S = \mu\,\mathrm{Id}$ and $\eta = 0$ [1], and nearly parallel $G_2$-manifolds are those with $\bar{S} = \lambda\,\mathrm{Id}$ [2], since the defining equation reduces then to the classical constraint for a Riemannian Killing spinor. If $S$ or $\bar{S}$ is symmetric (and, in dimension 6, additionally $\eta = 0$), this is the equation defining generalised Killing spinors, which are known to correspond to half-flat structures [3] and cocalibrated $G_2$-structures [4]. For all other classes, Theorems 3.13 and 4.8 provide new information concerning $\phi$ and $\bar{\phi}$. To mention but one example, we shall characterise in Theorem 3.7 Riemannian spin 6-manifolds admitting a harmonic spinor of constant length. Theorem 4.8 states the analogue fact for $G_2$-manifolds.

We begin by reviewing algebraic aspects of the dimensions 6 and 7—and explain why it is more convenient to use, in the former case, real spinors instead of complex spinors. In Section 3 we carefully relate the various geometric quantities cropping up in special Hermitian geometry, with particular care regarding: the vanishing or (anti-)symmetry of $S$, $\eta$, the intrinsic torsion, induced differential forms and Nijenhuis tensor, Lee and Kähler forms, and the precise spinorial PDE for $\phi$. We introduce a connection well suited to describe the geometry, and its relationship to the more familiar characteristic connection. The same programme is then carried out in Section 4 for $G_2$-manifolds. The first major application of this set-up occupies Section 5: our results can be used to study embeddings of SU(3)-manifolds in $G_2$-manifolds and describe different types of cones (Section 6). The latter results complement the first and last author's work [5]. This leads to the inception of a more unified picture relating the host of special spinor fields occurring in different parts of the literature: Riemannian Killing spinors, generalised Killing spinors, quasi-Killing spinors, Killing spinors with torsion etc. What we show in Section 7 is that all those turn out to be special instances of the characterising spinor field equations for $\phi$ and $\bar{\phi}$ that we started with, and although looking, in general, quite different, these equations can be drastically simplified in specific situations.

The pattern that emerges here clearly indicates that the spinorial approach is not merely the overhaul of an established theory. Our point is indeed that it should be used to describe efficiently these and other types of geometries, like SU(2)- or Spin(7)-manifolds, and that it provides more information than previously known. Additionally, the explicit formulas furnish a working toolkit for understanding many different concrete examples, and for further study.

## 2. Spin linear algebra

The real Clifford algebras in dimensions 6, 7 are isomorphic to $\mathrm{End}(\mathbb{R}^8)$ and $\mathrm{End}(\mathbb{R}^8)\oplus \mathrm{End}(\mathbb{R}^8)$ respectively. The spin representations are real and 8-dimensional, so they coincide as vector spaces, and we denote this common space by $\Delta := \mathbb{R}^8$. By fixing an orthonormal basis $e_1, \dots, e_7$ of the Euclidean space $\mathbb{R}^7$, one choice for the real representation of the Clifford algebra on $\Delta$ is

$$\begin{aligned} e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46},\\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48},\\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67},\\ e_7 &= +E_{12} - E_{34} - E_{56} + E_{78}, \end{aligned}$$

where the matrices $E_{ij}$ denote the standard basis elements of the Lie algebra $\mathfrak{so}(8)$, i. e. the endomorphisms mapping $e_i$ to $e_j$, $e_j$ to $-e_i$ and everything else to zero.

We begin by discussing the 6-dimensional case. Albeit real, the spin representation admits a Spin(6)-invariant complex structure $j : \Delta \to \Delta$ defined be the formula

$$j := e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot e_5 \cdot e_6.$$

Indeed, $j^2 = -1$ and $j$ anti-commutes with the Clifford multiplication by vectors of $\mathbb{R}^6$; this reflects the fact that Spin(6) is isomorphic to SU(4). The complexification of $\Delta$ splits,

$$\Delta \otimes_{\mathbb{R}} \mathbb{C} = \Delta^+ \oplus \Delta^-,$$

a consequence of the fact that $j$ is a real structure making $(\Delta, j)$ complex-(anti)-linearly isomorphic to either $\Delta^\pm$, via $\phi \mapsto \phi \pm i \cdot j(\phi)$. Any real spinor $0 \neq \phi \in \Delta$, furthermore, decomposes $\Delta$ into three pieces,

$$\Delta = \mathbb{R}\phi \oplus \mathbb{R}j(\phi) \oplus \{X \cdot \phi : X \in \mathbb{R}^6\}. \tag{2.1}$$

In particular, $j$ preserves the subspaces $\{X \cdot \phi : X \in \mathbb{R}^6\} \subset \Delta$, and the formula

$$J_\phi(X) \cdot \phi := j(X \cdot \phi)$$

defines an orthogonal complex structure $J_\phi$ on $\mathbb{R}^6$ that depends on $\phi$. Moreover, the spinor determines a 3-form by means of

$$\psi_\phi(X, Y, Z) := -(X \cdot Y \cdot Z \cdot \phi,\ \phi)$$

where the brackets indicate the inner product on $\Delta$. The pair $(J_\phi, \psi_\phi)$ defines an SU(3)-structure on $\mathbb{R}^6$, and any such arises in this fashion from some real spinor. In certain cases this is an established construction: a nearly Kähler structure may be recovered from the Riemannian Killing spinor [1], for instance. All this can be summarised in the known fact that SU(3)-structures on $\mathbb{R}^6$ correspond one-to-one with real spinors of length one (mod$\mathbb{Z}_2$),

$$\mathrm{SO}(6)/\mathrm{SU}(3) \cong \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

**Example 2.1.** Consider the spinor $\phi = (0, 0, 0, 0, 0, 0, 0, 1) \in \Delta = \mathbb{R}^8$. With the basis chosen on p. 2, then, $J_\phi$ and $\psi_\phi$ read

$$J_\phi e_1 = -e_2, \qquad J_\phi e_3 = e_4, \qquad J_\phi e_5 = e_6, \qquad \psi_\phi = e_{135} - e_{146} + e_{236} + e_{245},$$

where $e_{135} = e_1 \wedge e_3 \wedge e_5$ &c. Throughout this article $e_i$ indicate tangent vectors and one-forms indifferently.

Below we summarise formulas expressing the action of $J_\phi$ and $\psi_\phi$, whose proof is an easy exercise in local coordinates and so omitted.

**Lemma 2.2.** *For any unit spinors $\phi$, $\phi^*$ and any vector $X \in \mathbb{R}^6$*

$$\psi_\phi \cdot \phi = -4 \cdot \phi, \qquad \psi_\phi \cdot j(\phi) = 4 \cdot j(\phi), \qquad \psi_\phi \cdot \phi^* = 0 \ \text{ if } \ \phi^* \perp \phi, j(\phi),$$
$$(X \lrcorner \psi_\phi) \cdot \phi = 2X \cdot \phi, \qquad J_\phi \phi = 3j(\phi), \qquad J_\phi (j(\phi)) = -3\phi.$$

In dimension 7 the space $\Delta$ does not carry an invariant complex structure akin to $j$. However, we still have a decomposition. If we take a non-trivial real spinor $0 \neq \phi \in \Delta$, we may split

$$\Delta = \mathbb{R}\phi \oplus \{X \cdot \phi \ : \ X \in \mathbb{R}^7\}, \tag{2.2}$$

and we can still define a 3-form

$$\Psi_\phi(X, Y, Z) := (X \cdot Y \cdot Z \cdot \phi, \phi).$$

It turns out that $\Psi_\phi$ is stable (its GL(7)-orbit is open), and its isotropy group inside $\mathrm{GL}(7, \mathbb{R})$ is isomorphic to the exceptional Lie group $\mathrm{G}_2 \subset \mathrm{SO}(7)$. Thus we recover the renowned fact that there is a one-to-one correspondence between positive stable 3-forms $\Psi \in \Lambda^3\mathbb{R}^7$ of fixed length and real lines in $\Delta$:

$$\mathrm{SO}(7)/\mathrm{G}_2 \cong \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

In analogy to Lemma 2.2, here are formulas to be used in the sequel.

**Lemma 2.3.** *Let $\Psi_\phi$ be a stable three-form on $\mathbb{R}^7$ inducing the spinor $\phi$, and suppose $\phi^*$ is a unit spinor orthogonal to $\phi$. Then*

$$\Psi_\phi \cdot \phi = 7\phi, \qquad \Psi_\phi \cdot \phi^* = -\phi^*, \qquad (X \lrcorner \Psi_\phi) \cdot \phi = -3X \cdot \phi.$$

**Remark 2.4.** The existence of the unit spinor $\phi$ on $M^6$ is a general fact. Any 8-dimensional real vector bundle over a 6-manifold admits a unit section, see e. g. [6, Ch. 9, Thm. 1.2]. Consequently, an oriented Riemannian 6-manifold admits a spin structure if and only if it admits a reduction from $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ to SU(3). The argument also applies to Spin(7)- and $\mathrm{G}_2$-structures, and was practised extensively in [2, Prop. 3.2].

The power of the approach presented in this paper is already manifest at this stage. Consider a 7-dimensional Euclidean space $\bar{U}$ equipped with a $\mathrm{G}_2$-structure $\Psi \in \Lambda^3\bar{U}$. The latter induces an SU(3)-structure on any codimension-one subspace $U$, which may be defined in two ways. One can restrict $\Psi$ to $U$, so that the inner product $V \lrcorner \Psi$ with a normal vector $V$ defines a complex structure on $U$. But it is much simpler to remark that both structures, on $\bar{U}$ and $U$, correspond to the same choice of the real spinor $\phi \in \Delta$.

## 3. Special Hermitian geometry

The premises now in place, an SU(3)-manifold will be a Riemannian spin manifold $(M^6, g, \phi)$ equipped with a global spinor $\phi$ of length one. We always denote its spin bundle by $\Sigma$ and the corresponding Levi-Civita connection by $\nabla$. The induced SU(3)-structure is determined by the 3-form $\psi_\phi$, while the 2-form $\omega(.\,,.) = g(.\,,J.)$ defines the underlying U(3)-structure. From now onwards we will drop the symbol for the Clifford product, so $X \cdot \phi$ will simply read $X\phi$.

**Definition 3.1.** By decomposition (2.1) there exist a unique one-form $\eta \in T^*M^6$ and a unique section $S \in \mathrm{End}(TM^6)$ such that

$$\nabla_X\phi = \eta(X)j(\phi) + S(X)\phi. \tag{3.1}$$

We call $S$ the *intrinsic endomorphism* and $\eta$ the *intrinsic* 1*-form* of the SU(3)-manifold $(M^6, g, \phi)$; this terminology will be fully justified by Proposition 3.3.

Recall that the geometric features of $M^6$ are captured [7] (see also [8] and [9]) by the intrinsic torsion $\Gamma$ which, under

$$\Lambda^2T^*M^6 \cong \mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp,$$

becomes a one-form with values in $\mathfrak{su}(3)^\perp$. For instance, nearly Kähler manifolds are those almost Hermitian manifolds for which $\Gamma$, identified with $\nabla\omega \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp$, is skew: $\nabla_X\omega(X, Y) = 0,\ \forall X, Y$. The aim is to recover the various SU(3)-classes (complex, symplectic, lcK...) essentially by reinterpreting the intrinsic torsion using $S$ and $\eta$. Besides $\psi_\phi$, we have a second, so-to-speak fundamental 3-form

$$\psi^J_\phi(X,Y,Z) := \psi_\phi(JX,JY,JZ) = -\psi_\phi(JX,Y,Z) = -(XYZ\phi, j(\phi)),$$

which gives the imaginary part of a $J_\phi$-holomorphic complex 3-form (the real part being $\psi_\phi$). As a first result, we prove that the intrinsic torsion can be expressed through $S$ and $\psi^J_\phi$, while $\eta$ is related to $\nabla\psi^J_\phi$—thus generalising the well-known definition of nearly Kähler manifolds cited above.

**Lemma 3.2.** *The intrinsic endomorphism $S$ and the intrinsic 1-form $\eta$ are related to $\nabla\omega$ and $\nabla\psi^J_\phi$ through ($X, Y, Z$ any vector fields)*

$$(\nabla_X\omega)(Y,Z) = 2\,\psi^J_\phi(S(X),Y,Z) \quad \textit{and} \quad 8\,\eta(X) = -(\nabla_X\psi^J_\phi)(\psi_\phi).$$

**Proof.** We immediately find $\eta = (\nabla\phi, j(\phi))$. Since $j$ can be thought of as the volume form, it is parallel under $\nabla$ and we conclude

$$\nabla_X(j(\phi)) = j\nabla_X\phi = jS(X)\phi + j\eta(X)j(\phi) = -S(X)j(\phi) - \eta(X)\phi.$$

With $\omega(X,Y) = -(X\phi, Yj(\phi))$ we get

$$\begin{aligned}-\nabla_X\omega(Y,Z) &= X(Y\phi, Zj(\phi)) - (\nabla_X Y\phi, Zj(\phi)) - (Y\phi, \nabla_X Zj(\phi))\\ &= (Y\nabla_X\phi, Zj(\phi)) + (Y\phi, Z\nabla_X j(\phi)) = (YS(X)\phi, Zj(\phi)) - (Y\phi, ZS(X)j(\phi))\\ &= -2\psi^J_\phi(S(X),Y,Z).\end{aligned}$$

Similarly, we compute

$$\begin{aligned}\nabla_X(\psi^J_\phi)(\psi_\phi) &= -X(\psi_\phi\phi, j(\phi)) + (\nabla_X\psi_\phi\phi, j(\phi))\\ &= -(\psi_\phi S(X)\phi, j(\phi)) + (\psi_\phi\phi, S(X)j(\phi)) - \eta(X)(\psi_\phi j(\phi), j(\phi)) + \eta(X)(\psi_\phi\phi,\phi)\\ &= 2\eta(X)(\psi_\phi\phi,\phi) = -8\eta(X).\end{aligned}$$

This finishes the proof. □

To understand the role of the pair $(S,\eta)$ better we shall employ the SU(3)-connection

$$\nabla^n_X Y = \nabla_X Y - \Gamma(X)(Y), \tag{3.2}$$

given by the Levi-Civita connection $\nabla$ minus the intrinsic torsion, see [7,9]. We shall always use only one symbol for covariant derivatives on the tangent bundle and their liftings to the spinor bundle $\Sigma$, whence for any spinor $\phi^*$

$$\nabla^n_X\phi^* = \nabla_X\phi^* - \frac{1}{2}\Gamma(X)\phi^*.$$

**Proposition 3.3.** *The intrinsic torsion of the* SU(3)*-structure $(M^6, g, \phi)$ is given by*

$$\Gamma = S\lrcorner\,\psi_\phi - \frac{2}{3}\eta\otimes\omega$$

*where $S\lrcorner\,\psi_\phi(X,Y,Z) := \psi_\phi(S(X),Y,Z)$.*

**Proof.** The spinor $\phi$ is parallel for $\nabla^n$, as $\mathrm{Stab}(\phi) = \mathrm{SU}(3)$, so $\nabla_X\phi = \frac{1}{2}\Gamma(X)\phi$. By Lemma 2.2 we know that $\omega\phi = -3j(\phi)$, hence

$$\nabla_X\phi = S(X)\phi + \eta(X)j(\phi) = \frac{1}{2}(S(X)\lrcorner\,\psi_\phi)\phi - \frac{1}{3}\eta(X)\omega\phi.$$

Since $(X\lrcorner\,\psi_\phi)(Y, J_\phi Z) = (X\lrcorner\,\psi_\phi)(J_\phi Y, Z)$ we see that $X\lrcorner\,\psi_\phi \in \mathfrak{su}(3)^\perp$, and as $\omega \in \mathfrak{su}(3)^\perp$ the 1-form $S\lrcorner\,\psi_\phi - \frac{2}{3}\eta\otimes\omega$ is the intrinsic torsion of the spin connection. □

**Notation 3.4.** The original approach to the classification of U(3)-manifolds in [10] was by the covariant derivative of the Kähler form. In analogy to their result, one calls the seven 'basic' irreducible modules of an SU(3)-manifold the *Gray–Hervella classes*. Throughout this paper they will be indicated $\chi_1^+,\ \chi_1^-,\ \chi_2^+,\ \chi_2^-,\ \chi_3,\ \chi_4,\ \chi_5$; for simplicity we often will write $\chi_j, \chi_{\bar{j}}$ for $\chi_j^+, \chi_j^-$ respectively, and shorten $\chi_1^+ \oplus \chi_2^- \oplus \chi_4$ to $\chi_{1\bar{2}4}$, &c. In [3] the Gray–Hervella classes of SU(3)-manifolds were derived in terms of the components of the intrinsic torsion, while their identification with the covariant derivatives of the Kähler form and of the complex volume form may be found in [11].

The following result links the intrinsic endomorphism $S$ and the intrinsic 1-form $\eta$ (and thus the spinorial field equation (3.1)) directly to the Gray–Hervella classes $\chi_i$.

**Lemma 3.5.** *The basic classes of an* SU(3)*-structure* $(M^6, g, \phi)$ *are determined as follows, where* $\lambda, \mu \in \mathbb{R}$*:*

| *Class* | *Description* | *Dimension* |
|---|---|---|
| $\chi_1$ | $S = \lambda J_\phi,\ \eta = 0$ | 1 |
| $\chi_{\bar{1}}$ | $S = \mu\,\mathrm{Id},\ \eta = 0$ | 1 |
| $\chi_2$ | $S \in \mathfrak{su}(3),\ \eta = 0$ | 8 |
| $\chi_{\bar{2}}$ | $S \in \{A \in S^2_0T^*M \mid AJ_\phi = J_\phi A\},\ \eta = 0$ | 8 |
| $\chi_3$ | $S \in \{A \in S^2_0T^*M \mid AJ_\phi = -J_\phi A\},\ \eta = 0$ | 12 |
| $\chi_4$ | $S \in \{A \in \Lambda^2(\mathbb{R}^6) \mid AJ_\phi = -J_\phi A\},\ \eta = 0$ | 6 |
| $\chi_5$ | $S = 0,\ \eta \neq 0$ | 6 |

In particular, the class is $\chi_{\bar{1}\bar{2}3}$ if and only if $S$ is symmetric and $\eta$ vanishes, recovering a result of [4].

### *3.1. Spinorial characterisation*

The description of SU(3)-structures in terms of $\phi$ is the main result of this section. To start with, we discuss geometric quantities that pertain the SU(3)-structure and how they correspond to $\phi$. Denote by $D$ the Riemannian Dirac operator.

**Lemma 3.6.** *The* $\chi_4$ *component of the intrinsic torsion of an* SU(3)*-manifold is determined by*

$$\delta\omega(X) = 2[(D\phi, Xj(\phi)) - \eta(X)],$$

*and in particular* $\delta\omega = 0$ *is equivalent to* $(D\phi, Xj(\phi)) = \eta(X)$*. The Lee form is given by*

$$\theta(X) = \delta\omega \circ J(X) = 2(D\phi, X\phi) - 2\eta \circ J(X).$$

**Proof.** We have

$$(\nabla_X\omega)(Y, Z) = (ZY\nabla_X\phi, j(\phi)) + (ZY\phi, \nabla_X j(\phi)) = -2(YZ\nabla_X\phi, j(\phi)) - 2g(Y, Z)\eta(X),$$

leading to

$$\begin{aligned}\delta\omega(X) &= -\sum_i(\nabla_{e_i}\omega)(e_i, X) = \sum_i(\nabla_{e_i}\omega)(X, e_i)\\ &= -2\sum_i((Xe_i\nabla_{e_i}\phi, j(\phi)) - g(X, e_i)\eta(e_i))\\ &= -2(XD\phi, j(\phi)) - 2\eta(X) = 2(D\phi, Xj(\phi)) - 2\eta(X). \quad \square\end{aligned}$$

We consider the space of all possible types $T^*M^6 \otimes \phi^\perp \ni \nabla\phi$, where $\phi^\perp = \mathbb{R}j(\phi) \oplus \{X\phi \mid X \in TM^6\}$ is the orthogonal complement of $\phi$. The Clifford multiplication restricts then to a map

$$m : T^*M^6 \otimes \phi^\perp \to \Sigma.$$

Let $\pi$ : Spin(6) $\to$ SO(6) be the usual projection. For any $h \in$ Spin(6) we have

$$m(\pi(h)\eta \otimes h\phi^*) = h\eta h^{-1}h\phi^* = hm(\eta \otimes \phi^*)$$

and $m$ is Spin(6)-equivariant and thus SU(3)-equivariant. Comparing the dimensions of the modules appearing in (2.1) and the ones of Lemma 3.5 we see that $\chi_{2\bar{2}3} \subset \mathrm{Ker}\,(m)$, and using

$$D\phi = 6\lambda j(\phi) \quad \text{for } S = \lambda J_\phi \quad \text{and} \quad D\phi = -6\mu\phi \quad \text{for } S = \mu\mathrm{Id}$$

we find correspondences

$$\chi_1 \to \mathbb{R}j(\phi) \quad \text{and} \quad \chi_{\bar{1}} \to \mathbb{R}\phi,$$

together with $(D\phi, j(\phi)) = 6\lambda$ and $(D\phi, \phi) = -6\mu$.

Let us look at $\chi_{45}$ closer: recall that $\{J_\phi e_i\phi, \phi, j(\phi)\}$, $i = 1, \ldots, 6$ is a basis of $\Sigma$ for some local orthonormal frame $e_i$, hence

$$D\phi = \sum_{i=1}^6 (D\phi, J_\phi e_i\phi)J_\phi e_i\phi + (D\phi, \phi)\phi + (D\phi, j(\phi))j(\phi).$$

With Lemma 3.6 we conclude that

$$D\phi = \sum_{i=1}^6 \left[\tfrac{1}{2}\delta\omega(e_i) + \eta(e_i)\right] e_ij(\phi) + 6\lambda j(\phi) - 6\mu\phi = \left(\tfrac{1}{2}\delta\omega + \eta\right) j(\phi) + 6\lambda j(\phi) - 6\mu\phi.$$

Therefore, as image of $m$, the component $\mathbb{R}^6$ of $\Sigma$ is determined by $\delta\omega + 2\eta$. This line of thought immediately proves

**Theorem 3.7.** *A 6-dimensional Riemannian spin manifold $(M, g)$ carries a unit spinor $\phi$ lying in the kernel of the Dirac operator*

$$D\phi = 0$$

*if and only if it admits an* SU(3)*-structure of class $\chi_{2\bar{2}345}$ with the restriction $\delta\omega = -2\eta$.*

*The 'complementary' torsion components $\chi_1$ and $\chi_{\bar{1}}$ are determined by the scalars*

$$\lambda = \frac{1}{6}(D\phi, j(\phi)) = -\frac{1}{6}\mathrm{tr}(J_\phi S) \quad \textit{and} \quad \mu = -\frac{1}{6}(D\phi, \phi) = \frac{1}{6}\mathrm{tr}(S).$$

One cannot but notice that harmonic spinors can exist on manifolds whose class is the opposite to that of nearly Kähler manifolds. The consequences of this observation remain – at this stage – to be seen, and will be addressed in forthcoming work.

The linear combination $\chi_4 + 2\chi_5$ vanishing in the theorem also shows up (up to a choice of volume) in [12] and plays a role in supersymmetric compactifications of heterotic string theory.

**Example 3.8.** Consider the Lie algebra $\mathfrak{g} = \mathrm{span}\{e_1, \dots, e_6\}$ with structure equations

$$\vec{d} = (e_{34} + 2e_{35}, e_{45}, 0, 0, 0, e_{51} + e_{23})$$

in terms of the Chevalley–Eilenberg differential $d\beta(e_i, e_j) = \beta[e_j, e_i], \forall \beta \in \mathfrak{g}$. Since the structure constants are rational the corresponding 1-connected Lie group $G$ contains a co-compact lattice $\Lambda$. We consider the spin structure on $M^6 = G/\Lambda$ determined by choosing $\phi = (1, 1, 0, 0, 0, 0, 0, 0)$. This gives us

$$S = -\frac{1}{4}\begin{bmatrix} 0 & 0 & & & & \\ -2 & 0 & & & & \\ & & 0 & 1 & -1 & 0 \\ & & 1 & 0 & 0 & -1 \\ & & -1 & 0 & 0 & -1 \\ & & 0 & 1 & 1 & 0 \end{bmatrix}, \qquad \eta = -\frac{1}{2}e_1$$

and it is not hard to see that $D\phi = 0$. The structure is of class $\chi_{2\bar{2}345}$, and the presence of component nr. 5 is reflected in the non-vanishing $\eta$.

**Notation 3.9.** Recalling Lemma 3.5 we decompose the intrinsic endomorphism into

$$S = \lambda J_\phi + \mu\,\mathrm{Id} + S_2 + S_{34}$$

where $J_\phi$ commutes with $S_2$, anti-commutes with $S_{34}$, and both $S_2$ and $J_\phi S_2$ are trace-free.

The next results discusses the Nijenhuis tensor $N_J(X, Y) = 8\mathrm{Re}\,[X^{1,0}, Y^{1,0}]^{0,1}$, whose vanishing tells that $M$ is a complex manifold. The customary trick in a Riemannian setting is to view it as a three-tensor $N(X, Y, Z) = g(N_J(X, Y), Z)$ by contracting with the metric.

**Lemma 3.10.** *The $\chi_{1\bar{1}2\bar{2}}$ component is controlled by the Nijenhuis tensor*

$$N(X, Y, Z) = -2[\psi^J_\phi((J_\phi S + SJ_\phi)X, Y, Z) - \psi^J_\phi((J_\phi S + SJ_\phi)Y, X, Z)].$$

*Therefore if the class is $\chi_{1\bar{1}345}$, the Nijenhuis tensor reads*

$$N(X, Y, Z) = 8[\lambda\psi^J_\phi(X, Y, Z) - \mu\psi_\phi(X, Y, Z)].$$

**Proof.** We have $g((\nabla_X J_\phi)Y, Z) = -(\nabla_X\omega)(Y, Z)$, and from Lemma 3.2

$$\begin{aligned} N(X, Y, Z) &= -(\nabla_X\omega)(J_\phi Y, Z) + (\nabla_Y\omega)(J_\phi X, Z) - (\nabla_{J_\phi X}\omega)(Y, Z) + (\nabla_{J_\phi Y}\omega)(X, Z) \\ &= 2[-\psi^J_\phi(SX, J_\phi Y, Z) + \psi^J_\phi(SY, J_\phi X, Z) - \psi^J_\phi(SJ_\phi X, Y, Z) + \psi^J_\phi(SJ_\phi Y, X, Z)] \\ &= 2[-\psi^J_\phi((J_\phi S + SJ_\phi)X, Y, Z) + \psi^J_\phi((J_\phi S + SJ_\phi)Y, X, Z)]. \end{aligned}$$

Furthermore for $S = \lambda J_\phi + \mu\mathrm{Id} + S_{34}$ we have

$$J_\phi S + SJ_\phi = J_\phi(S_{34} + \lambda J_\phi + \mu\mathrm{Id}) + (S_{34} + \lambda J_\phi + \mu\mathrm{Id})J_\phi = -2\lambda\mathrm{Id} + 2\mu J_\phi,$$

as claimed. □

Eventually, $\chi_{1\bar{1}34}$ depends on $d\omega$ in the following way:

**Lemma 3.11.** *Retaining Notation 3.9 we have*

$$d\omega(X, Y, Z) = 6\lambda\psi_\phi(X, Y, Z) + 6\mu\psi^J_\phi(X, Y, Z) + 2\overset{XYZ}{\mathfrak{S}}\psi^J_\phi(S_{34}(X), Y, Z).$$

**Table 3.1**
Correspondence of SU(3)-structures and spinorial field equations (see Theorem 3.13).

| Class | Spinorial equations |
|---|---|
| $\chi_1$ | $\nabla_X\phi = \lambda Xj(\phi)$ for $\lambda \in \mathbb{R}$ |
| $\chi_{\bar 1}$ | $\nabla_X\phi = \mu X\phi$ for $\mu \in \mathbb{R}$ |
| $\chi_2$ | $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi), (Y\nabla_X\phi, j(\phi)) = +(X\nabla_Y\phi, j(\phi)),\ \lambda = \eta = 0$ |
| $\chi_{\bar 2}$ | $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi), (Y\nabla_X\phi, j(\phi)) = -(X\nabla_Y\phi, j(\phi)),\ \mu = \eta = 0$ |
| $\chi_3$ | $(J_\phi Y\nabla_X\phi, \phi) = +(Y\nabla_{J_\phi X}\phi, \phi), (Y\nabla_X\phi, j(\phi)) = +(X\nabla_Y\phi, j(\phi)), \eta = 0$ |
| $\chi_4$ | $(J_\phi Y\nabla_X\phi, \phi) = +(Y\nabla_{J_\phi X}\phi, \phi), (Y\nabla_X\phi, j(\phi)) = -(X\nabla_Y\phi, j(\phi)), \eta = 0$ |
| $\chi_5$ | $\nabla_X\phi = (\nabla_X\phi, j(\phi))j(\phi)$ |
| $\chi_{1\bar 1}$ | $\nabla_X\phi = \lambda Xj(\phi) + \mu X\phi$ |
| $\chi_{2\bar 2}$ | $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi), \lambda = \mu = 0$ and $\eta = 0$ |
| $\chi_{2\bar 2 5}$ | $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$ and $\lambda = \mu = 0$ |
| $\chi_{1\bar 1 2\bar 2}$ | $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$ and $\eta = 0$ |
| $\chi_{1\bar 1 2\bar 2 5}$ | $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$ |
| $\chi_{2\bar 2 3}$ | $D\phi = 0$ and $\eta = 0$ |
| $\chi_{1\bar 1 2\bar 2 3}$ | $(D\phi, X\phi) = 0$ and $\eta = 0$ |
| $\chi_{1\bar 1 2\bar 2 34}$ | $(\nabla_X\phi, j(\phi)) = 0$ |
| $\chi_{2\bar 2 35}$ | $(D\phi, Xj(\phi)) = \eta(X)$ and $\lambda = \mu = 0$ |
| $\chi_{1\bar 1 2\bar 2 35}$ | $(D\phi, Xj(\phi)) = \eta(X)$ |
| $\chi_{34}$ | $(J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi)$ and $\eta = 0$ |
| $\chi_{345}$ | $(J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi)$ |
| $\chi_{2\bar 2 345}$ | $\lambda = \mu = 0$ |
| $\chi_{\bar 1 \bar 2 3}$ | $(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$ and $\eta = 0$ |

**Proof.** We have $d\omega(X,Y,Z) = \overset{XYZ}{\mathfrak{S}} (\nabla_X\omega)(Y,Z)$, and the fact that $\overset{XYZ}{\mathfrak{S}} \psi^J_\phi(S_2(X), Y, Z)$ vanishes corresponds to $d\omega = 0$ in $\chi_{2\bar 2 5}$. □

To attain additional equations in terms of $\phi$, thus completing the picture, we need one last technicality.

**Lemma 3.12.** *The intrinsic tensors $(S, \eta)$ of a Riemannian spin manifold $(M^6, g, \phi)$ satisfy the following properties:*

$$\begin{aligned}
S, J_\phi \text{ commute} &\iff (J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi),\\
S, J_\phi \text{ anti-commute} &\iff (J_\phi Y\nabla_X\phi, \phi) = (Y\nabla_{J_\phi X}\phi, \phi),\\
S \text{ is symmetric} &\iff (X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi),\\
S \text{ is skew-symmetric} &\iff (X\nabla_Y\phi, \phi) = -(Y\nabla_X\phi, \phi).
\end{aligned}$$

**Proof.** As $(J_\phi S(X)\phi, Y\phi) = (SJ_\phi(X)\phi, Y\phi)$ if and only if $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi(X)}\phi, \phi)$, the first two equivalences are clear.

Since both $\phi, j(\phi)$ are orthogonal to $Y\phi$, for any $Y \in TM^6$, we obtain

$$g(S(X), Y) = (\nabla_X\phi, Y\phi) \quad \text{and} \quad g(X, S(Y)) = (\nabla_Y\phi, X\phi)$$

and hence the remaining formulas. □

**Theorem 3.13.** *The classification of* SU(3)*-structures in terms of the defining spinor $\phi$ is contained in Table 3.1, where*

$$\eta(X) := (\nabla_X\phi, j(\phi))$$

*and $\lambda = \frac{1}{6}(D\phi, j(\phi))$, $\mu = -\frac{1}{6}(D\phi, \phi)$ (as of Theorem 3.7).*

**Proof.** We first prove that $\lambda$ and $\mu$ in $\chi_1$ and $\chi_{\bar 1}$ are constant. In $\chi_1$ we have $S = \lambda J_\phi$ and thus $\nabla_X(\phi + j(\phi)) = -\lambda X(\phi + j(\phi))$. Since a nearly Kähler structure (type $\chi_{1\bar 1 5}$) is given by a Killing spinor [1], the function $\lambda$ must be constant. In the case $\chi_{\bar 1}$ the spinors $\phi$ and $j(\phi)$ themselves are Killing spinors with Killing constants $\mu, -\mu$.

We combine the results of Lemma 3.12 as follows. By Lemma 3.5, a structure is of type $\chi_2$ if $S$ is skew-symmetric, it commutes with $J_\phi$, and the trace of $J_\phi S$ and $\eta$ vanish. The first statement of Lemma 3.12 gives us the condition for $S$ and $J_\phi$ to commute, and the last states that skew-symmetry is equivalent to $(X\nabla_Y\phi, \phi) = -(Y\nabla_X\phi, \phi)$, under which condition the equation $(J_\phi Y\nabla_X\phi, \phi) = -(Y\nabla_{J_\phi X}\phi, \phi)$ is equivalent to

$$(Y\nabla_X\phi, j(\phi)) = (X\nabla_Y\phi, j(\phi)).$$

The other classes can be calculated similarly, making extensive use of Lemmas 3.6, 3.12. □

It makes little sense to compute all possible combinations (in principle, $2^7$), so we listed only those of some interest. Others can be inferred by arguments of the following sort. Suppose we want to show that class $\chi_{124}$ has $(X\nabla_Y\phi, \phi) = -(Y\nabla_X\phi, \phi)$ and $\eta = 0$ as defining equations. ¿From Lemma 3.5 we know $\chi_{124}$ is governed by the skew-symmetry of $S$, and at the same time $\eta$ controls $\chi_5$, whence the claim is straightforward. Another example: assume we want to show that

$$3\lambda\psi_\phi(X,Y,Z) + 3\mu\psi^J_\phi(X,Y,Z) + \overset{XYZ}{\mathfrak{S}} (YZ\nabla_X\phi, j(\phi)) + \overset{XYZ}{\mathfrak{S}} \eta(X)g(Y,Z) = 0$$

is an alternative description of $\chi_{1\bar{1}2\bar{2}5}$. ¿From Lemma 3.11 we know that $d\omega = 6\lambda\psi_\phi + 6\mu\psi_\phi^J$ defines that class, so we conclude by using $d\omega(X, Y, Z) = \overset{XYZ}{\mathfrak{S}} (\nabla_X\omega)(Y, Z)$ and the first equality in the proof of Lemma 3.6.

**Remarks 3.14.** (i) The proof above shows that the real Killing spinors of an SU(3)-structure of class $\chi_{1\bar{1}}$ (with Killing constants $\pm|\lambda|$) necessarily have the form $\phi \pm j(\phi)$ in case $\chi_1$, and $\phi$, $j(\phi)$ in case $\chi_{\bar{1}}$. Now notice that a rotation of $\phi$ to $\phi\cos\alpha + j(\phi)\sin\alpha$, for some function $\alpha$, affects the intrinsic tensors as follows:

$$S \rightsquigarrow S\cos(2\alpha) + J_\phi \circ S\sin(2\alpha), \qquad \eta \rightsquigarrow \eta + d\alpha.$$

The $\chi_5$ component varies, and $\chi_i^\pm$, $i = 1, 2$ change, too [3].

In class $\chi_{\bar{1}23}$ we have the constraint $D\phi = f\phi$, so $\phi$ is an eigenspinor with eigenfunction $f$. (One can alter $\phi$ so to have it in $\chi_{1\bar{1}2\bar{2}3}$.) Therefore, if we are after a Killing spinor (class $\chi_{1\bar{1}}$), the eigenfunction $f$ necessarily determines the fifth component $\eta = -d\alpha$. In Section 6 we will treat cases where $f = h$ is a constant map.

(ii) It is fairly evident (cf. [3]) that the effect of modifying $S \rightsquigarrow JS$ is to exchange $\chi_j^+$ and $\chi_j^-$, $j = 1, 2$, whilst the other components remain untouched. As such it corresponds to a $\pi/2$-rotation in the fibres of the natural circle bundle $\mathbb{RP}^7 \longrightarrow \mathbb{CP}^3$.

**Example 3.15.** The twistor spaces $M^6 = \mathbb{CP}^3$, $\mathrm{U}(3)/\mathrm{U}(1)^3$ of the self-dual Einstein manifolds $S^4$ and $\mathbb{CP}^2$ are very interesting from the spinorial point of view. As is well known, both carry a one-parameter family of metrics $g_t$ compatible with two almost complex structures $\Omega^{\mathrm{K}}$, $\Omega^{\mathrm{nK}}$, in such a way that in a suitable, but pretty standard normalisation $(M^6, g_{1/2}, \Omega^{\mathrm{nK}})$ is nearly Kähler and $(M^6, g_1, \Omega^{\mathrm{K}})$ is Kähler [13]. The two almost complex structures differ by an orientation flip on the two-dimensional fibres. Here is a short and uniform description of both instances. We choose the spin representation used in [14, Sect. 5.4], whereby the Riemannian scalar curvature of $g_t$ is

$$\mathrm{Scal}_t = 2c(6 - t + 1/t)$$

where $c$ is a constant (equal to 1 for $\mathbb{CP}^3$ and $c = 2$ for $\mathrm{U}(3)/\mathrm{U}(1)^3$, due to an irrelevant yet nasty factor of 2 in standard normalisations). Using an appropriate orthonormal frame the orthogonal almost complex structures read

$$\Omega^{\mathrm{K}} = e_{12} - e_{34} - e_{56}, \qquad \Omega^{\mathrm{nK}} = e_{12} - e_{34} + e_{56}.$$

There exist two linearly independent and isotropy-invariant real spinors $\phi_\varepsilon$ in $\Delta$ ($\varepsilon = \pm1$), which define global spinor fields on the two spaces. On can prove directly that the $\phi_\varepsilon$ induce the same $J_\phi$, corresponding to $\Omega^{\mathrm{nK}}$, and also the 3-forms

$$\psi_\varepsilon := \psi_{\phi_\varepsilon} = \varepsilon(e_{135} + e_{146} - e_{236} + e_{245}) =: \varepsilon\Psi.$$

When $t = 1/2$, $\phi_\varepsilon$ are known to be Killing spinors. For a generic $t \neq 0$ let us define the symmetric endomorphisms $S_\varepsilon : TM^6 \to TM^6$

$$S_\varepsilon = \varepsilon\sqrt{c}\cdot \mathrm{diag}\left(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}}\right).$$

An explicit calculation shows that $\phi_\varepsilon$ solve

$$\nabla_X\phi_\varepsilon = S_\varepsilon(X)\,\phi_\varepsilon,$$

making them generalised Killing spinors. In particular, $S_\varepsilon$ are the intrinsic endomorphisms and $\eta = 0$; observe that $S_\varepsilon$ commute with $\Omega^{\mathrm{nK}}$ due to their block structure. By Lemma 3.5 the SU(3)-structure defined by $\phi_\varepsilon$ is therefore of class $\chi_{\bar{1}\bar{2}}$ for $t \neq 1/2$, and reduces to class $\chi_{\bar{1}}$ when $t = 1/2$.

The spinors $\phi_\varepsilon$ are eigenspinors of the Riemannian Dirac operator $D$ with eigenvalues $6\mu = \mathrm{tr}S_\varepsilon = \varepsilon\sqrt{c}\,\frac{t+1}{\sqrt{t}}$; they coincide, as they should, with the limiting values of Friedrich's general estimate [15] when $t = 1/2$, and Kirchberg's estimate for Kähler manifolds [16] for $t = 1$.

A further routine calculation shows that

$$\nabla_{e_i}\Omega^{\mathrm{nK}} = \begin{cases} -\sqrt{ct}Je_i \lrcorner\Psi & 1 \leqslant i \leqslant 4 \\ -\dfrac{\sqrt{c}\,(1-t)}{\sqrt{t}}Je_i \lrcorner\Psi & i = 5, 6. \end{cases}$$

Hence, we conclude that $\nabla_X\Omega^{\mathrm{nK}} = -2JS_\varepsilon(X)\lrcorner\psi_\varepsilon$ holds, as it should by Lemma 3.2.

Let us finish with a comment on the Kähler structures ($t = 1$). Kirchberg's equality is attained in odd complex dimensions by a pair of so-called Kählerian Killing spinors, basically $\phi_1$, $\phi_{-1}$ [17]. These, however, do *not* induce $\Omega^{\mathrm{K}}$, rather the 'wrong' almost complex structure $\Omega^{\mathrm{nK}}$. This means two things: first, the Kähler structure cannot be recovered from the two Kählerian Killing spinors; secondly, it reflects the fact that the Killing spinors do not define a 'compatible' SU(3)-structure. For the projective space this stems from our description of $\mathbb{CP}^3$ as $SO(5)/U(2)$, on which there is no invariant spinor inducing $\Omega^{\mathrm{K}}$. In the other case the reason is that every almost Hermitian structure on the flag manifold is SU(3)-invariant [18].

### 3.2. Adapted connections

Let $(M^6, g, \phi)$ be an SU(3)-manifold with Levi-Civita connection $\nabla$. As we are interested in non-integrable structures, $\nabla\phi \neq 0$, we look for a metric connection that preserves the SU(3)-structure. The *canonical connection* defined in (3.2) is one such instance.

The space of metric connections is isomorphic to the space of (2, 1)-tensors $\mathcal{A}^g := TM^6 \otimes \Lambda^2(TM^6)$ by $\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$. Define the map

$$\Xi : TM^6 \oplus \mathrm{End}(TM^6) \to \mathcal{A}^g,\ (\eta, S) \mapsto -S\lrcorner\psi_\phi + \frac{2}{3}\eta \otimes \omega$$

where $S, \eta$ are the intrinsic tensors of the SU(3)-structure on $M^6$. Then $\nabla^n_X Y := \nabla_X Y + \Xi(\eta, S)$ is a metric connection on $M^6$, and we get

$$\begin{aligned}\nabla^n_X\phi &= \nabla_X\phi - \psi_\phi \tfrac{1}{2}(S(X), ., .)\cdot\phi + \tfrac{1}{3}\eta(X)\omega\cdot\phi \\ &= S(X)\cdot\phi + \eta(X)j(\phi) - S(X)\cdot\phi - \eta(X)j(\phi) = 0\end{aligned}$$

by Lemma 2.2, showing that $\nabla^n$ is an SU(3)-connection. The space $\mathcal{A}^g$ splits under the representation of SO($n$) (see [19, p. 51] and [20]) into

$$\mathcal{A}^g = TM^6 \oplus \Lambda^3(TM^6) \oplus \mathcal{T},$$

whose summands are referred to as *vectorial*, *skew-symmetric* and *cyclic traceless* connections. A computer algebra system calculates the map $\Xi$ at one point and gives

**Lemma 3.16.** *The 'pure' classes of an* SU(3)*-manifold* $M^6$ *correspond to* $\nabla^n$ *in:*

| *Class of* $M^6$ | $\chi_{1\bar{1}}$ | $\chi_{2\bar{2}}$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|---|---|---|---|---|---|
| *Type of* $\nabla^n$ | $\Lambda^3$ | $\mathcal{T}$ | $\Lambda^3 \oplus \mathcal{T}$ | $TM^6 \oplus \Lambda^3 \oplus \mathcal{T}$ | $TM^6 \oplus \Lambda^3 \oplus \mathcal{T}$ |

The projection to the skew-symmetric part of the torsion given in the previous lemma generates the so-called *characteristic connection* $\nabla^c$. This is a metric connection that preserves the SU(3)-structure and additionally has the same geodesics as $\nabla$. If an SU(3)-manifold admits such a connection, we know from [21] that the $\chi_{2\bar{2}}$ part of the intrinsic torsion vanishes.

We are interested in finding out whether and when an SU(3)-manifold $(M^6, g, \phi)$ admits a characteristic connection, that is to say when

$$\nabla^c\psi_\phi = 0.$$

Any connection doing that must be the (unique) characteristic connection of the underlying U(3)-structure, so to begin with the SU(3)-class must necessarily be $\chi_{1\bar{1}345}$. What is more,

**Lemma 3.17.** *Given an* SU(3)*-manifold* $(M^6, g, \phi)$*, a connection with skew torsion* $\tilde{\nabla}$ *is characteristic if and only if it preserves the spinor* $\phi$.

**Proof.** Obvious, but just for the record: $\nabla^c$ is an SU(3)-connection, and SU(3) = Stab($\phi$) forces $\phi$ to be parallel. Conversely, if $\phi$ is $\tilde{\nabla}$-parallel, the connection must preserve any tensor arising in terms of the spinor, like $\omega$ and $\psi_\phi$, cf. Lemma 3.2. To conclude, just recall that the characteristic connection is unique [21,22]. □

To obtain the ultimate necessary & sufficient condition we need to impose an additional constraint on $\chi_4$, $\chi_5$:

**Theorem 3.18.** *A Riemannian spin manifold* $(M^6, g, \phi)$ *admits a characteristic connection if and only if it is of class* $\chi_{1\bar{1}345}$ *and* $4\eta = \delta\,\omega$.

**Proof.** Let $\nabla^c$ be the U(3)-characteristic connection, $T$ its torsion. We shall determine in which cases $\nabla^c\phi = \nabla^c j(\phi) = 0$. First of all

$$0 = (\nabla^c_X\omega)(Y, Z) = -2(\nabla^c_X\phi, ZYj(\phi)) - 2g(Y, Z)(\nabla^c_X\phi, j(\phi)).$$

Consequently $(\nabla^c_X\phi, ZYj(\phi)) = 0$ if $Y \perp Z$. But as $Y \perp Z$ vary, the spinors $YZj(\phi)$ span $\phi^\perp$. In conclusion, $\nabla^c$ is characteristic for the SU(3)-structure iff $(\nabla^c_X\phi, j(\phi)) = 0$. Now choose a local adapted basis $e_1, \ldots, e_6$ with $J_\phi e_i = -e_{i+1}$, $i = 1, 3, 5$. Using the formula

$$\nabla^c_X\phi = \nabla_X\phi + \frac{1}{4}(X\lrcorner T)\phi$$

and $\omega(X,Y) = -(XY\phi, j(\phi))$ we arrive at $4\eta(X) = -(X\lrcorner T\phi, j(\phi)) = \omega(X\lrcorner T) = T(\omega, X) = -1/2\sum T(e_i, J_\phi e_i, X)$, and eventually

$$4\eta(X) = -\tfrac{1}{2}\sum_{i=1}^{6} T(e_i, Je_i, X) = -\sum_{i=1}^{6}(\nabla_{e_i}\omega)(e_i, X) = \delta\omega(X)$$

because $0 = (\nabla^c_X\omega)(Y,Z) = (\nabla_X\omega)(Y,Z) - \frac{1}{2}(T(X, J_\phi Y, Z) + T(X, Y, J_\phi Z))$. □

The next theorem gives an explicit formula for the torsion of $\nabla^c$. It relies on the computation for the Nijenhuis tensor of Lemma 3.10.

Suppose $M^6$ is of class $\chi_{1\bar{1}345}$, and decompose the intrinsic endomorphism into

$$S = \lambda J_\phi + \mu \mathrm{Id} + S_{34},$$

as explained in Notation 3.9.

**Theorem 3.19.** *Suppose $(M^6, g, \phi)$ is of class $\chi_{1\bar{1}345}$. Then the characteristic torsion of the induced* $\mathrm{U}(3)$*-structure reads*

$$T(X,Y,Z) = 2\lambda\psi^J_\phi(X,Y,Z) - 2\mu\psi_\phi(X,Y,Z) - 2\overset{XYZ}{\mathfrak{S}}\psi_\phi(S_{34}(X), Y, Z).$$

*If $\eta = \frac{1}{4}\delta\omega$, $T$ is the characteristic torsion of the* $\mathrm{SU}(3)$*-structure as well.*

**Proof.** From Lemma 3.11 we infer

$$d\omega \circ J_\phi(X,Y,Z) = 6\lambda\psi^J_\phi(X,Y,Z) - 6\mu\psi_\phi(X,Y,Z) + 2\overset{XYZ}{\mathfrak{S}}\psi_\phi(S_{34}(X), Y, Z).$$

The formula $T = N - d\omega\circ J$ (see [21]) together with Lemma 3.10 allows to conclude. □

**Remark 3.20.** For the class $\chi_{1\bar{1}}$, the torsion $T^c$ of the characteristic connection is parallel (for nearly Kähler manifolds, compare [23,24]). For such $G$-manifolds, the 4-form $\sigma_T := \frac{1}{2}\sum_i (e_i\lrcorner T)\wedge(e_i\lrcorner T)$ encodes a lot of geometric information. It is indeed equal to $dT/2$, it measures the non-degeneracy of the torsion, and it appears in the Bianchi identity, the Nomizu construction, and the identity for $T^2$ in the Clifford algebra (see [25] where all these aspects are addressed). For the class $\chi_{1\bar{1}}$, an easy computation shows

$$\sigma_T = \lambda\, d\psi^J_\phi - \mu\, d\psi_\phi = 12(\lambda^2+\mu^2) * \omega,$$

thus confirming the statement that $\sigma_T$ encodes much of the geometry: it is basically given by the Kähler form.

**Example 3.21.** Take the real 6-manifold $M = \mathrm{SL}(2,\mathbb{C})$ viewed as the reductive space

$$\frac{\mathrm{SL}(2,\mathbb{C})\times\mathrm{SU}(2)}{\mathrm{SU}(2)} = G/H$$

with diagonal embedding. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of $G$ and $H$, and set $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$, so that

$$\mathfrak{m} = \{(A,B)\in\mathfrak{g} \mid A - \bar{A}^t = 0,\ \mathrm{tr}A = 0,\ B + \bar{B}^t = 0,\ \mathrm{tr}B = 0\}.$$

The almost complex structure

$$J(A,B) = (iA, iB)$$

defines a $\mathrm{U}(3)$-structure of class $\chi_3$, see [24]. The characteristic connection $\nabla^c = \nabla + \frac{1}{2}T$ preserves a spinor $\phi$, so $\nabla^c$ is also characteristic for the induced $\mathrm{SU}(3)$-structure, which is of class $\chi_{35}$. By Theorem 3.18 we have $\eta = 0$, so actually the $\mathrm{SU}(3)$-class is $\chi_3$. But then $\phi$ is harmonic.

The following result shows that this reflects a more general fact:

**Corollary 3.22.** *Whenever $\nabla^c$ exists,*

$$\phi\in\mathrm{Ker}\,D \iff T\phi = 0 \iff \text{the } \mathrm{SU}(3)\text{-class is } \chi_3.$$

**Proof.** By Lemma 3.17, $\phi$ is $\nabla^c$-parallel; since the Riemannian Dirac operator and the Dirac operator $D^c$ of $\nabla^c$ are related by $D^c = D + \frac{3}{4}T$, the first equivalence follows. The equivalence of the first and the last statement is a direct consequence of Theorems 3.7, 3.18. □

Example 3.21 satisfies $T\phi = 0$, as shown in [24], so the first condition should be employed if more convenient. This example also shows that there exist $\mathrm{SU}(3)$-structures different from type $\chi_{1\bar{1}5}$ (namely, $\chi_3$) whose torsion is parallel.

## 4. $G_2$ geometry

Let $(M^7, g, \phi)$ be a Riemannian manifold with a globally defined unit spinor $\phi$, inducing a $G_2$-structure $\Psi_\phi$ and the cross product $\times$:

$$\Psi_\phi(X, Y, Z) := (XYZ\phi, \phi) =: g(X \times Y, Z).$$

We recall two standard properties (see [26] or [5]):

**Lemma 4.1.** *The cross product and the 3-form $\Psi_\phi$ satisfy the identities*

(1) $(X \times Y)\phi = -XY\phi - g(X, Y)\phi$
(2) $*\Psi_\phi(V, W, X, Y) = \Psi_\phi(V, W, X \times Y) - g(V, X)g(W, Y) + g(V, Y)g(W, X)$.

Motivated by the fact that $\{X\phi \mid X \in TM^7\} = \phi^\perp$, cf. (2.2), we have

**Definition 4.2.** There exists an endomorphism $S$ of $TM^7$ satisfying

$$\nabla_X \phi = S(X)\phi \tag{4.1}$$

for every tangent vector $X$ on $M^7$, called the *intrinsic endomorphism* of $(M^7, g, \phi)$.

**Lemma 4.3.** *The intrinsic endomorphism $S$ satisfies*

$$(\nabla_V \Psi_\phi)(X, Y, Z) = 2 * \Psi_\phi(S(V), X, Y, Z).$$

**Proof.** We calculate

$$\begin{aligned}(\nabla_V \Psi_\phi)(X, Y, Z) &= (XYZ\nabla_V\phi, \phi) + (XYZ\phi, \nabla_V\phi)\\ &= (XYZS(V)\phi, \phi) - (S(V)XYZ\phi, \phi)\\ &= 2(XYZS(V)\phi, \phi) - 2g(S(V), Z)g(X, Y) + 2g(S(V), Y)g(X, Z) - 2g(S(V), X)g(Y, Z).\end{aligned}$$

With Lemma 4.1 we get

$$\begin{aligned}&2 * \Psi_\phi(S(V), X, Y, Z)\\ &\quad = -2[(XY(Z \times S(V))\phi, \phi) - g(X, Z)g(S(V), Y) + g(X, S(V))g(Y, Z)]\\ &\quad = 2(XYZS(V)\phi, \phi) - 2g(Z, S(V))g(X, Y) + 2g(X, Z)g(S(V), Y) - 2g(X, S(V))g(Y, Z). \quad \square\end{aligned}$$

**Proposition 4.4.** *The intrinsic torsion of the $G_2$-structure $\Psi_\phi$ is*

$$\Gamma = -\frac{2}{3} S \lrcorner \Psi_\phi$$

*where* $S \lrcorner \Psi_\phi(X, Y, Z) := \Psi_\phi(S(X), Y, Z)$.

**Proof.** Immediate from Lemma 2.3, for $\frac{1}{2}\Gamma(X)\phi = \nabla_X\phi = S(X)\phi = -\frac{1}{3}(S(X)\lrcorner\Psi_\phi)\phi$. $\square$

To classify $G_2$-structures one looks at endomorphisms of $\mathbb{R}^7$

$$\mathrm{End}(\mathbb{R}^7) = \mathbb{R} \oplus S_0^2\mathbb{R}^7 \oplus \mathfrak{g}_2 \oplus \mathbb{R}^7,$$

where $S_0^2\mathbb{R}^7$ denotes symmetric, traceless endomorphisms of $\mathbb{R}^7$. The original approach to the classification of $G_2$-structures by Fernández–Gray [26] was by the covariant derivative of the 3-form $\Psi_\phi$. In [9] it was explained how the intrinsic torsion of $G_2$-manifolds can be identified with $\mathrm{End}(\mathbb{R}^7)$, thus yielding an alternative approach to the Fernández–Gray classes. The following result links the intrinsic endomorphism (and thus the spinorial field equation (4.1)) directly to the Fernández–Gray classes.

**Lemma 4.5.** *$G_2$-structures fall into four basic types:*

| *Class* | *Description* | *Dimension* |
|---|---|---|
| $\mathcal{W}_1$ | $S = \lambda\,\mathrm{Id}$ | 1 |
| $\mathcal{W}_2$ | $S \in \mathfrak{g}_2$ | 14 |
| $\mathcal{W}_3$ | $S \in S_0^2\mathbb{R}^7$ | 27 |
| $\mathcal{W}_4$ | $S \in \{V\lrcorner\Psi_\phi \mid V \in \mathbb{R}^7\}$ | 7 |

*In particular, $S$ is symmetric if and only if $S \in \mathcal{W}_1 \oplus \mathcal{W}_3$ and skew iff it belongs in $\mathcal{W}_2 \oplus \mathcal{W}_4$.*

**Table 4.1**
Correspondence of $G_2$-structures and spinorial field equations (see Theorem 4.8).

| Class | Spinorial equation |
|---|---|
| $\mathcal{W}_1$ | $\nabla_X\phi = \lambda X\phi$ |
| $\mathcal{W}_2$ | $\nabla_{X\times Y}\phi = Y\nabla_X\phi - X\nabla_Y\phi + 2g(Y, S(X))\phi$ |
| $\mathcal{W}_3$ | $(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$ and $\lambda = 0$ |
| $\mathcal{W}_4$ | $\nabla_X\phi = XV\phi + g(V, X)\phi$ for some $V \in TM^7$ |
| $\mathcal{W}_{12}$ | $\nabla_{X\times Y}\phi = Y\nabla_X\phi - X\nabla_Y\phi + g(Y, S(X))\phi - g(X, S(Y))\phi - \lambda(X \times Y)\phi$ |
| $\mathcal{W}_{13}$ | $(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$ |
| $\mathcal{W}_{14}$ | $\exists V, W \in TM^7\colon \nabla_X\phi = XVW\phi - (XVW\phi, \phi)\phi$ |
| $\mathcal{W}_{23}$ | $S\phi = 0$ and $\lambda = 0$, or $D\phi = 0$ |
| $\mathcal{W}_{24}$ | $(X\nabla_Y\phi, \phi) = -(Y\nabla_X\phi, \phi)$ |
| $\mathcal{W}_{34}$ | $3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi)$ and $\lambda = 0$ |
| $\mathcal{W}_{123}$ | $(S\phi, X\phi) = 0$, or $D\phi = -7\lambda\phi$ |
| $\mathcal{W}_{124}$ | $(Y\nabla_X\phi, \phi) + (X\nabla_Y\phi, \phi) = -2\lambda g(X, Y)$ |
| $\mathcal{W}_{134}$ | $3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi) - 7\lambda g(X, Y)$ |
| $\mathcal{W}_{234}$ | $\lambda = 0$ |

### 4.1. Spin formulation

By identifying $TM^7 \cong \phi^\perp$ we obtain the isomorphism $T^*M^7 \otimes TM^7 \cong T^*M^7 \otimes \phi^\perp$, given explicitly by

$$\eta \otimes X \mapsto \eta \otimes X\phi.$$

This enables us to describe the tensor product directly, through $\phi$.

As in the SU(3) case we will shorten $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ to $\mathcal{W}_{134}$ and so on. The restricted Clifford product $m : T^*M^7 \otimes \phi^\perp \to \Delta$ decomposes the space $\mathcal{W}_{1234}$, as prescribed by the next result.

**Theorem 4.6.** *Let $(M^7, g, \phi)$ be a Riemannian spin manifold with unit spinor $\phi$. Then $\phi$ is harmonic*

$$D\phi = 0$$

*if and only if the underlying $G_2$-structure is of class $\mathcal{W}_{23}$.*

**Proof.** First of all, the spin representation splits as $\Delta = \mathbb{R}\phi \oplus \phi^\perp = \mathcal{W}_{14}$, so we may write the intrinsic-torsion space as

$$TM^7 \otimes \phi^\perp = \Delta \oplus \mathcal{W}_{23}.$$

Yet the multiplication $m$ is $G_2$-equivariant, so $\operatorname{Ker} m = \{\sum_{ij} a_{ij} e_i \otimes e_j\phi \mid (a_{ij}) \in S^2_0\mathbb{R}^7\} = \mathcal{W}_{23}$, and the assertion follows from the definition of $D = m \circ \nabla$. $\square$

**Lemma 4.7.** *In terms of $\phi$ the module $\mathcal{W}_{24}$ depends on*

$$\tfrac{1}{2}\delta\Psi_\phi(X, Y) = (X\phi, \nabla_Y\phi) - (Y\phi, \nabla_X\phi) + (D\phi, XY\phi) + g(X, Y)(D\phi, \phi).$$

**Proof.** To prove the claim we simply calculate, in some orthonormal basis $e_1, \ldots, e_7$,

$$\begin{aligned}\delta\Psi_\phi(X, Y) &= -\sum(\nabla_{e_i}\Psi_\phi)(e_i, X, Y) = -\sum[(XYe_i\nabla_{e_i}\phi, \phi) + (XYe_i\phi, \nabla_{e_i}\phi)]\\ &= -(XYD\phi, \phi) - \sum[-2g(e_i, Y)(X\phi, \nabla_{e_i}\phi) + 2g(e_i, X)(Y\phi, \nabla_{e_i}\phi) + (e_iXY\phi, \nabla_{e_i}\phi)]\\ &= 2(D\phi, XY\phi) + 2(X\phi, \nabla_Y\phi) - 2(Y\phi, \nabla_X\phi) + 2g(X, Y)(D\phi, \phi). \quad \square\end{aligned}$$

At this point the complete picture is at hand.

**Theorem 4.8.** *The basic classes of $G_2$-manifolds are described by the spinorial field equations for $\phi$ as in Table 4.1. Here, $\lambda := -\frac{1}{7}(D\phi, \phi) : M \to \mathbb{R}$ is a real function, $\times$ denotes the cross product relative to $\Psi_\phi$, and*

$$S\phi := \sum_{i,j} g(e_i, S(e_j))e_ie_j\phi.$$

**Proof.** The proof relies on standard properties, like the fact that $S$ is symmetric if and only if $(X\nabla_Y\phi, \phi) = (Y\nabla_X\phi, \phi)$. It could be recovered by going through the original argument of [26], but we choose an alternative approach.

For $\mathcal{W}_1$ there is actually nothing to prove, for the given equation is nothing but the Killing spinor equation characterising this type of manifolds [27,14].

The endomorphism $S$ lies in $\mathcal{W}_2$ if and only if $S(X \times Y) = S(X) \times Y + X \times S(Y)$. Then

$$\begin{aligned}\nabla_{X\times Y}\phi &= (-Y \times S(X) + X \times S(Y))\phi \\ &= YS(X)\phi + g(Y, S(X))\phi - XS(Y)\phi - g(X, S(Y))\phi \\ &= Y\nabla_X\phi - X\nabla_Y\phi + 2g(S(X), Y)\phi.\end{aligned}$$

By taking the dot product with $\phi$ we re-obtain that $S$ is skew-symmetric.

For $\mathcal{W}_3$ we use Lemma 4.7:

$$\tfrac{1}{2}\delta\Psi_\phi(X, Y) = (D\phi, XY\phi) + g(X, Y)(D\phi, \phi) + (X\phi, \nabla_Y\phi) - (Y\phi, \nabla_X\phi).$$

This fact together with $\operatorname{tr} S = -(D\phi, \phi)$ allows to conclude.

Suppose $S \in \mathcal{W}_4$. The vector representation $\mathbb{R}^7$ is $\{V \times . \mid V \in \mathbb{R}^7\}$, so if $S$ is represented by $V$ we have $\nabla_X\phi = (V \times X)\phi = -VX\phi - g(V, X)\phi = XV\phi + g(V, X)\phi$.

As for the remaining 'mixed' types, we shall only prove what is not obvious. For type $\mathcal{W}_{12}$, $S$ is of the form $S = \lambda \mathrm{Id} + S'$, where $S' \in \mathfrak{g}_2$. Thus,

$$\nabla_{X\times Y}\phi = \lambda(X \times Y)\phi + S'(X \times Y)\phi.$$

The first term may be rewritten using Lemma 4.1(1), while we deal with the second basically as we did in the pure $\mathcal{W}_2$ case. A clever rearrangement of terms then yields the desired identity.

By [26] a structure is of type $\mathcal{W}_{23}$ if $(D\phi, \phi) = 0$ and $0 = \frac{1}{2}\sum_{i,j}\delta\Psi_\phi(e_i, e_j)\Psi_\phi(e_i, e_j, X)$. This is equivalent to

$$\begin{aligned}0 &= \sum_{i,j}[(D\phi, e_ie_j\phi) - 7g(e_i, e_j)\lambda + (e_i\phi, S(e_j)\phi) - (e_j\phi, S(e_i)\phi)](e_ie_jX\phi, \phi) \\ &= -\sum_{i,j}(D\phi, e_ie_j\phi)(e_ie_j\phi, X\phi) + 2\sum_{i,j}(e_i\phi, S(e_j)\phi)(e_ie_jX\phi, \phi).\end{aligned}$$

As $\{e_ie_j\phi \mid i, j = 1, \dots, 7\}$ spans $\Delta$, we obtain $\sum_{i,j}(\phi^*, e_ie_j\phi)e_ie_j\phi = 6\phi_1 + (\phi^*, \phi)\phi$. Define

$$S\phi := \sum_{i,j} g(e_i, S(e_j))e_ie_j\phi$$

and get $0 = -6(D\phi, X\phi) - 2(S\phi, X\phi)$. Therefore $3D\phi = -S\phi$ holds on $\phi^\perp$. If $\lambda = 0$ we then have

$$(S\phi, \phi) = \sum g(e_i, S(e_j))(e_ie_j\phi, \phi) = -\sum g(e_i, S(e_i)) = (D\phi, \phi) = 0.$$

A structure is of type $\mathcal{W}_{34}$ if $(D\phi, \phi) = 0$ and

$$3\delta\Psi_\phi(X, Y) = \tfrac{1}{2}\sum_{i,j}\delta\Psi_\phi(e_i, e_j)\Psi_\phi(e_i, e_j, X \times Y).$$

Due to the calculation above, the right-hand side equals

$$-6(D\phi, (X \times Y)\phi) - 2(S\phi, (X \times Y)\phi) = 6(D\phi, XY\phi) - 42g(X, Y)\lambda + 2(S\phi, XY\phi) + 2g(X, Y)(S\phi, \phi).$$

As

$$3\delta\Psi_\phi(X, Y) = 6(D\phi, XY\phi) - 42g(X, Y)\lambda + 6(X\phi, \nabla_Y\phi) - 6(Y\phi, \nabla_X\phi),$$

the defining equation is equivalent to

$$3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi) - 7g(X, Y)\lambda$$

and if $\lambda = 0$ we get $3(X\phi, \nabla_Y\phi) - 3(Y\phi, \nabla_X\phi) = (S\phi, XY\phi)$.

As for $\mathcal{W}_{124}$, note that $S$ satisfies $(Y\nabla_X\phi, \phi) + (X\nabla_Y\phi, \phi) = -2g(X, Y)\lambda$ if it is skew. If symmetric, instead, it satisfies the equation iff $g(X, S(Y)) = g(X, Y)\lambda$, i. e. if $S = \lambda\,\mathrm{Id}$. □

**Remark 4.9.** The spinorial equation for $\mathcal{W}_4$ defines a connection with vectorial torsion [28]. This is a $G_2$-connection, since $\phi$ is parallel by construction.

### *4.2. Adapted connections*

Let $(M^7, g, \phi)$ be a 7-dimensional spin manifold. As usual we identify (3, 0)- and (2, 1)-tensors using $g$. The prescription

$$\nabla^n := \nabla + \frac{2}{3}S \lrcorner \Psi_\phi$$

defines a natural $G_2$-connection, since $(X \lrcorner \Psi_\phi)\phi = -3X\phi$, $\Psi_\phi\phi = 7\phi$ and $\Psi_\phi\phi^* = -\phi^*, \forall\phi^* \perp \phi$. Abiding by Cartan's formalism, the set of metric connections is isomorphic to

$$\underbrace{\mathbb{R}^7}_{\Lambda^1\mathbb{R}^7} \oplus \underbrace{(\mathbb{R} \oplus \mathbb{R}^7 \oplus S^2_0\mathbb{R}^7)}_{\Lambda^3\mathbb{R}^7} \oplus \underbrace{(\mathfrak{g}_2 \oplus S^2_0\mathbb{R}^7 \oplus \mathbb{R}^{64})}_{\mathcal{T}}$$

under $G_2$, and this immediately yields the analogous statement to Lemma 3.16:

**Lemma 4.10.** *The 'pure' classes of a* $G_2$*-manifold* $(M^7, g, \phi)$ *correspond to* $\nabla^n$ *in:*

| *Class of* $M^7$ | $\mathcal{W}_1$ | $\mathcal{W}_2$ | $\mathcal{W}_3$ | $\mathcal{W}_4$ |
|---|---|---|---|---|
| *type of* $\nabla^n$ | $\Lambda^3$ | $\mathcal{T}$ | $\Lambda^3 \oplus \mathcal{T}$ | $TM^7 \oplus \Lambda^3$ |

Connections of type $\mathcal{T}$ are rarely considered, although calibrated $G_2$-structures ($\mathcal{W}_2$) have an adapted connection of this type [29].

Among $G_2$-connections there exists at most one connection $\nabla^c$ with skew-symmetric torsion $T^c$. Therefore we may write

$$S = \lambda\,\mathrm{Id} + S_3 + S_4 \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$$

with $S_3 \in S^2_0TM^7$ and $S_4 = V \lrcorner \Psi_\phi$ for some vector $V$.

**Proposition 4.11.** *Let* $(M^7, g, \phi)$ *be a* $G_2$*-manifold of type* $\mathcal{W}_{134}$*. The characteristic torsion reads*

$$T^c(X, Y, Z) = -\frac{1}{3} \overset{XYZ}{\mathfrak{S}} \Psi_\phi((2\lambda\mathrm{Id} + 9S_3 + 3S_4)X, Y, Z).$$

**Proof.** Consider the projections

$$T^*M^7 \otimes \mathfrak{g}_2^\perp \xrightarrow{\kappa} \Lambda^3(T^*M^7) \xrightarrow{\Theta} T^*M^7 \otimes \mathfrak{g}_2^\perp$$
$$\Psi_\phi(SX, Y, Z) \overset{\kappa}{\longmapsto} \frac{1}{3} \overset{XYZ}{\mathfrak{S}} \Psi_\phi(SX, Y, Z), \qquad T \overset{\Theta}{\longmapsto} \sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{g}_2^\perp}.$$

A little computation shows that the composite $\Theta \circ \kappa$ is the identity map, with eigenvalues $1, 0, 2/9, 2/3$ on the four summands $\mathcal{W}_i$. But from [21] we know that if $-2\Gamma = \Theta(T)$ for some 3-form $T$, then $T$ is the characteristic torsion. □

## 5. Hypersurface theory

Let $(\bar{M}^7, \bar{g}, \phi)$ be a $G_2$-manifold and $M^6$ a hypersurface with transverse unit direction $V$

$$T\bar{M}^7 = TM^6 \oplus \langle V \rangle\,. \tag{5.1}$$

By restriction the spinor bundle $\bar{\Sigma}$ of $\bar{M}^7$ gives a Spin(6)-bundle $\Sigma$ over $M^6$, and so the Clifford multiplication $\cdot$ of $M^6$ reads

$$X \cdot \phi = VX\phi$$

in terms of the one on $\bar{M}^7$ (whose symbol we suppress, as usual). This implies, in particular, that any $\sigma \in \Lambda^{2k}M^6 \subset \Lambda^{2k}\bar{M}^7$ of even degree will satisfy $\sigma \cdot \phi = \sigma\phi$. This notation was used in [30] to describe almost Killing spinors (see Section 7). Caution is needed because this is not the same as $X \cdot \phi = X\phi$ described in Section 2 for comparing Clifford multiplications.

The second fundamental form $g(W(X), Y)$ of the immersion ($W$ is the Weingarten map) accounts for the difference between the two Riemannian structures, and in $\bar{\Sigma}$ we can compare

$$\bar{\nabla}_X\phi = \nabla_X\phi - \frac{1}{2}VW(X)\phi.$$

A global spinor $\phi$ on $\bar{M}^7$ (a $G_2$-structure) restricts to a spinor $\phi$ on $M^6$ (an SU(3)-structure). The next lemma explains how both the almost complex structure and the spin structure are, essentially, induced by $\phi$ and the unit normal $V$.

**Lemma 5.1.** *For any section* $\phi^* \in \Sigma$ *and any vector* $X \in TM^6$

(1) $V\phi^* = j(\phi^*)$
(2) $VX\phi = (J_\phi X)\phi$.

**Proof.** The volume form $\sigma_7$ satisfies $\sigma_7\phi^* = -\phi^*$ for any $\phi^* \in \Sigma$. Therefore $Vj(X\phi) = \sigma_7(X\phi) = -X\phi$. □

This lemma is, at the level of differential forms, prescribing the rule $V \lrcorner \Psi_\phi = -\omega$.

**Proposition 5.2.** *With respect to decomposition (5.1) the intrinsic* $G_2$*-endomorphism of* $\bar{M}^7$ *has the form*

$$\bar{S} = \begin{bmatrix} J_\phi S - \frac{1}{2} J_\phi W & * \\ \eta & ** \end{bmatrix} \tag{5.2}$$

*where* $(S, \eta)$ *are the intrinsic tensors of* $M^6$*,* $J_\phi$ *the almost complex structure,* $W$ *the Weingarten map of the immersion.*

**Proof.** This result was first proved in [4] using the Cartan–Kähler machinery. Our argument is much simpler: the definitions imply $\nabla_X \phi = VS(X)\phi + \eta(X)V\phi$, and invoking Lemma 5.1 we infer $\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} VW(X)\phi = J_\phi S(X)\phi - \frac{1}{2} J_\phi W(X)\phi + \eta(X)V\phi$. $\square$

The starred terms in (5.2) should point to the half-obvious fact that the derivative $\nabla_V \phi$ cannot be reconstructed from $S$ and $\eta$. As a matter of fact, later we will show that the bottom row of $\bar{S}$ is controlled by the product $(\nabla\phi, V\phi)$, so that the entry $**$ vanishes when $\nabla_V \phi = 0$.

Now we are ready for the main results, which explain how to go from $M^6$ to $\bar{M}^7$ (Theorem 5.4) and backwards (Theorem 5.5). The run-up to those requires a preparatory definition.

Recall that the Weingarten endomorphism $W$ is symmetric if the SU(3)-structure is half-flat (Lemma 3.5). Motivated by this

**Definition 5.3.** We say that a hypersurface $M^6 \subset \bar{M}^7$ has

(0) *type zero* if $W$ is the trivial map (meaning $\bar{\nabla} = \nabla$),
(I) *type one* if $W$ is of class $\chi_{\bar{1}}$,
(II) *type two* if $W$ is of class $\chi_{\bar{2}}$,
(III) *type three* if $W$ is of class $\chi_3$.

Due to the freedom in choosing entries in (5.2), we will take the easiest option (probably also the most meaningful one, geometrically speaking) and consider only embeddings where $\nabla_V \phi = 0$.

**Theorem 5.4.** *Embed* $(M^6, g, \phi)$ *in some* $(\bar{M}^7, \bar{g}, \phi)$ *as in (5.1), and suppose the* $G_2$*-structure is parallel in the normal direction:* $\bar{\nabla}_V \phi = 0$.

*Then the classes* $\mathcal{W}_\alpha$ *of* $(\bar{M}^7, \bar{g}, \phi)$ *depend on the column position (the class of* $M^6$*) and the row position (the Weingarten type of* $M^6$*) as in the table*

| | $\chi_1$ | $\chi_{\bar{1}}$ | $\chi_2$ | $\chi_{\bar{2}}$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|---|---|---|---|---|---|---|---|
| 0 | $\mathcal{W}_{13}$ | $\mathcal{W}_4$ | $\mathcal{W}_3$ | $\mathcal{W}_2$ | $\mathcal{W}_3$ | $\mathcal{W}_{24}$ | $\mathcal{W}_{234}$ |
| I | $\mathcal{W}_{134}$ | $\mathcal{W}_4$ | $\mathcal{W}_{34}$ | $\mathcal{W}_{24}$ | $\mathcal{W}_{34}$ | $\mathcal{W}_{24}$ | $\mathcal{W}_{234}$ |
| II | $\mathcal{W}_{123}$ | $\mathcal{W}_{24}$ | $\mathcal{W}_{23}$ | $\mathcal{W}_2$ | $\mathcal{W}_{23}$ | $\mathcal{W}_{24}$ | $\mathcal{W}_{234}$ |
| III | $\mathcal{W}_{13}$ | $\mathcal{W}_{34}$ | $\mathcal{W}_3$ | $\mathcal{W}_{23}$ | $\mathcal{W}_3$ | $\mathcal{W}_{234}$ | $\mathcal{W}_{234}$ |

**Proof.** Let $A$ be an endomorphism of $\mathbb{R}^6$ and $\theta$ a covector. Then $\bar{A} = \begin{bmatrix} J_\phi A & 0 \\ \theta & 0 \end{bmatrix}$ is of type $\mathcal{W}_4$ iff $\theta = 0$ and $A$ is a multiple of the identity, since $J_\phi$ is given by $g(X, J_\phi Y) = \frac{1}{2}\Psi_\phi(V, X, Y)$.

With similar, easy arguments one shows that the type of $\bar{A} = \begin{bmatrix} J_\phi A & 0 \\ \theta & 0 \end{bmatrix}$ is determined by the class of the intrinsic tensors $(A, \theta)$ on $M^6$ in the following way:

| $(A, \theta) \in$ | $\chi_1$ | $\chi_{\bar{1}}$ | $\chi_2$ | $\chi_{\bar{2}}$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|---|---|---|---|---|---|---|---|
| $(J_\phi A, \theta) \in$ | $\chi_{\bar{1}}$ | $\chi_1$ | $\chi_{\bar{2}}$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
| $\bar{A} \in$ | $\mathcal{W}_{13}$ | $\mathcal{W}_4$ | $\mathcal{W}_3$ | $\mathcal{W}_2$ | $\mathcal{W}_3$ | $\mathcal{W}_{24}$ | $\mathcal{W}_{234}$ |

Now the theorem can be proved thus: consider for example $(S, \eta)$ of class $\chi_3$ on a hypersurface of type I. Then $\begin{bmatrix} J_\phi S & 0 \\ \eta & 0 \end{bmatrix}$ has class $\mathcal{W}_3$, and since $W$ is a multiple of the identity $\begin{bmatrix} J_\phi W & 0 \\ 0 & 0 \end{bmatrix}$ has class $\mathcal{W}_4$. This immediately gives $\bar{S} = \begin{bmatrix} J_\phi S - \frac{1}{2} J_\phi W & 0 \\ \eta & 0 \end{bmatrix}$, so the class of the $G_2$-structure is $\mathcal{W}_{34}$. All other cases are analogous. $\square$

With that in place we can now do the opposite: start from the ambient space $(\bar{M}^7, \bar{g}, \phi)$ and infer the structure of its codimension-one submanifolds $M^6$. By inverting formula (5.2) we immediately see

$$S = -J_\phi \bar{S}\big|_{TM^6} + \frac{1}{2} W, \qquad \eta(X) = g(\bar{S}X, V)$$

for any $X \in TM^6$. The next, final result on hypersurfaces can be found, in a different form, in [31, Sect. 4].

**Theorem 5.5.** *Let $(\bar{M}^7, \bar{g}, \phi)$ be a Riemannian spin manifold of class $\mathcal{W}_\alpha$. Then a hypersurface $M^6$ with normal $V \in T\bar{M}^7$ carries an induced spin structure $\phi$: its class is an entry in the matrix below that is determined by the column (Weingarten type) and row position ($\mathcal{W}_\alpha$)*

| | $\mathcal{W}_1$ | $\mathcal{W}_2$ | $\mathcal{W}_3$ | $\mathcal{W}_4$ |
|---|---|---|---|---|
| 0 | $\chi_1$ | $\chi_{\bar{1}245}$ | $\chi_{1235}$ | $\chi_{\bar{1}45}$ |
| I | $\chi_{1\bar{1}}$ | $\chi_{\bar{1}245}$ | $\chi_{1\bar{1}235}$ | $\chi_{\bar{1}45}$ |
| II | $\chi_{1\bar{2}}$ | $\chi_{\bar{1}245}$ | $\chi_{12\bar{2}35}$ | $\chi_{\bar{1}245}$ |
| III | $\chi_{13}$ | $\chi_{\bar{1}2345}$ | $\chi_{1235}$ | $\chi_{\bar{1}345}$ |

**Proof.** In order to proceed as in Theorem 5.4, we prove that the class of an endomorphism $\bar{A} = \begin{bmatrix} J_\phi A & * \\ \theta & * \end{bmatrix}$ on $\mathbb{R}^7$ determines the class of $(A, \theta)$ on $\mathbb{R}^6$ in the following way:

| $\bar{A} \in$ | $\mathcal{W}_1$ | $\mathcal{W}_2$ | $\mathcal{W}_3$ | $\mathcal{W}_4$ |
|---|---|---|---|---|
| $(A, \theta) \in$ | $\chi_1$ | $\chi_{\bar{1}245}$ | $\chi_{1235}$ | $\chi_{\bar{1}45}$ |

If $\bar{A} \in \mathcal{W}_1$ we have $\bar{A} = \lambda\,\mathrm{Id}$ and hence $\theta = 0$ and $A = \lambda J_\phi$.

If $\bar{A} \in \mathcal{W}_2$ then $J_\phi A$ is skew-symmetric, and $A$ has type $\chi_{\bar{1}24}$.

If $\bar{A}$ is of type $\mathcal{W}_3$ it follows $\bar{S} = \begin{bmatrix} J_\phi A & \eta \\ \eta & -\mathrm{tr}(J_\phi A) \end{bmatrix}$ for some symmetric $J_\phi A$. Therefore $JA$ is of type $\chi_{\bar{1}23}$, implying the type $\chi_{123}$ for $A$.

Suppose $\bar{A} \in \mathcal{W}_4$, so there is a vector $Z$ such that $g(X, \bar{A}Y) = \Psi_\phi(Z, X, Y)$, whence

$$(XYZ\phi, \phi) = (\bar{A}Y\phi, X\phi)$$

for every $X, Y \in \mathbb{R}^7$. Restrict this equation to $X, Y \in \mathbb{R}^6$ and put $Z = \lambda V + Z_1, Z_1 \in \mathbb{R}^6$. Then $J_\phi A = \lambda J_\phi + A_1$ with $(XYZ_1\phi, \phi) = (A_1 Y\phi, X\phi)$. Since $A_1$ is skew we have

$$\begin{aligned} g(X, A_1 J_\phi Y) &= (Z_1 X J_\phi Y\phi, \phi) = (Z_1 XVY\phi, \phi) = -(Z_1 YVX\phi, \phi) \\ &= -(Z_1 Y J_\phi X\phi, \phi) = -g(Y, A_1 J_\phi X) = -g(X, J_\phi A_1 Y), \end{aligned}$$

so $A_1 J_\phi = -J_\phi A_1$ and $A_1$ has type $\chi_4$. Eventually, $J_\phi A \in \chi_{14}$. □

The above table explains why we cannot have a $\mathcal{W}_1$-manifold if the derivative of $\phi$ along $V$ vanishes. Moreover, in case $\nabla_V \phi = 0$ the $\chi_5$ component disappears everywhere, simplifying the matter a little.

Theorems 5.4 and 5.5 amend a petty mistake in [3, Thm 3.1] that was due to a (too) special choice of local basis.

## 6. Spin cones

We wish to explain how one can construct $G_2$-structures, of any desired class, on cones over an SU(3)-manifold. The recipe, which is a generalisation of the material presented in [5], goes as follows.

As usual, start with $(M^6, g, \phi)$ with intrinsic torsion $(S, \eta)$. Choose a complex-valued function $h : I \to S^1 \subset \mathbb{C}$ defined on some real interval $I$. Setting

$$\phi_t := h(t)\phi := \mathrm{Re}\, h(t)\phi + \mathrm{Im}\, h(t) j(\phi)$$

gives a new family of SU(3)-structures on $M^6$ depending on $t \in I$, and $j(\phi)_t = j(\phi_t) = h(t) j(\phi)$. The product of a complex number $a \in \mathbb{C}$ with an endomorphism $A \in \mathrm{End}(TM)$ is defined as $aA = (\mathrm{Re}\, a)A + (\mathrm{Im}\, a) J_\phi A$. Then $h(A(X)\phi^*) = (hA)(X)\phi^* = A(X)\bar{h}\phi^*$ for any spinor $\phi^*$. The first observation is that the intrinsic torsion of $(M^6, g, \phi_t)$ is given by $(h^2 S, \eta)$ (cf. Remarks 3.14(ii), with $f = h$ constant on $M^6$), because

$$\begin{aligned} \nabla_X \phi_t &= h\nabla_X \phi = h(S(X)\cdot\phi) + h\eta(X) j(\phi) \\ &= (hS)(X)\cdot(\bar{h}h\phi) + \eta(X) j(\phi)_t = (h^2 S)(X)\cdot\phi_t + \eta(X) j(\phi)_t. \end{aligned}$$

If we rescale the metric conformally by some positive function $f : I \to \mathbb{R}_+$, we may consider

$$M^6_t := (M^6, f(t)^2 g, \phi_t).$$

Note that $M^6$ and $M^6_t$ have the same Levi-Civita connection and spin bundle $\Sigma$, but distinct Clifford multiplications $\cdot\,, \ \cdot_t$ , albeit related by $X\cdot\phi^* = \frac{1}{f(t)} X \cdot_t \phi^*, \forall\phi^*$. As

$$\nabla_X \phi_t = h^2 S(X)\cdot\phi_t + \eta(X) j(\phi)_t = \frac{h^2}{f} S(X)\cdot_t \phi_t + \eta(X) j(\phi)_t,$$

the intrinsic torsion of $M^6_t$ gets rescaled as $(\frac{h^2}{f} S, \eta)$.

**Definition 6.1.** The metric cone

$$(\bar{M}^7, \bar{g}) = (M^6 \times I, f(t)^2 g + dt^2)$$

equipped with spin structure $\bar{\phi} := \phi_t$ will be referred to as the *spin cone* over $M^6$. The article [5] considered a version of this construction where $f(t) = t$.

The Levi-Civita connection $\bar{\nabla}^t$ of the cone reads

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{f'(t)}{f(t)} \bar{g}(X, Y) \partial_t$$

for $X, Y \in TM^6$, whence the Weingarten map is $W = -\frac{f'}{f}\mathrm{Id}$. Furthermore,

$$\bar{\nabla}_{\partial_t} \bar{\phi} = \bar{\nabla}_{\partial_t} h\phi = h'\phi = -ih' j(\phi) = -i\frac{h'}{h} hV\phi = -i\frac{h'}{h} V\bar{\phi}.$$

To sum up, the intrinsic torsion of $\bar{M}^7$ is encoded in

$$\bar{S} = \begin{bmatrix} \dfrac{h^2}{f} J_\phi S + \dfrac{f'}{2f} J_\phi & 0 \\ \eta & -i\frac{h'}{h} \end{bmatrix}.$$

By decomposing $S = \lambda J_\phi + \mu \mathrm{Id} + R \in \chi_1 \oplus \chi_{\bar{1}} \oplus \chi_{2\bar{2}345}$, the upper-left term in the matrix $\bar{S}$ can be written as

$$\frac{-\lambda \mathrm{Im}\, h^2 + \mu \mathrm{Re}\, h^2 + f'/2}{f} J_\phi - \frac{\lambda \mathrm{Re}\, h^2 + \mu \mathrm{Im}\, h^2}{f} \mathrm{Id} + \frac{\mathrm{Re}\, h^2}{f} J_\phi R - \frac{\mathrm{Im}\, h^2}{f} R.$$

Let us see what happens for specific choices of hypersurface structure.

Suppose we require $\bar{M}^7$ to be a nearly integrable $\mathrm{G}_2$-manifold (class $\mathcal{W}_1$): since $\bar{S}$ is then a multiple of the identity, we need $h'/h$ to be constant, so $h(t) = \exp(i(ct + d)),\ c, d \in \mathbb{R}$. The easiest instance of this situation is the following:

**The sine cone**. Start with an SU(3)-manifold $(M^6, g, \phi)$ of type $\chi_{\bar{1}}$ with $S = -\frac{1}{2}\mathrm{Id}$. The choice $h = e^{it/2}$ produces a cone

$$(M^6 \times (0, \pi),\ \sin(t)^2 g + dt^2,\ e^{it/2}\phi)$$

for which $\bar{S} = \frac{1}{2}\mathrm{Id}$. This construction was introduced in [32], see also [33,34].

**Cones of pure class**. To obtain other classes of $\mathrm{G}_2$-manifolds we start this time by fixing the function $h = 1$, so that $\bar{\phi} = \phi$ and

$$\bar{S} = \begin{bmatrix} \dfrac{\mu + \frac{1}{2}f'}{f} J_\phi - \dfrac{\lambda}{f}\mathrm{Id} + \dfrac{1}{f} J_\phi R & 0 \\ \eta & 0 \end{bmatrix},$$

and only now we prescribe the SU(3)-structure.

(a) Take $M^6$ to be $\chi_{\bar{1}2}$, say $S = \mu\, \mathrm{Id} + R$, and $\mu < 0$ constant: the cone

$$(M^6 \times \mathbb{R}_+,\ 4\mu^2 t^2 g + dt^2,\ \phi)$$

has $\bar{S} = \begin{bmatrix} -\frac{1}{2\mu t} J_\phi R & 0 \\ 0 & 0 \end{bmatrix}$, and so it carries a calibrated $\mathrm{G}_2$-structure (class $\mathcal{W}_2$).

(b) On $M^6$ of type $\chi_{\bar{1}23}$ with $\mu < 0$ constant, we can build the same cone as in (a), but now the resulting $\mathrm{G}_2$-structure will be balanced (class $\mathcal{W}_3$).

(c) Take a $\chi_{\bar{1}}$-manifold ($S = \mu\, \mathrm{Id}$). Since $\begin{bmatrix} k(t) J_\phi & 0 \\ 0 & 0 \end{bmatrix}$ is of type $\mathcal{W}_4$ irrespective of the map $k(t)$, the cone

$$(M^6 \times I,\ f(t)^2 g + dt^2,\ \phi)$$

is always $\mathcal{W}_4$, since $R$ and $\lambda$ vanish. When $\mu < 0$, the special choice $f(t) = -2\mu t$ will additionally give $\bar{S} = 0$. This Ansatz was used in [35] to manufacture a parallel $\mathrm{G}_2$-structure (trivial class $\{0\}$) on the cone.

Other choices of SU(3)-class on $M^6$ and functions $h$, $f$ will allow, along these lines, to construct any desired $\mathrm{G}_2$-class on a suitable cone.

## 7. Killing spinors with torsion

Let $(\bar{M}^7, \bar{g}, \phi)$ be a $G_2$-manifold with characteristic connection $\bar{\nabla}^c$ and torsion $\bar{T}$, and suppose $(M^6, g, \phi)$ is a submanifold of type I or III such that $V \lrcorner \bar{T} = 0$, cf. (5.1). The latter equation warrants that $\bar{T}$ restricts to a 3-form on $M^6$; observe that the condition is more restrictive than assuming $\bar{\nabla}_V \phi = 0$, which implies only $(V \lrcorner \bar{T})\phi = 0$.

We decompose the Weingarten map $W = \mu \,\mathrm{Id} + W_3$ with $JW_3 = -W_3 J$ and prove

**Lemma 7.1.** *The differential form*

$$L(X, Y, Z) := -\overset{XYZ}{\mathfrak{S}}\ \psi_\phi(W_3(X), Y, Z) - \mu\psi_\phi(X, Y, Z)$$

*satisfies* $(X \lrcorner L)\phi = -2W(X)\phi$.

**Proof.** In an arbitrary orthonormal basis $e_1, \ldots, e_6$ the torsion is $-\sum_i (e_i \lrcorner T)_{\mathfrak{su}(3)^\perp} \otimes e_i = 2\Gamma$, where $(e_i \lrcorner T)_{\mathfrak{su}(3)^\perp}$ denotes the projection of $e_i \lrcorner T$ under $\mathfrak{so}(6) \to \mathfrak{su}(3)^\perp$. It is not hard to see that the maps

$$T^*M^6 \otimes \mathfrak{su}(3)^\perp \xrightarrow{\kappa} \Lambda^3(T^*M^6) \xrightarrow{\Theta} T^*M^6 \otimes \mathfrak{su}(3)^\perp$$
$$S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega \overset{\kappa}{\longmapsto} \frac{1}{3}\,\mathfrak{S}\,(S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega), \qquad T \overset{\Theta}{\longmapsto} \sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{su}(3)^\perp}$$

satisfy $\Theta \circ \kappa_{|\chi_3} = \frac{1}{3}\mathrm{Id}_{\chi_3}$ and $\Theta \circ \kappa_{|\chi_1} = \mathrm{Id}_{\chi_1}$. But since SU(3) is the stabiliser of $\phi$, for any $R \in \Lambda^3 T^*M^6$ we have $R(X)\,\phi = \Theta(R)(X)\,\phi$, so

$$(X \lrcorner L)\,\phi = -(\psi_\phi \lrcorner W)\,\phi = -2W(X)\,\phi,$$

proving the lemma. □

For $X \in TM^6$ we have

$$0 = \bar{\nabla}^c_X \phi = \bar{\nabla}_X \phi + \tfrac{1}{4}(X \lrcorner \bar{T})\phi = \nabla_X \phi + \tfrac{1}{4}(X \lrcorner \bar{T})\,\phi - \tfrac{1}{2}W(X)\,\phi.$$

So if we define

$$T := \bar{T}_{|M^6} + L,$$

then $\nabla^c := \nabla + T$ is characteristic for $(M^6, g, \phi)$. This means that if $\bar{M}^7$ and $M^6$ admit characteristic connections, their difference must be $L$.

**Definition 7.2.** Consider the one-parameter family of metric connections

$$\nabla^s := \nabla + 2sT$$

passing through $\nabla^c$ at $s = 1/4$ and $\nabla$ at the origin. A spinor $\phi^*$ is called a *generalised Killing spinor with torsion* (gKST) if

$$\nabla^s_X \phi^* = A(X)\,\phi^*$$

for some symmetric $A : TM^6 \to TM^6$. This notion captures many old acquaintances: taking $s = 0$ will produce generalised Killing spinors (without torsion) [30,36], and quasi-Killing spinors on Sasaki manifolds for special $A$ [37]. Killing spinors with torsion correspond to $A = \mathrm{Id}$, $s \neq 0$ [38], while ordinary Killing spinors arise of course from $s = 0$ and $A = \mathrm{Id}$ [15,14]. Our treatment intends to subsume all these notions into one and shed light on the mutual relationships.

**Example 7.3.** In view of Lemma 4.5, any cocalibrated $G_2$-manifold (class $\mathcal{W}_{13}$) is defined by a gKS. For example, the standard $G_2$-structure of a 7-dimensional 3-Sasaki manifold is cocalibrated, and indeed the *canonical spinor* is generalised Killing [39].

Suppose that $\phi^*$, restricted to $M^6$, is a gKST. Then at any point of $M^6$

$$\begin{aligned}\bar{\nabla}^s_X \phi^* &= \bar{\nabla}_X \phi^* + s(X \lrcorner \bar{T})\phi^* = \nabla_X \phi^* + s(X \lrcorner \bar{T})\phi^* - \tfrac{1}{2}VW(X)\phi^* \\ &= \nabla^s_X \phi^* + s(X \lrcorner (\bar{T} - T))\phi^* - \tfrac{1}{2}VW(X)\phi^* \\ &= V(A - \tfrac{1}{2}W)(X)\phi^* - s(X \lrcorner L)\phi^*.\end{aligned}$$

Picking $A = \frac{1}{2}W$ annihilates the first term, so we are left with $\bar{\nabla}^s_X \phi^* = -s(X \lrcorner L)\phi^*$. Conversely, any $\bar{\nabla}^s$-parallel spinor on $\bar{M}^7$ satisfies

$$0 = \bar{\nabla}^s_X \phi^* = \nabla^s_X \phi^* + s(X \lrcorner (\bar{T} - T))\,\phi^* - \tfrac{1}{2}W(X)\,\phi^*.$$

To sum up,

**Theorem 7.4.** *Let $(\bar{M}^7, \bar{g}, \phi)$ be a $G_2$-manifold with characteristic connection $\bar{\nabla}^c$ and torsion $\bar{T}$. Take a hypersurface $M^6 \subset \bar{M}^7$ of type one or three such that $V \lrcorner \bar{T} = 0$. Then*

(1) *$(M^6, g = \bar{g}_{|TM^6}, \phi)$ is an SU(3)-manifold with characteristic connection $\nabla + \bar{T} + L$;*

(2) *any solution $\phi^*$ on $\bar{M}^7$ to the gKST equation $\nabla^s_X \phi^* = \frac{1}{2} W(X)\, \phi^*$ on $M^6$ must satisfy*

$$\bar{\nabla}^s_X \phi^* = -s(X \lrcorner L)\phi^*;$$

(3) *vice versa, if $\phi^*$ is $\bar{\nabla}^s$-parallel on $\bar{M}^7$, it solves*

$$\nabla^s_X \phi^* = -sX \lrcorner (\bar{T} - T)\, \phi^* + \frac{1}{2} W(X)\, \phi^*.$$

**Example 7.5.** Given $(M^6, g)$ we build the twisted cone

$$(\bar{M}^7 := M^6 \times \mathbb{R}, \bar{g} := a^2 t^2 g + dt^2)$$

for some $a > 0$. From the submanifold $M^6 \cong M^6 \times \{\frac{1}{a}\} \subset \bar{M}^7$ we can only infer the Clifford multiplication of $\bar{M}^7$ at points of $M^6 \times \{\frac{1}{a}\}$. Therefore we consider, as in Section 6, the hypersurface $M^6_t := (M^6, a^2t^2 g) \cong M^6 \times \{\frac{t}{a}\} \subset \bar{M}^7$. At any point in $M^6_t$ the spinor bundles of $M^6_t$ and $\bar{M}^7$ are the same and can be identified with the spinor bundle of $M^6$. Hence $X\,\phi^* = \frac{1}{at}\partial_t X \phi^*$. Since the metric of $M^6_t$ is just a rescaling of that of $M^6$, the Levi-Civita connections $\nabla$ coincide. For the Riemannian connection $\bar{\nabla}$ on $\bar{M}^7$ we have

$$\bar{\nabla}_X \phi^* = \nabla_X \phi^* + \frac{1}{2t} \partial_t X \phi^* = \nabla_X \phi^* + \frac{a}{2} X\, \phi^*,$$

as $W(X) = -\frac{1}{t} X$. Therefore the submanifolds $M^6_t$ are of type I, and one can determine the possible structures using Theorems 5.4, 5.5. Any 2-form $\sigma$ on $M^6$ is a 2-form on $\bar{M}^7$ with $\partial_t \lrcorner \sigma = 0$, and in addition

$$\sigma \cdot \phi^* = a^2 t^2 \sigma \phi^*$$

for any spinor $\phi^*$.

Let $\phi$ be an SU(3)-structure on $M^6$ and consider the $G_2$-structure on $\bar{M}^7$ given by $\phi$. Then $\partial_t \lrcorner \Psi_\phi = -a^2 t^2 \omega$. If $M^6$ has characteristic connection $\nabla^c$ with torsion $T$,

$$\begin{aligned} 0 &= \nabla^c_X \phi = \nabla_X \phi + \tfrac{1}{4}(X \lrcorner T)\, \phi = \bar{\nabla}_X \phi - \tfrac{a}{2} X\, \phi + \tfrac{1}{4}(X \lrcorner T)\, \phi \\ &= \bar{\nabla}_X \phi - \tfrac{a}{4}(X \lrcorner \psi_\phi)\, \phi + \tfrac{1}{4}(X \lrcorner T)\, \phi = \bar{\nabla}_X \phi + \tfrac{1}{4}(X \lrcorner (T - a\psi_\phi))\, \phi \\ &= \bar{\nabla}_X \phi + \tfrac{1}{4}(X \lrcorner a^2 t^2 (T - a\psi_\phi))\phi, \end{aligned}$$

showing that $\bar{T} = a^2 t^2 (T - a\psi_\phi)$ is the characteristic torsion of $\bar{M}^7$.

Given a $\bar{\nabla}^s$-parallel spinor $\phi^*$

$$\begin{aligned} 0 &= \bar{\nabla}_X \phi^* + s(X \lrcorner \bar{T})\phi^* = \nabla_X \phi^* + \tfrac{a}{2} X\, \phi^* + \tfrac{s}{a^2 t^2}(X \lrcorner \bar{T})\, \phi^* \\ &= \nabla_X \phi^* + \tfrac{a}{2} X\, \phi^* + s(X \lrcorner (T - a\psi_\phi))\, \phi^* = \nabla^s_X \phi^* + \tfrac{a}{2} X\, \phi^* - as(X \lrcorner \psi_\phi)\, \phi^*, \end{aligned}$$

from which

$$\nabla^s_X \phi^* - as(X \lrcorner \psi_\phi)\, \phi^* = -\tfrac{a}{2} X\, \phi^*.$$

Consider the differential form on $\bar{M}^7$

$$\bar{\psi}_\phi(X, Y, Z) := a^3 t^3 \psi^-(X, Y, Z) \quad \text{for } X, Y, Z \in TM^6 \text{ and } \partial_t \lrcorner \bar{\psi}_\phi = 0.$$

For a Killing spinor solving $\nabla^s_X \phi^* = -\frac{a}{2} X\, \phi^*$ we then have

$$\begin{aligned} 0 &= \nabla^s_X \phi^* + \tfrac{a}{2} X\, \phi^* = \nabla_X \phi^* + \tfrac{a}{2} X\, \phi^* + s(X \lrcorner T)\, \phi^* \\ &= \bar{\nabla}_X \phi^* + s a^2 t^2 (X \lrcorner T)\phi^* = \bar{\nabla}_X \phi^* + s a^3 t^2 (X \lrcorner \psi_\phi)\phi^* + s(X \lrcorner \bar{T})\phi^*. \end{aligned}$$

Consequently

$$0 = \bar{\nabla}^s_X \phi^* + \frac{s}{t}(X \lrcorner \bar{\psi}_\phi)\phi^*.$$

**Example 7.6.** Let $(M^7, g, \xi, \eta, \psi)$ be an Einstein–Sasaki manifold with Killing vector $\xi$, Killing 1-form $\eta$ and almost complex structure $\psi$ on $\xi^\perp$. The Tanno deformation $(t > 0)$

$$g_t := tg + (t^2 - t)\eta \otimes \eta, \qquad \xi_t := \frac{1}{t}\xi, \qquad \eta_t := t\eta$$

has the property that $(M^7, g_t, \xi_t, \eta_t, \psi)$ remains Sasaki for all values of $t$. Call $\nabla^{g_t}$ the Levi-Civita connection of $(M, g_t, \xi_t, \eta_t, \psi)$ and $T^{g_t}$ the characteristic torsion of the almost contact structure (a characteristic connection exists since the manifold is Sasaki). Becker-Bender proved [40, Thm. 2.22] the existence of a Killing spinor with torsion for

$$\nabla^{g_t}_X + (\tfrac{1}{2t} - \tfrac{1}{2})(X \lrcorner T^{g_t}).$$

Quasi Killing spinors [41] are special instances of Definition 7.2 and produce gKST on the deformed Sasaki manifold $(M^7, g_t, \xi_t, \eta_t, \psi)$. As proved in [40], in this example generalised Killing spinors with torsion and Killing spinors with torsion are the same. Since the $A$ of a gKS is symmetric, the $G_2$-structure given by this spinor is cocalibrated ($\mathcal{W}_{13}$).

**Example 7.7.** In [38] it was proved that on a nearly Kähler manifold the sets of $\nabla^c$-parallel spinors, Riemannian Killing spinors, and Killing spinors with torsion coincide.

To conclude, the different existing notions of (generalised) Killing spinors (with torsion) are far from being disjoint and are best described, at least in dimensions 6 and 7, using the characterising spinor of the underlying $G$-structure as presented in this article.

**Remark 7.8.** At last note that the sign of the Killing constant may be reversed by choosing $j(\phi^*)$ instead of $\phi^*$.

## Acknowledgements

SGC was supported by an 'IN dAM-COFUND Fellowship in Mathematics and/or Applications for experienced researchers cofunded by Marie Curie actions', and thanks the FB12 at Philipps-Universität Marburg for the tremendous stay. He is indebted to Simon Salamon for sharing his insight at an early stage, and acknowledges his unwavering influence.

The authors thank the anonymous reviewer for reading the manuscript very carefully and checking almost all computations—his or her comments were highly appreciated.

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