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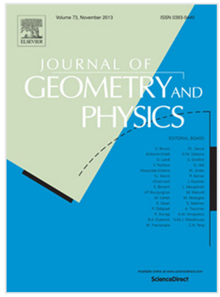

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SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN METRIC LIE GROUPS

PIERRE BAYARD, JULIEN ROTH AND BERENICE ZAVALA JIMÉNEZ

Abstract. In this paper we give a spinorial representation of submanifolds of any dimension and codimension into Lie groups equipped with left invariant metrics. As applications, we get a spinorial proof of the Fundamental Theorem for submanifolds into Lie groups, we recover previously known representations of submanifolds in $\mathbb{R}^n$ and in the 3-dimensional Lie groups $\mathbb{S}^3$ and $E(\kappa,\tau)$, and we get a new spinorial representation for surfaces in the 3-dimensional semi-direct products: this achieves the spinorial representations of surfaces in the 3-dimensional homogeneous spaces. We finally indicate how to recover a Weierstrass-type representation for CMC-surfaces in 3-dimensional metric Lie groups recently given by Meeks, Mira, Perez and Ros.

Keywords: Spin geometry, metric Lie groups, isometric immersions, Weierstrass representation.

2010 Mathematics Subject Classification: 53C27, 53C40.

1. Introduction

The purpose of this paper is to give a spinorial representation of an isometric immersion of a Riemannian manifold M into a Lie group G equipped with a left invariant metric. The result is roughly the following: if M is a simply connected Riemannian manifold, E is a real vector bundle on M equipped with a fiber metric and a compatible connection, and $B : TM \times TM \to E$ is bilinear and symmetric, then an isometric immersion of M into G with normal bundle E and second fundamental form B is equivalent to the existence of a spinor field φ solution of a Killing-type equation on M; the spinor bundle of G is constructed from the Clifford algebra of the metric Lie algebra $\mathcal{G}$ of the group, and the immersion is explicitly obtained by the integration of a $\mathcal{G}$-valued 1-form on M defined in terms of the spinor field φ. We state here the main result and refer to Section 2 for the precise definitions of the spinor bundle and the various objects defined on it.

Theorem 1. *Let M be a simply connected Riemannian manifold, E a real vector bundle on M equipped with a fiber metric and a compatible connection, and $B : TM \times TM \to E$ a bilinear and symmetric map. We assume that E and TM are oriented and spin, with given spin structures and that the compatibility conditions (21) and (22) are statisfied. Then, the following statements are equivalent:*

(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that

$$\nabla_X\varphi = -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot B(X,e_j)\cdot\varphi + \frac{1}{2}\Gamma(X)\cdot\varphi \qquad (1)$$

for all $X \in TM$.

(2) There exists an isometric immersion $F: M \to G$ with normal bundle E and second fundamental form B.

More precisely, if φ is a solution of (1), replacing φ by $\varphi \cdot a$ for some $a \in Spin(\mathcal{G})$ if necessary, and considering the $\mathcal{G}-$valued 1-form ξ defined by

$$\xi(X) := \langle\langle X \cdot \varphi, \varphi \rangle\rangle \tag{2}$$

for all $X \in TM$, the formula $F = \int \xi$ defines an isometric immersion in G with normal bundle E and second fundamental form B. Here $\int$ stands for the Darboux integral, i.e. $F = \int \xi : M \to G$ is such that $F^*\omega_G = \xi$, where $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G defined in (3). Reciprocally, an isometric immersion $M \to G$ with normal bundle E and second fundamental form B may be written in that form.

Note that in this statement the spinor bundles Σ, $U\Sigma$ and the bilinear map $\langle\langle ., . \rangle\rangle : \Sigma \times \Sigma \to Cl(\mathcal{G})$ are introduced in Section 2.4. We also want to point out that the compatibility conditions (21) and (22) are explicitely defined in Section 2.6 and are necessary to write down the equations of Gauss, Codazzi and Ricci and get a Fundamental Theorem for immersions into a general metric Lie group: they indeed allow the definition of the map Γ, which corresponds in the abstract setting to the Levi-Civita connection of the metric Lie group; see Section 2.

The explicit representation formula of the immersion in terms of the spinor field, $F = \int \xi$ where ξ is defined by (2), may be considered as a generalized Weierstrass representation formula for manifolds into metric Lie groups.

This theorem generalizes the main result of [4] to a Lie group equipped with a left invariant metric.

We then give some applications of this result. We first obtain an easy proof of a theorem by Piccione and Tausk [18]: under suitable hypotheses, the necessary equations of Gauss, Codazzi and Ricci are also sufficient to obtain an immersion of a simply connected manifold into a metric Lie group (Corollary 1). We then show how our general result permits to recover the known spinorial representation for submanifolds in $\mathbb{R}^n$ (Theorem 2) proved by Bayard, Lawn and Roth in [4] (see also [3] for $\mathbb{R}^4$). We also obtain a new spinorial representation for submanifolds in $\mathbb{H}^n$ considered as a metric Lie group (Theorem 3). Note that this gives an alternative representation to the one obtained by Morel [16] for surfaces into $\mathbb{H}^3$. We finally study more precisely the case of surfaces in a 3-dimensional metric Lie group (Theorems 4 and 5): we recover the known spinorial representations in $\mathbb{S}^3$ by Morel [16] and $E(\kappa, \tau)$ by Roth [19], and obtain a new spinorial representation of surfaces in a general semi-direct product (Theorem 6); this especially includes the cases of surfaces into the groups Sol_3 and $\mathbb{H}^2 \times \mathbb{R}$, which achieves the spinorial representations of surfaces into the 3-dimensional homogeneous spaces initiated in [8, 16, 19]. We also deduce alternative proofs of the Fundamental Theorems for surfaces in $E(\kappa, \tau)$ by Daniel [7] (Corollary 2) and in Sol_3 by Lodovici [11]. We finish the paper showing how the general spinorial representation formula permits to recover the recent Weierstrass-type representation formula by Meeks, Mira, Perez and Ros [13, Theorem 3.12] concerning constant mean curvature surfaces in 3-dimensional metric Lie groups. The main result of the paper thus gives a general framework for a variety of Weierstrass-type representation formulas existing in the literature, and is also a tool to get representation formulas in new contexts.

We also mention the following related papers for completeness of the historical context. Spinorial representations were also studied in pseudo-Riemannian spaces, by Lawn in $\mathbb{R}^{2,1}$ [9], Lawn and Roth in 3-dimensional Lorentzian space forms [10], Bayard in $\mathbb{R}^{3,1}$ [1], Bayard and Patty [5] and Patty [17] in $\mathbb{R}^{2,2}$. Close to the purpose of the paper, Berdinskii and Taimanov gave in [6] a spinorial representation for a surface in a 3-dimensional metric Lie group.

The outline of the paper is as follows: Section 2 is dedicated to preliminaries concerning notation and spin geometry of a submanifold in a metric Lie group, Section 3 to the proof of the main theorem, and Section 4 to a spinorial proof of the Fundamental Theorem for submanifolds in a metric Lie group. We then give further applications in Section 5: we study the cases of a submanifold in $\mathbb{R}^n$ and $\mathbb{H}^n$, and of a hypersurface in a general metric Lie group, specifying further to the cases of a surface in $\mathbb{S}^3$, $E(\kappa,\tau)$ and a semi-direct product, as Sol_3 and $\mathbb{H}^2\times\mathbb{R}$. We finally consider the case of a CMC-surface in a 3-dimensional metric Lie group. An appendix ends the paper concerning the links between the Clifford product and some natural operations on skew-symmetric operators.

2. Preliminaries

2.1. **Notations.** Let G be a Lie group, endowed with a left invariant metric $\langle .,.\rangle$, and $\mathcal{G}$ its Lie algebra: $\mathcal{G}$ is the space of the left invariant vector fields on G, equipped with the Lie bracket $[.,.]$ and is identified to the linear space tangent to G at the identity. We consider the Maurer-Cartan form $\omega_G\in\Omega^1(G,\mathcal{G})$ defined by

$$\omega_G(v)=L_{g^{-1}*}(v)\qquad\in\ \mathcal{G}\tag{3}$$

for all $v\in T_gG$, where $L_{g^{-1}}$ denotes the left multiplication by g^{-1} on G and $L_{g^{-1}*}:T_gG\to\mathcal{G}$ is its differential. This form induces a bundle isomorphism

$$\begin{array}{ccc} TG & \to & G\times\mathcal{G}\\ (g,v) & \mapsto & (g,\omega_G(v))\end{array}\tag{4}$$

which preserves the fiber metrics. We note that a vector field $X\in\Gamma(TG)$ is left invariant if, by (4), $X:G\to\mathcal{G}$ is a constant map. Let us consider the Levi-Civita connection ∇^G of $(G,\langle .,.\rangle)$ and the linear map

$$\begin{array}{cccc}\Gamma: & \mathcal{G} & \to & \Lambda^2\mathcal{G}\\ & X & \mapsto & \Gamma(X)\end{array}$$

such that, for all $X,Y\in\mathcal{G}$

$$\nabla^G_XY=\Gamma(X)(Y).\tag{5}$$

By the Koszul formula, Γ is determined by the metric as follows: for all $X,Y,Z\in\mathcal{G}$,

$$\langle\Gamma(X)(Y),Z\rangle=\frac12\langle[X,Y],Z\rangle+\frac12\langle[Z,X],Y\rangle-\frac12\langle[Y,Z],X\rangle.\tag{6}$$

Since ∇^G is without torsion, we have, for all $X,Y\in\mathcal{G}$,

$$\Gamma(X)(Y)-\Gamma(Y)(X)=[X,Y].\tag{7}$$

We note that the curvature of ∇^G is given by

$$R^G(X,Y)=[\Gamma(X),\Gamma(Y)]-\Gamma([X,Y])\qquad\in\Lambda^2\mathcal{G}\tag{8}$$

for all $X, Y \in \mathcal{G}$. In the formula the first brackets stand for the commutator of the endomorphisms.

2.2. **The spinor bundle of G.** Let us denote by $Cl(\mathcal{G})$ the Clifford algebra of $\mathcal{G}$ with its scalar product, and let us consider the representation

$$\begin{array}{rccl} \rho: & Spin(\mathcal{G}) & \to & GL(Cl(\mathcal{G})) \\ & a & \mapsto & \xi \mapsto a\xi. \end{array}$$

This representation is a real representation and is not irreducible in general: it is a sum of irreducible representations [12]. By (4) the principal bundle Q_G of the positively oriented and orthonormal frames of G is trivial

$$Q_G \simeq G \times SO(\mathcal{G}),$$

and we may consider the trivial spin structure

$$\tilde{Q}_G := G \times Spin(\mathcal{G})$$

and the corresponding spinor bundle

$$\Sigma := \tilde{Q}_G \times_\rho Cl(G) \simeq G \times Cl(\mathcal{G}).$$

We will say that a spinor field $\varphi \in \Gamma(\Sigma)$ is *left invariant* if it is constant as a map $G \to Cl(\mathcal{G})$. The covariant derivative of a left invariant spinor field is

$$\nabla^G_X \varphi = \frac{1}{2}\Gamma(X) \cdot \varphi \tag{9}$$

where $\Gamma(X) \in \Lambda^2\mathcal{G} \subset Cl(\mathcal{G})$ and the dot "$\cdot$" stands for the Clifford product.

2.3. **The spin representation of $Spin(p) \times Spin(q)$.** Let us assume that $p+q = n$, and fix an orthonormal basis $e_1^o, e_2^o, \ldots, e_n^o$ of $\mathcal{G}$; this gives a splitting $\mathcal{G} = \mathbb{R}^p \oplus \mathbb{R}^q$ (the first factor corresponds to the first p vectors, and the second factor to the last q vectors of the basis) and a natural map

$$Spin(p) \times Spin(q) \to Spin(\mathcal{G}), \qquad (a_p, a_q) \mapsto a := a_p \cdot a_q$$

associated to the isomorphism

$$Cl(\mathcal{G}) = Cl_p \hat{\otimes} Cl_q.$$

We thus also have a representation, still denoted by ρ,

$$\begin{array}{rccl} \rho: & Spin(p) \times Spin(q) & \to & GL(Cl(\mathcal{G})) \\ & (a_p, a_q) & \mapsto & \xi \mapsto a\xi. \end{array} \tag{10}$$

2.4. **The twisted spinor bundle.** We consider a p-dimensional Riemannian manifold M and a bundle $E \to M$ of rank q, with a fiber metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures

$$\tilde{Q}_M \xrightarrow{2:1} Q_M \quad \text{and} \quad \tilde{Q}_E \xrightarrow{2:1} Q_E$$

where Q_M and Q_E are the bundles of positively oriented orthonormal frames of TM and E, and we set

$$\tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E;$$

this is a $Spin(p) \times Spin(q)$ principal bundle. We define

$$\Sigma := \tilde{Q} \times_\rho Cl(\mathcal{G})$$

and
$$U\Sigma := \tilde{Q}\times_\rho Spin(\mathcal{G}) \qquad \subset\ \Sigma$$
where ρ is the representation (10). Similarly to the usual construction in spin geometry, if we consider the representation
$$Ad: \qquad Spin(p)\times Spin(q)\rightarrow Spin(\mathcal{G})\overset{2:1}{\rightarrow} SO(\mathcal{G})\rightarrow GL(Cl(\mathcal{G}))$$
and the Clifford bundle
$$Cl(TM\oplus E)=\tilde{Q}\times_{Ad} Cl(\mathcal{G}),$$
there is a Clifford action of $Cl(TM\oplus E)$ on Σ; this action will be denoted below by a dot " $\cdot$ ". The vector bundle Σ is moreover equipped with the covariant derivative ∇ naturally associated to the spinorial connections on $\tilde{Q}_M$ and $\tilde{Q}_E$. Let us denote by $\tau: Cl(\mathcal{G})\rightarrow Cl(\mathcal{G})$ the anti-automorphism of $Cl(\mathcal{G})$ such that
$$\tau(x_1\cdot x_2\cdots x_k)=x_k\cdots x_2\cdot x_1$$
for all $x_1,x_2,\ldots,x_k\in\mathcal{G}$, and set
$$\begin{array}{rrcl}\langle\langle .,.\rangle\rangle: & Cl(\mathcal{G})\times Cl(\mathcal{G}) & \rightarrow & Cl(\mathcal{G})\\ & (\xi,\xi') & \mapsto & \tau(\xi')\xi.\end{array} \tag{11}$$
This map is $Spin(\mathcal{G})-$invariant: for all $\xi,\xi'\in Cl(\mathcal{G})$ and $g\in Spin(\mathcal{G})$ we have
$$\langle\langle g\xi,g\xi'\rangle\rangle=\tau(g\xi')g\xi=\tau(\xi')\tau(g)g\xi=\tau(\xi')\xi=\langle\langle\xi,\xi'\rangle\rangle,$$
since $Spin(\mathcal{G})\subset\{g\in Cl^0(\mathcal{G}):\ \tau(g)g=1\}$; this map thus induces a $Cl(\mathcal{G})-$valued map
$$\begin{array}{rrcl}\langle\langle .,.\rangle\rangle: & \Sigma\times\Sigma & \rightarrow & Cl(\mathcal{G})\\ & (\varphi,\varphi') & \mapsto & \langle\langle[\varphi],[\varphi']\rangle\rangle\end{array} \tag{12}$$
where $[\varphi]$ and $[\varphi']\in Cl(\mathcal{G})$ represent φ and φ' in some spinorial frame $\tilde{s}\in\tilde{Q}$.

Lemma 2.1. *The map $\langle\langle .,.\rangle\rangle:\Sigma\times\Sigma\rightarrow Cl(\mathcal{G})$ satisfies the following properties: for all $\varphi,\psi\in\Gamma(\Sigma)$ and $X\in\Gamma(TM\oplus E)$,*
$$\langle\langle\varphi,\psi\rangle\rangle=\tau\langle\langle\psi,\varphi\rangle\rangle \tag{13}$$
and
$$\langle\langle X\cdot\varphi,\psi\rangle\rangle=\langle\langle\varphi,X\cdot\psi\rangle\rangle. \tag{14}$$

Proof. We have
$$\langle\langle\varphi,\psi\rangle\rangle=\tau[\psi]\ [\varphi]=\tau(\tau[\varphi]\ [\psi])=\tau\langle\langle\psi,\varphi\rangle\rangle$$
and
$$\langle\langle X\cdot\varphi,\psi\rangle\rangle=\tau[\psi]\ [X][\varphi]=\tau([X][\psi])[\varphi]=\langle\langle\varphi,X\cdot\psi\rangle\rangle$$
where $[\varphi]$, $[\psi]$ and $[X]\ \in Cl(\mathcal{G})$ represent φ,ψ and X in some given frame $\tilde{s}\in\tilde{Q}$. $\square$

Lemma 2.2. *The connection ∇ is compatible with the product $\langle\langle .,.\rangle\rangle$:*
$$\partial_X\langle\langle\varphi,\varphi'\rangle\rangle=\langle\langle\nabla_X\varphi,\varphi'\rangle\rangle+\langle\langle\varphi,\nabla_X\varphi'\rangle\rangle$$
for all $\varphi,\varphi'\in\Gamma(\Sigma)$ and $X\in\Gamma(TM)$.

Proof. If $\varphi=[\tilde{s},[\varphi]]$ is a section of $\Sigma=\tilde{Q}\times_\rho Cl(\mathcal{G})$, we have

$$\nabla_X\varphi=[\tilde{s},\partial_X[\varphi]+\rho_*(\tilde{s}^*\alpha(X))([\varphi])]\,,\qquad \forall X\in\ TM, \tag{15}$$

where ρ is the representation (10) and α is the connection form on $\tilde{Q}$; the term $\rho_*(\tilde{s}^*\alpha(X))$ is an endomorphism of $Cl(\mathcal{G})$ given by the multiplication on the left by an element belonging to $\Lambda^2\mathcal{G}\subset Cl(\mathcal{G})$, still denoted by $\rho_*(\tilde{s}^*\alpha(X))$. Such an element satisfies

$$\tau\left(\rho_*(\tilde{s}^*\alpha(X))\right)=-\rho_*(\tilde{s}^*\alpha(X)),$$

and we have

$$\begin{aligned}\langle\langle\nabla_X\varphi,\varphi'\rangle\rangle+\langle\langle\varphi,\nabla_X\varphi'\rangle\rangle &= \tau\{[\varphi']\}\left(\partial_X[\varphi]+\rho_*(\tilde{s}^*\alpha(X))[\varphi]\right)\\ &\quad+\tau\left\{\partial_X[\varphi']+\rho_*(\tilde{s}^*\alpha(X))[\varphi']\right\}[\varphi]\\ &= \tau\{[\varphi']\}\partial_X[\varphi]+\tau\left\{\partial_X[\varphi']\right\}[\varphi]\\ &= \partial_X\langle\langle\varphi,\varphi'\rangle\rangle.\end{aligned}$$

□

We finally note that there is a natural action of $Spin(\mathcal{G})$ on $U\Sigma$, by right multiplication: for $\varphi=[\tilde{s},[\varphi]]\in U\Sigma=\tilde{Q}\times_\rho Spin(\mathcal{G})$ and $a\in Spin(\mathcal{G})$ we set

$$\varphi\cdot a:=[\tilde{s},[\varphi]\cdot a]\ \in U\Sigma. \tag{16}$$

2.5. **The spin geometry of a submanifold of G.** We keep the notation of the previous section, assuming moreover here that M is a submanifold of a Lie group G and that $E\to M$ is its normal bundle. If we consider spin structures on TM and on E whose sum is the trivial spin structure of $TM\oplus E$ [15], we have

$$\Sigma=\tilde{Q}\times_\rho Cl(\mathcal{G})\simeq M\times Cl(\mathcal{G}),$$

where the last bundle is the spinor bundle of G restricted to M. Two connections are thus defined on Σ, the connection ∇ and the connection ∇^G; they satisfy the following Gauss formula:

$$\nabla^G_X\varphi=\nabla_X\varphi+\frac{1}{2}\sum_{j=1}^p e_j\cdot B(X,e_j)\cdot\varphi \tag{17}$$

for all $\varphi\in\Gamma(\Sigma)$ and all $X\in\Gamma(TM)$, where $B:TM\times TM\to E$ is the second fundamental form of M into G and $e_1,\ldots,e_p$ is an orthonormal basis of TM. We refer to [1] for the proof (in a slightly different context). Since the covariant derivative of a left invariant spinor field is given by (9), the restriction to M of such a spinor field satisfies

$$\nabla_X\varphi=-\frac{1}{2}\sum_{j=1}^p e_j\cdot B(X,e_j)\cdot\varphi+\frac{1}{2}\Gamma(X)\cdot\varphi \tag{18}$$

for all $X\in TM$.

2.6. **Compatibility condition.** We finish this section of preliminaries by giving the following two assumptions which are needed to establish the main result of the present paper.

(1) There exists a bundle isomorphism

$$f: \ TM \oplus E \to M \times \mathcal{G} \tag{19}$$

which preserves the metrics; this mapping permits to define a bundle map

$$\Gamma: \ TM \oplus E \to \Lambda^2(TM \oplus E) \tag{20}$$

such that, for all $X, Y \in \Gamma(TM \oplus E)$,

$$f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y)) \tag{21}$$

where on the right-hand side Γ is the map defined on $\mathcal{G}$ by (5), together with the following notion: a section $Z \in \Gamma(TM \oplus E)$ will be said to be left invariant if $f(Z) : M \to \mathcal{G}$ is a constant map.

(2) The covariant derivative of a left invariant section $Z \in \Gamma(TM \oplus E)$ is given by

$$\nabla_X Z = \Gamma(X)(Z) - B(X, Z^T) + B^*(X, Z^N) \tag{22}$$

for all $X \in TM$, where $Z = Z^T + Z^N$ in $TM \oplus E$ and $B^* : TM \times E \to TM$ is the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X, Y), N\rangle = \langle Y, B^*(X, N)\rangle.$$

These two assumptions are equivalent to the assumptions made in [11, 18]: they are necessary to write down the equations of Gauss, Codazzi and Ricci in a general metric Lie group, and to obtain a Fundamental Theorem for immersions in that context; see Section 4.

Remark 1. *Sometimes it is convenient to write these assumptions in some local frames. For sake of simplicity, we assume that E is a trivial line bundle, oriented by a unit section ν. Let $(e_1^o, e_2^o, \ldots, e_n^o)$ be an orthonormal basis of $\mathcal{G}$ and $\Gamma_{ij}^k \in \mathbb{R}$, $1 \le i, j, k \le n$, be such that*

$$\Gamma(e_i^o)(e_j^o) = \sum_{k=1}^{n} \Gamma_{ij}^k \ e_k^o.$$

We set, for $i = 1, \ldots, n$, $\underline{e}_i \in \Gamma(TM \oplus E)$ such that $f(\underline{e}_i) = e_i^o$, and $f_i \in C^\infty(M)$, $T_i \in \Gamma(TM)$ such that $\underline{e}_i = T_i + f_i \nu$. Since f preserves the metrics, the vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$ are orthonormal, and we have

$$\langle T_i, T_j\rangle + f_i f_j = \delta_{ij} \tag{23}$$

for all $i, j = 1, \ldots, n$. The assumption (22) then reads as follows: for all $X \in TM$ and $j = 1, \ldots, n$,

$$\nabla_X T_j = \sum_{i,k} \Gamma_{ij}^k \langle X, T_i\rangle T_k + f_j S(X), \tag{24}$$

$$df_j(X) = \sum_{i,k} \Gamma_{ij}^k f_k \langle X, T_i\rangle - h(X, T_j) \tag{25}$$

where $S(X) = B^*(X, \nu)$ and $h(X, Y) = \langle B(X, Y), \nu\rangle$. Conversely, if vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$, $1 \le i \le n$, are given such that (23), *(24) and (25) hold, we may define a bundle isomorphism $f : TM \oplus E \to M \times \mathcal{G}$ preserving the metrics and such that (22) holds: setting $\underline{e}_i = T_i + f_i\nu$, we define f such that $f(\underline{e}_i) = e_i^o$, $i = 1, \ldots, n$.*

3. Proof of the main result

Now we will give the proof of Theorem 1. For this, we need the following two propositions. First, we have

Proposition 3.1. *Assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (1) and define ξ by (2). Then*

(1) ξ takes its values in $\mathcal{G} \subset Cl(\mathcal{G})$;
(2) there exists $T \in SO(\mathcal{G})$ such that $\xi = T \circ f$;
(3) replacing φ by $\varphi \cdot a$ where $a \in Spin(\mathcal{G})$ is such that $Ad(a) = T$, we have $\xi = f$, and ξ satisfies the structure equation

$$d\xi + [\xi, \xi] = 0. \tag{26}$$

Proof. (1). By the very definition of ξ, we have

$$\xi(X) = \tau[\varphi][X][\varphi]$$

for all $X \in TM$, where $[X]$ and $[\varphi]$ represent X and φ in a given frame $\tilde{s}$ of $\tilde{Q}$. Since $[X]$ belongs to $\mathcal{G} \subset Cl(\mathcal{G})$ and $[\varphi]$ is an element of $Spin(\mathcal{G})$, $\xi(X)$ belongs to $\mathcal{G}$.
(2). Let us first show that for every left invariant section $Z \in \Gamma(TM \oplus E)$, the map $\xi(Z) : M \to \mathcal{G}$ is constant: if $Z \in \Gamma(TM \oplus E)$ is left invariant, we compute, for $X \in TM$,

$$\partial_X\ \xi(Z) = \langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X \varphi \rangle\rangle.$$

But, by (1),

$$\begin{aligned} \langle\langle Z \cdot \nabla_X\varphi, \varphi\rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X\varphi\rangle\rangle &= \langle\langle [-\Gamma(X) + \sum_{j=1}^{p} e_j \cdot B(X, e_j), Z] \cdot \varphi, \varphi \rangle\rangle \\ &= \langle\langle \{-\Gamma(X)(Z) + B(X, Z^T) - B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle\rangle \qquad (27) \end{aligned}$$

where the brackets $[.,.]$ stand here for the commutator in $Cl(TM \oplus E)$ and where we use Lemmas A.1 and A.3 in the last step. Thus $\partial_X\ \xi(Z) = 0$ by (22), and $\xi(Z) : M \to \mathcal{G}$ is constant. Now, if $(e_1^o, \ldots, e_n^o)$ is a fixed orthonormal basis of $\mathcal{G}$ and denoting by $\underline{e}_1, \ldots, \underline{e}_n$ the left invariant sections of $TM \oplus E$ such that $f(\underline{e}_i) = e_i^o$, $i = 1, \ldots, n$, we have, for all section $Z = \sum_i Z_i \underline{e}_i \in \Gamma(TM \oplus E)$,

$$\xi(Z) = \sum_{i=1}^{n} Z_i\ \xi(\underline{e}_i)$$

where $(\xi(\underline{e}_1), \ldots, \xi(\underline{e}_n))$ is a constant orthonormal basis of $\mathcal{G}$. Considering the orthogonal transformation $T : \mathcal{G} \to \mathcal{G}$ such that $T(e_i^o) = \xi(\underline{e}_i)$, $i = 1, \ldots, n$, we get

$$\xi(Z) = \sum_{i=1}^{n} Z_i\ T(e_i^o) = T\left(\sum_{i=1}^{n} Z_i e_i^o\right) = T(f(Z)),$$

i.e. $\xi = T\circ f$.
(3). For all $a\in Spin(\mathcal{G})$ and $X\in TM$, we have

$$\begin{array}{rcl}\langle\langle X\cdot(\varphi\cdot a),\varphi\cdot a\rangle\rangle & = & \tau([\varphi]a)[X][\varphi]a\\ & = & \tau(a)\ \langle\langle X\cdot\varphi,\varphi\rangle\rangle\ a\\ & = & Ad(a^{-1})(\xi(X))\\ & = & Ad(a^{-1})(T\circ f(X));\end{array}$$

thus, replacing φ by $\varphi\cdot a$ where $a\in Spin(\mathcal{G})$ is such that $Ad(a)=T$ we get $\xi=f$. By the computation in (27), we have, for $X,Y\in\Gamma(TM)$ such that $\nabla X=\nabla Y=0$ at x_0,

$$\begin{array}{rcl}\partial_X\ \xi(Y) & = & \langle\langle Y\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Y\cdot\varphi,\nabla_X\varphi\rangle\rangle\\ & = & \langle\langle\{-\Gamma(X)(Y)+B(X,Y)\}\cdot\varphi,\varphi\rangle\rangle\end{array}$$

and thus

$$\begin{array}{rcl}d\xi(X,Y) & = & \partial_X\ \xi(Y)-\partial_Y\ \xi(X)\\ & = & -\langle\langle\{\Gamma(X)(Y)-\Gamma(Y)(X)\}\cdot\varphi,\varphi\rangle\rangle\\ & = & -\xi(\Gamma(X)(Y)-\Gamma(Y)(X))\\ & = & -[\xi(X),\xi(Y)],\end{array}$$

since $\xi=f$, Γ satisfies (21), and by (7). $\square$

We keep the assumption and notation of Proposition 3.1, and moreover assume that M is simply connected; we consider

$$F: M\to G$$

such that $F^*\omega_G=\xi$ (assuming that φ is chosen in such a way that ξ satisfies the structure equation (26)). The next proposition follows from the properties of the Clifford product:

Proposition 3.2. *1. The map $F:M\to G$ is an isometry.*
2. The map

$$\begin{array}{rrcl}\Phi_E: & E & \to & F(M)\times\mathcal{G}\\ & X\in E_m & \mapsto & (F(m),\xi(X))\end{array}$$

is an isometry between E and the normal bundle of $F(M)$ into G, preserving connections and second fundamental forms. Here, for $X\in E$, $\xi(X)$ still stands for the quantity $\langle\langle X\cdot\varphi,\varphi\rangle\rangle$.

Proof. For $X,Y\in\Gamma(TM\oplus E)$, we have

$$\begin{array}{rcl}\langle\xi(X),\xi(Y)\rangle & = & -\frac{1}{2}\left(\xi(X)\xi(Y)+\xi(Y)\xi(X)\right)\\ & = & -\frac{1}{2}\left(\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi]+\tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi]\right)\\ & = & -\frac{1}{2}\tau[\varphi]\left([X][Y]+[Y][X]\right)[\varphi]\\ & = & \langle X,Y\rangle,\end{array}$$

since $[X][Y]+[Y][X]=-2\langle[X],[Y]\rangle=-2\langle X,Y\rangle$. This implies that F is an isometry, and that Φ_E is a bundle map between E and the normal bundle of $F(M)$ into G which preserves the metrics of the fibers. Let us denote by B_F and ∇'^F the second fundamental form and the normal connection of the immersion F; the aim is now to prove that

$$\xi(B(X,Y)) = B_F(\xi(X),\xi(Y)) \quad \text{and} \quad \xi(\nabla'_X N) = \nabla'^F_{\xi(X)}\xi(N) \tag{28}$$

for $X,Y\in\Gamma(TM)$ and $N\in\Gamma(E)$. First,

$$B_F(\xi(X),\xi(Y)) = (\nabla^G_{\xi(X)}\xi(Y))^N = \{\partial_X\ \xi(Y) + \Gamma(\xi(X))(\xi(Y))\}^N$$

where the superscript N means that we consider the component of the vector which is normal to the immersion. We fix a point $x_0\in M$, assume that $\nabla Y = 0$ at x_0, and compute, using (27):

$$\begin{aligned}\partial_X\ \xi(Y) &= \langle\langle Y\cdot\nabla_X\varphi,\varphi\rangle\rangle + \langle\langle Y\cdot\varphi,\nabla_X\varphi\rangle\rangle\\ &= \langle\langle B(X,Y)\cdot\varphi,\varphi\rangle\rangle - \langle\langle \Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle.\end{aligned}$$

Since $\langle\langle B(X,Y)\cdot\varphi,\varphi\rangle\rangle = \xi(B(X,Y))$ is normal to the immersion, we get

$$\{\partial_X\ \xi(Y)\}^N = \xi(B(X,Y)) - \langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle^N,$$

and thus

$$\begin{aligned}B_F(\xi(X),\xi(Y)) &= \xi(B(X,Y)) - \langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi,\rangle\rangle^N + \Gamma(\xi(X))(\xi(Y))^N\\ &= \xi(B(X,Y))\end{aligned}$$

since

$$\begin{aligned}\langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle &= \xi(\Gamma(X)(Y))\\ &= f(\Gamma(X)(Y))\\ &= \Gamma(f(X))(f(Y)) \quad \text{(by definition of } \Gamma \text{ on } TM\oplus E)\\ &= \Gamma(\xi(X))(\xi(Y)).\end{aligned}$$

We finally show the second identity in (28): we have

$$\begin{aligned}\nabla'^F_{\xi(X)}\xi(N) &= (\nabla^G_{\xi(X)}\xi(N))^N\\ &= (\partial_X\ \xi(N) + \Gamma(\xi(X))(\xi(N)))^N\\ &= \langle\langle\nabla'_X N\cdot\varphi,\varphi\rangle\rangle^N + \langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N + \langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N\\ &\quad +\Gamma(\xi(X))(\xi(N))^N.\end{aligned}$$

The first term in the right-hand side is $\xi(\nabla'_X N)$, and we only need to show that

$$\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N + \langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N + \Gamma(\xi(X))(\xi(N)))^N = 0. \tag{29}$$

From (27), we have

$$\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle + \langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle = -\langle\langle B^*(X,N)\cdot\varphi,\varphi\rangle\rangle - \langle\langle\Gamma(X)(N)\cdot\varphi,\varphi\rangle\rangle,$$

which gives (29) since $\langle\langle B^*(X,N)\cdot\varphi,\varphi\rangle\rangle$ is tangent to the immersion ($B^*(X,N)$ belongs to TM) and

$$\langle\langle\Gamma(X)(N)\cdot\varphi,\varphi\rangle\rangle = \Gamma(\xi(X))(\xi(N))$$

(see the first part of the proof above). □

Now, we can prove the main result of the present paper.

Proof of Theorem 1

First, the fact that (1) ⇒ (2) is a direct consequence of Propositions 3.1 and 3.2. For the converse statement (2) ⇒ (1) : we suppose that $F : M\to G$ is an isometric immersion with normal bundle E and second fundamental form B, we consider the orthonormal frame $s_o = 1_{SO(\mathcal{G})}$ of $\mathcal{G}$, and the spinor frame $\tilde{s}_o = 1_{Spin(\mathcal{G})}$ (recall that $Q_G = G \times SO(\mathcal{G})$ and $\tilde{Q}_G = G \times Spin(\mathcal{G})$; see Section 2). The spinor field $\varphi = [\tilde{s}_o, 1_{Cl(\mathcal{G})}]$ satisfies (1) as a consequence of the Gauss formulas (17)-(18); moreover, its associated 1-form is, for all $X \in TM$,

$$\xi(X) = \langle\langle F_*X \cdot \varphi, \varphi \rangle\rangle = \tau[\varphi]\ [F_*X]\ [\varphi] = [F_*X],$$

where $[F_*X] \in \mathcal{G}$ represents F_*X in s_o, that is $[F_*X] = \omega_G(F_*X)$ ($\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G). Thus $\xi = F^*\omega_G$, that is $F = \int \xi$ which conludes the proof. $\square$

Remark 2. *(1) If φ is a solution of (1) and a belongs to $Spin(\mathcal{G})$, $\varphi' := \varphi \cdot a$ is also a solution of (1) (see (16) for the definition of $\varphi \cdot a$). Moreover the associated 1-forms ξ_φ and $\xi_{\varphi'}$ are linked by*

$$\xi_{\varphi'} \ = \ \tau(a)\ \xi_\varphi\ a = Ad(a^{-1}) \circ \xi_\varphi. \tag{30}$$

Let us recall that a 1-form $\xi \in \Omega^1(M, \mathcal{G})$ is Darboux integrable if and only if it satisfies the structure equation $d\xi + [\xi, \xi] = 0$ (M is simply connected). The theorem thus says that if φ is a solution of (1), it is possible to find an other solution φ' of this equation such that $\xi_{\varphi'}$ is Darboux integrable and $F = \int \xi_{\varphi'}$ is an immersion with normal bundle E and second fundamental form B. The proof of (1) $\Rightarrow$ (2) in the theorem in fact followed these lines.

(2) We proved in Proposition 3.2 that if $\varphi \in \Gamma(U\Sigma)$ is a solution of (1) such that ξ_φ satisfies the structure equation (26) then $F = \int \xi_\varphi$ is an immersion with normal bundle E and second fundamental form B. By (30) it is clear that if $a \in Spin(\mathcal{G})$ is such that $Ad(a^{-1}) : \mathcal{G} \to \mathcal{G}\ \in SO(\mathcal{G})$ is an automorphism of Lie algebra, then $\xi_{\varphi \cdot a}$ satisfies the structure equation too; in fact, the corresponding immersions $F_\varphi = \int \xi_\varphi$ and $F_{\varphi \cdot a} = \int \xi_{\varphi \cdot a}$ are linked by the following formula: if $\Phi_a : G \to G$ is the automorphism of G such that $d(\Phi_a)_e = Ad(a^{-1})$, then Φ_a is also an isometry for the left invariant metric, and

$$F_{\varphi \cdot a} = L_b \circ \Phi_a \circ F_\varphi \tag{31}$$

for some b belonging to G. This relies on the following formula: if $\Phi : G \to G$ is an automorphism, $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G and $F : M \to G$ is a smooth map, then

$$(\Phi \circ F)^* \omega_G = d(\Phi)_e \circ (F^*\omega_G).$$

This formula applied to $\Phi = \Phi_a$ and $F = F_\varphi$ shows that $\Phi_a \circ F_\varphi$ is a solution of the Darboux equation associated to the form $\xi_{\varphi \cdot a}$; thus, by uniqueness of a solution of the Darboux equation, (31) holds for some b belonging to G.

(3) Setting

$$\vec{H} = \frac{1}{2} \sum_{j=1}^{p} B(e_j, e_j)\ \in E \qquad \text{and} \qquad \gamma = \frac{1}{2} \sum_{j=1}^{p} e_j \cdot \Gamma(e_j)\ \in Cl(TM \oplus E)$$

where $e_1, \ldots, e_p$ is an orthonormal basis of TM, a solution φ of (1) is a solution of the Dirac equation

$$D\varphi := \sum_{j=1}^{p} e_j \cdot \nabla_{e_j} \varphi = \left(\vec{H} + \gamma\right) \cdot \varphi. \tag{32}$$

This equation will be especially interesting for the representation of a surface in a 3-dimensional Lie group (see Section 5).

4. An application: the Fundamental Theorem for immersions in a metric Lie group

We now show that the equations of Gauss, Ricci and Codazzi on B are exactly the integrability conditions of (1). We recall these equations for immersions in the metric Lie group G: if R^G denotes the curvature tensor of $(G, \langle .,. \rangle)$, and if R^T and R^N stand for the curvature tensors of the connections on TM and on E (M is a submanifold of G and E is its normal bundle), then we have, for all $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(E)$,

(1) the Gauss equation

$$(R^G(X,Y)Z)^T = R^T(X,Y)Z - B^*(X, B(Y,Z)) + B^*(Y, B(X,Z)), \tag{33}$$

(2) the Ricci equation

$$(R^G(X,Y)N)^N = R^N(X,Y)N - B(X, B^*(Y,N)) + B(Y, B^*(X,N)), \tag{34}$$

(3) the Codazzi equation

$$(R^G(X,Y)Z)^N = \tilde{\nabla}_X B(Y,Z) - \tilde{\nabla}_Y B(X,Z); \tag{35}$$

in the last equation, $\tilde{\nabla}$ denotes the natural connection on $T^*M \otimes T^*M \otimes E$.

These equations make sense if M is an abstract manifold and $E \to M$ is an abstract bundle, if we assume the existence of the bundle map f in (19), since f permits to define Γ on $TM \oplus E$ by (21), and R^G may be written in terms of Γ only (see (7)-(8)). We prove the following:

Proposition 4.1. *We assume that M is simply connected. There exists $\varphi \in \Gamma(U\Sigma)$ solution of (1) if and only if $B : TM \times TM \to E$ satisfies the Gauss, Ricci and Codazzi equations.*

Proof. We first prove that the Gauss, Ricci and Codazzi equations are necessary if we have a non-trivial solution of (1). We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (1) and compute the curvature

$$R(X,Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X,Y]}\varphi.$$

We fix a point $x_0 \in M$, and assume that $\nabla X = \nabla Y = 0$ at x_0. We have

$$\begin{aligned}
\nabla_X \nabla_Y \varphi &= -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot \left(\tilde{\nabla}_X B(Y,e_j)\cdot\varphi + B(Y,e_j)\cdot\nabla_X\varphi\right) \\
&\quad +\frac{1}{2}\left(\nabla_X \Gamma(Y)\cdot\varphi + \Gamma(Y)\cdot\nabla_X\varphi\right) \\
&= -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot \tilde{\nabla}_X B(Y,e_j)\cdot\varphi - \frac{1}{4}\sum_{j,k=1}^{p} e_j\cdot e_k\cdot B(Y,e_j)\cdot B(X,e_k) \\
&\quad -\frac{1}{4}\sum_{j=1}^{p} e_j\cdot B(Y,e_j)\cdot\Gamma(X)\cdot\varphi + \frac{1}{2}\nabla_X\Gamma(Y)\cdot\varphi - \frac{1}{4}\Gamma(Y)\cdot\sum_{j=1}^{p} e_j\cdot B(X,e_j)\cdot\varphi \\
&\quad +\frac{1}{4}\Gamma(Y)\cdot\Gamma(X)\cdot\varphi.
\end{aligned}$$

Thus

$$\begin{aligned} R(X,Y)\varphi &= -\frac{1}{2}\sum_{j=1}^{p} e_j\cdot\left(\tilde{\nabla}_X B(Y,e_j)-\tilde{\nabla}_Y B(X,e_j)\right)\cdot\varphi \\ &\quad \underbrace{+\frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)-B(Y,e_j)\cdot B(X,e_k)\right)\cdot\varphi}_{\mathcal{A}} \\ &\quad \underbrace{-\frac{1}{4}\sum_{j=1}^{p}\left(B(X,e_j)\cdot B(Y,e_j)-B(Y,e_j)\cdot B(X,e_j)\right)\cdot\varphi}_{\mathcal{B}} \\ &\quad \underbrace{+\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(X,e_j),\Gamma(Y)\right]\cdot\varphi}_{\mathcal{C}_1}+\underbrace{-\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(Y,e_j),\Gamma(X)\right]\cdot\varphi}_{\mathcal{C}_2} \\ &\quad \underbrace{+\frac{1}{2}\left(\nabla_X\Gamma(Y)-\nabla_Y\Gamma(X)\right)\cdot\varphi}_{\mathcal{C}_3}+\underbrace{-\frac{1}{2}[\Gamma(X),\Gamma(Y)]\cdot\varphi}_{\mathcal{C}_4} \end{aligned} \tag{36}$$

where the brackets stand for the commutator in the Clifford bundle $Cl(TM\oplus E)$: $\forall\eta,\xi\ \in Cl(TM\oplus E)$,

$$[\eta,\xi]=\frac{1}{2}\left(\eta\cdot\xi-\xi\cdot\eta\right).$$

We computed the second and the third terms in [4]; we only recall the result here:

Lemma 4.2. [4] *We have*

$$\mathcal{A}=\frac{1}{2}\sum_{j<k}\left\{\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle\right\}e_j\cdot e_k$$

and

$$\mathcal{B}=\frac{1}{2}\sum_{k<l}\langle B(X,B^*(Y,n_k))-B(Y,B^*(X,n_k)),n_l\rangle\, n_k\cdot n_l,$$

where $e_1,\ldots,e_p$ and $n_1,\ldots,n_q$ are orthonormal bases of TM and E.

We now compute the other terms in (36). We first compute the covariant derivative of Γ, considering Γ as a map

$$\Gamma:\quad TM\oplus E\rightarrow End(TM\oplus E).$$

Lemma 4.3. *If $X,Y\in TM$ and $Z\in TM\oplus E$,*

$$\begin{aligned} (\nabla_X\Gamma)(Y)Z &= \{\Gamma(X)\circ\Gamma(Y)-\Gamma(Y)\circ\Gamma(X)\}(Z)-\Gamma(\Gamma(X)Y)(Z)+\Gamma(B(X,Y))(Z) \\ &\quad -B(X,(\Gamma(Y)Z)^T)+B^*(X,(\Gamma(Y)Z)^N)+\Gamma(Y)(B(X,Z^T)-B^*(X,Z^N)). \end{aligned}$$

Proof. Since the expression is tensorial, we may assume that $X,Y,Z\in\Gamma(TM\oplus E)$ are left invariant vector fields. By definition,

$$(\nabla_X\Gamma)(Y)Z=\nabla_X(\Gamma(Y)Z)-\Gamma(\nabla_XY)Z-\Gamma(Y)(\nabla_XZ). \tag{37}$$

Since X,Y and Z are left invariant vector fields, so are $\Gamma(Y)Z$, ∇_XY and ∇_XZ, and, by (22),

$$\nabla_X(\Gamma(Y)Z)=\Gamma(X)(\Gamma(Y)Z)-B(X,(\Gamma(Y)Z)^T)+B^*(X,(\Gamma(Y)Z)^N),$$

$$\Gamma(Y)(\nabla_X Z) = \Gamma(Y)(\Gamma(X)Z) - \Gamma(Y)B(X, Z^T) + \Gamma(Y)B^*(X, Z^N)$$

and

$$\Gamma(\nabla_X Y)(Z) = \Gamma(\Gamma(X)Y)Z - \Gamma(B(X, Y^T))Z + \Gamma(B^*(X, Y^N))Z.$$

Plugging these formulas in (37) and using finally that Y belongs to TM (ie $Y^T = Y$ and $Y^N = 0$), we get the result. □

We now regard Γ as a map

$$\Gamma : \quad TM \oplus E \to \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E),$$

and compute the term $\mathcal{C}_3$ in (36). According to Lemma A.1, for all $X, Y \in TM \oplus E$,

$$\Gamma(X)(Y) = [\Gamma(X), Y].$$

Lemma 4.4. *If $X, Y \in TM$,*

$$\begin{aligned}\frac{1}{2}\left((\nabla_X\Gamma)(Y) - (\nabla_Y\Gamma)(X)\right) &= [\Gamma(X), \Gamma(Y)] - \frac{1}{2}\Gamma([\Gamma(X), Y] - [\Gamma(Y), X]) \\ &\quad -\frac{1}{2}\left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y)\right] + \frac{1}{2}\left[\sum_{j=1}^p e_j \cdot B(Y, e_j), \Gamma(X)\right].\end{aligned}$$

Here the brackets stand for the commutator in $Cl(TM \oplus E)$.

Proof. By Lemmas A.1 and A.2 in the appendix, the linear maps $\Gamma(X) \circ \Gamma(Y) - \Gamma(Y) \circ \Gamma(X)$, $Z \mapsto \Gamma(\Gamma(X)Y)Z$ and $Z \mapsto \Gamma(B(X, Y))Z$ appearing in Lemma 4.3 are respectively represented by the bivectors $[\Gamma(X), \Gamma(Y)]$, $\Gamma([\Gamma(X), Y])$ and $\Gamma(B(X, Y))$. Moreover, by Lemma A.4 applied to the linear maps $B(X, .) : TM \to E$ and $\Gamma(Y) : TM \oplus E \to TM \oplus E$, the map

$$Z \mapsto -B^*(X, (\Gamma(Y)Z)^N) + \Gamma(Y)(B^*(X, Z^N)) + B(X, (\Gamma(Y)Z)^T) - \Gamma(Y)(B(X, Z^T))$$

is represented by the bivector

$$\left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y)\right] \quad \in Cl(TM \oplus E).$$

The result follows. □

We readily deduce the sum of the last four terms in (36):

Lemma 4.5. *Let us set, for $X, Y \in TM$,*

$$R^G(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma\left\{[\Gamma(X), Y] - [\Gamma(Y), X]\right\} \ \in \Lambda^2(TM \oplus E),$$

the curvature tensor of G, pulled-back to $TM \oplus E$ by the bundle isomorphism f introduced in (19). Then

$$\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 = \frac{1}{2}R^G(X, Y).$$

We thus get from (36) the formula

$$\begin{aligned}R(X, Y)\varphi &= -\frac{1}{2}\sum_{j=1}^p e_j \cdot \left(\tilde\nabla_X B(Y, e_j) - \tilde\nabla_Y B(X, e_j)\right) \cdot \varphi \\ &\quad + \mathcal{A} \cdot \varphi + \mathcal{B} \cdot \varphi + \frac{1}{2}R^G(X, Y) \cdot \varphi\end{aligned} \tag{38}$$

where $\mathcal{A}$ and $\mathcal{B}$ are computed in Lemma 4.2 and R^G may be conveniently written in the form

$$\begin{aligned} R^G(X,Y) &= \sum_{1\leq j<k\leq p}\langle R^G(X,Y)(e_j),e_k\rangle e_j\cdot e_k \\ &\quad+\sum_{j=1}^{p}\sum_{r=1}^{q}\langle R^G(X,Y)(e_j),n_r\rangle e_j\cdot n_r \\ &\quad+\sum_{1\leq r<s\leq q}\langle R^G(X,Y)(n_s),n_r\rangle n_r\cdot n_s. \end{aligned}$$

On the other hand, the curvature of the spinorial connection is given by

$$R(X,Y)\varphi = \frac{1}{2}\left(\sum_{1\leq j<k\leq p}\langle R^T(X,Y)(e_j),e_k\rangle\ e_j\cdot e_k + \sum_{1\leq r<s\leq q}\langle R^N(X,Y)(n_r),n_s\rangle\ n_r\cdot n_s\right)\cdot\varphi. \tag{39}$$

We now compare the expressions (38) and (39): since in a given frame $\tilde{s}$ belonging to $\tilde{Q}$, φ is represented by an element which is invertible in $Cl(\mathcal{G})$ (it is in fact represented by an element belonging to $Spin(\mathcal{G})$), we may identify the coefficients and get

$$\langle R^T(X,Y)(ej),e_k\rangle=\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle+\langle R^G(X,Y)(e_j),e_k\rangle,$$

$$\langle R^N(X,Y)(n_r),n_s\rangle=\langle B(X,B^*(Y,n_r)),n_s\rangle-\langle B(Y,B^*(X,n_r)),n_s\rangle+\langle R^G(X,Y)(n_r),n_s\rangle$$

and

$$\langle\tilde{\nabla}_X B(Y,e_j)-\tilde{\nabla}_Y B(X,e_j),n_r\rangle=\langle R^G(X,Y)(e_j),n_r\rangle$$

for all the indices. These equations are the equations of Gauss, Ricci and Codazzi.

We now prove that the equations of Gauss, Ricci and Codazzi are also sufficient to get a solution of (1). The calculations above in fact show that the connection on Σ defined by

$$\nabla'_X\varphi:=\nabla_X\varphi+\frac{1}{2}\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi-\frac{1}{2}\Gamma(X)\cdot\varphi \tag{40}$$

for all $\varphi\in\Gamma(\Sigma)$ and $X\in\Gamma(TM)$ is flat if and only if the equations of Gauss, Ricci and Codazzi hold. But if this connection is flat there exists a solution $\varphi\in\Gamma(U\Sigma)$ of (1); this is because ∇' may be also interpreted as a connection on $U\Sigma$ regarded as a principal bundle (of group $Spin(\mathcal{G})$, acting on the right): indeed, ∇ defines such a connection (since it comes from a connection on $\tilde{Q}$), and the right hand side term in (40) defines a linear map

$$\begin{aligned} TM &\to \chi_V^{inv}(U\Sigma) \\ X &\mapsto \varphi\mapsto\frac{1}{2}\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi-\frac{1}{2}\Gamma(X)\cdot\varphi \end{aligned}$$

from TM to the vector fields on $U\Sigma$ which are vertical and invariant under the action of the group (these vector fields are of the form $\varphi\mapsto\eta\cdot\varphi$, $\eta\in\Lambda^2(TM\oplus E)\subset$

$Cl(TM \oplus E)$). Assuming that the equations of Gauss, Codazzi and Ricci hold, we thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (1). ☐

The considerations above give a spinorial proof of the Fundamental Theorem of submanifold theory in the metric Lie group G (see [18] for another proof). We keep the hypotheses and notation of Theorem 1.

Corollary 1. *We moreover assume that $B : TM \times TM \to E$ satisfies the equations of Gauss, Codazzi and Ricci (33)-(35). Then there is an isometric immersion of M into G with normal bundle E and second fundamental form B. The immersion is unique up to a rigid motion in G, that is up to a transformation of the form*

$$L_b \circ \Phi_a : \begin{array}{ccc} G & \to & G \\ g & \mapsto & b\Phi_a(g) \end{array} \tag{41}$$

where $a \in Spin(\mathcal{G})$ is such that $Ad(a) :\ \mathcal{G} \to \mathcal{G}$ is an automorphism of Lie algebra, $\Phi_a : G \to G$ is the group automorphism such that $d(\Phi_a)_e = Ad(a)$, and b belongs to G.

Proof. The equations of Gauss, Codazzi and Ricci are the integrability conditions of (1). We thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (1); with such a spinor field at hand, $F = \int \xi$ where ξ is defined in (2) is the immersion. Finally, a solution of (1) is unique up to the right action of an element of $Spin(\mathcal{G})$; the right multiplication of φ by $a \in Spin(\mathcal{G})$ and the left multiplication by $b \in G$ in the last integration give also an immersion, if $Ad(a) : \mathcal{G} \to \mathcal{G}$ is moreover an automorphism of Lie algebra. This immersion is obtained from the immersion defined by φ by a rigid motion, as described in (41). ☐

Remark 3. *In $\mathbb{R}^n$, a rigid motion as in (41) is a transformation of the form*

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & ax+b, \end{array}$$

with $a \in SO(n)$ and $b \in \mathbb{R}^n$.

5. Special cases

5.1. **Submanifolds in $\mathbb{R}^n$.** If the metric Lie group is $\mathbb{R}^n$ with its natural metric, we recover the main result of [4]. We suppose that M is a p-dimensional Riemannian manifold, $E \to M$ a bundle of rank q, with a fiber metric and a compatible connection. We assume that TM and E are oriented and spin with given spin structures, and that $B : TM \times TM \to E$ is bilinear and symmetric.

Theorem 2. [4] *We moreover assume that M is simply connected. The following statements are equivalent:*

(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that

$$\nabla_X \varphi = -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \tag{42}$$

for all $X \in TM$.

(2) *There exists an isometric immersion $F: M \to \mathbb{R}^n$ with normal bundle E and second fundamental form B.*
Moreover, $F = \int \xi$ where ξ is the $\mathbb{R}^n-$valued 1-form defined by

$$\xi(X) := \langle\langle X \cdot \varphi, \varphi \rangle\rangle \tag{43}$$

for all $X \in TM$.

Proof. We only prove (1) $\Rightarrow$ (2). This will be a consequence of Theorem 1 if we may define a bundle map f as in (19) such that (22) holds. We assume that φ is a solution of (42), and set

$$\begin{array}{rccc} f: & TM \oplus E & \to & M \times \mathbb{R}^n \\ & Z & \mapsto & \langle\langle Z \cdot \varphi, \varphi \rangle\rangle. \end{array}$$

The map Γ defined by (21) is $\Gamma = 0$. We now show that (22) is satisfied for every $Z \in \Gamma(TM \oplus E)$ such that $f(Z): M \to \mathbb{R}^n$ is a constant map: for all $X \in TM$, we have $\partial_X\{f(Z)\} = 0$, which reads

$$\langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X \varphi \rangle\rangle = 0.$$

But (42) gives

$$\langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X \varphi \rangle\rangle = \langle\langle \{B(X, Z^T) - B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle\rangle$$

(see the computations in (27) with $\Gamma = 0$). Thus

$$\langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle = \langle\langle \{-B(X, Z^T) + B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle\rangle$$

and

$$\nabla_X Z = -B(X, Z^T) + B^*(X, Z^N),$$

which is (22) with $\Gamma = 0$. □

5.2. **Submanifolds in $\mathbb{H}^n$.** Spinor representations of submanifolds in $\mathbb{H}^n$ with its natural metric were already given in [16, 3, 4]. We give here another representation using the group structure of $\mathbb{H}^n$, with an arbitrary left invariant metric. Let us set

$$\mathbb{H}^n = \{a = (a', a_n) \in \mathbb{R}^n : a_n > 0\},$$

and, for $a \in \mathbb{H}^n$, the transformation

$$\begin{array}{rccc} \varphi_a: & \mathbb{R}^{n-1} & \to & \mathbb{R}^{n-1} \\ & x & \mapsto & a_n x + a'; \end{array}$$

φ_a is an homothety composed by a translation. The homotheties composed by translations naturally form a group under composition, and the bijection

$$\begin{array}{rccc} \varphi: & \mathbb{H}^n & \to & \{\text{homotheties-translations } \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}\} \\ & a & \mapsto & \varphi_a \end{array}$$

induces a group structure on $\mathbb{H}^n$: it is such that

$$ab = (a_n b' + a', a_n b_n) \tag{44}$$

for all $a, b \in \mathbb{H}^n$; the identity element is $e = (0, 1) \in \mathbb{H}^n$. Let us denote by $(e_1^o, e_2^o, \ldots, e_n^o)$ the canonical basis of $T_e\mathbb{H}^n = \mathbb{R}^n$ and keep the same letters to denote the corresponding left invariant vector fields on $\mathbb{H}^n$. The Lie bracket may be easily seen to be given by

$$[e_i^o, e_j^o] = 0 \quad \text{and} \quad [e_n^o, e_i^o] = e_i^o$$

for $i, j = 1, \ldots, n-1$. This may also be written in the form

$$[X, Y] = l(X)Y - l(Y)X \tag{45}$$

for all $X, Y \in \mathbb{R}^n$, where $l : \mathbb{R}^n \to \mathbb{R}$ is the linear form such that $l(e_i^o) = 0$ if $i \leq n-1$ and $l(e_n^o) = 1$. This property implies that every left invariant metric on $\mathbb{H}^n$ has constant negative curvature $-|l|^2$ [14, 13].

We suppose that a left invariant metric $\langle ., . \rangle$ is given on $\mathbb{H}^n$, and consider the vector $U_o \in T_e\mathbb{H}^n$ such that $l(X) = \langle U_o, X \rangle$ for all $X \in T_e\mathbb{H}^n$. We have $|U_o| = |l|$, and, by the Koszul formula (6),

$$\Gamma(X)(Y) = -\langle Y, U_o \rangle X + \langle X, Y \rangle U_o \tag{46}$$

for all $X, Y \in T_e\mathbb{H}^n$.

We keep the hypotheses made at the beginning of Section 5.1. We suppose moreover that $U \in \Gamma(TM \oplus E)$ is given such that $|U| = |l|$ and, for all $X \in TM$,

$$\nabla_X U = -|U|^2 X + \langle X, U \rangle U - B(X, U^T) + B^*(X, U^N). \tag{47}$$

We set, for $X \in TM$ and $Y \in TM \oplus E$,

$$\Gamma(X)(Y) = -\langle Y, U \rangle X + \langle X, Y \rangle U. \tag{48}$$

Remark 4. *Equation (47) implies the following:*

1. *U is a solution of (22), with the definition (48) of Γ.*
2. *The norm of U is constant, since, by a straightforward computation,*

   $$d|U|^2(X) = 2\langle \nabla_X U, U \rangle = 0$$

   for all $X \in TM$. The additional hypothesis $|U| = |l|$ is thus not very restrictive.

We note that it is not necessary to assume the existence of U solution of (47) to get a spinor representation of a submanifold in $\mathbb{H}^n$ if $\mathbb{H}^n$ is regarded as the set of unit vectors in Minkowski space $\mathbb{R}^{n,1}$ [16, 3, 4]. Nevertheless, this hypothesis seems necessary if we consider $\mathbb{H}^n$ as a group, since the group structure introduces an anisotropy: the vector $e_n \in T_e\mathbb{H}^n$ is indeed a special direction for the group structure.

Let us construct the spinor bundles Σ and $U\Sigma$ on M as in Section 2.4 with here $\mathcal{G} = T_e\mathbb{H}^n$.

Theorem 3. *We assume that M is simply connected. The following statements are equivalent:*

1. *There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (1) where Γ is defined by (48).*
2. *There exists an isometric immersion $M \to \mathbb{H}^n$ with normal bundle E and second fundamental form B.*

Proof. We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (1) where Γ is defined by (48), and define $f : TM \oplus E \to M \times T_e\mathbb{H}^n$ by

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi \rangle\rangle$$

for all $Z \in TM \oplus E$. Let us first observe that if Z is a vector field solution of (22), then $f(Z)$ is constant: we have, for all $X \in TM$,

$$\partial_X f(Z) = \langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X \varphi \rangle\rangle;$$

this is 0, by (22), (1) and the computation (27). Since U is a solution of (22) (see Remark 4), we deduce that $f(U) \in T_e\mathbb{H}^n$ is a constant, and, since $|f(U)| = |U| = |U_o|$, replacing φ by $\varphi \cdot a$ for some $a \in Spin(T_e\mathbb{H}^n)$ if necessary, we may suppose that $f(U) = U_o$. Since Γ is defined on $T_e\mathbb{H}^n$ by (46) and on $TM \oplus E$ by (48), and since f preserves the metrics, it is straightforward to see that $f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y))$ for all $X, Y \in TM \oplus E$. Finally, (22) holds for all $Z \in \Gamma(TM \oplus E)$ such that $f(Z)$ is constant: this is the same argument as in the proof of Theorem 2 in Section 5.1, just adding the term Γ. The result then follows from Theorem 1. $\square$

5.3. **Hypersurfaces in a metric Lie group.** We assume that G is a simply connected n-dimensional metric Lie group, M is a p-dimensional Riemannian manifold, $n = p+1$, and E is the trivial line bundle on M, oriented by a unit section $\nu \in \Gamma(E)$. We moreover suppose that M is simply connected and that $h : TM \times TM \to \mathbb{R}$ is a given symmetric bilinear form, and that the hypotheses (1) and (2) of Section 2.6 with $B = h\nu$ hold. According to Theorem 1, an isometric immersion of M into G with second fundamental form h is equivalent to a section φ of $\Gamma(U\Sigma)$ solution of the Killing equation (1). Note that $Q_E \simeq M$ and the double covering $\tilde{Q}_E \to Q_E$ is trivial, since M is assumed to be simply connected. Fixing a section $\tilde{s}_E$ of $\tilde{Q}_E$ we get an injective map

$$\begin{array}{ccc} \tilde{Q}_M & \to & \tilde{Q}_M \times_M \tilde{Q}_E =: \tilde{Q} \\ \tilde{s}_M & \mapsto & (\tilde{s}_M, \tilde{s}_E). \end{array}$$

Using

$$Cl_p \simeq Cl^0_{p+1} \subset Cl_{p+1}$$

(induced by the Clifford map $\mathbb{R}^p \to Cl_{p+1}$, $X \mapsto X \cdot e_{p+1}$), we deduce a bundle isomorphism

$$\begin{array}{ccccc} \tilde{Q}_M \times_\rho Cl_p & \to & \tilde{Q} \times_\rho Cl^0_{p+1} & \subset \Sigma \\ \psi & \mapsto & \psi^*. & \end{array} \tag{49}$$

It satisfies the following properties: for all $X \in TM$ and $\psi \in \tilde{Q}_M \times_\rho Cl_p$,

$$(X \cdot \psi)^* = X \cdot \nu \cdot \psi^* \quad \text{and} \quad \nabla_X(\psi^*) = (\nabla_X \psi)^*. \tag{50}$$

To write down the Killing equation (1) in the bundle $\tilde{Q}_M \times_\rho Cl_p$, we need to decompose the Clifford action of $\Gamma(X)$ into its tangent and its normal parts:

Lemma 5.1. *Recall the notation introduced in Remark 1. Then, for all $X \in TM$,*

$$\Gamma(X) = \sum_{i=1}^n \langle X, T_i \rangle \sum_{1 \le j < k \le n} \Gamma_{ij}^k \left(\frac{1}{2}(T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu \right). \tag{51}$$

Proof. We have

$$X = \sum_{i=1}^n \langle X, \underline{e}_i \rangle \underline{e}_i = \sum_{i=1}^n \langle X, T_i \rangle \underline{e}_i,$$

$$\begin{aligned}\Gamma(X)(\underline{e}_j) &= \sum_{i=1}^{n}\langle X,T_i\rangle\Gamma(\underline{e}_i)(\underline{e}_j)\\ &= \sum_{i=1}^{n}\langle X,T_i\rangle\sum_{k=1}^{n}\Gamma_{ij}^{k}\underline{e}_k\\ &= \sum_{1\leq i,k\leq n}\Gamma_{ij}^{k}\langle X,T_i\rangle(T_k+f_k\nu),\end{aligned}$$

and thus

$$\begin{aligned}\Gamma(X) &= \frac{1}{2}\sum_{j=1}^{n}\underline{e}_j\cdot\Gamma(X)(\underline{e}_j)\\ &= \frac{1}{2}\sum_{j=1}^{n}(T_j+f_j\nu)\cdot\sum_{1\leq i,k\leq n}\Gamma_{ij}^{k}\langle X,T_i\rangle(T_k+f_k\nu)\\ &= \frac{1}{2}\sum_{1\leq i,j,k\leq n}\Gamma_{ij}^{k}\langle X,T_i\rangle(T_j+f_j\nu)\cdot(T_k+f_k\nu).\end{aligned}$$

Now

$$(T_j+f_j\nu)\cdot(T_k+f_k\nu)=T_j\cdot T_k+f_kT_j\cdot\nu-f_jT_k\cdot\nu-f_jf_k,$$

and the result follows since $\Gamma_{ij}^{k}=-\Gamma_{ik}^{j}$. □

The section $\varphi\in\Gamma(U\Sigma)$ solution of (1) thus identifies to a section ψ of $\tilde{Q}_M\times_\rho Cl_p$ solution of

$$\begin{aligned}\nabla_X\psi &= -\frac{1}{2}\sum_{j=1}^{p}h(X,e_j)e_j\cdot\psi+\frac{1}{2}\tilde{\Gamma}(X)\cdot\psi\\ &= -\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}\tilde{\Gamma}(X)\cdot\psi\end{aligned}$$

for all $X\in TM$, where

$$\tilde{\Gamma}(X)=\sum_{i=1}^{n}\langle X,T_i\rangle\sum_{1\leq j<k\leq n}\Gamma_{ij}^{k}\left(\frac{1}{2}\left(T_j\cdot T_k-T_k\cdot T_j\right)+\left(f_kT_j-f_jT_k\right)\right)\tag{52}$$

and $S:TM\rightarrow TM$ is the symmetric operator associated to h. We deduce the following result:

Theorem 4. *Let $S:TM\rightarrow TM$ be a symmetric operator. The following two statements are equivalent:*

(1) there exists an isometric immersion of M into G with shape operator S;
(2) there exists a normalized spinor field $\psi\in\Gamma(\tilde{Q}_M\times_\rho Cl_p)$ solution of

$$\nabla_X\psi=-\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}\tilde{\Gamma}(X)\cdot\psi\tag{53}$$

for all $X\in TM$, where $\tilde{\Gamma}$ is defined in (52).

Here, a spinor field $\psi\in\Gamma(\tilde{Q}_M\times_\rho Cl_p)$ is said to be normalized if it is represented in some frame $\tilde{s}\in\tilde{Q}_M$ by an element $[\psi]\in Cl_p\simeq Cl_{p+1}^0$ belonging to $Spin(p+1)$.

We will see below explicit representation formulas in the cases of the dimensions 3 and 4.

5.4. **Surfaces in a 3-dimensional metric Lie group.** Since $Cl_2 \simeq \Sigma_2$ we have

$$\tilde{Q}_M \times_\rho Cl_2 \simeq \Sigma M,$$

and φ is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (53) and such that $|\psi| = 1$. Moreover, the explicit representation formula $F = \int \xi$ may be written in terms of ψ : it may be proved by a computation that

$$\langle\langle X \cdot \varphi, \varphi \rangle\rangle = i2\mathcal{R}e\langle X \cdot \psi^+, \psi^- \rangle + j \left(\langle X \cdot \psi^+, \alpha(\psi^+) \rangle - \langle X \cdot \psi^-, \alpha(\psi^-) \rangle \right) \tag{54}$$

where the brackets $\langle .,. \rangle$ stand here for the natural hermitian product on Σ_2 and $\alpha : \Sigma_2 \to \Sigma_2$ is the natural quaternionic structure. If $G = \mathbb{R}^3$, this is the explicit representation formula given in [8] (see also [3]).

We also note that the expression (52) of $\tilde{\Gamma}$ simplifies if the Lie group is 3-dimensional:

Lemma 5.2. *If j, k, $j \neq k$, belong to $\{1, 2, 3\}$, let us denote by $l_{jk} \in \{1, 2, 3\}$ the number such that (j, k, l_{jk}) is a permutation of $\{1, 2, 3\}$ and by $\epsilon_{jk} = \pm 1$ the sign of this permutation. Then, for all $X \in TM$,*

$$\tilde{\Gamma}(X) = \sum_{i=1}^{3} \langle X, T_i \rangle \sum_{1 \leq j < k \leq 3} \Gamma_{ij}^k \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}}) \cdot \omega$$

where $\omega \in Cl(TM)$ is the area element of M.

Proof. Keeping the notation introduced above, we note that

$$\underline{e}_j \cdot \underline{e}_k \cdot \underline{e}_{l_{jk}} = \epsilon_{jk}\ \omega \cdot \nu,$$

which yields

$$\underline{e}_j \cdot \underline{e}_k = -\epsilon_{jk}\ \omega \cdot \nu \cdot \underline{e}_{l_{jk}}.$$

Thus

$$\begin{aligned} T_j \cdot T_k + (f_k T_j - f_j T_k) \cdot \nu - f_j f_k &= -\epsilon_{jk}\ \omega \cdot \nu \cdot (T_{l_{jk}} + f_{l_{jk}} \nu) \\ &= \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega \end{aligned}$$

since $T_{l_{jk}} \cdot \nu = -\nu \cdot T_{l_{jk}}$, $T_{l_{jk}} \cdot \omega = -\omega \cdot T_{l_{jk}}$ and $\omega \cdot \nu = \nu \cdot \omega$. Switching the indices j and k we also get

$$\begin{aligned} T_k \cdot T_j + (f_j T_k - f_k T_j) \cdot \nu - f_k f_j &= \epsilon_{kj} (f_{l_{kj}} - T_{l_{kj}} \cdot \nu) \cdot \omega \\ &= -\epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega \end{aligned}$$

since $\epsilon_{kj} = -\epsilon_{jk}$ and $l_{kj} = l_{jk}$. We deduce that

$$\frac{1}{2} (T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu = \epsilon_{jk} (f_{l_{jk}} - T_{l_{jk}} \cdot \nu) \cdot \omega.$$

The result is then a consequence of Lemma 5.1 together with the relation

$$\left(\tilde{\Gamma}(X) \cdot \psi \right)^* = \Gamma(X) \cdot \psi^*$$

and the first property in (50). $\square$

5.4.1. *The metric Lie group* $\mathbb{S}^3$. A spinor representation of a surface immersed in $\mathbb{S}^3$ was already given in [16] (see also [3, 4]). We give here a spinor representation relying on the group structure; it appears that it coincides with the result in [16].

We regard the sphere $\mathbb{S}^3$ as the set of the unit quaternions, with its natural group structure. The Lie algebra of $\mathbb{S}^3$ identifies to $\mathbb{R}^3$, with the bracket $[X,Y] = 2X \times Y$ for all $X, Y \in \mathbb{R}^3$ ($\times$ is the usual cross product). By the Koszul formula (6), for all $X, Y \in \mathbb{R}^3$,

$$\Gamma(X)(Y) = X \times Y.$$

As a bivector, for all $X = X_1 e_1^o + X_2 e_2^o + X_3 e_3^o \in \mathbb{R}^3$,

$$\begin{aligned}
\Gamma(X) &= \frac{1}{2}\left(e_1^o \cdot \Gamma(X)(e_1^o) + e_2^o \cdot \Gamma(X)(e_2^o) + e_3^o \cdot \Gamma(X)(e_3^o)\right) \\
&= X_1 e_2^o \cdot e_3^o + X_2 e_3^o \cdot e_1^o + X_3 e_1^o \cdot e_2^o \\
&= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o).
\end{aligned}$$

Thus, if $\varphi \in \tilde{Q} \times_\rho Cl_3^0$ represents an immersion of an oriented surface M in $\mathbb{S}^3$ and if $\psi \in \Gamma(\Sigma M)$ is such that $\varphi = \psi^*$, then, for all $X \in TM$,

$$\begin{aligned}
\Gamma(X) \cdot \varphi &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o) \cdot \varphi \\
&= -X \cdot \omega \cdot \nu \cdot \varphi \\
&= (X \cdot \nu) \cdot \omega \cdot \varphi \\
&= (X \cdot \omega \cdot \psi)^*
\end{aligned}$$

where ω is the area form of M, and ν is the vector normal to M in $\mathbb{S}^3$. Since $\varphi \in \Gamma(U\Sigma)$ is a solution of (1), $\psi \in \Gamma(\Sigma M)$ is a solution of

$$\nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} X \cdot \omega \cdot \psi$$

and satisfies $|\psi| = 1$. Taking the trace, we get

$$\begin{aligned}
D\psi &= e_1 \cdot \nabla_{e_1}\psi + e_2 \cdot \nabla_{e_2}\psi \\
&= H\psi - \omega \cdot \psi
\end{aligned}$$

where (e_1, e_2) is a positively oriented and orthonormal basis of TM. Now, setting $\overline{\psi} = \psi^+ - \psi^-$ and since $\omega \cdot \psi = -i\overline{\psi}$ (recall that $i\omega$ acts as the identity on $\Sigma^+ M$ and as -identity on $\Sigma^- M$), we get

$$D\psi = H\psi - i\overline{\psi},$$

which is also the spinor characterization given by Morel in [16].

5.4.2. *Surfaces in the 3-dimensional metric Lie groups* $E(\kappa, \tau)$, $\tau \neq 0$. We recover here a spinor characterization of immersions in the 3-dimensional homogeneous spaces $E(\kappa, \tau)$; this result was obtained by the second author in [19], using a characterization of immersions in these spaces by Daniel [7]. We give here an independent proof, and rather obtain the result of Daniel as a corollary.

The metric Lie group $E(\kappa, \tau)$, $\tau \neq 0$, is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the vectors e_1^o, e_2^o, e_3^o of the canonical basis by

$$[e_1^o, e_2^o] = 2\tau e_3^o, \qquad [e_2^o, e_3^o] = \sigma e_1^o, \qquad [e_3^o, e_1^o] = \sigma e_2^o$$

where $\sigma = \frac{\kappa}{2\tau}$. The metric on $\mathcal{G}$ is the canonical metric, ie the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. The Levi-Civita connection is then given by

$$\Gamma(X)(Y) = \{\tau(X - \langle X, e_3^o\rangle e_3^o) + (\sigma - \tau)\langle X, e_3^o\rangle e_3^o\} \times Y \tag{55}$$

for $X, Y \in \mathcal{G}$; see e.g. [7].

Let $S : TM \to TM$ be a symmetric operator. We assume that a vector field $T \in \Gamma(TM)$ and a function $f \in C^\infty(M, \mathbb{R})$ are given such that

$$|T|^2 + f^2 = 1, \tag{56}$$

$$\nabla_X T = f(S(X) - \tau JX) \tag{57}$$

and

$$df(X) = -\langle S(X) - \tau JX, T\rangle \tag{58}$$

for all $X \in TM$, where $J : TM \to TM$ denotes the rotation of angle $+\pi/2$ in the tangent planes.

Theorem 5. [19] *If M is simply connected, the following two statements are equivalent:*

1. *There exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and*
$$\nabla_X\psi = -\frac{1}{2}S(X)\cdot\psi + \frac{1}{2}\{(2\tau - \sigma)\langle X, T\rangle\,(T - f) - \tau X\}\cdot\omega\cdot\psi \tag{59}$$
for all $X \in TM$.
2. *There exists an isometric immersion of M into $E(\kappa, \tau)$, with shape operator S.*

Proof. We consider the trivial line bundle $E = \mathbb{R}\nu$, where ν is a unit section. The bundle $TM \oplus E$ is of rank 3, and is assumed to be oriented by the orientation of TM and by ν. We suppose that it is endowed with the natural product metric. Let us denote by $\times$ the natural cross product in the fibers. We set

$$\underline{e}_3 = T + f\nu,$$

and, for all $X, Y \in TM \oplus E$,

$$\Gamma(X)(Y) = \{\tau(X - \langle X, \underline{e}_3\rangle \underline{e}_3) + (\sigma - \tau)\langle X, \underline{e}_3\rangle \underline{e}_3\} \times Y. \tag{60}$$

Defining $B : TM \times TM \to E$ and its adjoint $B^* : TM \times E \to TM$ by

$$B(X, Y) = \langle S(X), Y\rangle\nu \qquad \text{and} \qquad B^*(X, \nu) = S(X) \tag{61}$$

for all $X, Y \in TM$, the equations (57) and (58) are equivalent to the single equation

$$\nabla_X \underline{e}_3 = \Gamma(X)(\underline{e}_3) - B(X, \underline{e}_3^T) + B^*(X, \underline{e}_3^N) \tag{62}$$

for all $X \in TM$, where ∇ is the sum of the Levi-Civita connection on TM and the trivial connection on E. This is (22) for $Z = \underline{e}_3$. We will need the following expression for Γ :

Lemma 5.3. *For all $X \in TM$, the linear map $\Gamma(X) : TM \oplus E \to TM \oplus E$ defined by (60) is represented by the bivector*

$$\Gamma(X) = \{(2\tau - \sigma)\,\langle X, T\rangle\,(T\cdot\nu - f) - \tau X\cdot\nu\}\cdot\omega.$$

Proof. The linear map $\Gamma(X)$ is represented by the bivector

$$\Gamma(X) = \frac{1}{2}\left(\underline{e}_1 \cdot \Gamma(X)(\underline{e}_1) + \underline{e}_2 \cdot \Gamma(X)(\underline{e}_2) + \underline{e}_3 \cdot \Gamma(X)(\underline{e}_3)\right)$$

where $\underline{e}_1, \underline{e}_2$ are such that $\underline{e}_1, \underline{e}_2, \underline{e}_3$ is a positively oriented and orthonormal basis of $TM \oplus E$ (see Lemma A.1); thus, a straightforward computation shows that $\Gamma(X)$ is represented by the bivector

$$\Gamma(X) = -\tau(X \times \underline{e}_3) \cdot \underline{e}_3 + (\sigma - \tau)\langle X, \underline{e}_3\rangle\ \underline{e}_1 \cdot \underline{e}_2. \tag{63}$$

The following formula may be checked by a direct computation: for $X, Y \in TM \oplus E$,

$$X \times Y = -\left(X \cdot Y + \langle X, Y\rangle\right) \underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3;$$

this gives

$$\begin{array}{rcl} (X \times \underline{e}_3) \cdot \underline{e}_3 & = & -\left(X \cdot \underline{e}_3 + \langle X, \underline{e}_3\rangle\right) \underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3 \cdot \underline{e}_3 \\ & = & \left(X - \langle X, \underline{e}_3\rangle \underline{e}_3\right) \underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3 \\ & = & \left(X - \langle X, T\rangle\,(T + f\nu)\right) \cdot \omega \cdot \nu \\ & = & \left(X \cdot \nu - \langle X, T\rangle\,(T \cdot \nu - f)\right) \cdot \omega. \end{array}$$

Moreover,

$$\begin{array}{rcl} \langle X, \underline{e}_3\rangle\ \underline{e}_1 \cdot \underline{e}_2 & = & \langle X, T\rangle\,(-\underline{e}_1 \cdot \underline{e}_2 \cdot \underline{e}_3 \cdot \underline{e}_3) \\ & = & \langle X, T\rangle\,(-\omega \cdot \nu \cdot (T + f\nu)) \\ & = & -\langle X, T\rangle\,(T \cdot \nu - f) \cdot \omega. \end{array}$$

Plugging these two formulas in (63) we get the result. $\square$

We deduce the following key lemma:

Lemma 5.4. *A spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (1) is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (59).*

Proof. We use the identification $\psi \in \Gamma(\Sigma M) \mapsto \psi^* \in \Gamma(\Sigma)$ described at the beginning of the section; we recall that, for all $X \in TM$,

$$(\nabla_X \psi)^* = \nabla_X(\psi^*) \quad \text{and} \quad (X \cdot \psi)^* = X \cdot \nu \cdot (\psi^*). \tag{64}$$

Thus, if $\varphi \in \Gamma(U\Sigma)$ is a solution of (1) and if $\psi \in \Gamma(\Sigma M)$ is such that $\psi^* = \varphi$, using (64) together with the formula

$$\sum_{j=1}^{p} e_j \cdot B(X, e_j) = \sum_{j=1}^{p} e_j \cdot \langle S(X), e_j\rangle \nu = S(X) \cdot \nu$$

and Lemma 5.3, we get:

$$\begin{array}{rcl} (\nabla_X \psi)^* & = & \nabla_X \varphi \\ & = & -\frac{1}{2} S(X) \cdot \nu \cdot \varphi + \frac{1}{2}\left\{(2\tau - \sigma)\,\langle X, T\rangle\,(T \cdot \nu - f) - \tau X \cdot \nu\right\} \cdot \omega \cdot \varphi \\ & = & \left(-\frac{1}{2} S(X) \cdot \psi + \frac{1}{2}\left\{(2\tau - \sigma)\,\langle X, T\rangle\,(T - f) - \tau X\right\} \cdot \omega \cdot \psi\right)^*. \end{array}$$

This gives (59). Reciprocally, if ψ is a solution of (59), the spinor field $\varphi = \psi^*$ satisfies (1). This proves the lemma. $\square$

Instead of $\psi \in \Gamma(\Sigma M)$ solution of (59) we may thus consider $\varphi \in \Gamma(U\Sigma)$ solution of (1). Theorem 5 will thus be a consequence of Theorem 1 if we can define a bundle isomorphism $f : TM \oplus E \to M \times \mathcal{G}$ such that (21) and (22) hold. Let us set

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi\rangle\rangle.$$

We first observe that $f(\underline{e}_3)$ is constant: indeed, for all $X \in TM$,

$$\partial_X(f(\underline{e}_3)) = \langle\langle \nabla_X \underline{e}_3 \cdot \varphi, \varphi\rangle\rangle + \langle\langle \underline{e}_3 \cdot \nabla_X\varphi, \varphi\rangle\rangle + \langle\langle \underline{e}_3 \cdot \varphi, \nabla_X\varphi\rangle\rangle = 0$$

in view of (62), (1) and the computation in (27). Moreover, since f preserves the norm of the vectors, $f(\underline{e}_3)$ is a unit vector. Replacing φ by $\varphi\cdot a$ for some $a \in Spin(\mathcal{G})$ if necessary, we may thus assume that $f(\underline{e}_3) = e_3^o$. We now check (21): since the map f is an orientation preserving isometry and using $f(\underline{e}_3) = e_3^o$, we have, for all $X, Y \in TM$,

$$\begin{aligned} f(\Gamma(X)(Y)) &= f\left(\{\tau(X - \langle X, \underline{e}_3\rangle \underline{e}_3) + (\sigma - \tau)\langle X, \underline{e}_3\rangle \underline{e}_3\} \times Y\right) \\ &= \{\tau(f(X) - \langle f(X), f(\underline{e}_3)\rangle f(\underline{e}_3)) + (\sigma - \tau)\langle f(X), f(\underline{e}_3)\rangle f(\underline{e}_3)\} \times f(Y) \\ &= \{\tau(f(X) - \langle f(X), e_3^o\rangle e_3^o) + (\sigma - \tau)\langle f(X), e_3^o\rangle e_3^o\} \times f(Y) \\ &= \Gamma(f(X))(f(Y)). \end{aligned}$$

Finally, the proof of (22) is very similar to the proof of this identity made in Section 5.1 for $G = \mathbb{R}^n$: we only have to add the term involving Γ which appears in the expression (1) of the covariant derivative of φ; we leave the details to the reader. □

Remark 5. *We also get an explicit representation formula: the immersion is given by the Darboux integral of $\xi : X \mapsto \langle\langle X \cdot \varphi, \varphi\rangle\rangle$, which may be written in terms of ψ by the formula (54).*

We deduce the following result, first obtained by Daniel in [7] using the moving frame method:

Corollary 2. *If S, T, f, κ and τ satisfy (56)-(58), the Gauss equation*

$$K = \det S + \tau^2 + \left(\kappa - 4\tau^2\right) f^2 \tag{65}$$

and the Codazzi equation

$$\nabla_X(SY) - \nabla_Y(SX) - S([X, Y]) = (\kappa - 4\tau^2) f(\langle Y, T\rangle X - \langle X, T\rangle Y), \tag{66}$$

then there exists an isometric immersion of M into $E(\kappa, \tau)$ with shape operator S. Moreover the immersion is unique up to a global isometry of $E(\kappa, \tau)$ preserving the orientations.

Proof. The equations (65) and (66) are equivalent to the Gauss and Codazzi equations (33) and (35) where B is defined by (61). They are thus exactly the integrability conditions for (1), and consequently also for (59). □

5.4.3. *Three-dimensional semi-direct products.* We consider here a semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

if (e_1^o, e_2^o, e_3^o) stands for the canonical basis of $\mathcal{G} = \mathbb{R}^2 \times \mathbb{R}$, the Lie bracket is given by

$$[e_1^o, e_2^o] = 0, \qquad [e_3^o, e_1^o] = ae_1^o + ce_2^o, \qquad [e_3^o, e_2^o] = be_1^o + de_2^o.$$

We equip $\mathbb{R}^2\rtimes_A\mathbb{R}$ with the left invariant metric such that (e_1^o, e_2^o, e_3^o) is orthonormal. By the Koszul formula, we get

$$\nabla_{e_1^o}e_1^o = a\ e_3^o,\qquad \nabla_{e_1^o}e_2^o=\frac{b+c}{2}\ e_3^o,\qquad \nabla_{e_1^o}e_3^o=-a\ e_1^o-\frac{b+c}{2}\ e_2^o, \tag{67}$$

$$\nabla_{e_2^o}e_1^o=\frac{b+c}{2}\ e_3^o,\qquad \nabla_{e_2^o}e_2^o=d\ e_3^o,\qquad \nabla_{e_2^o}e_3^o=-\frac{b+c}{2}\ e_1^o-d\ e_2^o \tag{68}$$

and

$$\nabla_{e_3^o}e_1^o=\frac{c-b}{2}\ e_2^o,\qquad \nabla_{e_3^o}e_2^o=\frac{b-c}{2}\ e_1^o,\qquad \nabla_{e_3^o}e_3^o=0, \tag{69}$$

and deduce

$$\Gamma(X)=\left(aX_1+\frac{b+c}{2}X_2\right)e_1^o\cdot e_3^o+\left(\frac{b+c}{2}X_1+dX_2\right)e_2^o\cdot e_3^o+\frac{c-b}{2}X_3e_1^o\cdot e_2^o$$

for all $X\in\mathcal{G}$. We first assume that M is an oriented surface in $G=\mathbb{R}^2\rtimes_A\mathbb{R}$. Recalling that

$$(T_j+f_j\nu)\cdot(T_k+f_k\nu)=\epsilon_{jk}(f_{l_{jk}}-T_{l_{jk}}\cdot\nu)\cdot\omega$$

(see the proof of Lemma 5.2), we obtain

$$\begin{aligned}\Gamma(X) &= -\left(aX_1+\frac{b+c}{2}X_2\right)(f_2-T_2\cdot\nu)\cdot\omega\\ &\quad+\left(\frac{b+c}{2}X_1+dX_2\right)(f_1-T_1\cdot\nu)\cdot\omega+\frac{c-b}{2}X_3(f_3-T_3\cdot\nu)\cdot\omega\end{aligned}$$

and

$$\begin{aligned}\tilde{\Gamma}(X) &= -\left(aX_1+\frac{b+c}{2}X_2\right)(f_2-T_2)\cdot\omega\\ &\quad+\left(\frac{b+c}{2}X_1+dX_2\right)(f_1-T_1)\cdot\omega+\frac{c-b}{2}X_3(f_3-T_3)\cdot\omega.\end{aligned} \tag{70}$$

Conversely, we consider an oriented Riemannian surface M, and a symmetric operator $S:TM\to TM$. We suppose that there exist tangent vector fields $T_i\in\Gamma(TM)$ and functions $f_i\in C^\infty(M)$ for $1\leqslant i\leqslant 3$ satisfying

$$\langle T_i,T_j\rangle+f_if_j=\delta_i^j \tag{71}$$

for all $1\le i,j\le 3$, and the equations (24) and (25) in Remark 1, with the coefficients Γ_{ij}^k given by (67)-(69). Theorem 4 then yields the following result:

Theorem 6. *If M is simply connected, the following two statements are equivalent:*

(1) there exists an isometric immersion of M into $\mathbb{R}^2\rtimes_A\mathbb{R}$ with shape operator S;

(2) there exists $\psi\in\Gamma(\Sigma M)$ such that $|\psi|=1$ and

$$\nabla_X\psi=-\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}\tilde{\Gamma}(X)\cdot\psi \tag{72}$$

for all $X\in TM$.

The metric Lie group Sol_3. Now, we describe the special case of a surface in Sol_3 : this achieves the spinor representation of immersions of surfaces into 3-dimensional Riemannian homogeneous spaces [19].

Let us recall that Sol_3 is the only metric Lie group whose isometry group is 3-dimensional. It is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the canonical basis (e_1^o, e_2^o, e_3^o) by

$$[e_1^o, e_2^o] = 0, \qquad [e_2^o, e_3^o] = -e_2^o, \qquad [e_3^o, e_1^o] = -e_1^o.$$

This is the semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $a = -1, b = c = 0, d = 1$. The metric on $\mathcal{G}$ is the canonical metric, i.e., the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. By the formulas (67)-(69), the Levi-Civita connection is then such that

$$\Gamma_{11}^3 = -\Gamma_{13}^1 = -1, \qquad \Gamma_{22}^3 = -\Gamma_{23}^2 = 1 \tag{73}$$

and $\Gamma_{ij}^k = 0$ for the other indices.

Let us consider an oriented Riemannian surface M, and a symmetric operator $S : TM \to TM$. We suppose that there exist tangent vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leqslant i \leqslant 3$ satisfying

$$\langle T_i, T_j\rangle + f_i f_j = \delta_i^j \tag{74}$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

$$\begin{aligned} \nabla_X T_i &= (-1)^i \langle X, T_i\rangle T_3 + f_i S(X), \\ df_i(X) &= (-1)^i \langle X, T_i\rangle f_3 - \langle SX, T_i\rangle \end{aligned} \tag{75}$$

for $1 \leqslant i \leqslant 2$,

$$\begin{aligned} \nabla_X T_3 &= \sum_{j=1}^{2} (-1)^{j+1} \langle X, T_j\rangle T_j + f_3 S(X), \\ df_3(X) &= \sum_{j=1}^{2} (-1)^{j+1} \langle X, T_j\rangle f_j - \langle S(X), T_3\rangle. \end{aligned} \tag{76}$$

The equations (75) and (76) are the equations (24) and (25) in Remark 1, with the coefficients Γ_{ij}^k given by (73). According to (70) with $a = -1, b = c = 0$ and $d = 1$ we set

$$\tilde{\Gamma}(X) = \{\langle X, T_1\rangle (f_2 - T_2) + \langle X, T_2\rangle (f_1 - T_1)\} \cdot \omega \tag{77}$$

for all $X \in TM$. Theorem 6 then gives a spinor characterization of an immersion in Sol_3.

As a corollary, we obtain a new proof of a result by Lodovici [11] concerning existence and uniqueness of isometric immersions in Sol_3, since equation (72) is solvable if and only if the equations of Gauss and Codazzi hold (see Section 4).

$\mathbb{H}^2 \times \mathbb{R}$ as a metric Lie group. Finally, viewing $\mathbb{H}^2 \times \mathbb{R}$ as a metric Lie group, we obtain a new spinor characterization of an immersion in $\mathbb{H}^2 \times \mathbb{R}$ which differs from [19] where the product point of view was used.
We recall that $\mathbb{H}^2 \times \mathbb{R}$ is the semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $a = 1, b = c = d = 0$.

The metric on $\mathcal{G}$ is the canonical metric, i.e., the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. Lie bracket is given by

$$[e_1^o, e_2^o] = 0, \qquad [e_3^o, e_1^o] = e_1^o, \qquad [e_3^o, e_2^o] = 0.$$

By the formulas (67)-(69), the Levi-Civita connection is then such that

$$\Gamma_{11}^3 = -\Gamma_{13}^1 = 1 \tag{78}$$

and $\Gamma_{ij}^k = 0$ for the other indices.
Let us consider an oriented Riemannian surface M, and a symmetric operator $S : TM \to TM$. We suppose that there exist tangent vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leqslant i \leqslant 3$ satisfying

$$\langle T_i, T_j \rangle + f_i f_j = \delta_i^j \tag{79}$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

$$\begin{aligned} \nabla_X T_1 &= \langle X, T_1 \rangle T_3 + f_1 S(X), \\ df_1(X) &= \langle X, T_1 \rangle f_3 - \langle SX, T_1 \rangle, \end{aligned} \tag{80}$$

$$\begin{aligned} \nabla_X T_2 &= f_2 S(X), \\ df_2(X) &= -\langle SX, T_2 \rangle, \end{aligned} \tag{81}$$

$$\begin{aligned} \nabla_X T_3 &= -\langle X, T_3 \rangle T_1 + f_3 S(X), \\ df_1(X) &= -\langle X, T_3 \rangle f_1 - \langle SX, T_3 \rangle. \end{aligned} \tag{82}$$

With these identities and according to (70) with $a = 1$,$b = c = d = 0$, we set

$$\tilde{\Gamma}(X) = -\langle X, T_1 \rangle (f_2 - T_2) \cdot \omega \tag{83}$$

for all $X \in TM$. Theorem 6 then gives a spinor characterization of an immersion in $\mathbb{H}^2 \times \mathbb{R}$.

5.5. **CMC-surfaces in a 3-dimensional metric Lie group.** The aim here is to show that the representation formula for CMC-surfaces in a 3-dimensional metric Lie group by Meeks, Mira, Perez and Ros [13, Theorem 3.12] may be obtained as a consequence of the general representation formula in Theorem 1. For sake of brevity we assume that the group G is unimodular and only give the principal arguments, without details. Under this hypothesis, there exists an orthonormal basis (e_1^o, e_2^o, e_3^o) of the Lie algebra $\mathcal{G}$ and constants $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that the Levi-Civita connection of G is given by

$$\begin{aligned} \Gamma(X)(e_1^o) &:= \nabla_X e_1^o = X_3 \mu_3 e_2^o - X_2 \mu_2 e_3^o, \\ \Gamma(X)(e_2^o) &:= \nabla_X e_2^o = -X_3 \mu_3 e_1^o + X_1 \mu_1 e_3^o, \\ \Gamma(X)(e_3^o) &:= \nabla_X e_3^o = X_2 \mu_2 e_1^o - X_1 \mu_1 e_2^o \end{aligned}$$

(see e.g. [13, Section 2.6]), i.e.

$$\begin{aligned} \Gamma(X) &= \frac{1}{2}\left(e_1^o \cdot \Gamma(X)(e_1^o) + e_2^o \cdot \Gamma(X)(e_2^o) + e_3^o \cdot \Gamma(X)(e_3^o)\right) \\ &= X_1 \mu_1 e_2^o \cdot e_3^o + X_2 \mu_2 e_3^o \cdot e_1^o + X_3 \mu_3 e_1^o \cdot e_2^o \end{aligned} \tag{84}$$

for all $X \in \mathcal{G}$. Following [13] we introduce the H-potential of the group G

$$R(g) = H\left(1 + |g|^2\right)^2 - \frac{i}{2}\left(\mu_1 |1 - g^2|^2 + \mu_2 |1 + g^2|^2 + 4\mu_3 |g|^4\right) \tag{85}$$

for all $g \in \overline{\mathbb{C}}$. The importance of this quantity appears in the following lemma, which will permit to express the right-hand side of the Dirac equation (32):

Lemma 5.5. *Let us consider a positively oriented and orthonormal basis e_1, e_2, ν of $\mathcal{G}$ and set, for $\nu = \nu_1 e_1^o + \nu_2 e_2^o + \nu_3 e_3^o$,*

$$T(\nu) = \mu_1\nu_1 e_1^o + \mu_2\nu_2 e_2^o + \mu_3\nu_3 e_3^o, \tag{86}$$

$A = \frac{1}{2}\langle e_2, T(\nu)\rangle$ and $B = -\frac{1}{2}\langle e_1, T(\nu)\rangle$. Then, if

$$g = \frac{\nu_1 + i\nu_2}{1+\nu_3}$$

is the stereographic projection of $\nu \in S^2$ with respect to the south pole $-e_3^o$ of S^2, we have

$$\begin{aligned} H\nu + \frac{1}{2}\left(e_1 \cdot \Gamma(e_1) + e_2 \cdot \Gamma(e_2)\right) \quad &= \quad \frac{1}{(1+|g|^2)^2}\left(\Re e\ R(g) - \Im m\ R(g)\ e_1 \cdot e_2\right) \cdot \nu \\ &\quad + Ae_1 + Be_2. \end{aligned}$$

Proof. For $i \in \{1, 2, 3\}$, let us denote by

$$p(e_i^o) := \langle e_i^o, e_1\rangle e_1 + \langle e_i^o, e_2\rangle e_2$$

the orthogonal projection of the vector e_i^o onto the plane generated by e_1 and e_2. By (84) we have

$$e_1 \cdot \Gamma(e_1) + e_2 \cdot \Gamma(e_2) \quad = \quad \mu_1\ p(e_1^o) \cdot e_2^o \cdot e_3^o + \mu_2\ p(e_2^o) \cdot e_3^o \cdot e_1^o + \mu_3\ p(e_3^o) \cdot e_1^o \cdot e_2^o.$$

The proof is then a direct and long computation using that $p(e_i^o) = e_i^o - \langle e_i^o, \nu\rangle\nu$ together with the formulas

$$\nu_1 = \frac{2\ \Re e\ g}{1+|g|^2}, \qquad \nu_2 = \frac{2\ \Im m\ g}{1+|g|^2}, \qquad \nu_3 = \frac{1-|g|^2}{1+|g|^2}. \tag{87}$$

$\square$

We consider the Clifford map

$$\begin{array}{ccc} \mathcal{G} & \to & \mathbb{H}(2) \\ x_1 e_1^o + x_2 e_2^o + x_3 e_3^o & \mapsto & \begin{pmatrix} ix_3 + j(x_1 - ix_2) & 0 \\ 0 & -ix_3 - j(x_1 - ix_2) \end{pmatrix} \end{array} \tag{88}$$

which identifies $\mathcal{G}$ to the imaginary quaternions so that

(89)
$$e_1^o \simeq \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \simeq j, \quad e_2^o \simeq \begin{pmatrix} -ji & 0 \\ 0 & ji \end{pmatrix} \simeq -ji, \quad e_3^o \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \simeq i.$$

It identifies $Cl(\mathcal{G})$ to the set of matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{H} \right\} \tag{90}$$

and $Spin(\mathcal{G})$ to the group of unit quaternions

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in \mathbb{H},\ |a| = 1 \right\} \simeq \{a \in \mathbb{H},\ |a| = 1\}.$$

We choose a conformal parameter $z = x + iy$ of the surface, and denote by μ the real function such that the metric is $\mu^2(dx^2 + dy^2)$. In a spinorial frame above the orthonormal frame $e_1 = \frac{1}{\mu}\partial_x$, $e_2 = \frac{1}{\mu}\partial_y$, the spinor field φ is represented by $[\varphi] = z_1 + jz_2$ where $z_1, z_2 \in \mathbb{C}$ are such that $|z_1|^2 + |z_2|^2 = 1$.

Lemma 5.6. *The Dirac equation (32) is equivalent to the system*

$$\frac{1}{\sqrt{\mu}}\partial_{\overline{z}}(\sqrt{\mu}\ \overline{z_1}) = i\frac{\mu}{2}\frac{\overline{R(g)}}{(1+|g|^2)^2}\overline{z_2}+\frac{\mu}{2}(A+iB)\overline{z_1} \tag{91}$$

$$\frac{1}{\sqrt{\mu}}\partial_{\overline{z}}(\sqrt{\mu}\ z_2) = -i\frac{\mu}{2}\frac{\overline{R(g)}}{(1+|g|^2)^2}z_1+\frac{\mu}{2}(A+iB)z_2. \tag{92}$$

Moreover, the $\mathcal{G}-$valued 1-form ξ in Theorem 1 is

$$\begin{aligned}\xi(X) &= i\left\{2x_1\ \Im m(z_1\overline{z_2})-2x_2\ \Re e(z_1\overline{z_2})+x_3\left(|z_1|^2-|z_2|^2\right)\right\}\\ &\quad +j\left\{x_1(z_1^2+z_2^2)-ix_2(z_1^2-z_2^2)-2ix_3z_1z_2\right\}\end{aligned} \tag{93}$$

for all $X=x_1e_1+x_2e_2+x_3\nu\in TM\oplus E$.

Proof. We use here the identification $\psi\in\Gamma(\Sigma M)\mapsto\psi^*\in\Gamma(\Sigma)$ satisfying the properties (64): according to Lemma 5.5, the spinor field $\psi\in\Gamma(\Sigma M)$ such that $\psi^*=\varphi$ is solution of

$$D\psi=\frac{1}{(1+|g|^2)^2}\left(\Re e\ R(g)-\Im m\ R(g)\ e_1\cdot e_2\right)\cdot\psi+(Ae_1+Be_2)\cdot\psi. \tag{94}$$

We identify Cl_2 to $\mathbb{H}$ using the Clifford map

$$\begin{array}{ccc}\mathbb{R}^2 & \rightarrow & \mathbb{H}\\ (x_1,x_2) & \mapsto & j(x_1-ix_2)\end{array} \tag{95}$$

so that, in the fixed spinorial frame above $e_1=\frac{1}{\mu}\partial_x$, $e_2=\frac{1}{\mu}\partial_y$,

$$[e_1]=j,\qquad [e_2]=-ji,\qquad [e_1\cdot e_2]=i.$$

Using moreover that

$$[\nabla_{\partial x}\psi]=\partial_x[\psi]-\frac{i}{2\mu}\partial_y\mu\ [\psi]\qquad [\nabla_{\partial y}\psi]=\partial_y[\psi]+\frac{i}{2\mu}\partial_x\mu\ [\psi]$$

(by (15), and the computation of the Christoffel symbols), the left-hand side of (94) is

$$[D\psi]=\frac{1}{\mu}j\left\{\partial_x[\psi]-\frac{i}{2\mu}\partial_y\mu\ [\psi]\right\}-\frac{1}{\mu}ji\left\{\partial_y[\psi]+\frac{i}{2\mu}\partial_x\mu\ [\psi]\right\}$$

whereas the right-hand side is

$$\left(\frac{\overline{R(g)}}{(1+|g|^2)^2}+j(A-iB)\right)[\psi].$$

We finally need to precise the identification $\psi\mapsto\psi^*$: in spinorial frames above e_1,e_2 and e_1,e_2,ν, since the second property in (64) is required and using the Clifford maps (88) and (95), it is not difficult to see that the map $\psi\mapsto\psi^*$ corresponds to the map

$$u+jv\mapsto\left(\begin{array}{cc}u+jiv & 0\\ 0 & u+jiv\end{array}\right);$$

ψ is thus represented by the quaternion $[\psi]=z_1-jiz_2$. Direct computations then give the system (91)-(92).

Expression (93) also follows from a direct computation: we have, in Cl_3,

$$\begin{aligned}\xi(X) &= \tau[\varphi][X][\varphi]\\ &\simeq (\overline{z_1}-j\overline{z_2})(ix_3+j(x_1-ix_2))(z_1+jz_2),\end{aligned}$$

which easily gives the result. $\square$

We set

$$g = i\frac{\overline{z_2}}{z_1}, \qquad f = -2\mu z_1^2. \tag{96}$$

The function g is the left invariant Gauss map of the surface, stereografically projected with respect to the south pole of S^2, since

$$\nu = i\left(|z_1|^2 - |z_2|^2\right) - 2jiz_1z_2$$

is a unit vector normal to the immersion ($x_1 = x_2 = 0$ and $x_3 = 1$ in (93)) and

$$\frac{\nu_1 + i\nu_2}{1+\nu_3} = \frac{2i\ \overline{z_1}\ \overline{z_2}}{1+|z_1|^2 - |z_2|^2} = \frac{2i\ \overline{z_1}\ \overline{z_2}}{2|z_1|^2} = i\frac{\overline{z_2}}{z_1}.$$

Direct computations then show that equations (91)-(92) are equivalent to

$$f = 4\frac{\partial_z g}{R(g)} \tag{97}$$

and

$$\frac{\partial_{\overline{z}} f}{f} = -\frac{2}{1+|g|^2}\partial_{\overline{z}}\overline{g}\ g + \mu(A+iB), \tag{98}$$

and that (93) reads

$$\xi = \Re e\left(\frac{1}{2}f(\overline{g}^2 - 1)dz,\ \frac{i}{2}f(\overline{g}^2+1)dz,\ f\overline{g}dz\right) \tag{99}$$

in (e_1^o, e_2^o, e_3^o) (recall (89)). This last formula is the Weierstrass-type representation given in [13, Theorem 3.15]. Using that

$$A = \left\langle \xi\left(\frac{\partial_y}{\mu}\right), T(\nu)\right\rangle \quad \text{and} \quad B = -\left\langle \xi\left(\frac{\partial_x}{\mu}\right), T(\nu)\right\rangle$$

(Lemma 5.5) together with (99) and (86) we get that

$$A + iB = -\frac{i}{4\mu}\overline{f}\left(\mu_1\nu_1(g^2-1) - i\mu_2\nu_2(g^2+1) + 2\mu_3\nu_3 g\right). \tag{100}$$

Differentiating (97) with respect to $\overline{z}$ and using (98) together with (100) and (87) we see by a further computation that g satisfies

$$g_{z\overline{z}} = \frac{R_g}{R}g_z g_{\overline{z}} + \left(\frac{R_{\overline{g}}}{R} - \frac{\overline{R_g}}{\overline{R}}\right)|g_z|^2, \tag{101}$$

which is the structure equation for the left invariant Gauss map in [13, Theorem 3.15].

Appendix A. Skew-symmetric operators and bivectors

We consider $\mathbb{R}^n$ endowed with its canonical scalar product. A skew-symmetric operator $u : \mathbb{R}^n \to \mathbb{R}^n$ naturally identifies to a bivector $\underline{u} \in \Lambda^2\mathbb{R}^n$, which may in turn be regarded as belonging to the Clifford algebra $Cl_n(\mathbb{R})$. We precise here the relations between the Clifford product in $Cl_n(\mathbb{R})$ and the composition of endomorphisms. If a and b belong to the Clifford algebra $Cl_n(\mathbb{R})$, we set

$$[a,b] = \frac{1}{2}(a\cdot b - b\cdot a),$$

where the dot $\cdot$ is the Clifford product. We denote by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}^n$.

Lemma A.1. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a skew-symmetric operator. Then the bivector*

$$\underline{u} = \frac{1}{2}\sum_{j=1}^{n} e_j \cdot u(e_j) \quad \in \ \Lambda^2\mathbb{R}^n \subset Cl_n(\mathbb{R}) \tag{102}$$

represents u, and, for all $\xi \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi).$$

In the paper, and for sake of simplicity, we will use the same letter u to denote $\underline{u}$.

Proof. For $i < j$, we consider the linear map

$$u : \quad e_i \mapsto e_j, \quad e_j \mapsto -e_i, \quad e_k \mapsto 0 \ \text{ if } \ k \neq i, j;$$

it is skew-symmetric and corresponds to the bivector $e_i \wedge e_j \in \Lambda^2\mathbb{R}^n$; it is thus naturally represented by $\underline{u} = e_i \cdot e_j = \frac{1}{2}\left(e_i \cdot e_j - e_j \cdot e_i\right)$, which is (102). We then compute, for $k = 1, \ldots, n$,

$$[\underline{u}, e_k] = \frac{1}{2}\left(e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j\right)$$

and easily get

$$[\underline{u}, e_k] = e_j \ \text{ if } \ k = i, \quad -e_i \ \text{ if } \ k = j, \quad 0 \ \text{ if } \ k \neq i, j.$$

The result follows by linearity. □

Lemma A.2. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}^n$ be two skew-symmetric operators, represented in $Cl_n(\mathbb{R})$ by*

$$u = \frac{1}{2}\sum_{j=1}^{n} e_j \cdot u(e_j) \quad \text{and} \quad v = \frac{1}{2}\sum_{j=1}^{n} e_j \cdot v(e_j)$$

respectively. Then $[u, v] \in \Lambda^2\mathbb{R}^n \subset Cl_n(\mathbb{R})$ represents $u \circ v - v \circ u$.

Proof. For $\xi \in \mathbb{R}^n$, the Jacobi equation yields

$$[[u, v], \xi] = [u, [v, \xi]] - [v, [u, \xi]].$$

Thus, using Lemma A.1 repeatedly, $[u, v]$ represents the map

$$\begin{array}{rcl} \xi \ \mapsto \ [[u, v], \xi] & = & [u, [v, \xi]] - [v, [u, \xi]] \\ & = & [u, v(\xi)] - [v, u(\xi)] \\ & = & (u \circ v - v \circ u)(\xi), \end{array}$$

and the result follows. □

We now assume that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, $p + q = n$.

Lemma A.3. *Let us consider a linear map $u : \mathbb{R}^p \to \mathbb{R}^q$ and its adjoint $u^* : \mathbb{R}^q \to \mathbb{R}^p$. Then the bivector*

$$\underline{u} = \sum_{j=1}^{p} e_j \cdot u(e_j) \quad \in \ \Lambda^2\mathbb{R}^n \subset Cl_n(\mathbb{R})$$

represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} : \quad \mathbb{R}^p \oplus \mathbb{R}^q \to \mathbb{R}^p \oplus \mathbb{R}^q,$$

we have

$$\underline{u} = \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) + \sum_{j=p+1}^{n} e_j \cdot (-u^*(e_j))\right) \tag{103}$$

and, for all $\xi = \xi_p + \xi_q \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi_p) - u^*(\xi_q).$$

As above, we will simply denote $\underline{u}$ *by* u.

Proof. In view of Lemma A.1, $\underline{u}$ represents the linear map $\xi \mapsto [\underline{u}, \xi]$. We compute, for $\xi \in \mathbb{R}^p$,

$$\begin{aligned}
[\underline{u}, \xi] &= \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^{p} e_j \cdot u(e_j)\right) \\
&= -\frac{1}{2}\sum_{j=1}^{p}(e_j \cdot \xi + \xi \cdot e_j) \cdot u(e_j) \\
&= \sum_{j=1}^{p}\langle \xi, e_j\rangle\ u(e_j) \\
&= u(\xi),
\end{aligned}$$

and, for $\xi \in \mathbb{R}^q$,

$$\begin{aligned}
[\underline{u}, \xi] &= \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^{p} e_j \cdot u(e_j)\right) \\
&= \frac{1}{2}\sum_{j=1}^{p} e_j \cdot (u(e_j) \cdot \xi + \xi \cdot u(e_j)) \\
&= -\sum_{j=1}^{p} e_j\ \langle u(e_j), \xi\rangle \\
&= -\sum_{j=1}^{p} e_j\ \langle e_j, u^*(\xi)\rangle \\
&= -u^*(\xi).
\end{aligned}$$

Finally,

$$\underline{u} = \sum_{j=1}^{p} e_j \cdot u(e_j) = \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) + \sum_{j=1}^{p} -u(e_j) \cdot e_j\right)$$

with

$$\begin{aligned}\sum_{j=1}^{p} -u(e_j)\cdot e_j &= -\sum_{i=p+1}^{p+q}\sum_{j=1}^{p}\langle u(e_j),e_i\rangle\ e_i\cdot e_j\\ &= \sum_{i=p+1}^{p+q} e_i\cdot\left(-\sum_{j=1}^{p}\langle e_j,u^*(e_i)\rangle\ e_j\right)\\ &= \sum_{i=p+1}^{p+q} e_i\cdot(-u^*(e_i)),\end{aligned}$$

which gives (103). □

Lemma A.4. *Let us consider two linear maps $u:\mathbb{R}^p\rightarrow\mathbb{R}^q$ and $v:\mathbb{R}^n\rightarrow\mathbb{R}^n$, with v skew-symmetric, and the associated bivectors*

$$\underline{u}=\sum_{j=1}^{p} e_j\cdot u(e_j),\qquad \underline{v}=\frac{1}{2}\sum_{j=1}^{n} e_j\cdot v(e_j).$$

Then $[\underline{u},\underline{v}]\in\Lambda^2\mathbb{R}^n$ represents the map

$$\xi=\xi_p+\xi_q\mapsto -u^*(v(\xi)_q)+v(u^*(\xi_q))+u(v(\xi)_p)-v(u(\xi_p)),$$

where the sub-indices p and q mean that we take the components of the vectors in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. In view of Lemma A.1, this may also be written in the form

$$[[\underline{u},\underline{v}],\xi]=-u^*(v(\xi)_q)+v(u^*(\xi_q))+u(v(\xi)_p)-v(u(\xi_p))$$

for all $\xi\in\mathbb{R}^n$.

Proof. From Lemmas A.2 and A.3, the bivector $[\underline{u},\underline{v}]\in\Lambda^2\mathbb{R}^n$ represents

$$\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix}\circ v - v\circ\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix},$$

that is the map

$$\begin{aligned}\xi\ &\mapsto \begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix}\begin{pmatrix}v(\xi)_p\\ v(\xi)_q\end{pmatrix}-v\begin{pmatrix}0 & -u^*\\ u & 0\end{pmatrix}\begin{pmatrix}\xi_p\\ \xi_q\end{pmatrix}\\ &=\begin{pmatrix}-u^*(v(\xi)_q)+v(u^*(\xi_q))\\ u(v(\xi)_p)-v(u(\xi_p))\end{pmatrix},\end{aligned}$$

which gives the result. □

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References

[1] C. Bär, *Extrinsic bounds for the eigenvalues of the Dirac operator*, Ann. Glob. Anal. Geom. **16** (1998) 573-596.

[2] P. Bayard, *On the spinorial representation of spacelike surfaces into 4-dimensional Minkowski space*, J. Geom. Phys. **74** (2013) 289-313.

[3] P. Bayard, M.A. Lawn & J. Roth, *Spinorial representation of surfaces in four-dimensional Space Forms*, Ann. Glob. Anal. Geom. **44:4** (2013) 433-453.

[4] P. Bayard, M.A. Lawn & J. Roth, *Spinorial representation of submanifolds in Riemannian space forms*, arXiv:1602.02919 (2016) 1-24.

[5] P. Bayard & V. Patty, *Spinor representation of Lorentzian surfaces in* $\mathbb{R}^{2,2}$, J. Geom. Phys. **95** (2015) 74-75.

[6] D.A. Berdinskii & I.A. Taimanov, *Surfaces in three-dimensional Lie groups*, Sibirsk Mat. Zh. **46:6** (2005) 1248-1264.

[7] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007) 87-131.

[8] T. Friedrich, *On the spinor representation of surfaces in Euclidean* 3*-space*, J. Geom. Phys. **28** (1998) 143-157.

[9] M.-A. Lawn, *A spinorial representation for Lorentzian surfaces in* $\mathbb{R}^{2,1}$, J. Geom. Phys. **58:6** (2008) 683-700.

[10] M.-A. Lawn & J. Roth, *Spinorial characterization of surfaces in pseudo-Riemannian space forms*, Math. Phys. Anal. and Geom. **14:3** (2011) 185-195.

[11] S.D. Lodovici, *An isometric immersion theorem in* Sol_3, Matemática Contemporânea **30** (2006) 109-123.

[12] P. Lounesto, *Clifford Algebras and Spinors*, London Mathematical Society, Lecture Note Series **286**, Cambridge University Press (2001).

[13] W. Meeks & J. Perez, *Constant mean curvature surfaces in metric Lie groups*, in "Geometric Analysis: Partial Differential Equations and Surfaces", Contemp. Math. **570** (2012) 25-110.

[14] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976) 293-329.

[15] J. Milnor, *Remarks concerning spin manifolds*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press (1965) 55-62.

[16] B. Morel, *Surfaces in* $\mathbb{S}^3$ *and* $\mathbb{H}^3$ *via spinors*, Actes du séminaire de théorie spectrale, Institut Fourier, Grenoble, **23** (2005) 9-22.

[17] V. Patty, *A generalized Weierstras representation of Lorentzian surfaces in* $\mathbb{R}^{2,2}$ *and applications*, Int. J. Geometric methods in modern Physics **13:6** (2016) 26 pages.

[18] P. Piccione & D.V. Tausk, *An existence theorem for G-structure preserving affine immersions*, Indiana Univ. Math. J. **57:3** (2008) 1431-1465.

[19] J. Roth, *Spinorial characterizations of surfaces into 3-homogeneous manifolds*, J. Geom. Phys. **60** (2010) 1045-1061.