



The exterior derivative of the Lee form of almost Hermitian manifolds

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ABSTRACT

The exterior derivative $d\theta$ of the Lee form θ of almost Hermitian manifolds is studied. If ω is the Kähler two-form, it is proved that the $\mathbb{R}\omega$ -component of $d\theta$ is always zero. Expressions for the other components, in $[\lambda_0^{1,1}]$ and in $[[\lambda^{2,0}]]$, of $d\theta$ are also obtained. They are given in terms of the intrinsic torsion. Likewise, it is described some interrelations between the Lee form and $U(n)$ -components of the Riemannian curvature tensor.

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1. Introduction

In [7] Gray and Hervella displayed a classification for almost Hermitian structures. Such a classification is based on the decomposition of the space possible intrinsic torsions into irreducible $U(n)$ -modules. Since they obtained a decomposition into four irreducible modules \mathcal{W}_i for $n > 2$, $i = 1, \dots, 4$, there are 2^4 classes of almost structure denoted by direct sums of \mathcal{W}_i determined for the non-zero components $\xi_{(i)}$ of the intrinsic torsion ξ . The component $\xi_{(4)}$ is determined by a one-form θ , usually called the *Lee form* in Ref. [10].

The exterior derivative $d\theta$ of the Lee form θ amounts interest because such a form is related with conformal changes of the almost Hermitian metric. For instance, if θ is closed, at least locally, it is possible to do a conformal change of metric such that $\xi_{(4)} = 0$ for the new almost Hermitian structure. In the particular case that $\xi_{(1)} = 0$, $\xi_{(3)} = 0$ and $n > 2$, i.e. the almost Hermitian structure is of type $\mathcal{W}_2 \oplus \mathcal{W}_4$, it is not hard to deduce that the Lee form θ must be closed. From this, a natural question arises: are there another types of almost Hermitian structures with $\theta \neq 0$ such that θ is necessarily closed for them?. The interest of this question is increased by the fact that the most examples in references with $\theta \neq 0$ are such that θ is closed. On the other hand, if the exterior derivative $d\theta$ of θ is non-zero, under action of $U(n)$, $d\theta$ is decomposed into $U(n)$ -components, when do some of them vanish?.

The initial purpose of the present text is to obtain answers for the mentioned questions. Thus we will give expressions for the $U(n)$ -components, $(d\theta)_{\mathbb{R}\omega}$, $(d\theta)_{[\lambda_0^{1,1}]}$ and $(d\theta)_{[[\lambda^{2,0}]]}$, of the exterior derivative $d\theta$ in terms of the intrinsic torsion. It

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is relevant that it is given a proof for the fact that the component $(d\theta)_{\mathbb{R}\omega}$, proportional to the Kähler two-form ω , always vanishes (see Proposition 3.4). The identity $(d\theta)_{\mathbb{R}\omega} = 0$ has already been obtained by Gauduchon in [5]. He gave another proof in the context of Hermitian structures (type $\mathcal{W}_3 \oplus \mathcal{W}_4$) which is also valid for general almost Hermitian structures.

In order to obtain expressions for $(d\theta)_{\mathbb{R}\omega}$, $(d\theta)_{[\lambda_0^{1,1}]}$ and $(d\theta)_{\|\lambda^{2,0}\|}$, we will make use of some interrelations among components of the intrinsic torsion which are consequences of the identity $d^2\omega = 0$. Such interrelations are interesting by their own. For instance, one of them has been used in [13] to explain the behavior of certain components of the Riemannian curvature. Likewise some of these interrelations have been applied in the study of harmonic $U(n)$ -structures (see [6]). Finally, we point out that the expressions obtained for $(d\theta)_{[\lambda_0^{1,1}]}$ and $(d\theta)_{\|\lambda^{2,0}\|}$ (see Proposition 3.4) allow us to say when some of them are zero. For instance, if the almost Hermitian manifold is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, then $(d\theta)_{[\lambda_0^{1,1}]}$ vanishes. Moreover, if the manifold is of dimension six, then θ is closed for such a type (see Proposition 3.6).

The Riemannian curvature tensor of almost Hermitian manifolds has been studied by Tricerri and Vanhecke [15], Falcitelli, Farinola and Salamon [4] and by Swann and the present author [13]. Section 4 could be considered as an application of the expressions for the $U(n)$ -components of $d\theta$ to the study of such a curvature tensor. Thus those expressions provide a better understanding of facts and properties shown in the mentioned references. For instance, it is explained the behavior of certain $U(n)$ -components of the Riemannian curvature tensor determined by means of the usual Ricci tensor Ric and another Ricci type tensor Ric^* given in [13]. Likewise, some results given in [4] are completed.

In the last section, some examples are studied to show that the two components of $d\theta$ orthogonal to ω , $(d\theta)_{[\lambda_0^{1,1}]}$ and $(d\theta)_{\|\lambda^{2,0}\|}$, can be non-zero. The first and second examples are four-dimensional manifolds equipped with a Hermitian structure for which both of these components are non-zero. The third example, which is also Hermitian and has previously been studied in some details by Abbena et al. [1], again has both components non-zero. In these examples computations relative to the Riemannian curvature are also done. This illustrates the study of facts and properties displayed in Section 4.

Finally, we point out that it is still an open question to find an example of almost Hermitian manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ such that $d\theta \neq 0$, i.e. $(d\theta)_{\|\lambda^{2,0}\|} \neq 0$. Note that such an example ought to be of dimension greater than six.

2. Preliminaries

An *almost complex* structure on a manifold M consists of a $(1, 1)$ -tensor J such that $J^2 = -I$. The manifold M must be of dimension $2n$. The presence of an almost complex structure is equivalent to say that there is a $GL(n, \mathbb{C})$ -structure defined on M . A manifold M is said to be *almost Hermitian*, if there is an almost complex structure and a Riemannian metric $\langle \cdot, \cdot \rangle$ defined on M such that they satisfy the compatibility condition $\langle JX, JY \rangle = \langle X, Y \rangle$. In this case it is said that there is a $U(n)$ -structure on M .

Associated with an almost Hermitian structure, the tensor $\omega = \langle \cdot, J\cdot \rangle$, called the *Kähler form*, is usually considered. Using ω , M can be oriented by fixing a constant multiple of ω^n as volume form. Under the action of $U(n)$, the cotangent space on each point T_m^*M is irreducible and it follows $\mathfrak{so}(2n) \cong \Lambda^2 T_m^*M = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp$, where $\mathfrak{u}(n)$ ($\mathfrak{u}(n)^\perp$) consists of those two-forms b such that $b(JX, JY) = b(X, Y)$ ($b(JX, JY) = -b(X, Y)$).

Denoting by ∇ the Levi Civita connection, the *minimal connection* $\nabla^{U(n)}$ is the unique $U(n)$ -connection on M such that $\xi = \nabla^{U(n)} - \nabla$ satisfies the condition $\xi \in T^*M \otimes \mathfrak{u}(n)^\perp$. The tensor ξ is referred to as the *intrinsic torsion* of the almost Hermitian structure [2]. In [7], Gray and Hervella showed that in general dimensions the space $T^*M \otimes \mathfrak{u}(n)^\perp$ of possible intrinsic torsions is decomposed into four irreducible $U(n)$ -modules providing a classification of $2^4 = 16$ classes or types of almost Hermitian structures. To be more precise, the space of possible intrinsic torsions $T^*M \otimes \mathfrak{u}(n)^\perp$ consists of those tensors ξ such that $J\xi_X Y + \xi_X JY = 0$ and, under the action of $U(n)$, is decomposed into:

- (1) if $n = 2$, $\xi \in T^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{W}_2 \oplus \mathcal{W}_4$;
- (2) if $n \geq 3$, $\xi \in T^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

The intrinsic torsion is explicitly determined by $\xi_X = -\frac{1}{2}J \circ \nabla_X J = \frac{1}{2}(\nabla_X J) \circ J$. The tensor $\xi_{(i)}$ will denote the component of ξ corresponding to \mathcal{W}_i . The component $\xi_{(4)}$ is determined by the vector field $\sum_{i=1}^{2n} \xi_{e_i} e_i$, where $\{e_1, \dots, e_{2n}\}$ is a local orthonormal frame field. Other expression for this vector field is $2 \sum_{i=1}^{2n} \xi_{e_i} e_i = -J(d^*\omega)^\sharp$, where d^* denotes coderivative and $\langle (d^*\omega)^\sharp, X \rangle = d^*\omega(X)$. Since the *Lee form*, considered in [7,10], is defined by $\theta = -\frac{1}{n-1}Jd^*\omega$, one has $\sum_{i=1}^{2n} (\xi_{e_i} e_i)^\flat = \frac{n-1}{2}\theta$, where $(\xi_{e_i} e_i)^\flat(X) = \langle \xi_{e_i} e_i, X \rangle$. We point out that, for a one-form a , $(Ja)(X) := -a(JX)$. The component $\xi_{(4)}$ is explicitly given by

$$4\xi_{(4)X} = X^\flat \otimes \theta^\sharp - \theta \otimes X - JX^\flat \otimes J\theta^\sharp + J\theta \otimes JX.$$

Remark 2.1 (Notations and Conventions). For using simpler and standard notation, we recall that $\lambda_0^{p,q}$ is a complex irreducible $U(n)$ -module coming from the (p, q) -part of the complex exterior algebra, and that its corresponding dominant weight in standard coordinates is given by $(1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$, where 1 and -1 are repeated p and q times, respectively. By analogy with the exterior algebra, there are also complex irreducible $U(n)$ -modules $\sigma_0^{p,q}$, with dominant weights $(p, 0, \dots, 0, -q)$ coming from the complex symmetric algebra. The notation $\|V\|$ stands for the real vector space

underlying a complex vector space V , and $[W]$ denotes a real vector space that admits W as its complexification. Thus for the $U(n)$ -modules above mentioned one has

$$\begin{aligned} T^*M &\cong \llbracket \lambda^{1,0} \rrbracket \cong \mathcal{W}_4, \quad u(n) \cong [\lambda^{1,1}], \quad \mathfrak{su}(n) \cong [\lambda_0^{1,1}], \quad u(n)^\perp \cong \llbracket \lambda^{2,0} \rrbracket, \\ \mathcal{W}_1 &\cong \llbracket \lambda^{3,0} \rrbracket, \quad \mathcal{W}_2 \cong \llbracket A \rrbracket, \quad \mathcal{W}_3 \cong \llbracket \lambda_0^{2,1} \rrbracket, \end{aligned}$$

where $A \subset \lambda^{1,0} \otimes \lambda^{2,0}$.

We will use the natural extension to forms of the metric $\langle \cdot, \cdot \rangle$. Thus, for all p -forms α, β ,

$$\langle \alpha, \beta \rangle = \frac{1}{p!} \sum_{i_1, \dots, i_p=1}^{2n} \alpha(e_{i_1}, \dots, e_{i_p}) \beta(e_{i_1}, \dots, e_{i_p}).$$

For instance, using this product, for a two-form α , one has $\alpha_{\mathbb{R}\omega} = \frac{1}{n} \langle \alpha, \omega \rangle \omega$. Another example using this product is the identity $Jd^*\omega = -\langle \lrcorner d\omega, \omega \rangle$, where \lrcorner denotes the interior product.

In the sequel, we will consider the orthonormal basis for tangent vectors $\{e_1, \dots, e_{2n}\}$. Likewise, we will use the summation convention to simplify notation. The repeated indexes will mean that the sum is extended from $i = 1$ to $i = 2n$. Otherwise, the sum will be explicitly written. We also point out that we will make reiterated use of the *musical isomorphisms* $\flat : TM \rightarrow T^*M$ and $\sharp : T^*M \rightarrow TM$, induced by $\langle \cdot, \cdot \rangle$, defined by $X^\flat = \langle X, \cdot \rangle$ and $\langle a^\sharp, \cdot \rangle = a$. Finally, if ψ is a $(0, s)$ -tensor, we write

$$J_{(i)}\psi(X_1, \dots, X_i, \dots, X_s) = -\psi(X_1, \dots, JX_i, \dots, X_s).$$

3. The components of the exterior derivative of the Lee form

In this section we will display several identities relating components of the intrinsic torsion which are consequences of the equalities $d^2\omega = 0$. They are interesting by their own and we will show some applications of them. For instance, from such identities we will obtain expressions for the $U(n)$ -components of the exterior derivative of the Lee form.

Lemma 3.1. *For an almost Hermitian manifold of dimension $2n$, $n > 1$, the following identities are satisfied:*

$$\begin{aligned} 0 &= \langle \nabla_{e_j}^{U(n)} \xi_{(4)e_i} e_i, J e_j \rangle, \\ 0 &= -\frac{n-2}{n-1} \langle \nabla_X^{U(n)} \xi_{(4)e_i} e_i, Y \rangle + \frac{n-2}{n-1} \langle \nabla_Y^{U(n)} \xi_{(4)e_i} e_i, X \rangle - 2 \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_X Y, e_i \rangle \\ &\quad + 2 \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_Y X, e_i \rangle - \frac{n-2}{n-1} \langle \nabla_{JX}^{U(n)} \xi_{(4)e_i} e_i, JY \rangle + \frac{n-2}{n-1} \langle \nabla_{JY}^{U(n)} \xi_{(4)e_i} e_i, JX \rangle \\ &\quad - 3 \langle \xi_{(1)X} e_i, \xi_{(2)Y} e_i \rangle + 3 \langle \xi_{(1)Y} e_i, \xi_{(2)X} e_i \rangle, \\ 0 &= 3 \langle (\nabla_{e_i}^{U(n)} \xi_{(1)})_{e_i} X, Y \rangle - \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_{e_i} X, Y \rangle + (n-2) \langle (\nabla_{e_i}^{U(n)} \xi_{(4)})_{e_i} X, Y \rangle \\ &\quad - \langle \xi_{(3)X} e_i, \xi_{(1)Y} e_i \rangle + \langle \xi_{(3)Y} e_i, \xi_{(1)X} e_i \rangle + \frac{1}{2} \langle \xi_{(3)X} e_i, \xi_{(2)Y} e_i \rangle - \frac{1}{2} \langle \xi_{(3)Y} e_i, \xi_{(2)X} e_i \rangle \\ &\quad - \frac{n-5}{n-1} \langle \xi_{(1)\xi_{(4)e_i} e_i} X, Y \rangle - \frac{n-2}{n-1} \langle \xi_{(2)\xi_{(4)e_i} e_i} X, Y \rangle + \langle \xi_{(3)\xi_{(4)e_i} e_i} X, Y \rangle. \end{aligned}$$

Remark 3.2. The third identity of Lemma 3.1 has already been shown in [13]. There it was used to explain some aspects of the behavior of the Riemannian curvature tensor.

Proof. Consider the Kähler form ω . Being a differential form it satisfies $d^2\omega = 0$. However, since the Levi-Civita connection ∇ is torsion-free, we may compute $d^2\omega$ using ∇ . Writing $\nabla = \nabla^{U(n)} - \xi$ and using $\nabla^{U(n)}\omega = 0$, we have first that

$$\frac{1}{2}d\omega(Y, Z, W) = \langle \xi_Y Z, JW \rangle + \langle \xi_W Y, JZ \rangle + \langle \xi_Z W, JY \rangle.$$

Now $d^2\omega = \mathbf{a}(\nabla^{U(n)}\omega) - \mathbf{a}(\xi d\omega)$, where $\mathbf{a} : T^*M \otimes \Lambda^3 T^*M \rightarrow \Lambda^4 T^*M$ is the alternation map. One computes that these two terms are the expressions obtained respectively by summing $\varepsilon \langle (\nabla_X^{U(n)} \xi)_Y Z, JW \rangle$ and $\varepsilon \langle \xi_{\xi_X Y} Z, JW \rangle$ over all permutations of (X, Y, Z, W) , where ε is the sign of the permutation. After doing all of this we obtain

$$\begin{aligned} d^2\omega(X_1, X_2, X_3, X_4) &= \sum_{1 \leq a < b \leq 4} (-1)^{a+b} \left(\left((\nabla_{X_a}^{U(n)} \xi)_{X_b} - (\nabla_{X_b}^{U(n)} \xi)_{X_a} \right) \omega \right) (X_c, X_d) \\ &\quad + \sum_{1 \leq a < b \leq 4} (-1)^{a+b} (\xi_{\xi_{X_a} X_b - \xi_{X_b} X_a} \omega) (X_c, X_d) \\ &\quad - \sum_{1 \leq a < b \leq 4} (-1)^{a+b} ([\xi_{X_a}, \xi_{X_b}] \omega) (X_c, X_d), \end{aligned} \quad (3.1)$$

where $c < d$, $\{c, d\} = \{1, \dots, 4\} - \{a, b\}$ in each case and $[\xi_{X_a}, \xi_{X_b}] = \xi_{X_a} \xi_{X_b} - \xi_{X_b} \xi_{X_a}$.

We have that

$$\Lambda^4 T^*M = [\lambda^{4,0}] + [\lambda^{3,1}] + [\lambda^{2,0}]\omega + [\lambda_0^{2,2}] + [\lambda_0^{1,1}]\omega + \mathbb{R}\omega^2,$$

so in order to compute the components in $[\lambda^{1,1}] = \mathbb{R} + [\lambda_0^{1,1}]$ and $[\lambda^{2,0}]$ of $d^2\omega$, we contract with ω on the first two arguments. Then we take the corresponding projections to $[\lambda^{1,1}]$ and $[\lambda^{2,0}]$, which are respectively the 1-eigenspace and (-1) -eigenspace of J acting on 2-forms. Using the symmetries of the components of ξ , one obtains the components in $[\lambda^{1,1}]$ and $[\lambda^{2,0}]$ of $d^2\omega$ written in terms of $\nabla^{U(n)}$ and ξ . Such components vanish because $d^2\omega = 0$. For the first identity, we do a contraction with ω on the component in $[\lambda^{1,1}]$. In this way we will obtain the equality

$$0 = 8\langle \nabla_{e_j}^{U(n)} \xi_{(4)e_i} e_i, J e_j \rangle - 3(n-1)\langle \xi_{(1)e_j} e_i, \xi_{(2)e_j} J e_i \rangle.$$

Since $\xi_{(2)} \circ J$ is still in \mathcal{W}_2 , $\xi_{(2)} \circ J$ is orthogonal to $\xi_{(1)}$. Hence $\langle \xi_{(1)e_j} e_i, \xi_{(2)e_j} J e_i \rangle = 0$ and the first identity follows. Taking this into account in the $[\lambda^{1,1}]$ -component, we will obtain the second identity. Finally, the third identity follows by considering the component in $[\lambda^{2,0}]$. \square

Remark 3.3. It is well known that the respective curvature tensors R and $R^{U(n)}$ of the connections ∇ and $\nabla^{U(n)}$ are related by

$$R(X, Y) = R^{U(n)}(X, Y) + (\nabla_X^{U(n)} \xi)_Y - (\nabla_Y^{U(n)} \xi)_X + \xi_{\xi_X Y} - \xi_{\xi_Y X} - [\xi_X, \xi_Y]$$

(see [2]). Using this identity in (3.1), it is obtained

$$0 = \sum_{1 \leq a < b \leq 4} (-1)^{a+b} ((R(X_a, X_b) - R^{U(n)}(X_a, X_b)) \omega)(X_c, X_d).$$

where $c < d$, $\{c, d\} = \{1, \dots, 4\} - \{a, b\}$ in each summand. We stress that $\nabla^{U(n)}\omega = 0$ is a key fact here.

Next we note that

$$\begin{aligned} (\nabla_X^{U(n)} (\xi_{(4)e_i} e_i)^b)(Y) - (\nabla_Y^{U(n)} (\xi_{(4)e_i} e_i)^b)(X) &= (d(\xi_{(4)e_i} e_i)^b)(X, Y) - \langle \xi_X Y - \xi_Y X, \xi_{(4)e_i} e_i \rangle, \\ (n-1)\langle \nabla_{e_i}^{U(n)} \xi_{(4)} e_i, X, Y \rangle &= (d(\xi_{(4)e_i} e_i)^b)_{[\lambda^{2,0}]}(X, Y) - 2\langle \xi_{(1)\xi_{(4)e_i} e_i} X, Y \rangle + \langle \xi_{(2)\xi_{(4)e_i} e_i} X, Y \rangle \end{aligned}$$

(for an explicit proof for this second identity, see Lemma 4.4 in [6]). Using these identities in previous Lemma, we will obtain the components of the exterior derivative of the Lee form θ .

Proposition 3.4. For almost Hermitian manifolds of dimension $2n$, $n > 1$, the following identities are satisfied:

$$\begin{aligned} (d\theta)_{\mathbb{R}\omega} &= 0, \\ \frac{n-2}{2}(d\theta)_{[\lambda_0^{1,1}]}(X, Y) &= -\langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_X Y, e_i \rangle + \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_Y X, e_i \rangle + \frac{n-2}{2} \langle \xi_{(3)X} Y - \xi_{(3)Y} X, \theta^\sharp \rangle \\ &\quad - \frac{3}{2} \langle \xi_{(1)X} e_i, \xi_{(2)Y} e_i \rangle + \frac{3}{2} \langle \xi_{(1)Y} e_i, \xi_{(2)X} e_i \rangle, \\ \frac{n-2}{2}(d\theta)_{[\lambda^{2,0}]}(X, Y) &= -3\langle (\nabla_{e_i}^{U(n)} \xi_{(1)})_{e_i} X, Y \rangle + \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_{e_i} X, Y \rangle + \langle \xi_{(3)X} e_i, \xi_{(1)Y} e_i \rangle \\ &\quad - \langle \xi_{(3)Y} e_i, \xi_{(1)X} e_i \rangle - \frac{1}{2} \langle \xi_{(3)X} e_i, \xi_{(2)Y} e_i \rangle + \frac{1}{2} \langle \xi_{(3)Y} e_i, \xi_{(2)X} e_i \rangle \\ &\quad + \frac{3(n-3)}{2} \langle \xi_{(1)\theta^\sharp} X, Y \rangle - \frac{n-1}{2} \langle \xi_{(3)\theta^\sharp} X, Y \rangle. \end{aligned}$$

Remark 3.5. The identity $(d\theta)_{\mathbb{R}\omega} = 0$ has already been obtained by Gauduchon in [5] in the context of Hermitian structures (type $\mathcal{W}_3 \oplus \mathcal{W}_4$). However, his proof is valid in the general context of almost Hermitian structures.

As a consequence of the expressions for the components of $d\theta$ given in Proposition 3.4, we have the following proposition whose part (i) is already well known.

Proposition 3.6. For almost Hermitian manifolds of dimension $2n$ with $n > 2$, we have:

- (i) If the structure is of type $\mathcal{W}_2 \oplus \mathcal{W}_4 \cong [\mathbb{A}] \oplus [\lambda^{1,0}]$, then the Lee form θ is closed.
- (ii) If the structure is of type $\mathcal{W}_1 \oplus \mathcal{W}_4 \cong [\lambda^{3,0}] \oplus [\lambda^{1,0}]$, then $(d\theta)_{[\lambda_0^{1,1}]}$ vanishes. In particular, if $n = 3$, $d\theta = 0$ for such a type. Moreover, if $\xi_{(1)} = 0$ on some point, then $\xi_{(1)} = 0$ on the whole corresponding connected component, i.e. the structure is of the type \mathcal{W}_4 called locally conformal Kähler structure. If $\xi_{(1)} \neq 0$, the Lee form is given by $\theta = d \ln \frac{1}{\|\xi_{(1)}\|^2}$ on the connected component. Thus, in this second case, the structure of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ is globally conformal to the type \mathcal{W}_1 and it makes sense to say that we have a globally conformal nearly Kähler structure.

Proof. As we have already said, (i) and the main part of (ii) are easily deduced from the expressions for $(d\theta)_{[\lambda_0^{1,1}]}$ and $(d\theta)_{[\lambda^{2,0}]}$ given in Proposition 3.4. It remains to verify the assertion for $n = 3$ in (ii). If the component $d\omega_{[\lambda^{3,0}]}$ in $[\lambda^{3,0}]$ of $d\omega$ is zero, then $d\theta = 0$ because of (i). Hence in the sequel we assume $d\omega_{[\lambda^{3,0}]} \neq 0$ and denote $w_1^+ = \frac{1}{6} \|d\omega_{[\lambda^{3,0}]}\|$.

Now, we fix as a complex volume form, $\Psi = \psi_+ + i\psi_-$, where $\psi_+ = \frac{1}{3w_1^+}d\omega_{[\lambda^{3,0}]}$ and $\psi_- = J_{(1)}\psi_+$. Then we are in the presence, at a least locally, of a $SU(3)$ -structure of type $\mathcal{W}_1^+ \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$ (see [12] for details). The component of the intrinsic $SU(3)$ -torsion in \mathcal{W}_1^+ is determined by the component of $d\omega$ in $\mathbb{R}\psi_+ \subseteq [\lambda^{3,0}] = \mathcal{W}_1$. On the other hand, the component of the intrinsic $SU(3)$ -torsion in \mathcal{W}_5 is determined by the one-form η which is computed by the identity

$$*(d\psi_+ \wedge \psi_+ + *d\psi_- \wedge \psi_-) = 4(3\eta - \theta),$$

where $*$ is the Hodge star operator with respect to the real volume form $\text{Vol} = -\frac{1}{4}\psi_+ \wedge \psi_- = \frac{1}{6}\omega \wedge \omega \wedge \omega$. In our situation we have the following exterior derivatives (see [12])

$$d\omega = 3w_1^+\psi_+ + \theta \wedge \omega, \quad d\psi_+ = (-3\eta + \theta) \wedge \psi_+, \quad d\psi_- = 2w_1^+\omega \wedge \omega + (-3\eta + \theta) \wedge \psi_-.$$

Doing again exterior differentiation, we have $0 = d^2\psi_+ = (-3d\eta + d\theta) \wedge \psi_+$. Hence $d\theta = (d\theta)_{[\lambda^{2,0}]} = 3(d\eta)_{[\lambda^{2,0}]}$. On the other hand, we obtain

$$0 = d^2\psi_- = 2(dw_1^+ + w_1^+(3\eta + \theta)) \wedge \omega \wedge \omega, \quad (3.2)$$

where we have used $\psi_+ \wedge \omega = 0$ and $(-3d\eta + d\theta) \wedge \psi_- = (-3d\eta + d\theta)_{[\lambda^{2,0}]} \wedge \psi_- = 0$. From Eq. (3.2), $d(\ln w_1^+) = -3\eta - \theta$ and $d\theta = -3d\eta = -3(d\eta)_{[\lambda^{2,0}]}$. This implies $d\theta = 0$.

Now, using the expression for $(d\theta)_{[\lambda^{2,0}]}$ in Proposition 3.4, we have $\langle (\nabla_{e_i}^{U(3)}\xi_{(1)})_{e_i}X, Y \rangle = 0$ and, in terms of $SU(3)$ -structure, here $\xi_{(1)} = \xi_{(1)}^+ = \frac{1}{2}w_1^+\psi_-$. Then

$$\langle (\nabla_{e_i}^{U(3)}\xi_{(1)})_{e_i}X, Y \rangle = \frac{1}{2}\psi_-((dw_1^+)^\sharp, X, Y) + \frac{1}{2}w_1^+(\nabla_{e_i}^{U(3)}\psi_-)(e_i, X, Y).$$

We recall $\nabla^{SU(3)} = \nabla^{U(3)} + \eta = \nabla + \xi + \eta$, where $\eta_X Y = (J\eta)(X)JY$, and $\nabla^{SU(3)}\psi_- = 0$. Therefore, $\nabla_X^{U(3)}\psi_- = -\eta_X\psi_- = 3(J\eta)_X\psi_+$. This implies

$$\begin{aligned} 0 &= \langle (\nabla_{e_i}^{U(3)}\xi_{(1)})_{e_i}X, Y \rangle = \frac{1}{2}\psi_-((dw_1^+)^\sharp, X, Y) + \frac{3}{2}w_1^+\psi_+(J\eta^\sharp, X, Y) \\ &= \frac{1}{2}\psi_-((dw_1^+)^\sharp - 3w_1^+\eta^\sharp, X, Y). \end{aligned}$$

Hence $d(\ln w_1^+) = 3\eta$ and, using $d(\ln w_1^+) = -3\eta - \theta$, we have $\theta = -2d \ln w_1^+ = d \ln \frac{1}{w_1^+}$. It is straightforward to check $w_1^{+2} = \|\xi_{(1)}^+\|^2$ and, in this situation, $\|\xi_{(1)}^+\|^2 = \|\xi_{(1)}\|^2$.

Finally, we will prove that if $\xi_{(1)} = 0$ for some point P , then $w_1^+ = \|\xi_{(1)}\|^2 = 0$ and $dw_1^+ = 0$ at P . Hence the function w_1^+ is constant and equal to zero on the whole connected component. In fact, if $dw_1^+ \neq 0$ on P , there exist a sequence $\{P_i\}_{i \in \mathbb{N}}$ of points converging to P such that $w_1^+(P_i) \neq 0$. Then we have $(dw_1^+)_{P_i} = -\frac{1}{2}w_1^+(P_i)\theta_{P_i}$ which converges to $(dw_1^+)_P = -\frac{1}{2}w_1^+(P)\theta_P = 0$, contradiction. \square

4. Lee form and Riemannian curvature

Next we will display relations between the Lee form and $U(n)$ -components of the Riemannian curvature R . Some of these components are determined by means of the Ricci tensor Ric and a Ricci type tensor Ric^* associated to the almost Hermitian structure. Ric^* is called the **-Ricci curvature tensor* and defined by $\text{Ric}^*(X, Y) = \langle R_{X, e_i}JY, Je_i \rangle$, where $R_{X, Y} = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$. Because $\text{Ric}^*(JX, JY) = \text{Ric}^*(Y, X)$, its symmetric part is in $[\lambda^{1,1}]$ and its skew-symmetric part is in $[\lambda^{2,0}]$.

4.1. Some components in the orthogonal complement of the space of Kähler curvatures

The components of R determined by the difference $\text{Ric} - \text{Ric}^*$ are in $\mathcal{K}_{-1} \cong \mathbb{R}$, $\mathcal{K}_{-2} \cong [\lambda_0^{1,1}]$, $\mathcal{C}_6 \cong [\lambda^{2,0}]$ and $\mathcal{C}_8 \cong [\sigma^{2,0}]$ (these notations have been fixed in [4]). Such components are included in the orthogonal complement \mathcal{K}^\perp of the space \mathcal{K} of those curvature tensors satisfying the same properties as the curvature of a Kähler manifold, i.e. $\langle R_{X, Y}Z, U \rangle = \langle R_{X, Y}JZ, JU \rangle$. Such an orthogonal complement is obtained in the space \mathcal{R} of possible Riemannian curvature tensors, i.e. $\mathcal{R} = \mathcal{K} \oplus \mathcal{K}^\perp$.

Because they are included in \mathcal{K}^\perp , the above mentioned components of R can be given in terms of the intrinsic torsion. Thus such an expression for $\text{Ric} - \text{Ric}^*$ (see [13]) is

$$\begin{aligned} \text{Ric}(X, Y) - \text{Ric}^*(X, Y) &= -2\langle (\nabla_{e_i}^{U(n)}\xi)X, Y, e_i \rangle + 2\langle (\nabla_X^{U(n)}\xi)_{e_i}Y, e_i \rangle \\ &\quad - 2\langle \xi_{\xi_{e_i}X}Y, e_i \rangle + 2\langle \xi_{\xi_X e_i}Y, e_i \rangle. \end{aligned} \quad (4.1)$$

From this identity, taking into account properties of $\xi_{(i)}$ and the fact that $\nabla_X^{U(n)}\theta$ is the Lee form of $\nabla_X^{U(n)}\xi$, it is long but straightforward to derive

$$\begin{aligned} (\text{Ric} - \text{Ric}^*)_{[\lambda^{1,1}]}(X, Y) &= -2\langle (\nabla_{e_i}^{U(n)}\xi_{(3)})X, Y, e_i \rangle - \frac{n-2}{2}\langle (\nabla_X\theta)(Y) + (\nabla_{JX}\theta)(JY) \rangle \\ &\quad + \frac{1}{2}\langle X, Y \rangle(d^*\theta + \frac{2n-3}{2}\|\theta\|^2) + 4\langle \xi_{(1)X}e_i, \xi_{(1)Y}e_i \rangle \\ &\quad - 2\langle \xi_{(2)e_i}X, \xi_{(2)e_i}Y \rangle - \frac{n-2}{4}\langle \theta(X)\theta(Y) + \theta(JX)\theta(JY) \rangle \\ &\quad - 2\langle \xi_{(1)X}e_i, \xi_{(2)Y}e_i \rangle + \langle \xi_{(1)Y}e_i, \xi_{(2)X}e_i \rangle + (n-2)\theta(\xi_{(3)X}Y). \end{aligned} \quad (4.2)$$

Since $(\text{Ric} - \text{Ric}^*)_{[\lambda^{1,1}]}$ is symmetric, the skew symmetric part of the right side in (4.2) must be zero. In fact, this is the case, because such a skew-symmetric part is given by one half of the expression in the right side of the second identity of Lemma 3.1 which is consequence of $d^2\omega = 0$. In other words, the above mentioned skew symmetric part is equal to $\frac{n-2}{2}(d\theta)_{[\lambda_0^{1,1}]} - A$, where A is the corresponding expression given in Proposition 3.4.

The component of the curvature in \mathcal{K}_{-1} is determined by the difference of the scalar curvatures s and s^* with respect to Ric and Ric^* . By using the previous identity for $\text{Ric} - \text{Ric}^*$, it is obtained the following result.

Lemma 4.1. *For an almost Hermitian manifold, we have*

$$s - s^* = 2(n-1)d^*\theta + (n-1)^2\|\theta\|^2 + 4\|\xi_{(1)}\|^2 - 2\|\xi_{(2)}\|^2, \quad (4.3)$$

where $\|\xi_{(a)}\|^2 = \langle \xi_{(a)e_i}e_j, \xi_{(a)e_i}e_j \rangle$, $a = 1, 2$.

Next we point out some immediate consequences of (4.3).

Proposition 4.2.

- (i) *An almost Kähler manifold ($\xi \in \mathcal{W}_2$) such that $R \in \mathcal{K}$ is Kähler.*
- (ii) *An almost Hermitian manifold such that $\xi \in \mathcal{W}_1 \oplus \mathcal{W}_4$, $d^*\theta \geq 0$ ($\text{div } \theta^\sharp \leq 0$) and $R \in \mathcal{K}$ is Kähler.*
- (iii) *A compact almost Hermitian manifold such that $\xi \in \mathcal{W}_1 \oplus \mathcal{W}_4$ and $R \in \mathcal{K}$ is Kähler.*

Claim (i) and a part of (ii) have been proved by Falcitelli et al. in [4, Prop. 5.5]. Other parts of (ii) and (iii) have been shown by Vaisman in [16, Theorem 2.1. (i)].

Next we will deduce an expression for the $[\lambda^{2,0}]$ -component $\text{Ric}^*_{[\lambda^{2,0}]} = -(\text{Ric} - \text{Ric}^*)_{[\lambda^{2,0}]}$ of Ric^* . From (4.1), taking into account properties of $\xi_{(i)}$ and the fact that $\nabla_X^{U(n)}\theta$ is the Lee form of $\nabla_X^{U(n)}\xi$, it is obtained

$$\begin{aligned} \text{Ric}^*_{[\lambda^{2,0}]}(X, Y) &= 2\langle (\nabla_{e_i}^{U(n)}\xi_{(1)})_{e_i}X, Y \rangle - \langle (\nabla_{e_i}^{U(n)}\xi_{(2)})_{e_i}X, Y \rangle + \frac{n-1}{2}(d\theta)_{[\lambda^{2,0}]}(X, Y) \\ &\quad - \langle \xi_{(1)\gamma}e_i, \xi_{(3)\gamma}e_i \rangle + \langle \xi_{(1)\gamma}e_i, \xi_{(3)\gamma}e_i \rangle - (n-3)\langle \xi_{(1)\theta^\sharp}X, Y \rangle \\ &\quad - \frac{1}{2}\langle \xi_{(2)\gamma}e_i, \xi_{(3)\gamma}e_i \rangle + \frac{1}{2}\langle \xi_{(2)\gamma}e_i, \xi_{(3)\gamma}e_i \rangle + \frac{n}{2}\langle \xi_{(2)\theta^\sharp}X, Y \rangle. \end{aligned} \quad (4.4)$$

Another expression for $\text{Ric}^*_{[\lambda^{2,0}]}$ has been obtained in [13, Lemma 3.8], it is given by

$$\text{Ric}^*_{[\lambda^{2,0}]}(X, Y) = \langle (\nabla_{e_i}^{U(n)}\xi)_{J e_i}X, Y \rangle - \langle \xi_{J e_i e_i}J X, Y \rangle.$$

Now, as before, taking into account properties of $\xi_{(i)}$, it is deduced

$$\begin{aligned} \text{Ric}^*_{[\lambda^{2,0}]}(X, Y) &= -\langle (\nabla_{e_i}^{U(n)}\xi_{(1)})_{e_i}X, Y \rangle - \langle (\nabla_{e_i}^{U(n)}\xi_{(2)})_{e_i}X, Y \rangle + \langle (\nabla_{e_i}^{U(n)}\xi_{(3)})_{e_i}X, Y \rangle \\ &\quad + \frac{1}{2}(d\theta)_{[\lambda^{2,0}]}(X, Y) + \frac{n-3}{2}\langle \xi_{(1)\theta^\sharp}X, Y \rangle + \frac{n}{2}\langle \xi_{(2)\theta^\sharp}X, Y \rangle \\ &\quad - \frac{n-1}{2}\langle \xi_{(3)\theta^\sharp}X, Y \rangle. \end{aligned} \quad (4.5)$$

If we take the difference between the identities (4.4) and (4.5), we will obtain $\frac{n-2}{2}(d\theta)_{[\lambda^{2,0}]} - B$, where B is the right side expression of the corresponding identity in Proposition 3.4. This was already noted in [13] and explains why both expressions agree.

Next, as consequences of the identity (4.4) (or (4.5)) and those ones given in Proposition 3.4, we display some relevant particular situations.

Proposition 4.3. *On almost Hermitian manifolds, relative to $\text{Ric}^*_{[\lambda^{2,0}]}$, we have:*

- (i) *If $\xi \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, then*

$$\text{Ric}^*_{[\lambda^{2,0}]}(X, Y) = -\langle (\nabla_{e_i}^{U(n)}\xi_{(2)})_{e_i}X, Y \rangle + \frac{n+1}{6}(d\theta)_{[\lambda^{2,0}]}(X, Y) + \frac{n}{2}\langle \xi_{(2)\theta^\sharp}X, Y \rangle.$$

In particular:

- (a) *if $\xi \in \mathcal{W}_1 \oplus \mathcal{W}_4$, then $\text{Ric}^*_{[\lambda^{2,0}]} = \frac{n+1}{6}d\theta$. Moreover, if $n = 3$, $\text{Ric}^*_{[\lambda^{2,0}]} = d\theta = 0$.*
- (b) *if $\xi \in \mathcal{W}_2 \oplus \mathcal{W}_4$ and $n > 2$, then*

$$\text{Ric}^*_{[\lambda^{2,0}]}(X, Y) = -\langle (\nabla_{e_i}^{U(n)}\xi_{(2)})_{e_i}X, Y \rangle + \frac{n}{2}\langle \xi_{(2)\theta^\sharp}X, Y \rangle;$$

if $n = 2$, then

$$\text{Ric}^*_{[\lambda^{2,0}]}(X, Y) = -\langle (\nabla_{e_i}^{U(n)}\xi_{(2)})_{e_i}X, Y \rangle + \frac{1}{2}(d\theta)_{[\lambda^{2,0}]}(X, Y) + \langle \xi_{(2)\theta^\sharp}X, Y \rangle.$$

- (ii) *If the structure is Hermitian ($\xi \in \mathcal{W}_3 \oplus \mathcal{W}_4$), then:*

(a) $\text{Ric}^*_{\|\lambda^{2,0}\|} = \frac{n-1}{2} d\theta_{\|\lambda^{2,0}\|}$; moreover, if $n > 2$, we also have the expression

$$\text{Ric}^*_{\|\lambda^{2,0}\|} = \frac{n-1}{n-2} \left(\langle (\nabla_{e_i}^{U(n)} \xi_{(3)}) e_i X, Y \rangle - \frac{n-1}{2} \langle \xi_{(3)\theta^\sharp} X, Y \rangle \right).$$

(b) If $n = 2$, $(\text{Ric} - \text{Ric}^*)_{[\lambda_0^{1,1}]} = 0$.

Part (ii) in Proposition completes the result given in [4, Proposition 7.2].

Another component of the curvature in the orthogonal complement \mathcal{K}^\perp is determined by the $\|\sigma^{2,0}\|$ -component $\text{Ric}_{\|\sigma^{2,0}\|} = (\text{Ric} - \text{Ric}^*)_{\|\sigma^{2,0}\|}$ of Ric. From (4.1), taking into account properties of $\xi_{(i)}$ and the fact that $\nabla_X^{U(n)} \theta$ is the Lee form of $\nabla_X^{U(n)} \xi$, it is obtained

$$\begin{aligned} \text{Ric}_{\|\sigma^{2,0}\|}(X, Y) &= -\langle (\nabla_{e_i}^{U(n)} \xi_{(2)})_X Y, e_i \rangle - \langle (\nabla_{e_i}^{U(n)} \xi_{(2)})_Y X, e_i \rangle \\ &\quad - \frac{n-1}{4} ((\nabla_X \theta)(Y) + (\nabla_Y \theta)(X) - (\nabla_{JX} \theta)(JY) - (\nabla_{JY} \theta)(JX)) \\ &\quad + \langle \xi_{(1)X} e_i, \xi_{(3)Y} e_i \rangle + \langle \xi_{(1)Y} e_i, \xi_{(3)X} e_i \rangle - \frac{1}{2} \langle \xi_{(2)X} e_i, \xi_{(3)Y} e_i \rangle \\ &\quad - \frac{1}{2} \langle \xi_{(2)Y} e_i, \xi_{(3)X} e_i \rangle + \frac{n-2}{2} \theta(\xi_{(2)X} Y + \xi_{(2)Y} X). \end{aligned}$$

As a consequence of this expression we have the following result.

Proposition 4.4. If an almost Hermitian manifold is such that $\xi \in \mathcal{W}_1 \oplus \mathcal{W}_4$ or $\xi \in \mathcal{W}_3 \oplus \mathcal{W}_4$, then

$$\text{Ric}_{\|\sigma^{2,0}\|}(X, Y) = -\frac{n-1}{4} ((\nabla_X \theta)(Y) + (\nabla_Y \theta)(X) - (\nabla_{JX} \theta)(JY) - (\nabla_{JY} \theta)(JX)).$$

Moreover, in such cases, if the vector field θ^\sharp is Killing, then $\text{Ric}_{\|\sigma^{2,0}\|} = 0$.

4.2. Ricci Forms

We would like to focus our attention to components of the curvature included in the space \mathcal{K} of type Kähler curvature tensors. Such space is decomposed into $\mathcal{K} = \mathcal{C}_3 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$, where $\mathcal{C}_3 \cong [\sigma_0^{2,2}]$, $\mathcal{K}_1 \cong \mathbb{R}$ and $\mathcal{K}_2 \cong [\lambda_0^{1,1}]$ (see [4]). The components in \mathcal{K}_1 and in \mathcal{K}_2 can be determined in terms of tensor Ric and Ric^* . More precisely, they are obtained by the tensor $(\text{Ric} + 3 \text{Ric}^*)_{[\lambda^{1,1}]}$. To derive expressions for such a tensor, we consider convenient to previously recall some notions relative to Ricci forms (see [5]).

Definition 4.5. For almost Hermitian manifolds, given a metric connection D with curvature tensor R_D , the first (second) Ricci form of D is the two-form $\rho_D(r_D)$ given by

$$\rho_D(X, Y) = -\frac{1}{2} \langle R_D(e_i, J e_i) X, Y \rangle \quad r_D(X, Y) = -\frac{1}{2} \langle R_D(X, Y) e_i, J e_i \rangle.$$

Next we compute the Ricci forms relative to the Levi Civita and the minimal connections.

Proposition 4.6. For almost Hermitian manifolds, we have:

(i) If D is a $U(n)$ -connection, then $\rho_D \in [\lambda^{1,1}]$ and r_D is closed.

(ii) $\text{Ric}^*(X, JY) = \rho_\nabla(X, Y) = r_\nabla(X, Y) = r_{\nabla^{U(n)}}(X, Y) + \langle \xi_X e_i, \xi_Y J e_i \rangle$

$$\begin{aligned} &= r_{\nabla^{U(n)}}(X, Y) + \sum_{a=1}^3 \langle \xi_{(a)X} e_i, \xi_{(a)Y} J e_i \rangle - \sum_{a=1}^3 J \theta(\xi_{(a)X} Y - \xi_{(a)Y} X) \\ &\quad + \sum_{1 \leq a < b \leq 3} (\langle \xi_{(a)X} e_i, \xi_{(b)Y} J e_i \rangle - \langle \xi_{(a)Y} e_i, \xi_{(b)X} J e_i \rangle) \\ &\quad - \frac{1}{4} \|\theta\|^2 \omega(X, Y) - \frac{1}{4} \theta \wedge J \theta(X, Y) \end{aligned}$$

(iii) $\rho_{\nabla[\lambda^{1,1}]}(X, Y) = \rho_{\nabla^{U(n)}}(X, Y) + \langle \xi_{e_i} X, \xi_{J e_i} Y \rangle$

$$\begin{aligned} &= \rho_{\nabla^{U(n)}}(X, Y) + \sum_{a=1}^3 \langle \xi_{(a) e_i} X, \xi_{(a) J e_i} Y \rangle - \frac{1}{8} \|\theta\|^2 \omega(X, Y) \\ &\quad + \sum_{1 \leq a < b \leq 3} (\langle \xi_{(a) e_i} X, \xi_{(b) J e_i} Y \rangle - \langle \xi_{(a) J e_i} Y, \xi_{(b) e_i} X \rangle) \\ &\quad - \frac{1}{2} J \theta(\xi_{(3)X} Y - \xi_{(3)Y} X) + \frac{n-2}{8} \theta \wedge J \theta(X, Y). \end{aligned}$$

Proof. If we consider an adapted local frame $\wp = \{e_\alpha, J e_\alpha = e_{\alpha'}\}$ to the $U(n)$ -structure, $\alpha' = \alpha + n$ and $\alpha = 1, \dots, n$, then $r_D(X, Y) = \sum_{\alpha=1}^n \Omega_{\alpha'}^{D\alpha}(\wp_\alpha X, \wp_\alpha Y)$, where $\Omega^D = (\Omega^{D i}_j)$ is the curvature two-form of D . If D is a $U(n)$ -connection

and $\varpi^D = (\varpi^{Dj}_i)$ is the connection one-form of D , then one has $\varpi^{D\alpha'}_{\beta'} = \varpi^{D\alpha}_{\beta} = -\varpi^{D\beta}_{\alpha}$ and $\varpi^{D\alpha'}_{\beta} = -\varpi^{D\alpha}_{\beta'} = \varpi^{D\beta'}_{\alpha}$. Hence

$$\begin{aligned} \sum_{\alpha=1}^n \sum_{i=1}^{2n} \varpi^{Di}_{\alpha'} \wedge \varpi^{D\alpha}_i &= \sum_{\alpha, \beta=1}^n (\varpi^{D\beta}_{\alpha'} \wedge \varpi^{D\alpha}_{\beta} + \varpi^{D\beta'}_{\alpha'} \wedge \varpi^{D\alpha}_{\beta'}) \\ &= \sum_{\alpha, \beta=1}^n (-\varpi^{D\alpha}_{\beta'} \wedge \varpi^{D\beta}_{\alpha} - \varpi^{D\alpha'}_{\beta'} \wedge \varpi^{D\beta}_{\alpha'}) = 0. \end{aligned}$$

Now using the structure equation, one has

$$\sum_{\alpha=1}^n \Omega^{D\alpha}_{\alpha'} = \sum_{\alpha=1}^n d\varpi^{D\alpha}_{\alpha'} - \sum_{\alpha=1}^n \sum_{i=1}^{2n} \varpi^{Di}_{\alpha'} \wedge \varpi^{D\alpha}_i = \sum_{\alpha=1}^n d\varpi^{D\alpha}_{\alpha'}.$$

From this, it follows that $r_D = \sum_{\alpha=1}^n \wp^* \Omega^{D\alpha}_{\alpha'} = d(\sum_{\alpha=1}^n \wp^* \varpi^{D\alpha}_{\alpha'})$ is closed. On the other hand, if D is a $U(n)$ -connection, then the matrix

$$(\rho_D(e_i, e_j)) = \left(\sum_{\alpha=1}^n \Omega^{Dj}_i(\wp_* e_{\alpha}, \wp_* J e_{\alpha}) \right)$$

belongs to the Lie algebra $\mathfrak{u}(n) \cong [\lambda^{1,1}]$ of $U(n)$.

Since D is a metric connection, then $D = \nabla + \xi^D$, where $\langle \xi^D_X Y, Z \rangle = -\langle \xi^D_X Z, Y \rangle$. Moreover, the corresponding curvature tensors R and R^D are related by

$$R(X, Y)Z = R^D(X, Y)Z + (D_X \xi^D)_Y Z - (D_Y \xi^D)_X Z + \xi^D_{\xi^D_X Y} Z - \xi^D_{\xi^D_Y X} Z - [\xi^D_X, \xi^D_Y]Z. \quad (4.6)$$

Therefore,

$$\begin{aligned} -2r_{\nabla}(X, Y) &= -2r_D(X, Y) + \langle (D_X \xi^D)_Y e_i, J e_i \rangle - \langle (D_Y \xi^D)_X e_i, J e_i \rangle \\ &\quad + \langle \xi^D_{\xi^D_Y X - \xi^D_X Y} e_i, J e_i \rangle - 2\langle \xi^D_X e_i, \xi^D_Y J e_i \rangle. \end{aligned} \quad (4.7)$$

In particular, when $D = \nabla^{U(n)}$, we will have

$$r_{\nabla}(X, Y) = r_{\nabla^{U(n)}}(X, Y) + \langle \xi_X e_i, \xi_Y J e_i \rangle.$$

From this, by using the properties of $\xi_{(a)}$, it follows the other expression for r_{∇} in (ii).

For (iii), from (4.6) we have

$$\rho_{\nabla}(X, Y) = \rho_{\nabla^{U(n)}}(X, Y) - \langle (\nabla^{U(n)}_{e_i} \xi)_{J e_i} X, Y \rangle + \frac{n-1}{2} \langle \xi_{J\theta} X, Y \rangle + \langle \xi_{e_i} X, \xi_{J e_i} Y \rangle.$$

Because $\nabla^{U(n)}$ is a $U(n)$ -connection, $\rho_{\nabla^{U(n)}}(X, Y) = \rho_{\nabla^{U(n)}}(X, Y) + \langle \xi_{e_i} X, \xi_{J e_i} Y \rangle$. Now, by using the properties of $\xi_{(a)}$, it follows the other expression for $\rho_{\nabla^{U(n)}}$ in (iii). \square

Remark 4.7. Note that in case of a presence of an $SU(n)$ -structure by [13, Lemma 3.3], one has $r_{\nabla^{U(n)}} = -nd\hat{\eta}$, where $\nabla^{SU(n)} = \nabla^{U(n)} + \eta$ and $\eta_X Y = \hat{\eta}(X)JY$, i.e. $\hat{\eta} = J\eta$. Hence $r_{\nabla^{U(n)}}$ would be exact. Anyway, since local $SU(n)$ -structures always exist on an almost Hermitian manifold, it follows that $r_{\nabla^{U(n)}}$ is locally the exterior derivative of a local one-form. This is an alternative argument confirming that $r_{\nabla^{U(n)}}$ is closed.

The first Chern class c_1 can be represented by $-\frac{1}{2\pi} r_D$ (in the de Rham cohomology group), where D is any $U(n)$ -connection. In particular, it is by $-\frac{1}{2\pi} r_{\nabla^{U(n)}}$. Note that in case of Kähler manifold, $\nabla^{U(n)} = \nabla$ and $r_{\nabla}(X, Y) = \text{Ric}(X, JY)$. This has motivated the name ‘Ricci form’. On other hand, the first Chern class c_1 is associated to the tangent bundle TM considered as a complex vector bundle (by means of J). Thus $c_1(M) = c_1(T_{\mathbb{C}}M = TM) = -c_1(\wedge^n(T_{\mathbb{C}}M)^*)$. The vanishing of c_1 is a necessary condition for the existence of a complex volume form globally defined on M , i.e. existence of a $SU(n)$ -structure.

In complex geometry there is a $GL(n, \mathbb{C})$ -connection which plays a relevant role, the *Chern connection*. It is the unique $GL(n, \mathbb{C})$ -connection ∇^h such that its torsion T^h satisfies $T^h(JX, Y) = JT^h(X, Y)$. In Hermitian geometry ($\xi \in \mathcal{W}_3 \oplus \mathcal{W}_4$), the Chern connection is given by $\nabla^h = \nabla + \xi^h$, where $\langle \xi^h_X Y, Z \rangle = \langle \xi_X Y + \xi_Y X, Z \rangle - \langle \xi_Z X, Y \rangle$. In almost Hermitian geometry, it is straightforward to check that ∇^h defined as before is a $U(n)$ -connection if and only if $\xi \in \mathcal{W}_3 \oplus \mathcal{W}_4$.

Proposition 4.8. For Hermitian manifolds, we have:

- (i) $r_{\nabla^h} = r_{\nabla^{U(n)}} + \frac{n-1}{2} dJ\theta$. Moreover, $r_{\nabla^h} = r_{\nabla^{U(n)}[\lambda^{1,1}]} + \frac{n-1}{2} (dJ\theta)_{[\lambda^{1,1}]}$.
- (ii) $\rho_{\nabla^h}(X, Y) = \rho_{\nabla^{U(n)}}(X, Y) - \langle (\nabla^{U(n)}_{e_i} \xi_{(3)})_X Y, J e_i \rangle + \langle (\nabla^{U(n)}_{e_i} \xi_{(3)})_Y X, J e_i \rangle$
 $- \frac{1}{2} (dJ\theta)_{[\lambda^{1,1}]}(X, Y) + \frac{1}{2} d^* \theta \omega(X, Y) + \frac{2n-1}{4} \|\theta\|^2 \omega(X, Y) + \frac{1}{4} \theta \wedge J\theta(X, Y)$
 $+ \frac{n}{2} J\theta(\xi_{(3)X} Y - \xi_{(3)Y} X) - 2\langle \xi_{(3)e_i} X, \xi_{(3)J e_i} Y \rangle + \langle \xi_{(3)X} e_i, \xi_{(3)Y} J e_i \rangle.$

Remark 4.9. Because the difference $r_{\nabla^h} - r_{\nabla^{U(n)}}$ is an exact two-form, the first Chern class is determined by $-\frac{1}{2\pi}r_{\nabla^h}$ or by $-\frac{1}{2\pi}r_{\nabla^{U(n)}}$ as it is expected.

Proof. We will use (4.7). Thus, we obtain the following identities by direct computation

$$\begin{aligned} \langle (\nabla_X^h \xi^h)_Y e_i, J e_i \rangle - \langle (\nabla_Y^h \xi^h)_X e_i, J e_i \rangle &= (n-1)dJ\theta(X, Y) + 2(n-1)\langle \xi_{J\theta} X, Y \rangle, \\ \langle \xi_{\xi^h Y}^h e_i, J e_i \rangle - \langle \xi_{\xi^h X}^h e_i, J e_i \rangle &= -2(n-1)\langle \xi_{J\theta} X, Y \rangle, \quad \langle \xi_X^h e_i, \xi_Y^h J e_i \rangle = \langle \xi_X e_i, \xi_Y J e_i \rangle. \end{aligned}$$

Hence we have $r_{\nabla}(X, Y) = r_{\nabla^h}(X, Y) - \frac{n-1}{2}dJ\theta + \langle \xi_X e_i, \xi_Y J e_i \rangle$. Now using Proposition 4.6 (ii), we get $r_{\nabla^h} = r_{\nabla^{U(n)}} + \frac{n-1}{2}dJ\theta$. Finally, since $\xi \in \mathcal{W}_3 \oplus \mathcal{W}_4$, using Proposition 4.3 (ii) and the properties of ξ , we obtain

$$r_{\nabla^{U(n)}}|_{\lambda^{2,0}}(X, Y) = r_{\nabla}|_{\lambda^{2,0}}(X, Y) = \frac{n-1}{2}(d\theta)|_{\lambda^{2,0}}(X, Y) = -\frac{n-1}{2}(dJ\theta)|_{\lambda^{2,0}}(X, Y).$$

Then $r_{\nabla^h} = r_{\nabla^{U(n)}}|_{\lambda^{1,1}} + \frac{n-1}{2}(dJ\theta)|_{\lambda^{1,1}}(X, Y)$.

For (ii), it is used the identity (4.6) for ∇^h , the facts that ρ_{∇^h} is in $[\lambda^{1,1}]$ and $\nabla^h = \nabla^{U(n)} - \xi + \xi^h$, the definition of ξ^h and the properties of the components $\xi_{(i)}$ of ξ . \square

4.3. Some components in the space of Kähler curvatures

As it was mentioned before, we will derive expressions for the tensor $(\text{Ric} + 3\text{Ric}^*)|_{\lambda^{1,1}}$. This tensor determines the components of the curvature in $\mathcal{K}_1 \cong \mathbb{R}$ and $\mathcal{K}_2 \cong [\lambda_0^{1,1}]$.

Proposition 4.10. For almost hermitian manifolds, we have

$$\begin{aligned} \frac{1}{2}(\text{Ric} + 3\text{Ric}^*)|_{\lambda^{1,1}}(X, Y) &= -2r_{\nabla^{U(n)}}|_{\lambda^{1,1}}(X, JY) - \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_X Y, e_i \rangle \\ &\quad - \frac{n-2}{4}((\nabla_X \theta)(Y) + (\nabla_Y \theta)(X)) + \frac{1}{4}(d^* \theta + \frac{2n-7}{2}\|\theta\|^2)(X, Y) \\ &\quad - 2\langle \xi_{(2)X} e_i, \xi_{(2)Y} e_i \rangle - \langle \xi_{(2)e_i} X, \xi_{(2)e_i} Y \rangle + 2\langle \xi_{(3)X} e_i, \xi_{(3)Y} e_i \rangle \\ &\quad - \frac{n-6}{8}(\theta(X)\theta(Y) + \theta(JX)\theta(JY)) + \langle \xi_{(1)X} e_i, \xi_{(2)Y} e_i \rangle \\ &\quad + \frac{5}{2}\langle \xi_{(1)Y} e_i, \xi_{(2)X} e_i \rangle + \frac{n-6}{2}\theta(\xi_{(3)X} Y) - 2\theta(\xi_{(3)Y} X). \end{aligned}$$

Proof. This identity directly follows by computing $4\text{Ric}^*|_{\lambda^{1,1}} + (\text{Ric} - \text{Ric}^*)|_{\lambda^{1,1}}$, using Proposition 4.6 (ii) and the identity (4.2). \square

The \mathcal{K}_1 -component of the curvature is determined by the metric contraction of $\text{Ric} + 3\text{Ric}^*$.

Corollary 4.11. For almost Hermitian manifolds we have

$$s + 3s^* = 8\langle r_{\nabla^{U(n)}}, \omega \rangle + 2(n-1)d^* \theta + (n-3)(n-1)\|\theta\|^2 - 6\|\xi_{(2)}\|^2 + 4\|\xi_{(3)}\|^2$$

Thus, from the identity (4.3) and the previous one, the following expressions for the scalar curvatures are obtained

$$\begin{aligned} s &= 2\langle r_{\nabla^{U(n)}}, \omega \rangle + 2(n-1)d^* \theta + \frac{1}{2}(2n-3)(n-1)\|\theta\|^2 + 3\|\xi_{(1)}\|^2 - 3\|\xi_{(2)}\|^2 + \|\xi_{(3)}\|^2, \\ s^* &= 2\langle r_{\nabla^{U(n)}}, \omega \rangle - \frac{1}{2}(n-1)\|\theta\|^2 - \|\xi_{(1)}\|^2 - \|\xi_{(2)}\|^2 + \|\xi_{(3)}\|^2. \end{aligned}$$

5. Examples

In this section we will display some examples showing that the components $d\theta|_{[\lambda_0^{1,1}]}$ and $d\theta|_{\lambda^{2,0}}$ of $d\theta$ can be non-zero. Also these examples will illustrate the formulae proved in the previous Section. For sake of simplicity, we will denote the wedge product by just juxtaposition of superindices, i.e. $e^{ij} = e^i \wedge e^j$. We also note that our convention for Nijenhuis tensor is $N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$. We recall that the vanishing of N characterizes the type $\mathcal{W}_3 \oplus \mathcal{W}_4$, called Hermitian structure.

Example 5.1. Let G be the four-dimensional simply-connected real solvable Lie group determined by the Lie algebra, displayed in [3] and denoted by M_0^3 there, generated by the basis $\{e^1, \dots, e^4\}$ of left-invariant one-forms such that

$$de^1 = e^{14}, \quad de^2 = 0, \quad de^3 = e^{24} + e^{34}, \quad de^4 = 0.$$

On G we consider the almost Hermitian structure such that $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 e^i \otimes e^i$ is the metric and $\omega = e^{31} + e^{42}$ is the Kähler form.

It is straightforward to check that the Nijenhuis tensor of the almost complex structure J is zero. On the other hand, one has the exterior derivative $d\omega = (-e^1 - 2e^4) \wedge \omega$. Therefore, the Lee form is $\theta = -e^1 - 2e^4$ and we are in the presence of a Hermitian structure (type \mathcal{W}_4 in the four-dimensional case). The non-zero components of $d\theta$ are given by

$$(d\theta)|_{[\lambda_0^{1,1}]} = -\frac{1}{2}(e^{14} + e^{23}), \quad (d\theta)|_{\lambda^{2,0}} = -\frac{1}{2}(e^{14} - e^{23}).$$

Because $n = 2$, the identities given in [Proposition 3.4](#) are trivial. For dimension 4, one has $\mathcal{W}_1 \oplus \mathcal{W}_4 = \mathcal{W}_4$ and, in this example, $d\theta|_{[\lambda_0^{1,1}]} \neq 0$ and $(d\theta)|_{[\lambda^{2,0}]} \neq 0$. Finally, we have

$$s - s^* = 2d^*\theta + \|\theta\|^2 = 9, \quad (\text{Ric} - \text{Ric}^*)_{\mathbb{R}} = \frac{9}{4}(\cdot, \cdot), \quad (\text{Ric} - \text{Ric}^*)|_{[\lambda_0^{1,1}]} = 0,$$

$$\text{Ric}^*|_{[\lambda^{2,0}]} = \frac{1}{2}(d\theta)|_{[\lambda^{2,0}]} = -\frac{1}{4}(e^{14} - e^{23}).$$

If $*$ is the Hodge star operator with respect to $\text{Vol} = -\frac{1}{2}\omega^2 = e^{1234}$, $d^*\theta = -*d*\theta = 2$. Now, we fix the complex volume form $\psi_+ + i\psi_-$, where $\psi_+ = e^{12} - e^{34}$ and $\psi_- = e^{32} + e^{14}$. For this $SU(2)$ -structure, we have

$$4\eta - \theta = (*d\psi_+ \wedge \psi_+ + *d\psi_- \wedge \psi_-) = 2e^4.$$

Then $\hat{\eta} = J\eta = -\frac{1}{4}e^3$ and it is globally defined. Thus, $r_{\nabla^{U(2)}} = -2d\hat{\eta} = \frac{1}{2}(e^{24} + e^{34})$ is exact. Hence, as it is expected by the existence of the $SU(2)$ -structure, the first Chern class vanishes $c_1 = 0$. The Ricci form of the Chern connection is given by $r_{\nabla^h} = r_{\nabla^{U(2)}} + \frac{1}{2}dJ\theta = 0$. The Lie brackets are given by $[e_1, e_4] = -e_1$, $[e_2, e_4] = -e_3$, $[e_3, e_4] = -e_3$ and $[e_i, e_j] = 0$, for the remaining pairs (i, j) . By Koszul's formula, the Levi Civita covariant derivatives are derived

$$\begin{aligned} \nabla_{e_1}e_1 &= \nabla_{e_3}e_3 = e_4, & \nabla_{e_1}e_4 &= -e_1, & \nabla_{e_2}e_3 &= \nabla_{e_3}e_2 = \frac{1}{2}e_4, \\ \nabla_{e_2}e_4 &= -\nabla_{e_4}e_2 = -\frac{1}{2}e_3, & \nabla_{e_3}e_4 &= -\frac{1}{2}e_2 - e_3, & \nabla_{e_4}e_3 &= -\frac{1}{2}e_2 \end{aligned}$$

and $\nabla_{e_i}e_j = 0$, for the remaining pairs (i, j) . These are needed in [Proposition 4.4](#) to obtain the following component of the Ricci tensor

$$\text{Ric}|_{[\sigma^{2,0}]} = \frac{1}{4}(e^1 \otimes e^4 + e^4 \otimes e^1 + e^2 \otimes e^3 + e^3 \otimes e^2).$$

Since $r_{\nabla^{U(2)}}|_{[\lambda^{1,1}]} = \frac{1}{2}e^{24} + \frac{1}{4}(e^{34} + e^{12})$ and we have all the information required in [Proposition 4.10](#) to compute the following Kähler component of the curvature

$$\begin{aligned} (\text{Ric} + 3\text{Ric}^*)|_{[\lambda^{1,1}]} &= -\frac{7}{4}e^1 \otimes e^1 - \frac{3}{4}e^2 \otimes e^2 - \frac{7}{4}e^3 \otimes e^3 - \frac{3}{4}e^4 \otimes e^4 \\ &\quad + 3e^1 \otimes e^4 + 3e^4 \otimes e^1 - 3e^2 \otimes e^3 - 3e^3 \otimes e^2. \end{aligned}$$

As a consequence (or by [Corollary 4.11](#)), for the scalar curvatures it is obtained

$$s + 3s^* = -5, \quad s = \frac{11}{2}, \quad s^* = -\frac{7}{2}.$$

Since one has the identities $\rho_{\nabla} = r_{\nabla} = -J_{(2)}\text{Ric}^*$, we obtain

$$\rho_{\nabla} = r_{\nabla} = \text{Ric}^*(\cdot, J\cdot) = -e^{31} - \frac{3}{4}e^{42} + \frac{1}{2}e^{12} + \frac{3}{2}e^{34}$$

This form is not closed, $d\rho_{\nabla} = -2e^{134} - \frac{3}{2}e^{124}$. Now we use [Proposition 4.6](#) (iii), to compute the first Ricci form of $\nabla^{U(2)}$ which is given by

$$\rho_{\nabla^{U(2)}} = -\frac{3}{8}e^{31} - \frac{1}{8}e^{42} + \frac{3}{4}e^{12} + \frac{3}{4}e^{34}.$$

It is in $[\lambda^{1,1}]$ but its exterior derivative is non-zero, $d\rho_{\nabla^{U(2)}} = -\frac{9}{8}e^{124} - \frac{3}{4}e^{134} \neq 0$. Likewise, by using [Proposition 4.8](#) (ii), the first Ricci form of the Chern connection is given by

$$\rho_{\nabla^h} = \frac{7}{2}e^{31} - \frac{5}{2}e^{42} + \frac{1}{2}e^{12} + \frac{1}{2}e^{34}.$$

It is also in $[\lambda^{1,1}]$ and has non-zero exterior derivative, $d\rho_{\nabla^h} = 3e^{124} + \frac{7}{2}e^{134} \neq 0$.

Example 5.2. In [8], Hasegawa determined all the complex surfaces which are diffeomorphic to compact solvmanifolds. Some of them are known as Inoue surfaces [9] of type S^+ or S^- . These can be written (up to finite covering) as $\Gamma \backslash G$, where Γ is a lattice of a simply connected solvable Lie group G (see [8] for details). The Lie algebra \mathfrak{g} of G is expressed as having a basis $\{X_1, X_2, X_3, X_4\}$ with the bracket multiplication $[X_2, X_3] = -X_1$, $[X_4, X_2] = X_2$, $[X_4, X_3] = -X_3$ and all other brackets vanish. The almost complex structure J is defined by

$$JX_1 = X_2, \quad -JX_2 = X_1, \quad JX_3 = X_4 - qX_2, \quad JX_4 = -X_3 - qX_1, \quad q \in \mathbb{R},$$

for which the Nijenhuis tensor vanishes. We will consider the metric $\langle \cdot, \cdot \rangle$ that makes orthonormal the basis

$$e_1 = X_1, \quad e_2 = X_2, \quad e_3 = X_3, \quad e_4 = X_4 - qX_2.$$

Such a metric is compatible with the complex structure J , then we are in the presence of a Hermitian structure (type \mathcal{W}_4 in the four-dimensional case). For the basis $\{e_1, e_2, e_3, e_4\}$, one has

$$[e_2, e_3] = -e_1, \quad [e_2, e_4] = -e_2, \quad [e_3, e_4] = -qe_1 + e_3,$$

and all other brackets vanish. The corresponding basis $\{e^1, e^2, e^3, e^4\}$ of left-invariant one-forms are such that

$$de^1 = e^{23} + qe^{34}, \quad de^2 = e^{24}, \quad de^3 = -e^{34}, \quad de^4 = 0.$$

The Hermitian metric is expressed as $\langle \cdot, \cdot \rangle = \sum_{i=1}^4 e^i \otimes e^i$ and the Kähler two-form as $\omega = e^{21} + e^{43}$. Its exterior derivative is given by $d\omega = (qe^2 - e^4) \wedge \omega$. Therefore, the Lee form is $\theta = qe^2 - e^4$. The non-zero components of $d\theta$ are given by

$$(d\theta)_{[\lambda_0^{1,1}]} = \frac{q}{2}(e^{13} + e^{24}), \quad (d\theta)_{[\lambda^{2,0}]} = \frac{q}{2}(-e^{13} + e^{24}).$$

Because $n = 2$, the identities given in Proposition 3.4 are trivial. For dimension 4, one has $\mathcal{W}_1 \oplus \mathcal{W}_4 = \mathcal{W}_4$ and, in this example, $d\theta_{[\lambda_0^{1,1}]} \neq 0$ and $(d\theta)_{[\lambda^{2,0}]} \neq 0$. Finally, we have

$$s - s^* = 2d^*\theta + \|\theta\|^2 = 1 + q^2, \quad (\text{Ric} - \text{Ric}^*)_{\mathbb{R}} = \frac{1+q^2}{4}\langle \cdot, \cdot \rangle, \quad (\text{Ric} - \text{Ric}^*)_{[\lambda_0^{1,1}]} = 0,$$

$$\text{Ric}^*_{[\lambda^{2,0}]} = \frac{1}{2}(d\theta)_{[\lambda^{2,0}]} = \frac{q}{4}(-e^{13} + e^{24}).$$

If $*$ is the Hodge star operator with respect to $\text{Vol} = -\frac{1}{2}\omega^2 = -e^{1234}$, $d^*\theta = -*d*\theta = 0$. Now, we fix the complex volume form $\psi_+ + i\psi_-$, where $\psi_+ = e^{13} - e^{24}$ and $\psi_- = e^{14} + e^{23}$. For this $SU(2)$ -structure, we have

$$4\eta - \theta = *(d\psi_+ \wedge \psi_+ + d\psi_- \wedge \psi_-) = -2e^4.$$

Then $\widehat{\eta} = J\eta = \frac{1}{4}(-qe^1 + 3e^3)$ and it is globally defined. Thus, $r_{\nabla U(2)} = -2d\widehat{\eta} = \frac{q}{2}qe^{23} + \frac{3+q^2}{2}e^{34}$ is exact. Hence, as it is expected by the existence of the $SU(2)$ -structure, the first Chern class c_1 vanishes. The Ricci form of the Chern connection is given by $r_{\nabla h} = r_{\nabla U(2)} + \frac{1}{2}dJ\theta = e^{34}$. By Koszul's formula, the Levi Civita covariant derivatives are derived

$$\begin{aligned} \nabla_{e_2}e_2 &= -\nabla_{e_3}e_3 = e_4, & \nabla_{e_1}e_2 &= \nabla_{e_2}e_1 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 &= \nabla_{e_3}e_1 = -\frac{1}{2}e_2 + \frac{1}{2}qe_4, \\ \nabla_{e_1}e_4 &= \nabla_{e_4}e_1 = -\frac{1}{2}qe_3, & \nabla_{e_2}e_3 &= -\nabla_{e_3}e_2 = -\frac{1}{2}e_1, & \nabla_{e_3}e_4 &= -\frac{q}{2}e_1 + e_3, & \nabla_{e_4}e_3 &= \frac{q}{2}e_1 \end{aligned}$$

and $\nabla_{e_i}e_j = 0$, for the remaining pairs (i, j) . These are needed in Proposition 4.4 to obtain the following component of the Ricci tensor

$$\text{Ric}_{[\sigma^{2,0}]} = \frac{1}{2}(e^1 \otimes e^1 - e^2 \otimes e^2 + e^3 \otimes e^3 - e^4 \otimes e^4).$$

Since $r_{\nabla U(2)[\lambda^{1,1}]} = \frac{3+q^2}{2}e^{34} + \frac{q}{4}(e^{23} - e^{14})$ and we have all the information required in Proposition 4.10 to compute the following Kähler component of the curvature

$$(\text{Ric} + 3\text{Ric}^*)_{[\lambda^{1,1}]} = \frac{-3+q^2}{4}(e^1 \otimes e^1 + e^2 \otimes e^2) - \frac{23+11q^2}{4}(e^3 \otimes e^3 + e^4 \otimes e^4).$$

As a consequence (or by Corollary 4.11), for the scalar curvatures it is obtained

$$s + 3s^* = -13 - 5q^2, \quad s = -\frac{5+q^2}{2}, \quad s^* = -\frac{7+3q^2}{2}.$$

Since one has the identities $\rho_{\nabla} = r_{\nabla} = -J_{(2)}\text{Ric}^*$, we obtain

$$\rho_{\nabla} = r_{\nabla} = \text{Ric}^*(\cdot, J\cdot) = \frac{1}{4}e^{12} + \frac{8+3q^2}{4}e^{34} + \frac{q}{4}(e^{14} + e^{23}).$$

This form is not closed, $d\rho_{\nabla} = \frac{q}{2}e^{234} - \frac{1}{4}e^{124}$. Now we use Proposition 4.6 (iii), to compute the first Ricci form of $\nabla^{U(2)}$ which is given by

$$\rho_{\nabla U(2)} = \frac{1-q^2}{8}e^{12} + \frac{5(3+q^2)}{8}e^{34}.$$

It is in $[\lambda^{1,1}]$ and its exterior derivative is $d\rho_{\nabla U(2)} = \frac{q(1-q^2)}{8}e^{234} - \frac{1-q^2}{8}e^{124}$. Likewise, by using Proposition 4.8 (ii), the first Ricci form of the Chern connection is given by

$$\rho_{\nabla h} = -\frac{1+q^2}{2}e^{12} + \frac{4+q^2}{2}e^{34} - \frac{q}{2}(e^{14} - e^{23}).$$

It is also in $[\lambda^{1,1}]$ and has non-zero exterior derivative, $d\rho_{\nabla h} = -\frac{q(2+q^2)}{2}e^{234} + \frac{1+q^2}{2}e^{124} \neq 0$.

Remark 5.3. An alternative way to see that our structure is complex is the one indicated in [12] for four-dimensional cases. In such cases one has $\nabla\omega = \xi_+ \otimes \psi_+ + \xi_- \otimes \psi_-$ and the structure is Hermitian (\mathcal{W}_4) if and only if $J\xi_+ = \xi_-$. The advantage is that the one-forms ξ_+ and ξ_- can be computed by means of exterior algebra using formulae in the mentioned reference. Thus $2\xi_+ = -e_1 - qe_3$ and $2\xi_- = -e_2 - qe_4$. In fact, $J\xi_+ = \xi_-$ in our case.

On other hand, Gauduchon in [5] claims that each conformal class of Hermitian metrics contains one metric, called *standard*, such that the corresponding Lee one-form θ is coclosed, $d^*\theta = 0$. This is the case in the present example.

Example 5.4. Let \mathfrak{g} be the Lie algebra with structure equations

$$de^i = 0, \quad 1 \leq i \leq 4, \quad de^5 = e^{12}, \quad de^6 = e^{14} + e^{23}.$$

This Lie algebra has been included in the list, given by Salamon in [14], of real 6-dimensional nilpotent Lie algebras for which the corresponding Lie group G has a left-invariant complex structure. Because the nilpotent Lie group G has rational structure constants, there is a discrete subgroup Γ such that $M = \Gamma \backslash G$ is a compact manifold [11].

Table 1
Levi Civita connection ∇ .

∇	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	$-\frac{1}{2}e_5$	0	$-\frac{1}{2}e_6$	$\frac{1}{2}e_2$	$\frac{1}{2}e_4$
e_2	$\frac{1}{2}e_5$	0	$-\frac{1}{2}e_6$	0	$-\frac{1}{2}e_1$	$\frac{1}{2}e_3$
e_3	0	$\frac{1}{2}e_6$	0	0	0	$-\frac{1}{2}e_2$
e_4	$\frac{1}{2}e_6$	0	0	0	0	$-\frac{1}{2}e_1$
e_5	$\frac{1}{2}e_2$	$-\frac{1}{2}e_1$	0	0	0	0
e_6	$\frac{1}{2}e_4$	$\frac{1}{2}e_3$	$-\frac{1}{2}e_2$	$-\frac{1}{2}e_1$	0	0

On M we consider the almost Hermitian structure such that the metric is the left-invariant on defined by $\langle \cdot, \cdot \rangle = \sum_{i=1}^6 e_i \otimes e_i$ and its Kähler form is given by

$$\omega = e^6 \wedge e^5 + (-\frac{1}{2}e^3 + \frac{\sqrt{3}}{2}e^4) \wedge e^1 + (\frac{1}{2}e^4 + \frac{\sqrt{3}}{2}e^3) \wedge e^2.$$

In [1] it was shown that this structure is Hermitian (see page 162 of [14], where $\omega = -c(\frac{2\pi}{3})$ because of our notations). The exterior derivative is given by $d\omega = e^{145} + e^{235} - e^{126}$. The corresponding Lee form is $\theta = -\frac{1}{2}Jd^*\omega = \frac{1}{2}\langle \cdot, \lrcorner d\omega \rangle = -\frac{\sqrt{3}}{2}e^5$. Then the exterior derivative of θ is expressed by $d\theta = -\frac{\sqrt{3}}{2}e^{12}$. Then the $U(3)$ -components of this two-form are given by

$$(d\theta)_{[\lambda_0^{1,1}]} = -\frac{\sqrt{3}}{4}(e^{12} - e^{34}), \quad (d\theta)_{[\lambda^{2,0}]} = -\frac{\sqrt{3}}{4}(e^{12} + e^{34}).$$

In order to see how the identities of Proposition 3.4 work out in this example, we will compute the intrinsic torsion ξ . In this case, because $N = 0$, it is given by $4\xi = (J_{(2)} + J_{(3)})d\omega$. Since components of the exterior derivative $d\omega$ are expressed by

$$\begin{aligned} (d\omega)_{W_4} &= \theta \wedge \omega = -\frac{\sqrt{3}}{4}e^{135} + \frac{3}{4}e^{145} + \frac{3}{4}e^{235} + \frac{\sqrt{3}}{4}e^{245}, \\ (d\omega)_{W_3} &= d\omega - \theta \wedge \omega = -e^{612} + \frac{\sqrt{3}}{4}e^{135} + \frac{1}{4}e^{145} + \frac{1}{4}e^{235} - \frac{\sqrt{3}}{4}e^{245}, \end{aligned}$$

the corresponding components of ξ are given by

$$\begin{aligned} 16\xi_{(4)} &= -2\sqrt{3}e^1 \otimes e^{15} - 2\sqrt{3}e^2 \otimes e^{25} - 2\sqrt{3}e^3 \otimes e^{36} - 2\sqrt{3}e^4 \otimes e^{45} \\ &\quad - \sqrt{3}e^1 \otimes e^{36} + 3e^1 \otimes e^{46} + 3e^2 \otimes e^{36} + \sqrt{3}e^2 \otimes e^{46} \\ &\quad + \sqrt{3}e^3 \otimes e^{16} - 3e^3 \otimes e^{26} - 3e^4 \otimes e^{16} - \sqrt{3}e^4 \otimes e^{26}, \\ 16\xi_{(3)} &= +2e^1 \otimes e^{25} - \sqrt{3}e^1 \otimes e^{36} - e^1 \otimes e^{46} - 2e^2 \otimes e^{15} + \sqrt{3}e^2 \otimes e^{46} - e^2 \otimes e^{36} \\ &\quad - \sqrt{3}e^3 \otimes e^{16} - e^3 \otimes e^{26} - 2e^3 \otimes e^{45} - e^4 \otimes e^{16} + \sqrt{3}e^4 \otimes e^{26} + 2e^4 \otimes e^{35} \\ &\quad - 4e^5 \otimes e^{12} - 4e^5 \otimes e^{34} - 2\sqrt{3}e^6 \otimes e^{13} - 2e^6 \otimes e^{14} - 2e^6 \otimes e^{23} + 2\sqrt{3}e^6 \otimes e^{24}. \end{aligned}$$

Now doing $\xi = \xi_{(3)} + \xi_{(4)}$, we obtain

$$\begin{aligned} 8\xi &= -\sqrt{3}e^1 \otimes e^{15} - \sqrt{3}e^2 \otimes e^{25} - \sqrt{3}e^3 \otimes e^{35} - \sqrt{3}e^4 \otimes e^{45} \\ &\quad + e^1 \otimes e^{25} - \sqrt{3}e^1 \otimes e^{36} + e^1 \otimes e^{46} - e^2 \otimes e^{15} + e^2 \otimes e^{36} + \sqrt{3}e^2 \otimes e^{46} \\ &\quad - 2e^3 \otimes e^{26} - e^3 \otimes e^{45} - 2e^4 \otimes e^{16} + e^4 \otimes e^{35} - 2e^5 \otimes e^{12} - 2e^5 \otimes e^{34} \\ &\quad - \sqrt{3}e^6 \otimes e^{13} - e^6 \otimes e^{14} - e^6 \otimes e^{23} + \sqrt{3}e^6 \otimes e^{24}. \end{aligned}$$

From the non-zero Lie brackets: $[e_1, e_2] = -e_5$ and $[e_1, e_4] = [e_2, e_3] = -e_6$, using Kozsul's formula, the Levi Civita connection is computed and given by Table 1. The minimal connection is now obtained as $\nabla^{U(3)} = \nabla + \xi$.

Finally, from all of this, it is straightforward to compute

$$\begin{aligned} 16\langle (\nabla_{e_i}^{U(3)}\xi_{(3)})_X Y, e_i \rangle &= \left(\frac{5}{2} \sum_{i=1}^4 e^i \otimes e^i - 5 \sum_{i=5}^6 e^i \otimes e^i + \frac{\sqrt{3}}{2}(e^{12} - e^{34}) \right) (X, Y), \\ \langle \xi_{(3)X} Y - \xi_{(3)Y} X, \xi_{(4)e_i} e_i \rangle &= -\frac{\sqrt{3}}{8}(e^{12} - e^{34})(X, Y), \\ \langle (\nabla_{e_i}^{U(3)}\xi_{(3)})_{e_i} X, Y \rangle &= 0, \end{aligned}$$

$$\langle \xi_{(3)\xi_{(4)}e_i} X, Y \rangle = \frac{\sqrt{3}}{8}(e^{12} + e^{34})(X, Y).$$

Hence one can see in this example how the identities given in Lemma 3.1 are satisfied.

Now we use the identities (4.2) and (4.3) to compute

$$s - s^* = 3, \quad (\text{Ric} - \text{Ric}^*)_{\mathbb{R}} = \frac{1}{2} \langle \cdot, \cdot \rangle, \quad (\text{Ric} - \text{Ric}^*)_{[\lambda_0^{1,1}]} = -\frac{1}{4} \sum_{i=1}^4 e^i \otimes e^i + \frac{1}{2} \sum_{i=5}^6 e^i \otimes e^i.$$

Propositions 4.3 and 4.4 are used to obtain

$$\text{Ric}^*_{\llbracket \lambda^{2,0} \rrbracket} = (d\theta)_{\llbracket \lambda^{2,0} \rrbracket} = -\frac{\sqrt{3}}{4}(e^{12} + e^{34}), \quad \text{Ric}_{\llbracket \sigma^{2,0} \rrbracket} = 0.$$

Next we fix $\Psi = \psi_+ + i\psi_-$, where

$$\psi_+ = e^{125} + e^{345} - \frac{1}{2}(e^{146} + e^{236}) + \frac{\sqrt{3}}{2}(e^{246} - e^{136}),$$

as complex volume form. Since $d\psi_+ = 0$, then $\eta = \frac{1}{6} * (d\psi_+ \wedge \psi_+) + \frac{1}{3}\theta = -\frac{\sqrt{3}}{6}e^5$. Hence $\hat{\eta} = J\eta = -\frac{\sqrt{3}}{6}e^6$, $r_{\nabla U(3)} = -3d\hat{\eta} = \frac{\sqrt{3}}{2}(e^{14} + e^{23})$ and $r_{\nabla h} = 0$. Therefore, $r_{\nabla U(3)[\lambda^{1,1}]}(X, JY) = \frac{3}{4} \sum_{i=1}^4 e^i \otimes e^i$. This with $16 \langle \xi_{(3)X} e_i, \xi_{(3)Y} e_i \rangle = \sum_{i=1}^4 e^i \otimes e^i$ and the above considerations complete the ingredients needed to compute, by Proposition 4.10,

$$(\text{Ric} + 3\text{Ric}^*)_{\mathbb{R}} = -\frac{11}{6} \langle \cdot, \cdot \rangle, \quad (\text{Ric} + 3\text{Ric}^*)_{[\lambda_0^{1,1}]} = -\frac{17}{12} \sum_{i=1}^4 e^i \otimes e^i + \frac{17}{6} \sum_{i=5}^6 e^i \otimes e^i.$$

Since we have already $s - s^* = 3$, it follows

$$s + 3s^* = -11, \quad s = -\frac{1}{2}, \quad s^* = -\frac{7}{2}.$$

From the fact, $\rho_{\nabla} = r_{\nabla} = -J_{(2)} \text{Ric}^*$, we obtain

$$\rho_{\nabla[\lambda^{1,1}]} = \frac{1}{4}e^{65} - \frac{1}{4}\omega, \quad \rho_{\nabla\llbracket \lambda^{2,0} \rrbracket} = \frac{\sqrt{3}}{2}(e^{14} + e^{23}).$$

Note that ρ_{∇} is closed. Finally, for sake of completeness, we use Proposition 4.6 (iii) to compute the first Ricci form of $\nabla^{U(3)}$ which is given by $\rho_{\nabla U(3)} = e^{65} - \frac{3}{4}\omega$. This Ricci form is in $[\lambda^{1,1}]$ but it is not closed. In fact, $d\rho_{\nabla U(3)} = \frac{1}{4}(e^{145} + e^{235} - e^{126}) = \frac{1}{4}d\omega \neq 0$. Likewise, by using Proposition 4.8 (ii), the first Ricci form of the Chern connection is obtained, $\rho_{\nabla h} = \frac{1}{2}e^{65} - \frac{1}{2}\omega$. This form $\rho_{\nabla h}$ is also in $[\lambda^{1,1}]$ and is closed, $d\rho_{\nabla h} = 0$.

Remark 5.5. By Proposition 3.6, for an almost Hermitian manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ with dimension $2n > 4$, the $[\lambda_0^{1,1}]$ component of $d\theta$ vanishes. It would interesting to find an example of the mentioned type with non-vanishing $\llbracket \lambda^{2,0} \rrbracket$ component of $d\theta$. Such an example must be of dimension $2n > 6$. Note that the type $\mathcal{W}_1 \oplus \mathcal{W}_4$ is specially rigid.

References

- [1] E. Abbena, A. Garbiero, S. Salamon, Almost Hermitian geometry on six dimensional nilmanifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 30 (1) (2001) 147–170.
- [2] R. Cleyton, A.F. Swann, Einstein metrics via intrinsic or parallel torsion, *Math. Z.* 247 (3) (2004) 513–528.
- [3] W.A. de Graaf, Classification of solvable Lie algebras, *Exp. Math.* 14 (1) (2005) 15–25.
- [4] M. Falcitelli, A. Farinola, S.M. Salamon, Almost-Hermitian geometry, *Differential Geom. Appl.* 4 (1994) 259–282.
- [5] P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte, *Math. Ann.* 267 (4) (1984) 495–518.
- [6] J.C. González Dávila, F. Martín Cabrera, Harmonic G-structures, *Math. Proc. Cambridge Phil. Soc.* 146 (2) (2009) 435–459, [arXiv:math.DG/0706.0116](https://arxiv.org/abs/math.DG/0706.0116).
- [7] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl. (4)* 123 (1980) 35–58.
- [8] K. Hasegawa, Complex and Kähler structures on compact solvmanifolds, *Conference on Symplectic Topology, J. Symplectic Geom.* 3 (4) (2005) 749–767.
- [9] M. Inoue, On surfaces of Class VII₀, *Invent. Math.* 24 (1974) 269–310.
- [10] H.C. Lee, A kind of even-dimensional differential geometry and its application to exterior calculus, *Amer. J. Math.* 65 (1943) 433–438.
- [11] A.I. Malcev, On a class of homogeneous spaces, *Amer. Math. Soc. Transl. Ser. 1* 9 (1962) 276–307, reprinted in.
- [12] F. Martín Cabrera, Special almost Hermitian geometry, *J. Geom. Phys.* 55 (4) (2005) 450–470.
- [13] F. Martín Cabrera, A. Swann, Curvature of special almost Hermitian manifolds, *Pacific J. Math.* 228 (1) (2006) 165–184.
- [14] S. Salamon, Complex structures on nilpotent Lie algebras, *J. Pure Appl. Algebra* 157 (2–3) (2001) 311–333.
- [15] F. Tricerri, L. Vanhecke, Curvature tensors on almost Hermitian manifolds, *Trans. Amer. Math. Soc.* 267 (1981) 365–398.
- [16] I. Vaisman, Some curvature properties of locally conformal Kähler manifolds, *Trans. Amer. Math. Soc.* 259 (1980) 439–447.