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LT-equivariant index from the viewpoint of KK-theory
A global analysis on the infinite-dimensional Heisenberg group

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ABSTRACT

Let T be a circle group, and LT be its loop group. We hope to establish an LT-equivariant index theory for Hamiltonian LT-spaces, from the viewpoint of KK-theory. We have already constructed several objects in the previous paper (Takata, arXiv:1701.06055), including a Hilbert space $\mathcal{H}$ which can be regarded as a "Hilbert space consisting of L^2-sections of a Spinor bundle on the infinite-dimensional manifold", an "LT-equivariant Dirac operator $\mathcal{D}$" acting on $\mathcal{H}$, and the "twisted group C^*-algebra of LT", in an ad hoc way.

In this paper, we will construct more sophisticated objects for the index problem, in terms of KK-theory. Concretely, we will define analytic index of $\mathcal{D}$ as a K-theory element of the "twisted group C^*-algebra of LT" mentioned above. Then, we will also define the "latter half" of the assembly map at the "index element" $(\mathcal{H}, \mathcal{D})$. Finally, we will prove that the value of the "latter half" of the assembly map at the "index element", coincides with the above analytic index of $\mathcal{D}$. This is an infinite-dimensional version of a part of the index theorem for complete Riemannian manifolds with proper cocompact actions of Kasparov.

We are mainly interested in the LT-equivariant index problem of infinite-dimensional manifolds, and so we use the language of representation of LT. However, our theory can be easily generalized to the setting of infinite-dimensional abstract Heisenberg groups equipped with some additional structures. This is potentially useful for researchers of other areas.

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1. Introduction

The overall goal of our research is the formulation and proof of an infinite-dimensional version of the Atiyah–Singer index theorem [3–5]. There have been several approaches to "index theories" for infinite-dimensional spaces, for example a formal argument [39] using the fixed-point formula of [3], a representation-theoretical approach [11,25,33] using the Peter–Weyl theorem, a probability-theoretical construction [15,16] by mathematical physicists, and so on. In our opinion, noncommutative geometry (NCG for short) offers very promising new approaches to this project. This is because NCG is a powerful tool for various generalizations of index theories [8,9,19–21]. In particular, KK-theory plays an essential role.

When we try to construct an index theory for infinite-dimensional spaces in terms of NCG, we soon encounter a very hard problem: There are no commutative C^*-algebras corresponding to infinite-dimensional manifolds. This is

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a consequence of a well-known theorem of Gelfand–Naimark: *Commutative C^*-algebras correspond to locally compact Hausdorff spaces [30]*. In this sense, **infinite-dimensional manifolds are non-commutative**.

If we could define a noncommutative C^*-algebra which plays a role of "function algebra of an infinite-dimensional manifold", we could formulate an index theorem as an equality in KK-theory of this C^*-algebra. Once such a satisfying theory was established, several related KK-elements would be derived, by the usual machinery of the index theory.

In the present paper, however, we will construct such related objects (in advance of such "function algebras"), which should be derived from the actual index theory in terms of the KK-theory of such "function algebras". In this sense, this paper will play a role of a guideline of such index theory. If we could establish an index theory using "function algebras of infinite-dimensional manifolds", the objects constructed in the present paper would be naturally derived.[1]

In this paper, we investigate an infinite-dimensional version of the following Kasparov index theorem for complete Riemannian manifold with a proper cocompact group action.

Let us explain the precise statement of it. Let Γ be a locally compact, second countable Hausdorff group (e.g. a finite-dimensional Lie group). We suppose that Γ is unimodular and amenable for simplicity (e.g. an abelian group, for details on amenability, see [31]). Let X be a finite-dimensional complete Riemannian manifold, equipped with an isometric proper cocompact Γ-action. Assume that a Γ-equivariant Clifford bundle $W \to X$ and a Γ-equivariant Dirac operator $D : L^2(X, W) \circlearrowleft$ are given. In this situation, it is natural to hope to consider the index problem of D. This operator can be non-Fredholm when X is non-compact, but the following makes sense even if X is non-compact.

Definition 1.1 (*[21]*)**.** The dense subspace $C_c(X, W) \subseteq L^2(X, W)$ has a pre-Hilbert $C_c(\Gamma)$-module structure given by the followings:

- For $s_1, s_2 \in C_c(X, W)$,
$$\left(s_1 \middle| s_2\right)(\gamma) := \int_X \left(s_1(x) \middle| \gamma.(s_2(\gamma^{-1}.x))\right)_W dx.$$
- For $s \in C_c(X, W)$ and $b \in C_c(\Gamma)$,
$$s * b(x) := \int_\Gamma \gamma.s(\gamma^{-1}.x) b(\gamma^{-1}) d\gamma.$$

The analytic index $\mathrm{ind}_{\Gamma \ltimes \mathbb{C}}(D)$ is defined by the completion of $C_c(X, W)$ with respect to the above inner product, where $\Gamma \ltimes \mathbb{C}$ denotes the group C^*-algebra of Γ.

We have defined a pair of a Hilbert $\Gamma \ltimes \mathbb{C}$-module and an operator. The Kasparov index theorem states that this pair is actually a Kasparov module.

In order to accept this definition as the analytic index, it is convenient to recall the classification of homotopy classes of Fredholm operators. For a separable Hilbert space H, the set of connected components of Fredholm operators is bijective to $\mathbb{Z}$ by the Fredholm index. Therefore, the Fredholm index of an operator T can be identified with the homotopy class of T.

There is an alternative way to define an element of $KK(\mathbb{C}, \Gamma \ltimes \mathbb{C})$ from the Dirac operator. Firstly, we recall that the pair $(L^2(X, W), D)$ defines an element of $KK_\Gamma(C_0(X), \mathbb{C})$ by the usual representation $C_0(X) \to \mathcal{B}(L^2(X, W))$ given by the multiplication operators. This KK-element is called the **index element**, and denoted by $[D]$. If we have a homomorphism from $KK_\Gamma(C_0(X), \mathbb{C})$ to $KK(\mathbb{C}, \Gamma \ltimes \mathbb{C})$, we can define a new element of $KK(\mathbb{C}, \Gamma \ltimes \mathbb{C})$ by the value of this homomorphism at the index element. In fact, the following homomorphism is the "alternative way", which is called the **assembly map** in the present paper. See [20,21] and Section 2.2 for details.

Definition 1.2. The assembly map
$$\mu^\Gamma : KK_\Gamma(C_0(X), \mathbb{C}) \to KK(\mathbb{C}, \Gamma \ltimes \mathbb{C})$$
is given by the composition of the "descent homomorphism" $j^\Gamma : KK_\Gamma(C_0(X), \mathbb{C}) \to KK(\Gamma \ltimes C_0(X), \Gamma \ltimes \mathbb{C})$ and the Kasparov product with the "Mishchenko line bundle" $[c] \in KK(\mathbb{C}, \Gamma \ltimes C_0(X))$, where $\Gamma \ltimes C_0(X)$ denotes the crossed product.

We can also define the following KK-elements: the "Clifford symbol element" $[\sigma_D]$ from topological data of the symbol of D, and the "Dirac element" $[d_X]$ from the geometrical data of X, which is independent of D itself (these KK-elements will not appear in the body of this paper, and we omit the details). In this situation, the following holds.

[1] In fact, a "function algebra of an infinite-dimensional Hilbert space" has been constructed in [14]. Then, it was generalized to more general manifolds in [10,12]. Such C^*-algebras must be essential for the index theory of infinite-dimensional manifolds. In fact, we have studied more KK-theoretical index theory in [34] after the present paper. We have found a weak connection between the result of the present paper and [34]. However, it is not completely satisfying.

In the present paper, we will concentrate on the "related objects".

Fact 1.3 ([19–21]). (1) The analytic index $\mathrm{ind}_{\Gamma\ltimes\mathbb{C}}(D)$ is indeed a Kasparov module, that is, D is "$\Gamma\ltimes\mathbb{C}$-Fredholm".

(2) The analytic index $\mathrm{ind}_{\Gamma\ltimes\mathbb{C}}(D)$ coincides with the value of the assembly map at the index element $[D]$, that is, $\mathrm{ind}_{\Gamma\ltimes\mathbb{C}}(D)=\mu^{\Gamma}([D])$.

(3) The equality $[D]=[\sigma_D]\otimes[d_X]$ holds. Therefore, the index element is completely determined by the topological data of the symbol of D.

Consequently, the analytic index is determined by topological data of the symbol of D.

As we have declared, our goal is to establish an infinite-dimensional version of the above Kasparov index theorem. For this goal, it is natural to begin with some special cases. In this paper, we try to replace Γ by the loop group LT of the circle group T, and replace X by a "proper LT-space" which is an obvious generalization of Hamiltonian LT-space. More precisely, we study the index theory for the following situation (see also [1,29] and [33] for example, for details on Hamiltonian loop group spaces).

Assumption 1.4. Let T be a circle group, and $LT:=C^\infty(S^1,T)$ be its loop group. It acts on the dual Loop algebra $L\mathfrak{t}^*$, via the gauge action $l.A:=A+l^{-1}dl$.

Let $\mathcal{M}$ be an infinite-dimensional manifold equipped with a smooth LT-action, a proper equivariant map $\Phi:\mathcal{M}\to L\mathfrak{t}^*$. Such manifold is called a **proper LT-space**.

Suppose that an admissible positive definite $U(1)$-central extension LT^τ of LT is given (see [11]). We suppose that $\mathcal{M}$ admits a τ-twisted LT-equivariant line bundle $\mathcal{L}\to\mathcal{M}$, where "$\tau$-twisted LT-equivariant" means that the restriction of the LT^τ-action on $\mathcal{L}$ to the added center $U(1)$ is given by the scalar multiplication.[2]

We further assume that $\mathcal{M}$ is "even-dimensional and $Spin^c$" in the sense of Section 4.1.

Although we have not completely established the index theory for this situation, in the present paper, we report our achievement so far.

In order to specify the problem, let us list necessary objects for the Kasparov index theorem for the classical cases: $C_0(X)$, $L^2(X,W)$, a nice dense subspace of it, a Dirac operator D, $\Gamma\ltimes\mathbb{C}$, a nice dense subspace of it, analytic index $\mathrm{ind}_{\Gamma\ltimes\mathbb{C}}(D)$, the index element $[D]$, $\Gamma\ltimes C_0(X)$, j^Γ, $[c]$, and the assembly map μ^Γ. If we can construct infinite-dimensional version of all of them, and if we can prove the parallel equalities, it will be called the *infinite-dimensional index theorem* for this case.

Before the explanation of the result of the present paper, let us write down the necessary result of the previous paper which we will use, in the notation of the present paper: *A substituting object is denoted with an underline*. We will summarize (potentially confusing) notations for infinite-dimensional objects in Section 5, especially all the underlined symbols.

Theorem 1.5 ([36]). *In the situation of Assumption 1.4, we can construct an LT-equivariant Spinor bundle $\mathcal{S}\to\mathcal{M}$ using [11], and we can construct the followings:*

- *A Hilbert space which plays a role of "$L^2(\mathcal{M},\mathcal{L}\otimes\mathcal{S})$", which is denoted by $\underline{L^2(\mathcal{M},\mathcal{L}\otimes\mathcal{S})}$.*
- *A "Dirac operator" $\mathcal{D}$ as an operator on $\underline{L^2(\mathcal{M},\mathcal{L}\otimes\mathcal{S})}$.*
- *A C^*-algebra which plays a role of the "τ-twisted group C^*-algebra of LT", which is denoted by $\underline{LT\ltimes_\tau\mathbb{C}}$.*[3]

We regard the pair $\underline{[\mathcal{D}]}:=(\underline{L^2(\mathcal{M},\mathcal{L}\otimes\mathcal{S})},\mathcal{D})$ as the "index element". This "Kasparov module" does not make sense, because we have not defined a C^*-algebra which plays a role of "$C_0(\mathcal{M})$".

However, we will define *actual* C^*-algebras and *actual* Kasparov modules which are related to this "index element". Concretely, we will prove the following result (The terminology "partial descent homomorphism" will be introduced in Section 2.3).

Theorem 1.6 (*Main Result*). *In the situation of Assumption 1.4, we can construct the followings:*

- *A Kasparov $(\mathbb{C},\underline{LT\ltimes_\tau\mathbb{C}})$-module which can be regarded as the "analytic index of $\mathcal{D}$", and which is denoted by $\underline{\mathrm{ind}_{LT\ltimes_\tau\mathbb{C}}(\mathcal{D})}$.*
- *A C^*-algebra which plays a role of "$LT\ltimes C_0(\mathcal{M})$", and it is denoted by $\underline{LT\ltimes C_0(\mathcal{M})}$.*
- *A Kasparov $(\underline{LT\ltimes C_0(\mathcal{M})},\underline{LT\ltimes_\tau\mathbb{C}})$-module which plays a role of the "image of the partial descent homomorphism at $\underline{[\mathcal{D}]}$", which is denoted by $\underline{j_\tau^{LT}([\mathcal{D}])}$.*
- *A Kasparov $(\mathbb{C},\underline{LT\ltimes C_0(\mathcal{M})})$-module which plays a role of the Mishchenko line bundle for $LT\circlearrowleft\mathcal{M}$, which is denoted by $\underline{[c_{\mathcal{M}}]}$.*

These objects enable us to define the "image of the assembly map at the index element" $\underline{\mu_\tau^{LT}([\mathcal{D}])}$ by the Kasparov product $\underline{[c_{\mathcal{M}}]}\otimes\underline{j_\tau^{LT}([\mathcal{D}])}$. This Kasparov product coincides with $\underline{\mathrm{ind}_{LT\ltimes_\tau\mathbb{C}}(\mathcal{D})}$ in $KK(\mathbb{C},\underline{LT\ltimes_\tau\mathbb{C}})$.

[2] The reason why a central extension of LT is required, is on the representation theory of loop groups. An extensively studied class of loop groups is **positive energy representations**. Such representations enjoy nice analytic properties, but, unfortunately, they are projective.

[3] The definition of twisted crossed products will be explained in Section 2.3.

Remark 1.7. This result is proved by factorizing the manifold into two parts: the locally compact part and a specific infinite-dimensional Heisenberg group part just like [36]. The essential part is the latter one, and the result on it can be easily generalized to more general infinite-dimensional Heisenberg group.

Precisely, if one has an infinite-dimensional S^1-invariant symplectic vector space (V, ω) with the infinitesimal S^1-action $d : V \circlearrowleft$ whose spectrum is discrete and bounded below, he can apply the argument of Section 4 in the present paper. The central extension associated to the cocycle $(v, w) \mapsto e^{i\omega(v,w)}$ is the infinite-dimensional Heisenberg group appearing in the subtitle of the present paper.

Such results and the techniques invented in the present paper, would be useful for researchers of other areas including the Higson–Kasparov theory or the Baum–Connes theory.

Now, let us explain the novelty of this result, comparing with some previous researches on infinite-dimensional index theory. Such previous researches are very useful for the construction of the "L^2-space" in the previous paper [36], but they are not enough for $\underline{j_\tau^{LT}([\mathcal{D}])}$.

Firstly, we begin with the representation theoretical approach. Notice that infinite-dimensional manifolds have no good measures, unlike finite-dimensional Riemannian manifolds. Consequently, we cannot define "L^2-spaces" on infinite-dimensional manifolds, in the usual way. However, according to the Peter–Weyl theorem, the L^2-space of a compact group can be completely described in the language of representation theory: $L^2(G) \cong \bigoplus_{\lambda \in \widehat{G}} V_\lambda \otimes V_\lambda^*$, where $\widehat{G}$ is the set of isomorphism classes of irreducible representations of G. Imitating this result, we can define an "algebraic L^2-space" for an arbitrary group by the right hand side of this formula. When we apply this construction for a loop group LG, the resulting Hilbert space is called the **Wess–Zumino–Witten model** (WZW model, for short). Since the infinitesimal representations correspond to the direction derivatives, we can also consider "differential operators" on this "algebraic L^2-space". Following this idea, "algebraic Dirac operators" have been studied in, for example, [18,22,26] and [37]. Then, Freed, Hopkins and Teleman used the "Dirac operator on a loop group"in order to construct an isomorphism between the Verlinde algebra of LG and the twisted equivariant K-theory of G, and then, Y. Song used their Dirac operator and the WZW model to construct a Hilbert space associated to a Hamiltonian LG-space. Our research is heavily motivated by this work by Song, and our Dirac operator is a special case of his Dirac operator.

Their "Dirac operator" is an infinite sum of unbounded operators, which is very hard to deal with. Then, why is this well-defined? The answer of [11] is quite clear: The infinite sum is in fact a finite sum vectorwisely, for a representation theoretical reason. The following metaphor completely explains this situation: *Consider the polynomial algebra with infinitely many variables* $\mathbb{C}[X_1, X_2, \ldots]$. *The Euler operator* $E := \sum X_k \frac{\partial}{\partial X_k}$ *is clearly an infinite sum. However, for an arbitrary* $f \in \mathbb{C}[X_1, X_2, \ldots]$, *the infinite sum* Ef *is actually a finite sum because* $\frac{\partial f}{\partial X_k} = 0$ *except for finitely many* k*'s.* We call such an argument an **algebraic technique**.

There is another example of algebraic techniques. In [14], Higson, Kasparov and Trout defined a Hilbert space which plays a role of the "L^2-space of a separable Hilbert space" using an inductive limit construction. Intuitively, an "L^2-function" of theirs is a "Gaussian up to a finite-dimensional sub space". Since the annihilation operators kill the Gaussian, they successfully defined the "Bott-Dirac operator" on this Hilbert space.

Note that a Bott-Dirac operator is, in a word, a "Dirac operator with a linear potential". For example, on a real line, the Bott-Dirac operator is defined by

$$D := c \otimes \frac{d}{dx} + \hat{c} \otimes x\times$$

as an operator on $[Cl(\mathbb{R}) \otimes \mathbb{C}] \otimes L^2(\mathbb{R})$, where c and $\hat{c}$ are operators on $Cl(\mathbb{R}) \otimes \mathbb{C}$, and $x\times$ is the multiplication operator by the function $x \mapsto x$. Thus, D is related to the annihilator, and hence the Gaussian naturally plays a special role.

These algebraic approaches are quite useful for infinite-dimensional situations. In fact, we have used this technique in the previous paper [36]. However, it is not enough for the construction of $\underline{j_\tau^{LT}([\mathcal{D}])}$. This is because we will encounter a situation where we would like to define a "Dirac operator without a potential". In this situation, there are no special functions like the Gaussian, because the direction derivatives are injective on $L^2(\mathbb{R}^N)$. Therefore, **we cannot use the algebraic technique**.

In the present paper, we will introduce a new technique to overcome this problem. Briefly speaking, we will consider a quantitative version of the algebraic technique: We will consider infinite tensor products of "functions tending to a constant function".

In this version, whether the infinite sum converges or not, is always highly non-trivial, and quantitative arguments are always required. We believe that this is natural to everyone, because we are studying infinite-dimensional spaces. In addition, we are convinced that this quantitative version is essential for the construction of the infinite-dimensional global analysis. In fact, such arguments appeared in [34] to study more detailed KK-elements.

Let us give several notational remarks.

Firstly, we will use the $\mathbb{Z}_2$-graded language:

- On a $\mathbb{Z}_2$-graded vector space (Hilbert space, vector bundle, C^*-algebra, a set of operators and so on) V, $\partial v \in \mathbb{Z}_2$ denotes the grading of a homogeneous element $v \in V$.

- An operator on a $\mathbb{Z}_2$-graded vector space is said to be *even* if it is grading preserving, and said to be *odd* if it is grading reversing.
- All the commutators are in the graded sense: $[F_0, F_1] \coloneqq F_0F_1 - (-1)^{\partial F_0 \partial F_1} F_1F_0$ for two homogeneous operators F_0 and F_1 on a $\mathbb{Z}_2$-graded vector space. For example, for odd operators D_0, D_1 on a $\mathbb{Z}_2$-graded Hilbert space, the commutator is defined by $D_0D_1 + D_1D_0$.
- All the tensor products of $\mathbb{Z}_2$-graded objects are taken in the graded sense. For example, given two operators $T_i : V_i \circlearrowleft$ on $\mathbb{Z}_2$-graded vector spaces (for $i = 1, 2$), the tensor product operator $T_1 \otimes T_2$ is defined by

$$T_1 \otimes T_2(v_1 \otimes v_2) \coloneqq (-1)^{\partial T_2 \partial v_1} T_1(v_1) \otimes T_2(v_2).$$

Secondly, we introduce notations $\otimes_1$ and $\otimes_2$ as follows: For three vector spaces V_1, V_2 and V_3, and operators $T : V_1 \otimes V_2 \circlearrowleft$ and $S : V_2 \otimes V_3 \circlearrowleft$, we define $T \otimes_2 \mathrm{id}$ and $\mathrm{id} \otimes_1 S$ by the obvious formula $T \otimes_2 \mathrm{id}(v_1 \otimes v_2 \otimes v_3) \coloneqq T(v_1 \otimes v_2) \otimes v_3$ and $\mathrm{id} \otimes_1 S(v_1 \otimes v_2 \otimes v_3) \coloneqq v_1 \otimes S(v_2 \otimes v_3)$.

Thirdly, we will use the both of bilinear forms and inner products (which are linear in the **second** variable, and anti-linear in the first variable). Bilinear forms are denoted by angle brackets $\langle \bullet, \bullet \rangle$ or $\langle\langle \bullet, \bullet \rangle\rangle$, and inner products are denoted by parentheses $\left(\bullet \middle| \bullet\right)$.

Finally, for the convenience, we introduce a notation for substituting objects (we have used to explain the main theorem). In this paper, something that substitutes some "virtual" objects is denoted by the symbol for the virtual object with an underline. For example, for a Fredholm operator $F : X_1 \to X_2$ between two Banach spaces, the index of F is defined by $\dim(\ker(F)) - \dim(\mathrm{coker}(F))$, but we allow to write it as $\underline{\dim(X_1) - \dim(X_2)}$. In the notation $\underline{\star}$, $\star$ itself probably does not make sense (e.g. $\dim(X_1) - \dim(X_2)$ does not make sense if X_1 and X_2 are infinite-dimensional), but the substituting object $\underline{\star}$ must make sense (e.g. $\dim(\ker(F)) - \dim(\mathrm{coker}(F))$ does make sense even if X_1 and X_2 are infinite-dimensional, if F is Fredholm). For example, in the expression $\underline{LT \ltimes_\tau \mathbb{C}}$, $LT \ltimes_\tau \mathbb{C}$ does not make sense since LT is infinite-dimensional, but $\underline{LT \ltimes_\tau \mathbb{C}}$ must make sense as a C^*-algebra. We will summarize definitions of all the underlined symbols in Section 5.

In addition, $\underline{\underline{[\mathcal{D}]}}$ and $\underline{\underline{[\not{\partial}_R]}}$ are just pairs of Hilbert spaces and operators. They are candidates of "K-homology elements of a functions algebra". However, such concepts do not make sense, because we have not constructed a C^*-algebras which play roles of function algebras of infinite-dimensional manifolds. In this sense, these pairs are "virtual K-homology elements". Thus we write them with double underlines. We seldom use this type of notations.

This paper is organized as follows.

In Section 2, we will prepare several KK-theoretical matters. The first subsection is just a collection of definitions and well-known results on KK-theory in the unbounded picture. In the second subsection, we prove the product formula of index maps in the language of unbounded Kasparov modules. In the third subsection we study the assembly maps for twisted equivariant vector bundles. For a group Γ, a central extension group Γ^τ itself is also a group, and hence we can apply the assembly map construction for the whole group Γ^τ. However, if a K-homology cycle satisfies a certain condition, such a construction is very wasteful. In this subsection, in order to find a more economical way to describe the assembly maps, we will introduce several subalgebras of crossed products. This observation seems to be trivial, but it will play an essential role in Section 4.

In Section 3, as a model of our problem, we will explicitly describe and compute the corresponding objects for finite-dimensional cases, as algebraically as possible.

In Section 4, we will construct several core objects imitating the result of Section 3. Firstly, we recall the precise setting and several manifold theoretical results on this setting from [36]. Secondly, we will give the review of the result of [36] which we will need for the construction. In the third subsection, we will encounter the "new quantitative argument" mentioned above, and we will construct the main object of this paper using this new technique. In the final subsection, we will prove the main result.

2. *KK*-theoretical preliminaries

We will prepare two stuffs in this section: the product formula of the assembly map and a slightly better description of assembly maps which fit our problem.

We will use the unbounded picture of equivariant KK-theory throughout this paper, and so we will begin with a brief review on this topic. For details on KK-theory, consult with [7,17,20], for example. Then, we will give the product formula in the language of unbounded Kasparov modules. In the final subsection, we will study a "better description" for the assembly maps for the $U(1)$-central extension of a group.

It is necessary to study a "better description" for the following reason. In our setting, the group acting on our space is LT. However, the group acting on the "L^2-space" is not LT but its $U(1)$-central extension LT^τ. The restriction of the action to the added $U(1)$ has a specific form: the center acts as scalar multiplication. Such situation is said to be "**τ-twisted LT-equivariant**" in this paper. Needless to say, we can study this situation as an LT^τ-equivariant problem, but it is more economical to reduce the problem to a "τ-twisted LT-equivariant one". This kind of reduction is just a small device for finite-dimensional cases, but it seems to be essential for infinite-dimensional cases.

2.1. Unbounded Kasparov modules

We begin with a review of KK-theory in the unbounded picture introduced in [6]. For simplicity, we deal with only trivially graded and separable C^*-algebras. Recall that a densely defined unbounded linear operator on a Hilbert module $(E, (\bullet|\bullet)_E)$, $D : \mathrm{dom}(D) \to E$ is said to be **adjointable** if there exists a densely defined operator D^* satisfying that $(De_1|e_2)_E = (e_1|D^*e_2)_E$ for $e_1 \in \mathrm{dom}(D)$ and $e_2 \in \mathrm{dom}(D^*)$. An adjointable operator D is said to be **regular** if $1 + D^*D$ has dense range. See also [7] and [24] for details.

Definition 2.1. For separable C^*-algebras A and B, an **unbounded Kasparov** (A, B)**-module** is a pair (E, D) such that:

- E is a countably generated $\mathbb{Z}_2$-graded Hilbert B-module equipped with an even[4] $*$-homomorphism $A \to \mathcal{L}_B(E)$. Using this homomorphism, we regard E as an (A, B)-bimodule.
- $D : \mathrm{dom}(D) \to E$ is a closed, self-adjoint, regular and adjointable B-module homomorphism.
- There exists a dense $*$-subalgebra A_1 satisfying the followings: For each $a \in A_1$, the operator given by a preserves $\mathrm{dom}(D)$, $[D, a]$ extends to a bounded operator, and $(1 + D^2)^{-1}a$ is a B-compact operator.

Let Γ be a locally compact, second countable Hausdorff group (say finite-dimensional Lie group). Suppose that A and B are Γ-C^*-algebras. An **unbounded** Γ**-equivariant Kasparov** (A, B)**-module** is an unbounded Kasparov (A, B)-module (E, D) such that:

- Γ acts on E satisfying that $\gamma.(aeb) = (\gamma.a) \cdot (\gamma.e) \cdot (\gamma.b)$ and $(\gamma.e_1|\gamma.e_2)_E = (e_1|e_2)_E$ for $a \in A$, $e, e_1, e_2 \in E$ and $b \in B$.
- D is Γ-equivariant.

Lemma 2.2 ([6]). *Let $\mathfrak{b} : \mathbb{R} \to \mathbb{R}$ be a function defined by $\mathfrak{b}(x) := x/\sqrt{1 + x^2}$. The correspondence $(E, D) \mapsto (E, \mathfrak{b}(D))$ maps an unbounded (equivariant) Kasparov module into a bounded (equivariant) Kasparov module. This correspondence is called the* **bounded transformation**.

Remarks 2.3. (1) There is a criterion to be essentially self-adjoint and regular [24]: If the both of $D \pm i$ have dense ranges, D is self-adjoint and regular. We will always use this criterion.

(2) Our definition of equivariant Kasparov module is sufficient to describe actually equivariant Dirac operators, and we can define a functor from the category of unbounded equivariant Kasparov modules (in the above sense) to the category of bounded equivariant Kasparov modules (in the sense of [20]). However, the above definition is probably insufficient to study the whole equivariant KK-theory. For example, the Bott-Dirac operator of [13] is not actually equivariant. In order to go further, we need relax the condition to be an equivariant Kasparov module.

One of the merits to study unbounded Kasparov modules is the simple formula for the exterior tensor product. Let A_i and B_i be Γ_i-C^*-algebras, and let (E_i, D_i) be a Γ_i-equivariant Kasparov module, for $i = 1, 2$. For simplicity we suppose that A_i's and B_i's are nuclear, and hence there is no ambiguity to define tensor products.

Proposition 2.4 ([6]). *The pair $(E_1 \otimes E_2, D_1 \otimes \mathrm{id} + \mathrm{id} \otimes D_2)$ defines a $\Gamma_1 \times \Gamma_2$-equivariant Kasparov $(A_1 \otimes A_2, B_1 \otimes B_2)$-module, and its bounded transformation coincides with the exterior Kasparov product $\mathfrak{b}(E_1, D_1) \otimes_{\mathbb{C}} \mathfrak{b}(E_2, D_2)$ in $KK_{\Gamma_1 \times \Gamma_2}(A_1 \otimes A_2, B_1 \otimes B_2)$.*

Thanks to this proposition, we may write $(E_1 \otimes E_2, D_1 \otimes \mathrm{id} + \mathrm{id} \otimes D_2)$ as $(E_1, D_1) \otimes_{\mathbb{C}} (E_2, D_2)$.

The following is a sufficient condition so that an unbounded Kasparov module is a Kasparov product of other two unbounded Kasparov modules.

Proposition 2.5 ([24]). *Let A, B, C be separable C^*-algebras. Let (E_1, D_1) be an unbounded Kasparov (A, B)-module, and (E_2, D_2) be an unbounded Kasparov (B, C)-module. A Kasparov (A, C)-module $(E_1 \otimes_B E_2, D)$ represents the Kasparov product of (E_1, D_1) and (E_2, D_2), if the following conditions are satisfied:*

- *For all e_1 in some dense subset of E_1, the graded commutator*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_{e_1} \\ T_{e_1}^* & 0 \end{pmatrix}\right]$$

  is bounded on $\mathrm{dom}(D) \oplus \mathrm{dom}(D_2)$, where $T_{e_1} : E_2 \to E_1 \otimes_B E_2$ is defined by $T_{e_1}(e_2) := e_1 \otimes_B e_2$.
- $\mathrm{dom}(D) \subseteq \mathrm{dom}(D_1) \otimes_B E_2$.
- *There is a real number κ such that $(D_1 \otimes \mathrm{id}(e)|D(e))_E + (D(e)|D_1 \otimes \mathrm{id}(e))_E \geq \kappa (e|e)_E$ for all $e \in \mathrm{dom}(D)$.*

Notice that this criterion is still valid for equivariant situation. This is because, in the bounded picture, an *equivariant* Kasparov module $z = (E_1 \otimes E_2, T)$ represents a Kasparov product of $x = (E_1, T_1)$ and $y = (E_2, T_2)$ in the *equivariant KK-theory*, if and only if z represents a Kasparov product of x and y in the *non-equivariant* sense.

[4] Since A is trivially graded, "even homomorphism" means that each operator given by $a \in A$ preserves the $\mathbb{Z}_2$-grading.

2.2. Assembly maps for unbounded cycles and the product formula

In this subsection, we will describe assembly maps in the unbounded picture, and verify the following product formula: *The value of the assembly map of an exterior product is the exterior product of the values of the assembly map.*

Let us begin with the descent homomorphism in the unbounded picture. Let Γ be a locally compact, second countable Hausdorff group. For simplicity, we further suppose that Γ is amenable and unimodular. Let A and B be separable Γ-C^*-algebras. Recall that the descent homomorphism $j^\Gamma : KK_\Gamma(A,B) \to KK(\Gamma \ltimes A, \Gamma \ltimes B)$ is defined by the crossed product construction on E [20, Theorem 3.11]. The same formula gives the descent homomorphism even in the unbounded picture.

Definition-Lemma 2.6. *Let (E,D) be a Γ-equivariant unbounded Kasparov (A,B)-module. $C_c(\Gamma,E)$ has a $(C_c(\Gamma,A), C_c(\Gamma,B))$-bimodule structure given by the followings:*

$$a * e(\gamma) := \int_\Gamma a(\gamma')\gamma'.\left(e(\gamma'^{-1}\gamma)\right) d\gamma',$$

$$e * b(\gamma) := \int_\Gamma e(\gamma')\gamma'.\left(b(\gamma'^{-1}\gamma)\right) d\gamma',$$

$$\left(e_1|e_2\right)_{\Gamma\ltimes E}(\gamma) := \int_\Gamma \left(e_1(\gamma')|e_2(\gamma'\gamma)\right)_E d\gamma',$$

for $a \in C_c(\Gamma,A)$, $b \in C_c(\Gamma,B)$ and $e, e_1, e_2 \in C_c(\Gamma,E)$. Let $\Gamma \ltimes E$ be the completion of $C_c(\Gamma,E)$ with respect to the above inner product $\left(\bullet|\bullet\right)_{\Gamma\ltimes E}$. The operator $\widetilde{D}$ is defined by $\widetilde{D}(e)(\gamma) := D(e(\gamma))$.

Then $(\Gamma \ltimes E, \widetilde{D})$ is an unbounded Kasparov $(\Gamma\ltimes A, \Gamma\ltimes B)$-module, and satisfies that

$$\mathfrak{b}(\Gamma\ltimes E,\widetilde{D}) = j^\Gamma[\mathfrak{b}(E,D)].$$

Thanks to this result, we may write the unbounded Kasparov module $(\Gamma\ltimes E,\widetilde{D})$ as $j^\Gamma(E,D)$.

Proof. Once we verify that $(\Gamma\ltimes E,\widetilde{D})$ defines an unbounded Kasparov module, the result is clear from the fact that the correspondence $D \mapsto \widetilde{D}$ preserves the composition of operators, which guarantees that $\mathfrak{b}(\widetilde{D}) = \mathfrak{b}(\widetilde{D})$.

$\widetilde{D}$ is essentially self-adjoint and regular: It suffices to check that the both of $\widetilde{D}\pm i\mathrm{id}$ have dense ranges. Note that an element of $\mathrm{dom}(D)\otimes^{\mathrm{alg}} C_c(\Gamma)$ defines an element of $C_c(\Gamma,E)$ by the usual formula $[\sum e_i\otimes f_i] : \gamma \mapsto \sum f_i(\gamma)e_i$. Then the restriction of $\widetilde{D}$ to $\mathrm{dom}(D)\otimes C_c(\Gamma)$ can be written as $D\otimes\mathrm{id}$, and the both of $\widetilde{D}\pm i\mathrm{id}|_{(\mathrm{dom}(D))\otimes C_c(\Gamma)} = (D\pm i\mathrm{id})\otimes\mathrm{id}|_{(\mathrm{dom}(D))\otimes C_c(\Gamma)}$ have dense ranges in $\Gamma\ltimes E$. This is because the both of $D\pm i\mathrm{id}$ have dense ranges in E, and $E\otimes^{\mathrm{alg}} C_c(\Gamma)$ is dense in $C_c(\Gamma,E)$. Clearly the ranges of $\widetilde{D}\pm i\mathrm{id}$ contain the ranges of $(D\pm i\mathrm{id})\otimes\mathrm{id}$, respectively, and hence $\widetilde{D}$ is essentially self-adjoint and regular.

There is a dense subalgebra $(\Gamma\ltimes A)_1$: Let A_1 be a dense subalgebra of A such that each $a\in A_1$ preserves $\mathrm{dom}(D)$ and $[D,a]$ is bounded. The subalgebra $A_1\otimes C_c(\Gamma)\subseteq \Gamma\ltimes A$ satisfies the required condition as follows. Take $a\otimes f\in A_1\otimes C_c(\Gamma)$ and $e\in C_c(\Gamma,\mathrm{dom}(D))$. Then,

$$[\widetilde{D}, a\otimes f]e(\gamma) = [D,a]\int f(\gamma')\gamma'.\left(e(\gamma'^{-1}\gamma)\right)d\gamma',$$

which is bounded in e.

$(1+\widetilde{D}^2)^{-1}a$ is compact: Since $(1+\widetilde{D}^2)^{-1} = 1-\mathfrak{b}(D)^2$, the argument in the bounded picture implies the statement. We omit the detail. □

This correspondence commutes with the tensor product thanks to Proposition 2.4. We describe it for simple cases. Let Γ_1 and Γ_2 be locally compact, second countable topological groups. We further suppose that they are amenable and unimodular. Let A_i and B_i be nuclear Γ_i-C^*-algebras $(i=1,2)$. Then the crossed products $\Gamma_i\ltimes A_i$'s are automatically nuclear [7, Theorem 15.8.2].

Lemma 2.7. *Let (E_i,D_i) be an unbounded Γ_i-equivariant Kasparov (A_i,B_i)-module for $i=1,2$. Then $j^{\Gamma_1\times\Gamma_2}[(E_1,D_1)\otimes_{\mathbb{C}}(E_2,D_2)]$ is isomorphic to the exterior tensor product of $j^{\Gamma_1}(E_1,D_1)$ and $j^{\Gamma_2}(E_2,D_2)$.*

This is clear from the isomorphism $(\Gamma_1\ltimes E_1)\otimes(\Gamma_2\ltimes E_2)\cong(\Gamma_1\times\Gamma_2)\ltimes(E_1\otimes E_2)$ and Proposition 2.4, and so we leave the proof to the reader.

In order to define the assembly map, we need one more stuff: the Mishchenko line bundle $[c]$. Let us recall it here. Let Γ be a locally compact, second countable Hausdorff group, and let X be a locally compact Hausdorff space equipped with a proper and cocompact action of Γ. Then there is a compactly supported continuous function $\mathfrak{c} : X\to\mathbb{R}_{\geq 0}$ such that $\int_\Gamma \mathfrak{c}(\gamma.x)d\gamma = 1$ for any $x\in X$. Such $\mathfrak{c}$ is called a **cut-off function**. The function $\mathfrak{c}$ defines a projection in $\Gamma\ltimes C_0(X)$ as follows, and consequently a K-theory element of $\Gamma\ltimes C_0(X)$.

Lemma 2.8. *Let c be a function from Γ to $C_0(X)$ defined by*

$$\gamma \mapsto \text{“}x \mapsto \sqrt{\mathfrak{c}(x)\mathfrak{c}(\gamma^{-1}.x)}\text{”}.$$

This function is compactly supported, and hence defines an element of $\Gamma \ltimes C_0(X)$.

Then, c is a projection (self-adjoint and idempotent), and thus defines a K-theory element of $KK(\mathbb{C}, \Gamma \ltimes C_0(X))$. The same symbol denotes the corresponding K-theory element.

In the Kasparov module language, $[c]$ is given by $(c * \big(\Gamma \ltimes C_0(X)\big), 0)$. This K-theory class is called the **Mishchenko line bundle associated to** *$\Gamma \curvearrowright X$.*

The proof of this lemma is an easy exercise to use the condition $\int \mathfrak{c}(\gamma^{-1}.x)d\gamma = 1$, and we leave it to the reader.

It is also compatible with the product of spaces. Consequently, we can prove a product formula of the assembly map. Let us precisely state it here. Suppose that two proper and cocompact group actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ are given. Then, the product of the groups acts on the product of spaces properly and cocompactly: $\Gamma_1 \times \Gamma_2 \curvearrowright X_1 \times X_2$. The Mishchenko line bundle associated to $\Gamma_1 \times \Gamma_2 \curvearrowright X_1 \times X_2$ is given by the following.

Lemma 2.9. *Let $\mathfrak{c}_1 : X_1 \to \mathbb{R}_{\geq 0}$ and $\mathfrak{c}_2 : X_2 \to \mathbb{R}_{\geq 0}$ be cut-off functions. Then $\mathfrak{c}_1 \otimes \mathfrak{c}_2 : X_1 \times X_2 \to \mathbb{R}_{\geq 0}$ defined by $\mathfrak{c}_1 \otimes \mathfrak{c}_2(x_1, x_2) = \mathfrak{c}_1(x_1)\mathfrak{c}_2(x_2)$ is a cut-off function associated to $(\Gamma_1 \times \Gamma_2) \curvearrowright (X_1 \times X_2)$. The associated projection*

$$(\gamma_1, \gamma_2, x_1, x_2) \mapsto \sqrt{\mathfrak{c}_1(x_1)\mathfrak{c}_2(x_2)\mathfrak{c}_1(\gamma_1^{-1}.x_1)\mathfrak{c}_2(\gamma_1^{-1}.x_2)}$$

is denoted by $c_1 \otimes c_2$, and the associated K-theory element is denoted by $[c_1 \otimes c_2]$.

Then, $[c_1 \otimes c_2] = [c_1] \otimes_{\mathbb{C}} [c_2] \in KK(\mathbb{C}, (\Gamma_1 \times \Gamma_2) \ltimes C_0(X_1 \times X_2))$ under the identification $\big(\Gamma_1 \ltimes C_0(X_1)\big) \bigotimes \big(\Gamma_2 \ltimes C_0(X_2)\big) \cong (\Gamma_1 \times \Gamma_2) \ltimes C_0(X_1 \times X_2)$.

Now, we can define the assembly map: It is defined by the formula $\mu^{\Gamma}(x) := [c] \otimes j^{\Gamma}(x)$ for $x \in KK_{\Gamma}(C_0(X), \mathbb{C})$. Combining lemmas prepared in this subsection and the associativity of the Kasparov product, one can easily prove the following product formula.

Proposition 2.10. *For an unbounded Γ_1-equivariant Kasparov $(C_0(X_1), \mathbb{C})$-module (E_1, D_1) and an unbounded Γ_2-equivariant Kasparov $(C_0(X_2), \mathbb{C})$-module (E_2, D_2), we have*

$$\mathfrak{b}\left\{\mu^{\Gamma_1 \times \Gamma_2}\left[(E_1, D_1) \otimes_{\mathbb{C}} (E_2, D_2)\right]\right\} = \mathfrak{b}\left\{\mu^{\Gamma_1}\left[E_1, D_1\right]\right\} \otimes_{\mathbb{C}} \mathfrak{b}\left\{\mu^{\Gamma_2}\left[(E_2, D_2)\right]\right\}.$$

2.3. The assembly map for a central extension of a group

Let us begin with an exposition of the typical (or classical) situation in which we would like to study the assembly map in this paper. Let X be a finite-dimensional, complete, Riemannian *Spin*^{c}-manifold, equipped with a proper, isometric, cocompact action of a locally compact, second countable and Hausdorff group Γ. Let $\pi : W \to X$ be a Γ-equivariant Clifford bundle over X. We suppose that a $U(1)$-central extension group Γ^{τ} of Γ

$$1 \to U(1) \xrightarrow{i} \Gamma^{\tau} \xrightarrow{p} \Gamma \to 1,$$

acts on W satisfying that $\pi(\gamma.w) = p(\gamma).\pi(w)$ for $\gamma \in \Gamma^{\tau}$ and $w \in W$. *We further suppose that $i(z).w = z^k w$ for $z \in U(1)$. Such actions (on vector bundles or vector spaces) are said to be $k\tau$-***twisted**.[5] Let D be a Dirac operator acting on $L^2(X, W)$ in a Γ^{τ}-equivariant way. In this situation, we have an equivariant Kasparov module $[D] := (L^2(X, W), D)$, which is an element of $KK_{\Gamma^{\tau}}(C_0(X), \mathbb{C})$. This element is called the **index element** of the Dirac operator D [21].

We would like to find a “better” way to describe the value of the assembly map $\mu^{\Gamma^{\tau}}([D]) \in KK(\mathbb{C}, \Gamma^{\tau} \ltimes \mathbb{C})$. This is because the whole construction, including $\Gamma^{\tau} \ltimes C_0(X)$ and $\Gamma^{\tau} \ltimes \mathbb{C}$, is very “wasteful”. In order to explain it, let us consider a trivial example: Γ is the trivial group $\{e\}$ and X is a compact manifold. The $U(1)$-central extension Γ^{τ} is $U(1)$ itself, and the $k\tau$-twisted Γ-action is given by $i(z).w = z^k w$. Then, a Dirac operator D is automatically equivariant, and its equivariant index (hence the value of the assembly map) is $[\mathbb{C}_k^{\dim(\ker(D))}] - [\mathbb{C}_k^{\dim(\operatorname{coker}(D))}]$, where $\mathbb{C}_k$ is the irreducible representation of $U(1)$ with weight k. Although the index is an element of $KK(\mathbb{C}, C^*(U(1)))$, the index lives in much smaller group $\mathbb{Z}[\mathbb{C}_k]$.

In order to find a more economical way to describe the assembly map for $k\tau$-twisted equivariant problems, we must begin with an operator algebraic formulation of “$k\tau$-twisted equivariant Kasparov modules”. Let Γ be a locally compact, second countable Hausdorff group, and suppose that a $U(1)$-central extension Γ^{τ} of Γ is given: $1 \to U(1) \xrightarrow{i} \Gamma^{\tau} \xrightarrow{p} \Gamma \to 1$. Let A and B be separable Γ-C^*-algebras. Then, A and B are automatically Γ^{τ}-C^*-algebras by the homomorphism $p : \Gamma^{\tau} \to \Gamma$.

[5] By the kth tensor power of τ, we can define a new $U(1)$-central extension which is denoted by $\tau^{\otimes k}$. Then, the two concepts “$k\tau$-twisted” and “$1\tau^{\otimes k}$-twisted” are equivalent. Moreover, two terminologies “1τ-twisted” given here and “τ-twisted” given in the beginning of this section, are equivalent. Therefore, if we write $\tau^{\otimes k}$ as $k\tau$, these two terminologies are compatible. From now on, we do not distinguish these two concepts.

Definition 2.11. An unbounded Γ^τ-equivariant Kasparov (A, B)-module (E, D) is said to be $k\tau$**-twisted** Γ**-equivariant** if the Γ^τ-action satisfies that $i(z).e = z^k e$ for $z \in U(1)$ and $e \in E$.

We can also define the concept of bounded $k\tau$-twisted Γ-equivariant Kasparov modules in the same way. The set of homotopy classes of $k\tau$-twisted Γ-equivariant Kasparov modules, is a subgroup of $KK_{\Gamma^\tau}(A, B)$. This subgroup is denoted by $KK_\Gamma^{k\tau}(A, B)$. Especially, when $k = 1$, this group is simply denoted by $KK_\Gamma^\tau(A, B)$.

Remark 2.12. For a bounded Γ^τ-equivariant Kasparov module (E, T), the operator T might be *not* actually equivariant, *but* "equivariant modulo locally compact operators" [20]. However, we can always assume that T is actually equivariant with respect to the $i(U(1))$-action thanks to the averaging procedure on the *compact* group $U(1)$.

In the above situation (X is a non-compact manifold, Γ is a locally compact group acting on X, Γ^τ is a $U(1)$-central extension of Γ, W is a $k\tau$-twisted Γ-equivariant Clifford bundle of X, and D is a Γ^τ-equivariant Dirac operator), the index element $[D]$ belongs to $KK_\Gamma^{k\tau}(C_0(X), \mathbb{C})$.

In order to describe the restriction of the descent homomorphism μ^{Γ^τ} to the subgroup $KK_\Gamma^{k\tau}(A, B)$, we introduce countably many "partial descent homomorphisms" $j_{k\tau}^\Gamma$ for $k \in \mathbb{Z}$. For this aim, we must begin with introducing subalgebras of crossed products.

Definition 2.13. For an integer m, a function f from Γ^τ satisfying that

$$f(i(z)g) = z^m f(g)$$

is said to be **at level** m.

Proposition 2.14. (1) *Let $\Gamma \ltimes_{m\tau} A$ be the closure, in $\Gamma^\tau \ltimes A$, of the subset consisting of compactly supported and continuous functions $f : \Gamma^\tau \to A$ at level m. Then $\Gamma \ltimes_{m\tau} A$ is a C^*-subalgebra. Moreover, unless $m = m'$, $\Gamma \ltimes_{m\tau} A$ and $\Gamma \ltimes_{m'\tau} A$ are orthogonal in the sense that $a * a' = 0$ for any $a \in \Gamma \ltimes_{m\tau} A$ and $a' \in \Gamma \ltimes_{m'\tau} A$. Briefly speaking, we have a direct sum decomposition as a C^*-algebra:*

$$\Gamma^\tau \ltimes A = \bigoplus_{m\in\mathbb{Z}} \Gamma \ltimes_{m\tau} A.$$

(2) *Similarly, for a $k\tau$-twisted Γ-equivariant (A, B)-bimodule E, we define $\Gamma \ltimes_{m\tau} E$ by the closure, in $\Gamma^\tau \ltimes E$, of the subset consisting of compactly supported and continuous functions $f : \Gamma^\tau \to E$ at level m. Then, for any $a \in \Gamma \ltimes_{m_1\tau} A$, $e \in \Gamma \ltimes_{m\tau} E$, and $b \in \Gamma \ltimes_{m_2\tau} B$,*

$$\begin{aligned} &a * e \text{ is at level } m, \text{ and it is } 0 \text{ unless } m_1 = m - k;\\ &e * b \text{ is at level } m, \text{ and it is } 0 \text{ unless } m_2 = m. \end{aligned}$$

Moreover, for $e_1 \in \Gamma \ltimes_{m\tau} E$ and $e_2 \in \Gamma \ltimes_{m'\tau} E$,

$$\left(e_1 \middle| e_2\right)_{\Gamma^\tau \ltimes E} \text{ is at level } m, \text{ and it is } 0 \text{ unless } m = m'.$$

Briefly speaking, $\Gamma^\tau \ltimes E$ is orthogonally decomposed into $\bigoplus \Gamma \ltimes_{m\tau} E$ as $(\Gamma^\tau \ltimes A, \Gamma^\tau \ltimes B)$-bimodules, and each component $\Gamma \ltimes_{m\tau} E$ is, in fact, a $(\Gamma \ltimes_{(m-k)\tau} A, \Gamma \ltimes_{m\tau} B)$-bimodule.

Proof. The proof is just an application of the formula $\frac{1}{2\pi}\int_{S^1} e^{ik\theta} d\theta = \delta_{k,0}$. We check only one statement: *For $a \in \Gamma \ltimes_{m_1\tau} A$ and $e \in \Gamma \ltimes_{m\tau} E$, $a * e$ is at level m, and the product is 0 unless $m_1 = m - k$.*

Let $a : \Gamma^\tau \to A$ be a continuous and compactly supported function at level m_1, and $e : \Gamma^\tau \to E$ be a continuous and compactly supported function at level m. Let $\gamma \in \Gamma^\tau$ and $z \in U(1)$. The product is at level m, because

$$a * e(i(z)\gamma) = \int a(\gamma')\gamma' . \left(e\left(\gamma'^{-1} i(z)\gamma\right)\right) d\gamma' = z^m a * e(\gamma).$$

At the same time, $a * e$ is at level $m_1 + k$. Let us check it. By the change of variables $\gamma' = i(z)\eta$,

$$\begin{aligned} a * e(i(z)\gamma) &= \int a(\gamma')\gamma' . \left(e\left(\gamma'^{-1} i(z)\gamma\right)\right) d\gamma' \\ &= \int a(i(z)\eta) i(z)\eta . \left[e\left(\eta^{-1} i(z)^{-1} i(z)\gamma\right)\right] d\eta \\ &= \int z^{m_1} a(\eta) \cdot z^k \left\{\eta . \left[e\left(\eta^{-1}\gamma\right)\right]\right\} d\eta = z^{m_1+k} a * e(\gamma), \end{aligned}$$

where we used the following conditions in the third line: $i(z)$ acts on E as $z^k \mathrm{id}_E$ and $i(z)$ belongs to the center of Γ^τ.

If a function f is at level m, and at the same time at level $m' \neq m$, we find the equality $z^m f(\gamma) = f(i(z)\gamma) = z^{m'} f(\gamma)$. It is impossible unless $f = 0$. □

The following is clear from the definition of $\widetilde{D}$.

Lemma 2.15. *$\widetilde{D}$ preserves the decomposition $\Gamma^\tau \ltimes E = \bigoplus \Gamma \ltimes_{n\tau} E$. The restriction to a direct summand $\Gamma \ltimes_{n\tau} E$ is denoted by $\widetilde{D}|_{\Gamma\ltimes_{n\tau}E}$, or simply by $\widetilde{D}$.*

Combining these formulas, we have reached the following definition.

Definition 2.16. The **partial descent homomorphism $j^\Gamma_{m\tau}$ at level m** is a correspondence from the category of unbounded $k\tau$-twisted Γ-equivariant Kasparov (A, B)-modules, to the category of unbounded Kasparov $(\Gamma \ltimes_{(m-k)\tau} A, \Gamma \ltimes_{m\tau} B)$-modules, given by $(E, D) \mapsto (\Gamma \ltimes_{m\tau} E, \widetilde{D}|_{\Gamma\ltimes_{m\tau}E})$.

The same formula defines a correspondence on bounded Kasparov modules, and it is compatible with homotopy. Consequently, we can define a homomorphism at the level of KK-theory:

$$j^\Gamma_{m\tau} : KK^{k\tau}_\Gamma(A, B) \to KK(\Gamma \ltimes_{(m-k)\tau} A, \Gamma \ltimes_{m\tau} B).$$

The original descent homomorphism for Γ^τ is given by the direct sum of the above homomorphisms. This is clear from the Fourier series theory.

Lemma 2.17. *For a $k\tau$-twisted Γ-equivariant Kasparov (A, B)-module (E, D),*

$$j^{\Gamma^\tau}(E, D) = \bigoplus_{m\in\mathbb{Z}} j^\Gamma_{m\tau}(E, D).$$

Remarks 2.18. (1) In the following, we will deal with only the case when $k = 1$ and $m = 1$. j^Γ_τ denotes the partial descent homomorphism at level 1 on $KK^\tau_\Gamma(A, B)$.

(2) The product formula is still valid for the partial descent homomorphisms.

The next step is to study of the Mishchenko line bundle $[c]$.

Firstly, we notice that $KK(\mathbb{C}, \Gamma \ltimes C_0(X)) \subseteq KK(\mathbb{C}, \Gamma^\tau \ltimes C_0(X))$. This is because, by the direct sum decomposition $\Gamma^\tau \ltimes C_0(X) = \bigoplus \Gamma \ltimes_{m\tau} C_0(X)$, we have a direct sum decomposition of the K-theory $KK(\mathbb{C}, \Gamma^\tau \ltimes C_0(X)) \cong \oplus_{m\in\mathbb{Z}} KK(\mathbb{C}, \Gamma \ltimes_{m\tau} C_0(X))$.

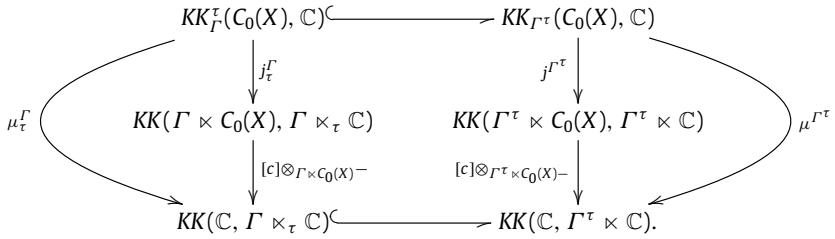
By definition of c: $\Gamma^\tau \ni \gamma \mapsto \sqrt{\mathfrak{c}(\bullet)\mathfrak{c}(p(\gamma)^{-1}.\bullet)} \in C_0(X)$, the function $c : \Gamma^\tau \to C_0(X)$ is at level 0.[6] Thus the Mishchenko line bundle $[c] \in KK(\mathbb{C}, \Gamma^\tau \ltimes C_0(X))$ is in fact an element of $KK(\mathbb{C}, \Gamma \ltimes C_0(X))$. In what follows, we regard $[c]$ as an element of $KK(\mathbb{C}, \Gamma \ltimes C_0(X))$, and at the same time an element of $KK(\mathbb{C}, \Gamma^\tau \ltimes C_0(X))$.

We introduce the **partial assembly map** by the following composition:

$$KK^\tau_\Gamma(C_0(X), \mathbb{C}) \xrightarrow{j^\Gamma_\tau} KK(\Gamma \ltimes C_0(X), \Gamma \ltimes_\tau \mathbb{C}) \xrightarrow{[c]\otimes_{\Gamma\ltimes C_0(X)}-} KK(\mathbb{C}, \Gamma \ltimes_\tau \mathbb{C}).$$

This is the correct (economical) form to study the assembly map for τ-twisted Γ-equivariant Kasparov modules. This is because the KK-groups appearing in the partial assembly map is much smaller than ones appearing in the original assembly map μ^{Γ^τ}, and the values of the two assembly maps coincide in the following sense.

Proposition 2.19. *By the direct sum decomposition $\Gamma^\tau \ltimes \mathbb{C} = \bigoplus \Gamma \ltimes_{m\tau} \mathbb{C}$, we can regard $KK(\mathbb{C}, \Gamma \ltimes_\tau \mathbb{C})$ as a subgroup of $KK(\mathbb{C}, \Gamma^\tau \ltimes \mathbb{C})$. Under this identification, $\mu^{\Gamma^\tau}(x) \in KK(\mathbb{C}, \Gamma \ltimes_\tau \mathbb{C})$ and $\mu^{\Gamma^\tau}(x) = \mu^\Gamma_\tau(x)$ for any $x \in KK^\tau_\Gamma(C_0(X), \mathbb{C})$. More precisely, the following diagram commutes:*



Proof. Let $x = [(E, T)] \in KK^\tau_\Gamma(C_0(X), \mathbb{C})$ be a τ-twisted Γ-equivariant K-homology element of $C_0(X)$. By Lemma 2.17, $j^{\Gamma^\tau}(x)$ can be written as a direct sum $\bigoplus_{m\in\mathbb{Z}}(\Gamma \ltimes_{m\tau} E, \widetilde{T})$. In addition, recall that $[c]$ is given by, in the Kasparov module language, $(c * (\Gamma \ltimes C_0(X)), 0)$.

[6] We are working on the situation where Γ acts on X properly and cocompactly. Through the homomorphism $p : \Gamma^\tau \to \Gamma$, the extended group Γ^τ acts on X. This action is also proper and cocompact, since $U(1)$ is compact. Since the $U(1)$-action on X is trivial, c is at level 0.

In order to compute the Kasparov product $[c]\otimes j^{\Gamma^\tau}(x)$, we need to begin with the tensor product of Hilbert modules $\{c*(\Gamma\ltimes C_0(X))\}\otimes_{\Gamma^\tau\ltimes C_0(X)}(\Gamma\ltimes_{m\tau}E)$. We check that the tensor product vanishes unless $m=1$. Take $c*a\in c*[\Gamma\ltimes C_0(X)]$ and $e\in\Gamma\ltimes_{m\tau}E$. Notice that $c*a=c*(c*a)$. Then, by definition of the tensor product over $\Gamma^\tau\ltimes C_0(X)$,

$$(c*a)\otimes e=(c*c)\otimes(a*e).$$

Thanks to Proposition 2.14, the right hand side is zero unless $m=1$. This means that $[c]\otimes j^\Gamma_{m\tau}(x)=0$ unless $m=1$. As a result, the sum $[c]\otimes\sum j^\Gamma_{m\tau}(x)$ is in fact the single Kasparov product $[c]\otimes j^\Gamma_\tau(x)$, which is nothing but $\mu^\Gamma_\tau(x)$. □

3. Finite-dimensional case

The purpose of this paper is to construct several core objects of the index theory for infinite-dimensional manifolds by imitating finite-dimensional cases. The classical construction of such objects is no longer valid for infinite-dimensional cases, but some alternative (algebraic) formulas can be applied. In this section, we compute such objects for a certain finite-dimensional model case, so that the answer is as algebraic as possible.

The concrete setting is the following: Let G be a (possibly non-compact) even-dimensional abelian Lie group; It acts on itself by the translation; Let $1\to U(1)\to G^\tau\to G\to 1$ be a $U(1)$-central extension of G, which induces a τ-twisted G-equivariant line bundle $\mathcal{L}:=G^\tau\times_{U(1)}\mathbb{C}=G^\tau\times\mathbb{C}/(g,v)\sim(gz,z^{-1}v)$; Take a G^τ-invariant inner product on $\mathfrak{g}^\tau$; Let (S_G,γ) be a Spinor module of $\mathfrak{g}$. We would like to study the value of the assembly map at the K-homology element associated to a Dirac operator for $\mathcal{L}\otimes S_G$.

We will describe the index element in the first subsection. In order to describe it, as a language, we will use functions on the extended group G^τ rather than sections of $\mathcal{L}$. Then, we will compute the partial descent homomorphism for this index element, as algebraically as possible. Finally, we will compute the assembly map for that K-homology element, and we will explicitly construct an isomorphism between the value of the assembly map and the analytic index.

3.1. A τ-twisted G-equivariant Kasparov module $(L^2(G,-\tau)\otimes S_G,\not{\partial}_R)$

In order to describe a Dirac operator in a systematic way, we describe sections of the line bundle $\mathcal{L}\to G$ as functions on G^τ. The following is fundamental.

Lemma 3.1. *There is a one to one correspondence between sections of $\mathcal{L}$ and functions on G^τ at level -1.*

Taking this fact into account, we write the Hilbert space consisting of L^2 sections of $\mathcal{L}$ as $L^2(G,-\tau)$. This Hilbert space admits a τ-twisted representation of G by the usual formula: $[g.\phi](x):=\phi(g^{-1}x)$. Moreover, it admits a $*$-representation of $C_0(G)$ by the multiplication, where we regard $C_0(G)$ as a subalgebra of $C_0(G^\tau)$ using the $*$-homomorphism p^*. These actions are compatible as follows:

$$[g.(f\cdot\phi)](x)=(f\cdot\phi)(g^{-1}x)=f(g^{-1}x)\phi(g^{-1}x)=[(g.f)\cdot(g.\phi)](x),$$

for $g,x\in G^\tau$, $f\in C_0(G)$ and $\phi\in L^2(G,-\tau)$. This construction extends to $L^2(G,-\tau)\otimes S_G$ by the obvious way.

Let us define the Dirac operator. Fix a splitting $s:\mathfrak{g}\to\mathfrak{g}^\tau$ of $dp:\mathfrak{g}^\tau\to\mathfrak{g}$. Let $R:G^\tau\to U(L^2(G,-\tau))$ be the right regular representation of G^τ: $[R(g)(f)](x):=f(xg)$. The infinitesimal representation of R is denoted by dR. More precisely, $dR_X(f)$ is given by $\frac{d}{dt}\big|_{t=0}R(e^{tX})f$ for $X\in\mathfrak{g}^\tau$ and $f\in L^2(G,-\tau)$. Take an orthonomal basis $\{e_i\}$ of $\mathfrak{g}$. Using them, we define a Dirac operator acting on $L^2(G,-\tau)\otimes S_G$ by the formula

$$\not{\partial}_R:=\sum_i dR_{s(e_i)}\otimes\gamma(e_i).$$

The following can be proved with a usual argument (including Leibniz rule, elliptic regularity, Rellich lemma and so on). Note that $\not{\partial}_R$ is G^τ-invariant with respect to the *left* regular representation, and the left regular representation on $L^2(G,-\tau)$ is τ-twisted.

Proposition 3.2. *The pair $(L^2(G,-\tau)\otimes S_G,\not{\partial}_R)$ is an unbounded τ-twisted G-equivariant Kasparov $(C_0(G),\mathbb{C})$-module. This pair is denoted by $[\not{\partial}_R]$ throughout this section, and it is called the* **index element**.

3.2. The value of the descent homomorphism

We compute the Kasparov $(G\ltimes C_0(G),G\ltimes_\tau\mathbb{C})$-module $j^G_\tau([\not{\partial}_R])$ explicitly. For this aim, we notice the isomorphism $G\ltimes C_0(G)\cong\mathcal{K}(L^2(G))$. It is given by, for $a:G\to C_0(G)$, $\phi\in L^2(G)$ and $x\in G$,

$$\pi(a)(\phi)(x):=\int_G a(g)(x)\phi(g^{-1}x)dg.$$

From now on, $\pi(a)(\phi)$ is simply denoted by $a*\phi$.

Firstly, we compute the bimodule $G \ltimes_\tau [L^2(G,-\tau)\otimes S_G]$. Since $C_0(G)$ and G^τ act on S_G-part trivially, it suffices to study $G \ltimes_\tau L^2(G,-\tau)$.

Let $C_c(G,\tau)$ be the set of compactly supported continuous functions of G^τ at level 1. Note that the completion of $C_c(G,\tau)$ is the twisted group C^*-algebra $G \ltimes_\tau \mathbb{C}$. Let us consider a linear map $m : C_c(G)\otimes C_c(G,\tau) \to G \ltimes_\tau L^2(G,-\tau)$ defined by

$$m(\phi_1\otimes\phi_2)(g)(x) := \phi_1(x)\phi_2(x^{-1}g),$$

where we regard g as an element of the *group* G^τ, and x as a point of the *space* G^τ. More precisely, for a given $g \in G^\tau$, we define an element $m(\phi_1\otimes\phi_2)(g)$ of $C_c(G,-\tau) \subseteq L^2(G,-\tau)$ by the formula $x \mapsto \phi_1(x)\phi_2(x^{-1}g)$.

Proposition 3.3. *The above linear map m is an isomorphism as $(G \ltimes C_0(G), G \ltimes_\tau \mathbb{C})$-bimodules.*

Proof. Let us verify that the above correspondence takes values in $G \ltimes_\tau L^2(G,-\tau)$. Firstly, for each element $g \in G^\tau$, the function $x \mapsto m(\phi_1\otimes\phi_2)(g)(x)$ is actually an L^2 function. This is because it is given by the product of an C_c (in particular L^2) function $\phi_1(\bullet)$ and a C_c function $\phi_2(\bullet^{-1}g)$. Secondly, the same function of x is at level -1 for each $g \in G^\tau$. This is because ϕ_1 is at level 0 and ϕ_2 is at level 1. Thirdly, the correspondence $g \mapsto m(\phi_1\otimes\phi_2)(g) \in L^2(G,-\tau)$ is at level 1 with respect to g. This is because ϕ_2 is at level 1. Finally, the same correspondence is compactly supported, that is, the L^2 function $m(\phi_1\otimes\phi_2)(g) \in L^2(G,-\tau)$ is zero if g lives outside a certain compact set. Let us check that. For a function ϕ, in general, $\mathrm{supp}(\phi)$ denotes the support of ϕ. The support of $m(\phi_1\otimes\phi_2)(\bullet, g)$ is given by $\mathrm{supp}(\phi_1) \cap \left\{x \in G^\tau \mid x^{-1}g \in \mathrm{supp}(\phi_2)\right\}$, or more symbolically,

$$\mathrm{supp}(\phi_1) \cap g\left(\mathrm{supp}(\phi_2)^{-1}\right).$$

Since the both sets $\mathrm{supp}(\phi_1)$ and $\mathrm{supp}(\phi_2)$ are compact, and the G^τ-action on G^τ is proper, the set of $g \in G^\tau$ satisfying that the above intersection is non-empty, is compact.

Let us verify that m is a $(G \ltimes C_0(G), G \ltimes_\tau \mathbb{C})$-bimodule isomorphism.

$G \ltimes C_0(G)$-homomorphism: For a compactly supported function $a : G \to C_0(G)$,

$$\begin{aligned}
a * m(\phi_1\otimes\phi_2)(g)(x) &= \int a(h)h.\left(m(\phi_1\otimes\phi_2)(h^{-1}g)\right)dh(x)\\
&= \int a(h,x)m(\phi_1\otimes\phi_2)(h^{-1}g)(h^{-1}x)dh\\
&= \int a(h,x)\phi_1(h^{-1}x)\phi_2(x^{-1}g)dh\\
&= a*\phi_1(x)\phi_2(x^{-1}g)\\
&= m\left((a*\phi_1)\otimes\phi_2\right)(g)(x).
\end{aligned}$$

$G \ltimes_\tau \mathbb{C}$-homomorphism: For a compactly supported function $b : G^\tau \to \mathbb{C}$,

$$\begin{aligned}
m(\phi_1\otimes\phi_2) * b(g)(x) &= \int \phi_1(x)\phi_2(x^{-1}h)(b(h^{-1}g))dh\\
&= \phi_1(x)\int \phi_2(x^{-1}h)b(h^{-1}g)dh\\
&= \phi_1(x)\left(\phi_2 * b\right)(x^{-1}g)\\
&= m\left(\phi_1\otimes(\phi_2*b)\right)(g)(x).
\end{aligned}$$

Inner product: For $\phi_1\otimes\phi_2,\ \psi_1\otimes\psi_2 \in C_c(G)\otimes C_c(G,\tau)$,

$$\begin{aligned}
\left(m(\phi_1\otimes\phi_2)\middle|m(\psi_1\otimes\psi_2)\right)_{G\ltimes_\tau L^2(G,-\tau)}(g) &= \int \left(m(\phi_1\otimes\phi_2)(h)\middle|m(\psi_1\otimes\psi_2)(hg)\right)_{L^2(G,-\tau)}dh\\
&= \int\int \overline{\phi_1(x)\phi_2(x^{-1}h)}\psi_1(x)\psi_2(x^{-1}hg)dxdh\\
&= \int \overline{\phi_1(x)}\psi_1(x)\int \overline{\phi_2(x^{-1}h)}\psi_2(x^{-1}hg)dhdx\\
&= \left(\phi_1\middle|\psi_1\right)_{L^2(G)}\int \phi_2^*(h^{-1})\psi_2(hg)dh\\
&= \left(\phi_1\middle|\psi_1\right)_{L^2(G)}\left(\phi_2^**\psi_2\right)(g)\\
&= \left(\phi_1\middle|\psi_1\right)_{L^2(G)}\left(\phi_2\middle|\psi_2\right)_{G\ltimes_\tau\mathbb{C}}(g),
\end{aligned}$$

where we have used Fubini's theorem, change of variables, right invariance of the measure (i.e. unimodularity of G), and the definition of the involution $*$ in $G \ltimes_\tau \mathbb{C}$: $f^*(g) = \overline{f(g^{-1})}$.

Moreover, the image of m is dense. This is proved by a usual bump function argument.

Since m is an isometric embedding with dense range, the induced map between the completions is an isomorphism. □

We can now describe $j^G_\tau([\not{\partial}_R]) = (G \ltimes_\tau L^2(G,-\tau)\otimes S_G, \widetilde{\not{\partial}_R})$ under the above isomorphism m (recall that $[\widetilde{\not{\partial}_R}\phi](g)$ is defined by $\not{\partial}_R[\phi(g)]$ for a continuous and compactly supported $\phi : G^\tau \to L^2(G,-\tau)$). For this aim, we need to introduce two operators D and $\not{\partial}_L$. Let $\{e_i\}$ be the orthonomal basis of $\mathfrak{g}$ which we have chosen in order to define $\not{\partial}_R$. e_i's are also regarded as elements of $\mathfrak{g}^\tau$ via the splitting $s : \mathfrak{g} \to \mathfrak{g}^\tau$.

(1) Let R' be the right regular representation of G (not of G^τ!), and let dR' be its infinitesimal representation. We define an operator acing on $C^\infty_c(G)\otimes S_G \subseteq L^2(G)\otimes S_G$ by

$$D := \sum dR'_{e_i}\otimes \gamma(e_i).$$

(2) Let L be the left regular representation of G^τ, and let dL be its infinitesimal representation. We define the operator acting on $S_G \otimes C^\infty_c(G,\tau) \subseteq S_G \otimes (G\ltimes_\tau \mathbb{C})$ by

$$\not{\partial}_L := \sum_i \gamma(e_i)\otimes dL_{s(e_i)}.$$

With these operators and the isomorphism m, we can describe the operator $\widetilde{\not{\partial}_R}$ here. This formula is obtained by an easy computation using the Leibniz rule, and so we leave it to the reader (note that the derivation is with respect to the variable x, and x appears in the both components $L^2(G)$ and $G\ltimes_\tau \mathbb{C}$).

Proposition 3.4. *For $\phi \in C^\infty_c(G)$, $b \in C^\infty_c(G,\tau)$, and $s \in S$,*

$$m^{-1}\circ \widetilde{\not{\partial}_R}\circ m(\phi\otimes s\otimes b) = D(\phi\otimes s)\otimes b + \phi\otimes \not{\partial}_L(s\otimes b).$$

Recall the notation $\otimes_1$ and $\otimes_2$: For vector spaces V_i $(i=1,2,3)$ and an operator F on $V_1\otimes V_2$, define a new operator $F\otimes_2 \mathrm{id}$ by $v_1\otimes v_2\otimes v_3 \mapsto F(v_1\otimes v_2)\otimes v_3$ for $v_i \in V_i$, and similarly for $\mathrm{id}\otimes_1 G$ for an operator G on $V_2\otimes V_3$. With this notation, we summarize the computations in this subsection as a theorem.

Theorem 3.5. *$j^G_\tau([\not{\partial}_R])$ is represented by*

$$(L^2(G)\otimes S_G\otimes (G\ltimes_\tau \mathbb{C}),\ D\otimes_2 \mathrm{id} + \mathrm{id}\otimes_1 \not{\partial}_L).$$

3.3. The Mishchenko line bundle, analytic index and the assembly map

In this subsection, we will compute the assembly map $[c]\otimes j^G_\tau([\not{\partial}_R])$. For this aim, we will firstly compute the Mishchenko line bundle $[c]$, under the identification $G\ltimes C_0(G) \cong \mathcal{K}(L^2(G))$. This isomorphism will be specified in the proof. Then, we will describe the analytic index of $\not{\partial}_R$ in our language. This index coincides with the Kasparov product $[c]\otimes j^G_\tau([\not{\partial}_R])$ according to Fact 1.3. We will explicitly prove it here. Then, we will add a remark on this isomorphism.

Before the computation of the K-theory element $[c]$, we note that $L^2(G)^*$ has a canonical Hilbert $\mathcal{K}(L^2(G))$-module structure. In fact, for $\phi,\phi_1,\phi_2 \in L^2(G)^*$ and $k \in \mathcal{K}(L^2(G))$,

$$\phi\cdot k := \phi\circ k \text{ and } \left(\phi_1\middle|\phi_2\right)_{L^2(G)^*} := \phi_1^*\circ\phi_2,$$

where ϕ_1^* is the element of $L^2(G)^{**} \cong L^2(G)$, and it is the Riesz dual of ϕ_1.[7] $\phi_1^*\circ\phi_2$ denotes the operator given by $L^2(G)\ni v \mapsto \phi_1^*\langle\phi_2, v\rangle \in L^2(G)$.

Lemma 3.6. *Let $\mathfrak{c}$ be a cut-off function associated to $G\curvearrowright G$. Then, under the isomorphism $G\ltimes C_0(G)\cong\mathcal{K}(L^2(G))$, the projection element $c \in G\ltimes C_0(G)$ gives the rank one projection onto $\mathbb{C}\sqrt{\mathfrak{c}}$.*

The Hilbert module $c*(G\ltimes C_0(G))$ is isomorphic to the above Hilbert module $L^2(G)^*$. Consequently, the Mishchenko line bundle $[c]\in KK(\mathbb{C}, G\ltimes C_0(G))$ can be rewritten, under the isomorphism $G\ltimes C_0(G)\cong\mathcal{K}(L^2(G))$, as $(L^2(G)^*, 0)$.

Proof. We have given the definition of the representation $\pi : G\ltimes C_0(G) \to \mathcal{K}(L^2(G))$ at the beginning of the previous subsection. Let us consider the case when $a \in C_c(G, C_c(G))$ is of the form $a(x,g) = a_1(x)\overline{a_2(g^{-1}x)}$ for $a_1, a_2 \in C_c(G)$, where we regard g as an element of the *group* G, and x as an element of the *space* G. For $\phi \in L^2(G)$,

$$\begin{aligned} a*\phi(x) &= \int_G a_1(x)\overline{a_2(h^{-1}x)}\phi(h^{-1}x)dh \\ &= a_1(x)\left(a_2\middle|\phi\right). \end{aligned}$$

[7] Recall that the Riesz dual map is a conjugate linear isomorphism.

Hence such a corresponds to the Schatten form θ_{a_1,a_2} (This is the reason why the representation π takes values in the set of compact operators).

Since $c(x,g)$ is define by $\sqrt{\mathfrak{c}(x)\mathfrak{c}(g^{-1}x)}$, and it is of the form the above formula, $[c]$ corresponds to the Schatten form $\theta_{\sqrt{\mathfrak{c}},\sqrt{\mathfrak{c}}}$, which is the rank one projection onto $\mathbb{C}\sqrt{\mathfrak{c}}$.

From now on, we identify $G\ltimes C_0(G)$ with $\mathcal{K}(L^2(G))$, and regard c as the rank one projection onto $\mathbb{C}\sqrt{\mathfrak{c}}$, which is denoted by $P_{\sqrt{\mathfrak{c}}}$. Under this identification, the product in $G\ltimes C_0(G)$ is denoted by $\circ$ instead of $*$.

Let us verify that $P_{\sqrt{\mathfrak{c}}}\circ\mathcal{K}(L^2(G))$ is isomorphic to $L^2(G)^*$ as Hilbert $\mathcal{K}(L^2(G))$-modules. The isomorphism is given by

$$P_{\sqrt{\mathfrak{c}}}\circ\mathcal{K}(L^2(G))\ni P_{\sqrt{\mathfrak{c}}}\circ k\mapsto\sqrt{\mathfrak{c}}^*\circ k\in L^2(G)^*,$$

where $\sqrt{\mathfrak{c}}^*$ is the functional given by $\phi\mapsto\left(\sqrt{\mathfrak{c}}\middle|\phi\right)$. This is an isometry as Hilbert $(G\ltimes C_0(G))$-modules, because

$$\begin{aligned}\left(\sqrt{\mathfrak{c}}^*\circ k_1\middle|\sqrt{\mathfrak{c}}^*\circ k_2\right)_{L^2(G)^*}&=k_1^*\circ\sqrt{\mathfrak{c}}\circ\sqrt{\mathfrak{c}}^*\circ k_2\\&=k_1^*\circ P_{\sqrt{\mathfrak{c}}}\circ P_{\sqrt{\mathfrak{c}}}\circ k_2\\&=\left(P_{\sqrt{\mathfrak{c}}}\circ k_1\middle|P_{\sqrt{\mathfrak{c}}}\circ k_2\right)_{P_{\sqrt{\mathfrak{c}}}\circ\mathcal{K}(L^2(G))},\end{aligned}$$

where we regarded $\sqrt{\mathfrak{c}}\in L^2(G)$ as a linear map $\mathbb{C}\to L^2(G)$, and we used the definition of the projection onto $\mathbb{C}\sqrt{\mathfrak{c}}$, that is, $\sqrt{\mathfrak{c}}\circ\sqrt{\mathfrak{c}}^*=P_{\sqrt{\mathfrak{c}}}$.

Moreover, this map is obviously a surjective $\mathcal{K}(L^2(G))$-module homomorphism. □

Next, we compute the analytic index of $\not{\partial}_R$.

Recall that, in general, the analytic index of an unbounded equivariant Kasparov $(C_0(X),\mathbb{C})$-module $(L^2(X,E),D)$ is given by the pair of the completion of $C_c^\infty(X,E)$ with respect to a certain $G\ltimes\mathbb{C}$-valued inner product and the extension of D (whose domain must be extended so that it is self-adjoint).

In the τ-twisted situation, the index is in fact a Kasparov $(\mathbb{C},G\ltimes_\tau\mathbb{C})$-module. In order to describe the index in a systematical way for the twisted cases, we recall a notation: $G\ltimes_{-\tau}\mathbb{C}$ is the completion of $C_c(G,-\tau)$ with respect to the C^*-norm. Although it is a C^*-algebra, it has a Hilbert $G\ltimes_\tau\mathbb{C}$-module structure. Let us explain it here. Firstly, we notice that the C^*-algebra $G\ltimes_{-\tau}\mathbb{C}$ is anti-isomorphic to $G\ltimes_\tau\mathbb{C}$ by the correspondence $C_c(G,\tau)\ni f\mapsto f^\vee\in C_c(G,-\tau)$, where $f^\vee(g):=f(g^{-1})$ for $g\in G^\tau$. The same formula $f\mapsto f^\vee$ gives the inverse. These correspondences extend to their completions (see Lemma 3.10). Using these anti-isomorphisms, we introduce a Hilbert $G\ltimes_\tau\mathbb{C}$-module structure on $G\ltimes_{-\tau}\mathbb{C}$. Concretely, for $f,f_1,f_2\in C_c(G,-\tau)$ and $b\in C_c(G,\tau)$,

$$f\cdot b:=b^\vee*f,\ \text{ and }\ \left(f_1\middle|f_2\right)_{G\ltimes_{-\tau}\mathbb{C}}:=(f_2*f_1^*)^\vee.$$

Consequently, $(G\ltimes_{-\tau}\mathbb{C})\otimes S_G$ naturally has a Hilbert $G\ltimes_\tau\mathbb{C}$-module structure: $[f\otimes s]\cdot b:=(f\cdot b)\otimes s$ and $\left(f_1\otimes s_1\middle|f_2\otimes s_2\right)_{S_G\otimes(G\ltimes_{-\tau}\mathbb{C})}:=\left(f_1\middle|f_2\right)_{G\ltimes_{-\tau}\mathbb{C}}\cdot\left(s_1\middle|s_2\right)_{S_G}$ for $f,f_1,f_2\in G\ltimes_{-\tau}\mathbb{C}$, $b\in G\ltimes_\tau\mathbb{C}$ and $s_1,s_2\in S_G$.

On the dense subset $C_c^\infty(G,-\tau)\otimes S_G$ of this Hilbert module, we can define the operator $\not{\partial}_R$ by the obvious formula. We prove that this operator is a symmetric Hilbert module homomorphism. For this aim, we need to verify that $\left(f_1\middle|dR_X(f_2)\right)_{G\ltimes_{-\tau}\mathbb{C}}=-\left(dR_X(f_1)\middle|f_2\right)_{G\ltimes_{-\tau}\mathbb{C}}$. For this purpose, it is sufficient to check that $\left(f_1\middle|R_g(f_2)\right)_{G\ltimes_{-\tau}\mathbb{C}}=\left(R_g^{-1}(f_1)\middle|f_2\right)_{G\ltimes_{-\tau}\mathbb{C}}$ for any $g\in G^\tau$. This statement is checked as follows:

$$\begin{aligned}\left(f_1\middle|R_g(f_2)\right)_{G\ltimes_{-\tau}\mathbb{C}}(x)&=[R_g(f_2)*f_1^*](x^{-1})\\&=\int f_2(hg)\overline{f_1([h^{-1}x^{-1}]^{-1})}dh\\&=\int f_2(h)\overline{f_1(xhg^{-1})}dh\\&=\left(R_{g^{-1}}(f_1)\middle|f_2\right)_{G\ltimes_{-\tau}\mathbb{C}}(x).\end{aligned}$$

Proposition 3.7. *The analytic index of $\not{\partial}_R$ is given by the above $((G\ltimes_{-\tau}\mathbb{C})\otimes S_G,\not{\partial}_R)$.*

Proof. The Hilbert $G\ltimes_\tau\mathbb{C}$-module for the analytic index is given by the completion of $C_c^\infty(G,-\tau)\otimes S_G$ with respect to the following pre-Hilbert module structure: For $f\otimes s,f_1\otimes s_1,f_2\otimes s_2\in C_c^\infty(G,-\tau)\otimes S_G$, and $b\in C_c(G,\tau)$,

$$\begin{aligned}(f\otimes s)\cdot b(x)&=\int_{G^\tau}g.[f\otimes s](g^{-1}.x)b(g^{-1})dg\\&=\int_{G^\tau}b^\vee(g)f(g^{-1}x)dgs=(f\cdot b)\otimes s(x).\end{aligned}$$

$$\left(f_1\otimes s_1\middle|f_2\otimes s_2\right)(g)=\int_{G^\tau}\left(f_1\otimes s_1(h)\middle|f_2\otimes s_2(g^{-1}h)\right)_{S_G}dh$$

$$
\begin{aligned}
&= \int_{G^\tau} \overline{f_1(h)} f_2(g^{-1}h) \left(s_1 \middle| s_2\right)_{S_G} dh \\
&= [f_2 * f_1^*](g^{-1}) \left(s_1 \middle| s_2\right)_{S_G} \\
&= \left(f_1 \middle| f_2\right)_{G \ltimes_{-\tau} \mathbb{C}} \left(s_1 \middle| s_2\right)_{S_G}.
\end{aligned}
$$

These formulas are the same with the above Hilbert $G \ltimes_\tau \mathbb{C}$-module structure on $(G \ltimes_{-\tau} \mathbb{C}) \otimes S_G$.

The operator *on the Hilbert* $G \ltimes_\tau \mathbb{C}$*-module* $(G \ltimes_{-\tau} \mathbb{C}) \otimes \mathbb{C}$ is nothing but the extension of the original operator $\not{\partial}_R$ *acting on* $C_c^\infty(G, -\tau) \otimes S_G \subseteq L^2(G, -\tau) \otimes S_G$. Thus, thanks to [21, Theorem 5.8], the above pair gives an unbounded Kasparov module, and it is the analytic index of $\not{\partial}_R$. □

We have known that the above analytic index gives the Kasparov product $[c] \otimes j_\tau^G([\not{\partial}_R])$. We explicitly prove it here.

Proposition 3.8. (1) *The tensor product module giving the Kasparov product* $[c] \otimes j_\tau^G([\not{\partial}_R])$

$$[L^2(G)]^* \otimes_{\mathcal{K}(L^2(G))} [L^2(G) \otimes S_G \otimes (G \ltimes_\tau \mathbb{C})]$$

is isomorphic to $S_G \otimes (G \ltimes_\tau \mathbb{C})$ *by* $\phi \otimes f \otimes s \otimes b \mapsto \langle \phi, f \rangle s \otimes b$ *for* $\phi \in L^2(G)^*$, $f \in L^2(G)$, $s \in S_G$ *and* $b \in G \ltimes_\tau \mathbb{C}$. *It is also isomorphic to* $(G \ltimes_{-\tau} \mathbb{C}) \otimes S_G$ *by the correspondence* $s \otimes b \mapsto b^\vee \otimes s$.

(2) *Under this isomorphism, the analytic index is a Kasparov product* $[c] \otimes j_G^\tau([\not{\partial}_R])$.

Proof. (1) Since $k \in \mathcal{K}(L^2(G))$ acts on $L^2(G) \otimes S_G \otimes (G \ltimes_\tau \mathbb{C})$ as $k \otimes \mathrm{id} \otimes \mathrm{id}$, in order to compute the whole tensor product, it is sufficient to verify that $L^2(G)^* \otimes_{\mathcal{K}(L^2(G))} L^2(G) \cong \mathbb{C}$ as Hilbert $(\mathbb{C}, \mathbb{C})$-modules by $\phi \otimes_{\mathcal{K}(L^2(G))} v \mapsto \langle \phi, v \rangle$ for $\phi \in L^2(G)^*$ and $v \in L^2(G)$, where $\langle \bullet, \bullet \rangle$ is the natural pairing between $L^2(G)^*$ and $L^2(G)$. This is verified as follows. Firstly, we notice that $\phi \in L^2(G)^*$ can be rewritten as $\sqrt{\mathfrak{c}}^* \circ [\sqrt{\mathfrak{c}} \otimes_{\mathbb{C}} \phi]$, where $\sqrt{\mathfrak{c}} \otimes_{\mathbb{C}} \phi$ is the rank one operator given by $y \mapsto \sqrt{\mathfrak{c}} \langle \phi, y \rangle$. Thus the tensor product $\sqrt{\mathfrak{c}}^* \circ [\sqrt{\mathfrak{c}} \otimes_{\mathbb{C}} \phi] \otimes_{\mathcal{K}(L^2(G))} v$ equals $\sqrt{\mathfrak{c}}^* \otimes_{\mathcal{K}(L^2(G))} (\sqrt{\mathfrak{c}} \otimes_{\mathbb{C}} \phi)(v) = \sqrt{\mathfrak{c}}^* \otimes_{\mathcal{K}(L^2(G))} \sqrt{\mathfrak{c}} \langle \phi, v \rangle$. Consequently, the Hilbert module $L^2(G)^* \otimes_{\mathcal{K}(L^2(G))} [L^2(G) \otimes S_G \otimes (G \ltimes_\tau \mathbb{C})]$ is isomorphic to $S_G \otimes (G \ltimes_\tau \mathbb{C})$.

Next, we define an isomorphism from $S_G \otimes (G \ltimes_\tau \mathbb{C})$ to $(G \ltimes_{-\tau} \mathbb{C}) \otimes S_G$ by $s \otimes f \mapsto f^\vee \otimes s$. For this aim, it is sufficient to verify that the correspondence $\Phi : G \ltimes_\tau \mathbb{C} \ni f \mapsto f^\vee \in G \ltimes_{-\tau} \mathbb{C}$ gives an isomorphism as Hilbert $G \ltimes_\tau \mathbb{C}$-modules. It is easily proved, and we omit the proof.

In what follows, to be precise, Φ denotes the isomorphism

$$L^2(G)^* \otimes_{\mathcal{K}(L^2(G))} [L^2(G) \otimes S_G \otimes (G \ltimes_\tau \mathbb{C})] \to (G \ltimes_{-\tau} \mathbb{C}) \otimes S_G$$

given by $\phi \otimes [v \otimes s \otimes b] \mapsto \langle \phi, v \rangle b^\vee \otimes s$.

(2) Recall Proposition 2.5. Since $[c]$ can be represented by $(L^2(G)^*, 0)$ in the language of Kasparov modules, the second and the third conditions are satisfied (this is because "D_1"$= 0$). Let us verify the first condition. Take a smooth and compactly supported function $\phi \in L^2(G)^*$ (Since $L^2(G)$ is anti-isomorphic to $L^2(G)$, we can regard an element of $L^2(G)^*$ as a function). The first condition can be described as follows:

$$\not{\partial}_R \circ \Phi \circ T_\phi - T_\phi \circ \Phi \circ [D \otimes_2 \mathrm{id} + \mathrm{id} \otimes_1 \not{\partial}_L] \text{ is bounded.}$$

One can easily prove that $\not{\partial}_R \circ \Phi \circ T_\phi = T_\phi \circ \Phi \circ (\mathrm{id} \otimes_1 \not{\partial}_L)$. Thus, it is sufficient to verify that $T_\phi \circ \Phi \circ [D \otimes_2 \mathrm{id}]$ is a bounded operator. Since $D = \sum dR_{e_i} \otimes \gamma(e_i)$ is a finite sum, it suffices to check the correspondence from $C_c^\infty(G) \otimes S_G \otimes C_c^\infty(G, \tau)$ to $C_c(G, -\tau) \otimes S_G$ given by the composition of

$$f \otimes s \otimes b \mapsto dR_{e_i}(f) \otimes \gamma(e_i)(s) \otimes b \mapsto \left\langle \phi, dR_{e_i}(f) \right\rangle b^\vee \otimes \gamma(e_i)(s)$$

is bounded. For this purpose, since $\gamma(e_i)$ is bounded, it is sufficient to check that $f \mapsto \left\langle \phi, dR_{e_i}(f) \right\rangle$ is bounded in the L^2-topology. This is proved as follows:

$$
\begin{aligned}
\left\langle \phi, dR_{e_i}(f) \right\rangle &= \int \phi(x) dR_{e_i}(f)(x) dx \\
&= -\int dR_{e_i}(\phi)(x) f(x) dx \\
&= -\left\langle dR_{e_i}(\phi), f \right\rangle.
\end{aligned}
$$

Since $dR_{e_i}^*(\phi)$ is again an L^2 function, the correspondence $f \mapsto \phi \left\langle dR_{e_i}^*(\phi), f \right\rangle$ is bounded. Hence the above correspondence is bounded. □

Remark 3.9. The above proof essentially contains an important argument: $k \circ dR : L^2 \to L^2$ is bounded for "smooth" k, nevertheless dR is unbounded.

Some analogy of this will appear in the next subsection.

We add a small remark on the correspondence $\Phi : G \ltimes_\tau \mathbb{C} \ni f \mapsto f^\vee \in G \ltimes_{-\tau} \mathbb{C}$.

We defined Φ in terms of *functions* on G^τ. However, we should understand it in terms of *operators* here for the next subsection. For this purpose, we begin with the $*$-representation of the twisted group C^*-algebra, occurring from a group representation. Let (V, σ) be a τ-twisted unitary representation on a Hilbert space of G at level 1: a homomorphism $\sigma : G^\tau \to U(V)$ satisfying that $\sigma(i(z)) = z\mathrm{id}_V$. It naturally induces a representation π_σ^- on $G \ltimes_{-\tau} \mathbb{C}$ by the formula

$$\pi_\sigma^-(f)(v) := \int f(g)\sigma(g)(v)dg \in \mathrm{End}(V)$$

for $f \in C_c(G, -\tau) \subseteq G \ltimes_{-\tau} \mathbb{C}$. Note that if f were at level $l \neq -1$, this integration would be always 0. The dual representation (V^*, σ^*) given by $\sigma^*(g) = {}^t\sigma(g^{-1})$, naturally gives a $*$-representation on $G \ltimes_\tau \mathbb{C}$ denoted by $\pi_\sigma^+ : G \ltimes_\tau \mathbb{C} \to \mathrm{End}(V^*)$:

$$\pi_\sigma^+(f)(\phi) := \int f(g)\sigma^*(g)(\phi)dg \in \mathrm{End}(V^*)$$

for $f \in C_c(G, \tau) \subseteq G \ltimes_\tau \mathbb{C}$.

The meaning of the phrase "*understanding Φ in terms of operators*" is, concretely, to describe $\pi_\sigma^-(\Phi(f))$ using the operator $\pi_\sigma^+(f)$. The answer is the following. Note that the bidual V^{**} of the representation space V is naturally isomorphic to V.

Lemma 3.10. *Let $\Psi : \mathrm{End}(V^*) \to \mathrm{End}(V)$ be the transpose of operators. Then, the following diagram commutes:*

$$\begin{array}{ccc} G \ltimes_\tau \mathbb{C} & \xrightarrow{\pi_\sigma^+} & \mathrm{End}(V^*) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ G \ltimes_{-\tau} \mathbb{C} & \xrightarrow{\pi_\sigma^-} & \mathrm{End}(V). \end{array}$$

Shortly, $\pi_\sigma^-(f^\vee) = {}^t(\pi_\sigma^-(f))$.

Proof. It is proved by a direct computation. For $f \in C_c(G, -\tau)$ and $\phi \in V^*$ and $v \in V$,

$$\begin{aligned} \pi_\sigma^+(\Phi(f))(\phi)(v) &= \left\langle \int f^\vee(g)\sigma^*(g)(\phi)dg, v \right\rangle_V \\ &= \int \left\langle f(g^{-1}){}^t\sigma(g^{-1})(\phi), v \right\rangle_V dg \\ &= \int \left\langle \phi, f(g^{-1})\sigma(g^{-1})v \right\rangle_V dg \\ &= \left\langle \phi, \pi_\sigma^-(f)(v) \right\rangle_V = {}^t[\pi_\sigma^-(f)](\phi)(v). \quad \square \end{aligned}$$

An infinite-dimensional Lie group $\mathcal{G}$ does not have a good measure, and hence the group C^*-algebra of $\mathcal{G}$ does not make sense. However, it is still worth trying to construct a substituting C^*-algebra using a representation theory of $\mathcal{G}$. In fact, we have constructed the "$\pm\tau$-twisted group C^*-algebra of LT" for positive definite central extension τ.

This C^*-algebra does not come from a function algebra. Thus the definition "$LT \ltimes_\tau \mathbb{C} \ni b \mapsto b^\vee \in LT \ltimes_{-\tau} \mathbb{C}$" is not clear. Nevertheless, thanks to the above observation, this operation still makes sense for LT.

4. Infinite-dimensional case

In this section, we will construct several core objects to study the assembly maps for some infinite-dimensional manifolds. The organization of this section is parallel to that of the previous section, except that we will give the precise setting of this paper firstly.

Firstly, we will give the precise setting of our problem, and we will explain some geometrical traits on it. As a result, our manifold turns out to be a product of a finite-dimensional non-compact manifold and a certain infinite-dimensional vector space denoted by U. Thanks to Section 2.3, we may concentrate on the U-part.

Secondly, we will recall the "L^2-space of U", the "Dirac operator" and the "twisted group C^*-algebra of U" from the previous paper [36]. The symbol $\underline{\underline{[\not{\partial}_R]}}$ denotes the pair of the "L^2-space" and the "Dirac operator". This pair should be regarded as an infinite-dimensional version of the "index element" of [21], although it has not been given a Kasparov module structure because the "function algebra of U" has not defined.

Thirdly, we will construct "the value of the assembly map at $\underline{\underline{[\not{\partial}_R]}}$" denoted by $\underline{j_\tau^U([\not{\partial}_R])}$, imitating the corresponding object in the finite-dimensional case. For this aim, we will introduce a Hilbert space which is regarded as "$L^2(U)$", an operator which is regarded as a "Dirac operator without potential" D, and a C^*-algebra which is regarded as "$U \ltimes C_0(U)$". Notice that "$\underline{j_\tau^U([\not{\partial}_R])}$" is actually a Kasparov module, although $\underline{\underline{[\not{\partial}_R]}}$ itself does not make sense as a Kasparov module.

Finally, we will define the "Mishchenko line bundle $\underline{[c]}$" and the "analytic index", and then we will prove that the index coincides with the Kasparov product of $\underline{[c]}$ and $\underline{j^U_\tau([\not{\partial}_R])}$.

Already existing "Dirac operators on infinite-dimensional spaces", including [13,14] and [11], converge for algebraic reasons: On some dense spaces, the infinite sums are in fact finite sums. However, the operator appearing in $\underline{j^U_\tau([\not{\partial}_R])}$ makes sense for some **analytic** reason. This is the newest point of this paper. We believe that this new technique will be essential for the construction of infinite-dimensional global analysis.

4.1. The setting, the factorization and the result

Let T be the circle group, and LT be its loop group. The dual Lie algebra has the gauge action of LT given by $l.A := A - l^{-1}dl$ for $l \in LT$ and $A \in L\mathfrak{t}^*$, where we regard $L\mathfrak{t}^*$ as the set of $\mathfrak{t}$-valued 1-forms on the circle, and we identify it with the set of connections on the trivial principal T-bundle on the circle. Briefly speaking, we study an index problem on infinite-dimensional manifolds which are "the same with $LT \curvearrowright L\mathfrak{t}^*$ up to compact sets". The concrete definition is the following.

Definition 4.1. An infinite-dimensional manifold $\mathcal{M}$ equipped with a smooth LT-action is called a **proper LT-space**, if it admits a proper equivariant map $\Phi : \mathcal{M} \to L\mathfrak{t}^*$.

A proper LT-space has a nice property, as proved in [36]. In order to summarize it here, we recall the canonical decomposition of LT:

$$LT = T \times \Pi_T \times U,$$

where T is the set of constant loops, Π_T is the set of rotation numbers, and U is the rest part. The precise isomorphism is given by the following steps: (1) The set of rotation numbers is nothing but the set of connected components of LT; (2) A loop in the identity component can be written as $\exp(f)$ for some $f \in C^\infty(S^1, \mathfrak{t})$; and (3) $\exp(\frac{1}{2\pi}\int f(\theta)d\theta) \in T$ and $\exp\left[f - \frac{1}{2\pi}\int f(\theta)d\theta\right] \in U$ are independent from the choice of f, and these give the right-hand-side counter part of the contractible loop $\exp(f)$. More concretely, U can be regarded as the Lie subgroup of LT corresponding to the subspace

$$\left\{\beta : S^1 \to \mathfrak{t} \;\middle|\; \int \beta(\theta)d\theta = 0\right\}.$$

This infinite-dimensional Lie group is the central object of this section.

Let us observe the LT-action on $L\mathfrak{t}^*$. LT has the subgroup $\Pi_T \times U$ (which is isomorphic to the fixed loop group ΩT). This subgroup acts on $L\mathfrak{t}^*$ freely, and the quotient space is diffeomorphic to T. In fact, the projection is given by the holonomy map $\mathrm{hol} : L\mathfrak{t}^* \to T$, identifying $L\mathfrak{t}^*$ with the set of connections. Since $\Phi : \mathcal{M} \to L\mathfrak{t}^*$ is equivariant, $\Pi_T \times U$ acts on $\mathcal{M}$ freely. The quotient space is denoted by M. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & L\mathfrak{t}^* \\ \downarrow & & \downarrow \\ M & \longrightarrow & T. \end{array}$$

Since the $\Pi_T \times U$-action is free and Φ is a proper map, M is a compact manifold. Conversely, given a compact T-manifold M with a T-equivariant map to T (T acts on itself trivially), we can construct a proper LT-space $\mathcal{M}$. These arguments are still valid for noncommutative loop groups. See [1] and [33] also.

For abelian loop groups, we have a slightly better result on this line. $L\mathfrak{t}^*$ has a canonical decomposition $\mathfrak{t} \times U$. This is because the map $\int : L\mathfrak{t}^* \to \mathfrak{t}$ gives a principal U-bundle, and the set of constant 1-forms $\mathfrak{t} \subseteq L\mathfrak{t}^*$ is a global slice. The holonomy map is given by the composition of $\int : L\mathfrak{t}^* \to \mathfrak{t}$ and $\exp : \mathfrak{t} \to T$. Let $\widetilde{M}$ be $\Phi^{-1}(\mathfrak{t})$. Since Φ is transverse to $\mathfrak{t}$, $\widetilde{M}$ is a submanifold of $\mathcal{M}$, thanks to the implicit function theorem. Since Φ is an equivariant map, $\widetilde{M}$ is a global slice. These arguments can be summarized as follows.

Proposition 4.2. (1) *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & L\mathfrak{t}^* \\ \downarrow & & \downarrow \\ \widetilde{M} & \longrightarrow & \mathfrak{t} \\ \downarrow & & \downarrow \\ M & \longrightarrow & T. \end{array}$$

(2) *$\mathcal{M}$ is diffeomorphic to the direct product $\widetilde{M} \times U$. This decomposition is compatible with the group action in the following sense: Let $l \in LT$ and $(t, n, u) \in T \times \Pi_T \times U$ be the corresponding element to l. Let $(m, v) \in \widetilde{M} \times U$, then*

$$(t, n, u).(m, v) = ((t, n).m, u \cdot v).$$

Example 4.3. Suppose that a compact symplectic manifold (M, ω) equipped with a T-action preserving ω, is given. Then, with a small perturbation of ω, there is a circle valued moment map $\phi : M \to T$ [28]. In particular, ϕ is a T-equivariant map. Considering the fiber product $\mathcal{M} := M \times_T Lt^*$, we obtain a Hamiltonian LT-space. More generally, if we have a compact T-manifold M, we can define a proper LT-space by the same formula.

In the context of representations of loop groups, only "admissible central extensions", defined in the following, are usually studied. Intuitively, they are $U(1)$-central extensions which are compatible with the rotation symmetry of loop groups given by $\theta_0.l(\theta) := l(\theta + \theta_0)$. In what follows, we use the notation $\mathbb{T}_{\mathrm{rot}}$ to denote the circle group acting on the loop groups by this rotation symmetry.

Definition 4.4 (*Definition 2.10 [11]*)**.** A $U(1)$-central extension $1 \to U(1) \xrightarrow{i} LT^\tau \xrightarrow{p} LT \to 1$ is admissible if the following conditions are satisfied:

(1) The $\mathbb{T}_{\mathrm{rot}}$-action on LT lifts to LT^τ, therefore, we can construct the semi-direct product $\mathbb{T}_{\mathrm{rot}} \ltimes LT^\tau$.
(2) There exists a $(\mathbb{T}_{\mathrm{rot}} \ltimes LT^\tau)$-invariant symmetric bilinear form $\langle\langle \bullet, \bullet \rangle\rangle_\tau$ on its Lie algebra such that $\langle\langle K, D \rangle\rangle_\tau = -1$ for any $D \in \mathrm{Lie}(\mathbb{T}_{\mathrm{rot}} \ltimes LG^\tau)$ projecting to the infinitesimal generator of $\mathbb{T}_{\mathrm{rot}}$, and K is the generator of the added circle $i(U(1)) \subseteq LT^\tau$.

An admissible central extension τ is said to be positive if the restriction of $\langle\langle \bullet, \bullet \rangle\rangle_\tau$ to Lt is positive definite.

Admissible central extensions have been classified. In particular, an admissible central extension of LT is in fact the exterior tensor product of a central extension of $T \times \Pi_T$ and a central extension of U [11, Proposition 2.27]. In this paper, we study the following extension.

Definition 4.5. For a fixed $k \in \mathbb{Z} - \{0\}$, let $\tau : LT \times LT \to U(1)$ be the 2-cocycle defined by the following formula under the canonical identification $LT = T \times \Pi_T \times U$:

- $\tau(T \times \Pi_T, U) = \tau(U, T \times \Pi_T) = 1$.
- $\tau((t_1, n_1), (t_2, n_2)) = t_2^{kn_1}$ for $t_1, t_2 \in T$ and $n_1, n_2 \in \Pi_T$.
- $\tau(\exp(l_1), \exp(l_2)) = \exp\left(i \int l_1 \frac{dl_2}{d\theta} d\theta\right)$, where $\exp(l_1), \exp(l_2) \in U$.

The $U(1)$-central extension of LT is given by $LT \times U(1)$ as a space, and

$$(l_1, z_1) \cdot (l_2, z_2) := (l_1 l_2, z_1 z_2 \tau(l_1, l_2))$$

as a group. LT^τ denotes the $U(1)$-central extension of LT given above.

Central extensions can be restricted to subgroups. In particular, we will deal with the restrictions to $T \times \Pi_T$ and U. These restrictions are denoted by $(T \times \Pi_T)^\tau$ and U^τ respectively.

Let us set up the problem precisely.

Problem 4.6. Suppose a proper LT-space $\mathcal{M}$ so that $\widetilde{M}$ is even-dimensional and $T \times \Pi_T$-equivariantly $Spin^c$, is given. We further suppose that a τ-twisted LT-equivariant line bundle $\mathcal{L} \to \mathcal{M}$ is given. In this setting, construct an LT-equivariant Spinor bundle $\mathcal{S} \to \mathcal{M}$, and study the index problem for the τ-twisted LT-equivariant Spinor bundle $\mathcal{L} \otimes \mathcal{S}$ in terms of KK-theory.

We have factorize the manifold $\mathcal{M}$ and the central extension τ. Vector bundles can be also factorized. The proof of the following is an easy exercise. See [35] for the proof.

Lemma 4.7. *A τ-twisted LT-equivariant line bundle $\mathcal{L}$ can be written as the exterior tensor product of the restriction of $\mathcal{L}$ to $\widetilde{M}$ and the line bundle*

$$U^\tau \times_{U(1)} \mathbb{C} = U \times \mathbb{C}/(u, v) \sim (uz, z^{-1}v)$$

over U.

The line bundle $U^\tau \times_{U(1)} \mathbb{C} \to U$ is denoted by $\mathcal{L}$ in the following.

For the Spinor bundle, we choose an easy way. For more geometrical construction, see [27]. Since $\mathcal{M}$ is the direct product $\widetilde{M} \times U$, it is natural to assume that these two factors are orthogonal. We have assumed that $\widetilde{M}$ has an equivariant Spinor bundle S. We will give a review on the Spinor bundle S_U over U following [11]. By the tensor product of these bundles, we define the Spinor bundle over $\mathcal{M}$ by the exterior tensor product $\mathcal{S} := S_{\widetilde{M}} \boxtimes S_U$.

Combining all observations of this subsection and the product formulas studied in Section 2, we reduce our main problem to that on U.

In order to explain the result, we recall the notation for substituting object from the end of the Introduction: The symbol $\underline{\star\star}$ is an actual object (a C^*-algebra, a Hilbert space or a Kasparov module) which substitutes "$\star\star$".

The following is the result of this section combined with [36].

Theorem 4.8. *For the τ-twisted U-equivariant Clifford bundle $\mathcal{L}\otimes S_U$, we can construct the followings:*

(1) a Hilbert space $\underline{L^2(U,\mathcal{L}\otimes S_U)}$.

(2) an operator which can be regarded as the "Dirac operator" $\not{\partial}_R$ acting on $\underline{L^2(U,\mathcal{L}\otimes S_U)}$. Consequently a pair $\underline{\underline{[\not{\partial}_R]}} := (\underline{L^2(U,\mathcal{L}\otimes S_U)},\underline{\not{\partial}_R})$ is defined.[8]

(3) a C^*-algebra $\underline{U\ltimes C_0(U)}$.

(4) an unbounded Kasparov $(\underline{U\ltimes C_0(U)},\underline{U\ltimes_\tau\mathbb{C}})$-module $\underline{j^U_\tau([\not{\partial}_R])}$, which is regarded as the "value of the partial descent homomorphism at $\underline{\underline{[\not{\partial}_R]}}$".

(5) an (unbounded) Kasparov $(\mathbb{C},\underline{U\ltimes C_0(U)})$-module $\underline{[c]}$ which is regarded as the Mishchenko line bundle for $U\curvearrowright U$.

(6) an unbounded Kasparov module $\underline{\mathrm{ind}(\not{\partial}_R)}$.

Moreover, $\underline{\mathrm{ind}(\not{\partial}_R)}$ gives a Kasparov product of $\underline{[c]}$ and $\underline{j^U_\tau([\not{\partial}_R])}$.

(1) and (2) have been constructed in [36], and we will give a brief review in Section 4.2. Then, we will construct (3) and (4) in Section 4.3. The remaining parts will be done in the final subsection.

4.2. The "index element" of U

We quickly recall the "index element" $(\underline{L^2(U,\mathcal{L}\otimes S_U)},\underline{\not{\partial}_R})$ and the "τ-twisted group C^*-algebra" $\underline{U\ltimes_\tau\mathbb{C}}$ from [36]. For this aim, we must begin with the definition of a certain class of representations of loop groups.

In order to detect some analytic (or geometric) information of loop groups from representations, it is natural to impose the following conditions: (1) they are infinite-dimensional; (2) they reflect the rotation symmetry; and (3) they satisfy some finiteness. There are no non-trivial ordinary representations satisfying these conditions, which is the reason why admissible central extensions are necessary.

Definition 4.9 (*[32]*)**.** Let G be a compact Lie groups, and LG^τ be an admissible $U(1)$-central extension of LG. A strongly continuous unitary representation $\rho: LG^\tau\to U(\mathcal{H})$ is said to be a **positive energy representation** (PER for short) at level τ if the following conditions are satisfied:

(1) The added center $i(U(1))$ acts on $\mathcal{H}$ as scalar multiplication, that is, ρ is a τ-twisted representation of LT.
(2) ρ lifts to the semi-direct product $\mathbb{T}_{\mathrm{rot}}\ltimes LG^\tau$.
(3) When we decompose $\mathcal{H}$ via the weight of the $\mathbb{T}_{\mathrm{rot}}$-action as $\mathcal{H}=\bigoplus_n\mathcal{H}_n$, each $\mathcal{H}_n$ is finite-dimensional and $\mathcal{H}_n=0$ for sufficiently small n, where $e^{i\theta}\in\mathbb{T}_{\mathrm{rot}}$ acts on $\mathcal{H}_n$ as $e^{in\theta}\mathrm{id}$.

Remarks 4.10. (1) This definition makes sense for topological groups equipped with a rotation symmetry. In particular, we can consider PERs of U.

(2) The infinitesimal generator of the rotation action on $L\mathfrak{g}$ is denoted by d in this paper, and hence the infinitesimal $\mathbb{T}_{\mathrm{rot}}$-action on a representation (V,ρ) is denoted by $d\rho(d)$. This is the same with the **energy operator** of [11] denoted by E up to multiplication by i: $E:=\frac{1}{i}d\rho(d)$.

Let us introduce finite-dimensional approximations of U.

Definition 4.11. U_N is the Lie subgroup corresponding to the subspace spanned by

$$\frac{\cos\theta}{\sqrt{\pi}},\frac{\sin\theta}{\sqrt{\pi}},\frac{\cos 2\theta}{\sqrt{2\pi}},\frac{\sin 2\theta}{\sqrt{2\pi}},\dots,\frac{\cos N\theta}{\sqrt{N\pi}},\frac{\sin N\theta}{\sqrt{N\pi}}$$

in the Lie algebra $\mathrm{Lie}(U)=\left\{\beta:S^1\to\mathfrak{t}\;\middle|\;\int\beta(\theta)d\theta=0\right\}$.

Remarks 4.12. (1) The coefficients on trigonometric functions are chosen to be an orthonomal bases with respect to the $L^2_{1/2}$-metric. We use this metric so that the central extension Definition 4.5 is continuous. It leads to the (slightly ugly) factor $\sqrt{n}$ in Definition 4.18.

[8] $\underline{\underline{[\not{\partial}_R]}}$ makes sense as a pair of a Hilbert space and an operator. However, it does not make sense as a Kasparov module, because there is no C^*-algebra substituting for "$C_0(U)$".

(2) Note that PERs of U_N make sense, because this subgroup is stable under the $\mathbb{T}_{\mathrm{rot}}$-action.
(3) The multiplication on $U_N^\tau = U_N \times U(1)$ is given by:

$$\left(a\frac{\cos k\theta}{\sqrt{k\pi}}, z_1\right)\cdot\left(b\frac{\sin k\theta}{\sqrt{k\pi}}, z_2\right) := \left(a\frac{\cos k\theta}{\sqrt{k\pi}} + b\frac{\sin k\theta}{\sqrt{k\pi}}, z_1z_2e^{\frac{i}{2}ab}\right)$$

and so on.

This subgroup is stable under the rotation symmetry. The infinitesimal action of d on $\mathrm{Lie}(U_N)$ is computed as follows:

$$d\left(\frac{\cos k\theta}{\sqrt{k\pi}}\right) = -k\frac{\sin k\theta}{\sqrt{k\pi}} \text{ and } d\left(\frac{\sin k\theta}{\sqrt{k\pi}}\right) = k\frac{\cos k\theta}{\sqrt{k\pi}}.$$

We define $J := d/|d|$, and then it gives a complex structure on $\mathrm{Lie}(U_N)$:

$$J\left(\frac{\cos k\theta}{\sqrt{k\pi}}\right) = -\frac{\sin k\theta}{\sqrt{k\pi}} \text{ and } J\left(\frac{\sin k\theta}{\sqrt{k\pi}}\right) = \frac{\cos k\theta}{\sqrt{k\pi}}.$$

We introduce a complex basis on $\mathrm{Lie}(U_N)\otimes\mathbb{C}$ by

$$z_n := \frac{1}{\sqrt{2}}\left(\frac{\cos(n\theta)}{\sqrt{n\pi}} + i\frac{\sin(n\theta)}{\sqrt{n\pi}}\right), \overline{z_n} := \frac{1}{\sqrt{2}}\left(\frac{\cos(n\theta)}{\sqrt{n\pi}} - i\frac{\sin(n\theta)}{\sqrt{n\pi}}\right).$$

Then we have the formulas

$$J(z_n) = iz_n \text{ and } J(\overline{z_n}) = -i\overline{z_n};$$
$$d(z_n) = inz_n \text{ and } d\overline{z_n} = -in\overline{z_n}.$$

Remarks 4.13. (1) The operations d and J, needless to say, make sense on $\mathrm{Lie}(U)$. Formulas on it are still valid: $d(z_n) = inz_n$ and so on.
(2) A finite linear combination of z_n's and $\overline{z_n}$'s is called a **finite energy vector**, and the set of finite energy vectors is denoted by $(\mathrm{Lie}(U)\otimes\mathbb{C})_{\mathrm{fin}}$. It is clearly a dense subspace of $\mathrm{Lie}(U)\otimes\mathbb{C}$.

Each U_N has only one irreducible PER on $L^2(\mathbb{R}^N)$ up to isomorphism (see [23] for example). This is because a PER of U_N is essentially the same with the canonical commutation relation which is a fundamental object of quantum mechanics [2].

This representation has another description on the Hilbert space completion of the symmetric algebra $\widehat{S}((\mathrm{Lie}(U_N)\otimes\mathbb{C})_+)$ (see [23] again). Although the Hilbert space "$L^2(\mathbb{R}^\infty)$" does not make sense, the symmetric algebra $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)$ does make sense, and it admits a PER of U [11,32]. This is the unique PER of U up to isomorphism.

We explicitly describe the (essentially unique) irreducible PER of U on $\widehat{S}((\mathrm{Lie}(U_N)\otimes\mathbb{C})_+)$ here. We do that at the infinitesimal level. Recall the commutation relations on $\mathrm{Lie}(U^\tau)\otimes\mathbb{C}$:

$$[z_n, z_m] = [\overline{z_n}, \overline{z_m}] = 0;$$
$$[z_n, \overline{z_m}] = 0 \text{ if } n \neq m; \text{ and}$$
$$[z_n, \overline{z_n}] = -iK,$$

where K is the infinitesimal generator of $i(U(1)) \subseteq U^\tau$.

Definition 4.14 ([32]). Let $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)_{\mathrm{fin}}$ be the symmetric algebra of $\bigoplus_{n>0}\mathbb{C}z_n$. It admits an inner product

$$\left\langle z_1^{k_1}z_2^{k_2}\cdots, z_1^{l_1}z_2^{l_2}\cdots\right\rangle := k_1!k_2!\cdots\delta_{k_1,l_1}\delta_{k_2,l_2}\cdots.$$

The completion of $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)_{\mathrm{fin}}$ with respect to this inner product is denoted by $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)$. On this Hilbert space, we define a PER of U by the followings:

$$d\rho(z_n)(z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n}\cdots) := z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n+1}\cdots;$$
$$d\rho(\overline{z_n})(z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n}\cdots) := -k_nz_1^{k_1}z_2^{k_2}\cdots z_n^{k_n-1}\cdots;$$
$$d\rho(K) := i\mathrm{id};$$
$$d\rho(d) := -i\sum md\rho(z_m)d\rho(\overline{z_m}).$$

One can prove that $d\rho$ gives a PER of U, that is, (1) it satisfies the commutation relation (for example $[d\rho(z_n), d\rho(\overline{z_n})] = \mathrm{id}$); (2) the restriction of $d\rho$ to $\mathrm{Lie}(U)$ take value in the set of skew symmetric operators on $\widehat{S}((\mathrm{Lie}(U_N)\otimes\mathbb{C})_+)_{\mathrm{fin}}$, and hence it defines a unitary representation; (3) $[d\rho(d), d\rho(z_n)] = ind\rho(z_n)$, $[d\rho(d), d\rho(\overline{z_n})] = -ind\rho(\overline{z_n})$, and hence the representation lifts to $\mathbb{T}_{\mathrm{rot}} \ltimes LT^\tau$; and (4) $\frac{1}{i}d\rho(d)$ has discrete and positive definite spectrum with finite multiplicities:

$$\frac{1}{i}d\rho(d)(z_1^{k_1}z_2^{k_2}\cdots) = (\sum jk_j)z_1^{k_1}z_2^{k_2}\cdots.$$

Let $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*$ be the dual Hilbert space of $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)$ realized as the completion of the symmetric tensor algebra of $\bigoplus\mathbb{C}\overline{z_n}$. It is equipped with a "negative energy representation at level $-\tau$" $\rho^*: U^\tau \to U(\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*)$ given by $\rho^*(g) := {}^t[\rho(g^{-1})]$.[9] We can also define the concept of finite energy vectors on $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*$, and $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*_{\mathrm{fin}}$ denotes the set of finite energy vectors.

Let us introduce the "L^2-space of U" in an ad hoc way. See [36] for the interpretation of this definition.

Definition 4.15. The vector space $\underline{L^2(U,-\tau)_{\mathrm{fin}}}$ is defined by the algebraic tensor product

$$\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)_{\mathrm{fin}} \otimes^{\mathrm{alg}} \widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*_{\mathrm{fin}}.$$

It has a naturally induced inner product, and its completion as a Hilbert space is denoted by $\underline{L^2(U,-\tau)}$.

Remark 4.16. This definition is the infinite-dimensional analogue of the actual isomorphism $L^2(U_N,-\tau)\cong L^2(\mathbb{R}^N)\otimes L^2(\mathbb{R}^N)^*$, where $L^2(\mathbb{R}^N)$ is the unique PER of U_N at level τ. This isomorphism is given by the matrix coefficient $v\otimes\phi : g \mapsto C\langle\rho(g^{-1})v,\phi\rangle$, where C is a certain normalization constant. Therefore, the representation $\rho\otimes\mathrm{id}$ on $\underline{L^2(U,-\tau)}$ corresponds to the "left regular representation", and its dual $\mathrm{id}\otimes\rho^*$ corresponds to the "right regular representation". Note that we are considering the central extension, and hence $d\rho_X$ is not just the direction derivative $\frac{\partial}{\partial X}$, but an operator $\frac{\partial}{\partial X}+$(some potential). This is the key feature so that the "Dirac operator" $\not{\partial}_R$ to be introduced later, has discrete spectrum (with infinite multiplicities).

In order to define a Dirac operator, we need a Clifford module. We use the Spinor constructed in [11]

Definition 4.17 ([11])**.** Let $S_{U,\mathrm{fin}}$ be the algebraic exterior algebra of $\bigoplus_{n>0}\mathbb{C}\overline{z_n}$. It has an inner product defined by $\langle\overline{z_{k_1}}\wedge\overline{z_{k_2}}\wedge\cdots\wedge\overline{z_{k_p}},\overline{z_{l_1}}\wedge\overline{z_{l_2}}\wedge\cdots\wedge\overline{z_{l_p}}\rangle := \delta_{k_1,l_1}\delta_{k_2,l_2}\cdots\delta_{k_p,l_p}$. This space admits a Clifford multiplication of $\mathrm{Lie}(U)\otimes\mathbb{C}$ by

$$\gamma(\overline{z_n}) := \sqrt{2}\overline{z_n}\wedge,$$
$$\gamma(z_n) := -\sqrt{2}\overline{z_n}\rfloor,$$

where $\rfloor$ denotes the interior product. Let S_U be the completion of $S_{U,\mathrm{fin}}$ with respect to the above inner product.

The exterior algebra has a specific vector "1". In order to distinguish it from the lowest weight vector belonging to the symmetric algebra, this specific vector "1" is denoted by $\mathbf{1}_f$, where "f" comes from "fermion".

Let N be the operator defined by $N(\overline{z_{k_1}}\wedge\overline{z_{k_2}}\wedge\cdots\wedge\overline{z_{k_p}}) := (\sum k_j)\overline{z_{k_1}}\wedge\overline{z_{k_2}}\wedge\cdots\wedge\overline{z_{k_p}}$. It satisfies the equality $N=-\frac{1}{2}\sum n\gamma(\overline{z_n})\gamma(z_n)$.

Note that N is an infinite sum. It is well-defined on $S_{U,\mathrm{fin}}$ because, for each $s\in S_{U,\mathrm{fin}}$, $\gamma(\overline{z_n})\gamma(z_n)s$ is zero except for finitely many n's. Since N has discrete spectrum, it has the self-adjoint extension.

Let us introduce the Dirac operator on U.

Definition 4.18. The "Dirac operator on U" acting on $\underline{L^2(U,\tau)_{\mathrm{fin}}}\otimes^{\mathrm{alg}} S_{U,\mathrm{fin}}$ is defined by

$$\not{\partial}_R := \sum\sqrt{n}\Big(\mathrm{id}\otimes d\rho^*(\overline{z_n})\otimes\gamma(z_n) + \mathrm{id}\otimes d\rho^*(z_n)\otimes\gamma(\overline{z_n})\Big).$$

We call an eigenvalue of $\not{\partial}_R^2$ an **energy**.

Remark 4.19. The infinite sum $\not{\partial}_R$ is well-defined. This is proved as follows: Take $v\otimes\phi\in L^2(U,\tau)_{\mathrm{fin}}$ and $s\in S_{U,\mathrm{fin}}$. We may assume that ϕ is an eigenvector of $d\rho^*(d)$ with eigenvalue λ. Then, $d\rho^*(z_n)v$ is again an eigenvector of $d\rho^*(d)$ with eigenvalue $\lambda+in$. Since ρ^* is of negative energy, the eigenvalue of a non-zero vector must be $i\times$(non-positive number). If n is large enough, it is impossible, and hence $d\rho^*(z_n)v$ must be zero. Therefore the infinite sum $\sum\sqrt{n}\mathrm{id}\otimes d\rho^*(z_n)\otimes\gamma(\overline{z_n})(v\otimes\phi\otimes s)$ is indeed a finite sum. On the other hand, $\gamma(z_n)s$ is zero except for finitely many n's, since $s\in S_{U,\mathrm{fin}}$. Therefore the infinite sum $\sum\sqrt{n}\mathrm{id}\otimes d\rho^*(\overline{z_n})\otimes\gamma(z_n)(v\otimes\phi\otimes s)$ is indeed a finite sum.

This proof also guarantees that $\not{\partial}_R$ preserves the subspace $\underline{L^2(U,\tau)_{\mathrm{fin}}}\otimes^{\mathrm{alg}} S_{U,\mathrm{fin}}$.

Finally, we recall one more terminology: the "τ-twisted group C^*-algebra of U". See [36] for the exposition from the view point of "functions on U".

Definition 4.20. We introduce two C^*-algebras by compact operators on irreducible representations of U,

$$\underline{U\ltimes_\tau\mathbb{C}} := \mathcal{K}(\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*);$$
$$\underline{U\ltimes_{-\tau}\mathbb{C}} := \mathcal{K}(\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)).$$

[9] The definition of the negative energy representation should be obvious. "At level $-\tau$" means that $i(z)\in i(U(1))\subseteq U^\tau$ acts as $z^{-1}\mathrm{id}$.

Remarks 4.21. (1) The representation space $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)$ has a specific vector 1, which is a unit vector of the kernel of $d\rho(d)$. This element is denoted by $\mathbf{1}_b$, where "b" comes from "boson". The rank one projection onto $\mathbb{C}\mathbf{1}_b$ is denoted by $P_{\mathbf{1}_b}\in \underline{U\ltimes_{-\tau}\mathbb{C}}$. Similarly, $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*$ has a specific vector $\mathbf{1}_b^*$, and the corresponding rank one projection is denoted by $P_{\mathbf{1}_b^*}\in \underline{U\ltimes_{\tau}\mathbb{C}}$.

(2) The C^*-algebra $\underline{U\ltimes_{-\tau}\mathbb{C}}$ has a dense subalgebra

$$\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)\otimes^{\mathrm{alg}}\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*.$$

This is the set of finite rank operators preserving the set of finite energy vectors. This subalgebra is denoted by $(\underline{U\ltimes_{-\tau}\mathbb{C}})_{\mathrm{fin}}$. The subalgebra $(\underline{U\ltimes_{\tau}\mathbb{C}})_{\mathrm{fin}}\subseteq\underline{U\ltimes_{\tau}\mathbb{C}}$ is defined in the same way. The specific elements $P_{\mathbf{1}_b}$ and $P_{\mathbf{1}_b^*}$ belong to these dense subalgebras, respectively.

4.3. The construction of "$j_\tau^U([\not{\partial}_R])$"

This subsection is the most essential in this paper.

We would like to define "$j_\tau^U([\not{\partial}_R])$", imitating Section 3.2, by

$$\text{"}(L^2(U)\otimes S_U\otimes(U\ltimes_\tau\mathbb{C}), D\otimes_2\mathrm{id}+\mathrm{id}\otimes_1\not{\partial}_L\text{)"},$$

as an unbounded Kasparov ("$U\ltimes C_0(U)$", $\underline{U\ltimes_\tau\mathbb{C}}$)-module. We have introduced $\underline{U\ltimes_\tau\mathbb{C}}$ and S_U. We now need to introduce "$L^2(U)$", "D", "$\not{\partial}_L$" and "$U\ltimes C_0(U)$". We have constructed a Hilbert space $\underline{L^2(U,-\tau)}$, but it cannot be used for "$L^2(U)$", because U should act on the latter in an untwisted way. After that, we will introduce "$U\ltimes C_0(U)$" using "$L^2(U)$". The operator "$\not{\partial}_L$" will be defined in a representation theoretical way. Notice that this operator acts on the Hilbert $\underline{U\ltimes\mathbb{C}}$-module $S_U\otimes\underline{U\ltimes_\tau\mathbb{C}}$. Finally, we will prove that the above pair is actually a Kasparov module. For this aim, we need some quantitative arguments, which have infinite-dimensional flavor.

The following is somehow ad hoc so as to make the estimates simpler. There can be much better way to define it. However, the essential point must be the same: considering the infinite tensor product of infinitely many kinds of functions tending to a constant function. *Roughly speaking, an "L^2-function" we need is an "asymptotically constant function".* For example, the Gaussian $e^{-\|x\|^2}=e^{-x_1^2}\otimes e^{-x_2^2}\otimes e^{-x_3^2}\otimes\cdots$ is not appropriate for our purpose. A "Gaussian with varying variance" $Ce^{-n_1x_1^2}\otimes e^{-n_2x_2^2}\otimes e^{-n_3x_3^2}\otimes\cdots$ would be appropriate, where n_k's are positive numbers tending to 0 sufficiently rapidly and C is a certain "renormalization constant".

Definition 4.22. Let χ_σ be the function on $\mathbb{R}^2$ defined by

$$\chi_\sigma(x)=\begin{cases}\frac{1}{\sqrt{\pi\sigma^2}} & |x|\le\sigma\\ 0 & |x|>\sigma,\end{cases}$$

for $\sigma>0$, and let Ξ_σ be the Fourier transform $\mathcal{F}(\chi_\sigma)$. χ_σ is a unit vector, and hence Ξ_σ is also. We construct a sequence of isometric embeddings

$$L^2(U_1)\hookrightarrow L^2(U_2)\hookrightarrow\cdots$$

by $i_N: L^2(U_N)\ni f\mapsto f\otimes\Xi_{\sigma_{N+1}}\in L^2(U_{N+1})$. Let $\{\sigma_N\}$ be a sequence of positive numbers satisfying that

$$\sum_N N\sigma_N<\infty. \tag{4.1}$$

We define $\underline{L^2(U)}$ by the inductive limit: $\underline{L^2(U)}:=\varinjlim L^2(U_N)$.

Let $L^2_\infty(U_N):=\left\{\mathcal{F}(\phi)\in L^2(U_N)\mid p\phi\in\underline{L^2(U_N)}\text{ for any polynomial }p\right\}$. The above embedding $L^2(U_N)\hookrightarrow L^2(U_{N+1})$ preserves subspaces $L^2_\infty(U_N)$. Let $\underline{L^2_\infty(U)}$ be the algebraic inductive limit $\varinjlim^{\mathrm{alg}}L^2_\infty(U_N)$.

Remarks 4.23. (1) The condition (4.1) is a key to guarantee that the pair which we are constructing is actually a Kasparov module. For example, $\sigma_k=2^{-k}$ satisfies the condition (4.1).

(2) An element of $\underline{L^2_\infty(U)}$ is regarded as an "**asymptotically constant function**".

(3) We will sometimes use a slightly loose symbol $\Xi=\Xi_{\sigma_{n+1}}\otimes\Xi_{\sigma_{n+2}}\otimes\cdots$ for *any* n to simplify the notations. For example, Ξ may be $\Xi_{\sigma_{n+1}}\otimes\Xi_{\sigma_{n+2}}\otimes\cdots$ in some place, and it may be $\Xi_{\sigma_1}\otimes\Xi_{\sigma_2}\otimes\cdots$ in another place.

Let us introduce an operator D. We use the complex coordinate, following the definition of $\not{\partial}_R$. Before that, recall the base of $\mathrm{Lie}(U)$: $\{\frac{\cos N\theta}{\sqrt{N\pi}},\frac{\sin N\theta}{\sqrt{N\pi}}\}_{N=1,2,\dots}$. It defines a coordinate on U denoted by $\{x_1,y_1,x_2,y_2,\dots\}$: More concretely,

$$U\ni\exp\left(\sum\left[x_k\frac{\cos k\theta}{\sqrt{k\pi}}+y_k\frac{\sin k\theta}{\sqrt{k\pi}}\right]\right)\mapsto(x_1,y_1,x_2,y_2,\dots)\in\mathbb{R}^\infty.$$

Let us introduce operators $dR_{z_n}, dR_{\overline{z_n}} : \underline{L^2_\infty(U)} \to \underline{L^2_\infty(U)}$. For this aim, we define dR_{z_n} on $L^2_\infty(U_N)$, for $n \leq N$, by the usual formula

$$\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_n} + i\frac{\partial}{\partial y_n}\right)$$

and $dR_{\overline{z_n}}$ by its complex conjugate $\frac{1}{\sqrt{2}}(\partial/\partial x_n - i\partial/\partial y_n)$. It extends to $\underline{L^2(U)}$ as follows. Firstly $j_N : L^2_\infty(U_N) \hookrightarrow \underline{L^2_\infty(U)}$ denotes the embedding defining the limit. Then, $dR_{z_n}(j_N(f))$ is defined by $j_N(dR_{z_n}(f))$. This is well-defined because dR_{z_n} commutes with the embedding $L^2_\infty(U_N) \to L^2_\infty(U_{N+1})$. $dR_{\overline{z_n}}$ is defined in the same way.

Definition 4.24. On the pre-Hilbert space $\underline{L^2_\infty(U)} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}}$, we introduce a "Dirac operator" by

$$D := \sum_{n=1}^{n=\infty} \sqrt{n}\Big(dR_{z_n} \otimes \gamma(\overline{z_n}) + dR_{\overline{z_n}} \otimes \gamma(z_n)\Big).$$

We also define its approximations by

$$D^M_N := \sum_{n=N}^{n=M} \sqrt{n}\Big(dR_{z_n} \otimes \gamma(\overline{z_n}) + dR_{\overline{z_n}} \otimes \gamma(z_n)\Big)$$

for $N < M \leq \infty$.

Remark 4.25. In this notation, for example, by considering the isomorphism $\underline{L^2(U)} \cong L^2(U_N) \otimes \underline{L^2(U_N^\perp)}$ and $S_U \cong S_{U_N} \otimes S_{U_N^\perp}$, we can rewrite D as $D^N_1 \otimes \mathrm{id} + \mathrm{id} \otimes D^\infty_{N+1}$.

Notice that D is an infinite sum, *even on the dense subspace* $\underline{L^2_\infty(U)} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}}$. The situation is completely different from that of $\partial\!\!\!/_R$. For example, it is far from being clear whether D is well-defined or not. In fact, D is essentially self-adjoint as follows.

Proposition 4.26. *For any element $\phi \in \underline{L^2_\infty(U)} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}}$, the infinite sum $D\phi$ converges in norm. Moreover, the operator $D : \underline{L^2_\infty(U)} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}} \to \underline{L^2(U)} \otimes S_U$ is essentially self-adjoint.*

Proof. We firstly verify that the infinite sum $D\phi$ converges. We may assume that ϕ is of the form $\big[f \otimes \Xi_{m+1} \otimes \Xi_{m+2} \otimes \cdots\big] \otimes s$ for $f \in L^2_\infty(U_m)$ and $s \in S_{U_m}$. Needless to say, each component $dR_{z_n} \otimes \gamma(\overline{z_n})(f \otimes \Xi) \otimes s$ is defined. Thus the problem is whether the infinite sum converges or not, and hence it suffices to study $D^\infty_{m+1}(\phi)$. Since dR_{x_n} corresponds to the multiplication by ix_n by the Fourier transform, $\|dR_{z_n}\Xi_{\sigma_n}\| \leq \sigma_n \|\Xi_{\sigma_n}\|$. Therefore, by the triangle inequality, and $\|\gamma(z_n)\|_{\mathrm{op}} \leq \sqrt{2}$,

$$\begin{aligned}
&\sum_{n=m+1}^{\infty} \sqrt{n}\|dR_{z_n} \otimes \gamma(\overline{z_n})[\phi] + dR_{\overline{z_n}} \otimes \gamma(z_n)[\phi]\| \\
&\quad \leq \sum_{n=m+1}^{\infty} 2\sqrt{2n}\sigma_n < \infty,
\end{aligned}$$

thanks to the key condition (4.1). Consequently, the infinite sum $D\phi$ converges.

Notice that D is a symmetric operator. This is because each component is symmetric. To be essentially self-adjoint, it suffices to check that the both of $D \pm i$ have dense ranges. Firstly, notice that the element 0 lies in the range of $D \pm i$. Suppose a non-zero element $f \otimes s \in \underline{L^2_\infty(U)} \otimes^{\mathrm{alg}} S_{U\mathrm{fin}}$ is given. For any positive ε, *we would like to find an approximate solution* $\phi_\pm \in \underline{L^2_\infty(U)} \otimes^{\mathrm{alg}} S_{U\mathrm{fin}}$ *satisfying that* $\|(D \pm i)\phi_\pm - f \otimes s\| < 2\varepsilon$. We may assume that $\varepsilon < \frac{\|f\otimes s\|}{2}$. Firstly find $M \in \mathbb{N}$ such that $\sum_{n>M} 2\sqrt{2n}\sigma_n < \frac{2\varepsilon}{3\|f\otimes s\|}$. It is always possible, thanks to the condition (4.1). We may assume that M is large enough so that f is of the form $f_0 \otimes \Xi_{\sigma_{M+1}} \otimes \Xi_{\sigma_{M+2}} \otimes \cdots$ for some $f_0 \in L^2_\infty(U_M)$, and s is in fact an element of S_{U_M}. Since D^M_1 is self-adjoint on $L^2(U_M) \otimes S_{U_M}$ by a finite-dimensional argument, there exists $\phi_{0,\pm} \in L^2(U_M)_{\mathrm{fin}} \otimes S_{U_M}$ satisfying that $\|(D^M_1 \pm i)\phi_{0,\pm} - f_0 \otimes s\| < \varepsilon$. Since $D^M_1 \pm i$ does not reduce the norm, $\|\phi_{0,\pm}\| < \|f_0 \otimes s\| + \epsilon < \frac{3\|f\otimes s\|}{2}$. Combining these estimates, we obtain the following[10]:

$$\begin{aligned}
&\|(D \pm i)\left(\phi_{0,\pm} \otimes (\Xi \otimes \mathbf{1}_f)\right) - (f_0 \otimes \Xi) \otimes s\| \\
&< \|[(D^M_1 \pm i)\phi_{0,\pm} - f_0 \otimes s] \otimes (\Xi \otimes \mathbf{1}_f)\| + \|(-1)^{\deg(\phi_{0,\pm})}\phi_{0,\pm} \otimes D^\infty_{M+1}(\Xi \otimes \mathbf{1}_f)\| \\
&< \|(D^M_1 \pm i)\phi_{0,\pm} - f_0 \otimes s\|\|\Xi \otimes \mathbf{1}_f\| + \|\phi_{0,\pm}\|\|D^\infty_{M+1}(\Xi \otimes \mathbf{1}_f)\| \\
&< 2\varepsilon. \quad \square
\end{aligned}$$

[10] Ξ means $\Xi_{\sigma_{M+1}} \otimes \Xi_{\sigma_{M+2}} \otimes \cdots$ in this paragraph.

For any locally compact Hausdorff group G, we know that $G \ltimes C_0(G) \cong \mathcal{K}(L^2(G))$ (see [38] for example). Thus it is natural to define "$U \ltimes C_0(U)$" by $\mathcal{K}(\underline{L^2(U)})$. Our "crossed product" is isomorphic to it, but we need some "smooth structure" in order to deal with unbounded Kasparov modules. Thus we define "$U \ltimes C_0(U)$" by the inductive limit construction.

Firstly, we notice that $U_N \ltimes C_0(U_N) \cong \mathcal{K}\left(L^2(U_N)\right)$ and $L^2(U_{N+1}) \cong L^2(U_N) \otimes L^2(U_{N+1} \ominus U_N)$, where $U_{N+1} \ominus U_N$ is the Lie subgroup corresponding to the orthogonal complement of $\mathrm{Lie}(U_N)$ in $\mathrm{Lie}(U_{N+1})$. We have chosen an embedding $L^2(U_N) \hookrightarrow L^2(U_{N+1})$, by the tensor product with the unit vector $\Xi_{\sigma_{N+1}}$ for each N. By using these embeddings, we define a sequence of C^*-algebras by

$$U_N \ltimes C_0(U_N) \cong \mathcal{K}\left(L^2(U_N)\right) \ni k \mapsto k \otimes P_{\Xi_{\sigma_{N+1}}} \in \mathcal{K}\left(L^2(U_{N+1})\right) \cong U_{N+1} \ltimes C_0(U_{N+1}),$$

where $P_{\Xi_{\sigma_{N+1}}}$ is the rank one projection onto $\mathbb{C}\Xi_{\sigma_{N+1}}$.

The C^*-algebra $\mathcal{K}\left(L^2(U_N)\right)$ has a dense subalgebra $L^2_\infty(U_N) \otimes^{\mathrm{alg}} L^2_\infty(U_N)^*$, which is denoted by $(U_N \ltimes C_0(U_N))_{\mathrm{fin}}$. The above embeddings clearly preserve these subalgebras, and hence the following makes sense.

Definition 4.27. We define the C^*-algebra $\underline{U \ltimes C_0(U)}$ by the *C^*-algebra inductive limit* of the above sequence: $\varinjlim_{N\to\infty} U_N \ltimes C_0(U_N)$.

This C^*-algebra has a dense $*$-subalgebra defined by the *algebraic inductive limit* of the specified dense subalgebras $\varinjlim^{\mathrm{alg}}_{N\to\infty}(U_N \ltimes C_0(U_N))_{\mathrm{fin}}$. This dense algebra is denoted by $\underline{U \ltimes C_0(U)}_{\mathrm{fin}}$.

Remark 4.28. The algebraic inductive limit $\varinjlim^{\mathrm{alg}}_{N\to\infty}(U_N \ltimes C_0(U_N))_{\mathrm{fin}}$ consists of finite rank operators on $\underline{L^2(U)}$ preserving the dense subspace $\underline{L^2_\infty(U)}$. Thus, our C^*-algebra $\underline{U \ltimes C_0(U)}$ is isomorphic to $\mathcal{K}(\underline{L^2(U)})$.

In order to define the Kasparov module "$j^U_\tau([\not{\partial}_R])$", we need one more operator $\not{\partial}_L$ acting on $S_U \otimes \underline{U \ltimes_\tau \mathbb{C}}$. However, an element of $\underline{U \ltimes_\tau \mathbb{C}}$ is not a set-theoretical function, and hence we cannot define the direction derivatives in the ordinary sense. This means that we need to define an appropriate substitute of the "direction derivative" dL_X for $X \in L\mathfrak{t}$ on $\underline{U \ltimes_\tau \mathbb{C}}$.

For this aim, we recall a finite-dimensional case. Let G be a locally compact Lie group and let f be a function on G. Let (V, ρ) be a unitary representation of G and let π_ρ be the induced representation $G \ltimes \mathbb{C} \to \mathcal{B}(V)$ on the group C^*-algebra. For an element $g \in G$, consider a new function $L_g(f)$. Then, the operator $\pi_\rho(L_g(f))$ can be computed as follows:

$$\begin{aligned}\pi_\rho(L_g(f)) &= \int f(g^{-1}\gamma)\rho(\gamma)d\gamma \\ &= \int f(\gamma)\rho(g)\rho(\gamma)d\gamma = \rho(g) \circ \pi_\rho(f).\end{aligned}$$

Therefore, $\pi_\rho(L_X(f)) = d\rho(X) \circ \pi_\rho(f)$.

Seeing this formula, we introduce the operator $\not{\partial}_L$ on $S_{U,\mathrm{fin}} \otimes^{\mathrm{alg}} (\underline{U \ltimes_\tau \mathbb{C}})_{\mathrm{fin}}$.

Definition 4.29. For $s \otimes b \in S_{U,\mathrm{fin}} \otimes^{\mathrm{alg}} (\underline{U \ltimes_\tau \mathbb{C}})_{\mathrm{fin}}$, $\not{\partial}_L(s \otimes b)$ is defined by

$$\not{\partial}_L(s \otimes b) := \sum \sqrt{n}\Big(\gamma(z_n)(s) \otimes d\rho^*(\overline{z_n}) \circ b + \gamma(\overline{z_n})(s) \otimes d\rho^*(z_n) \circ b\Big).$$

Remarks 4.30. (1) This infinite sum is a finite sum for each finite energy vector, just like $\not{\partial}_R$. In fact, $d\rho^*(z_n) \circ b$ vanishes except for finitely many n's. This is because $d\rho^*(z_n)$ increases the energy by n, the representation is of negative energy, the range of b is contained in the set of finite energy vectors and b is of finite rank. Moreover, $\gamma(z_n)(s)$ vanishes except for finitely many n's. Thus the above infinite sum is in fact a finite sum.

(2) This operator has one dimensional kernel spanned by $\mathbf{1}_f \otimes P_{\mathbf{1}^*_b}$.

The tensor product $\underline{L^2(U)} \otimes S_U \otimes (\underline{U \ltimes_\tau \mathbb{C}})$ has the natural Hilbert $\underline{U \ltimes_\tau \mathbb{C}}$-module structure:

$$\begin{aligned}(\phi \otimes s \otimes a) \cdot b &:= \phi \otimes s \otimes (a \circ b), \\ \big(\phi_1 \otimes s_1 \otimes a_1 \big| \phi_2 \otimes s_2 \otimes a_2\big) &:= \big(\phi_1 \big| \phi_2\big)_{\underline{L^2(U)}} \big(s_1 \big| s_2\big)_{S_U} a_1^* \circ a_2\end{aligned}$$

for $\phi_1, \phi_2 \in \underline{L^2(U)}$, $s_1, s_2 \in S_U$ and $a_1, a_2, a, b \in \underline{U \ltimes_\tau \mathbb{C}}$.

The following is the main target of this subsection.

Theorem 4.31. *The operator* $\widetilde{\not{D}}_R := D \otimes_2 \mathrm{id} + \mathrm{id} \otimes_1 \not{D}_L$ *acting on* $\underline{L^2(U)_{\mathrm{fin}}} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}} \otimes^{\mathrm{alg}} \underline{(U \ltimes_\tau \mathbb{C})_{\mathrm{fin}}}$ *is essentially self-adjoint and regular. The closure of this operator is denoted by the same symbol. Then the pair*

$$(\underline{L^2(U)} \otimes S_U \otimes \underline{(U \ltimes_\tau \mathbb{C})}, \widetilde{\not{D}}_R)$$

is a Kasparov $(\underline{U \ltimes C_0(U)}, \underline{U \ltimes_\tau \mathbb{C}})$*-module.*

Definition 4.32. The unbounded Kasparov $(\underline{U \ltimes C_0(U)}, \underline{U \ltimes_\tau \mathbb{C}})$-module $(\underline{L^2(U)} \otimes S_U \otimes \underline{(U \ltimes_\tau \mathbb{C})}, \widetilde{\not{D}}_R)$ is denoted by $\underline{j_\tau^U([\not{D}_R])}$. It is regarded as "the value of the partial descent homomorphism j_τ^U at the index element".

The rest of this subsection is devoted to the proof of this theorem. We prove it by dividing three steps: (a) $\widetilde{\not{D}}_R$ is self-adjoint and regular; (b) $[\widetilde{\not{D}}_R, a \otimes_1 \mathrm{id}]$ is bounded for all $a \in (\underline{U \ltimes C_0(U)})_{\mathrm{fin}}$; (c) $(1 + \widetilde{\not{D}}_R^{\,2})^{-1} a \otimes_1 \mathrm{id}$ is $\underline{U \ltimes_\tau \mathbb{C}}$-compact for all $a \in \underline{U \ltimes C_0(U)}$.

Lemma 4.33. $\widetilde{\not{D}}_R$ *is self-adjoint and regular.*

Proof. It is sufficient to check that the both of $\widetilde{\not{D}}_R \pm i$ have dense ranges. We follow the same argument of the proof of Proposition 4.26. More concretely, for any element $\psi \in \underline{L^2(U)_{\mathrm{fin}}} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}} \otimes^{\mathrm{alg}} \underline{(U \ltimes_\tau \mathbb{C})_{\mathrm{fin}}}$, we try to find an "approximate pre-image of ψ" $\phi_\pm$ satisfying that $\| \left(\widetilde{\not{D}}_R \pm i\right) \phi_\pm - \psi \|$ is arbitrary small. Let us consider the isomorphism

$$\begin{aligned} &\underline{L^2(U)} \otimes S_U \otimes \underline{(U \ltimes_\tau \mathbb{C})} \\ &\cong \left[L^2(U_N) \otimes S_{U_N} \otimes (U_N \ltimes_\tau \mathbb{C})\right] \bigotimes \left[\underline{L^2(U_N^\perp)} \otimes S_{U_N^\perp} \otimes \underline{(U_N^\perp \ltimes_\tau \mathbb{C})}\right] \end{aligned}$$

for sufficiently large N, where $U_N^\perp$ is the Lie subgroup corresponding to the orthogonal complement of $\mathrm{Lie}(U_N)$ in $\mathrm{Lie}(U)$. Under this isomorphism, we can divide $\widetilde{\not{D}}_R$ into two parts: $\widetilde{\not{D}}_{R1} \otimes \mathrm{id} + \mathrm{id} \otimes \widetilde{\not{D}}_{R2}$. We can choose so large N that $\widetilde{\not{D}}_{R2}(\Xi \otimes \mathbf{1}_f \otimes \mathbf{1}_b)$ is small enough, thanks to the same technique of Proposition 4.26. The first part is self-adjoint, since it can be written as $\widetilde{D}$ for a certain Dirac operator D for a finite-dimensional manifold U_N, and hence an approximate solution can be found. With the same technique of Proposition 4.26, we can find an approximate solution on the whole Hilbert space. □

Lemma 4.34. $[\widetilde{\not{D}}_R, a \otimes_1 \mathrm{id}]$ *is a bounded operator for each* $a \in (\underline{U \ltimes C_0(U)})_{\mathrm{fin}}$.

Proof. Since $a \otimes_1 \mathrm{id}$ commutes with $\mathrm{id} \otimes_1 \not{D}_L$, it is sufficient to deal with $[a \otimes \mathrm{id}, D]$. Hence we can focus on the $\underline{L^2(U)} \otimes S_U$-part. We may assume that a is a Schatten form $\phi \otimes \psi^*$, and ϕ and ψ are of the form $\phi_1 \otimes \Xi_{\sigma_{M+1}} \otimes \Xi_{\sigma_{M+2}} \otimes \cdots$ and $\psi_1 \otimes \Xi_{\sigma_{M+1}} \otimes \Xi_{\sigma_{M+2}} \otimes \cdots$ respectively, where $\phi_1, \psi_1 \in L^2_\infty(U_M)$. Then a can be written as $(\phi_1 \otimes \psi_1^*) \bigotimes P_{\Xi_{\sigma_{M+1}} \otimes \Xi_{\sigma_{M+2}} \otimes \cdots}$, and we write it as $a_1 \otimes P$ simply.

We check that the both of $D \circ a \otimes \mathrm{id}$ and $a \otimes \mathrm{id} \circ D$ are bounded.

For the former one, let us firstly compute $D \circ a \otimes \mathrm{id}(f \otimes s)$:

$$\begin{aligned} D \circ a \otimes \mathrm{id}(f \otimes s) &= D[(\phi \otimes \psi^*)(f) \otimes s] \\ &= D(\phi \otimes s) \langle \psi, f \rangle \\ &= \sum_n \sqrt{n} \left[dR_{z_n}(\phi) \otimes \gamma(\overline{z_n})(s) + dR_{\overline{z_n}}(\phi) \otimes \gamma(z_n)(s) \right] \langle \psi, f \rangle \,. \end{aligned}$$

The final form is an infinite sum of bounded operators. We need to estimate the norm of each operator. By definition of γ, we notice that $\|\gamma(z_n)\|, \|\gamma(\overline{z_n})\| \leq \sqrt{2}$, and then each operator norm is at most

$$\sqrt{2n} \left(\|dR_{z_n}(\phi)\| + \|dR_{\overline{z_n}}(\phi)\| \right) \|\psi\|.$$

Thanks to the condition (4.1), the infinite sum absolutely converges and we obtain the desired estimate.

One can prove the latter statement with the same argument of the above, noticing Remark 3.9. □

Lemma 4.35. $\left(1 + \widetilde{\not{D}}_R^{\,2}\right)^{-1} a \otimes_1 \mathrm{id}$ *is a* $\underline{U \ltimes_\tau \mathbb{C}}$*-compact operator for all* $a \in \underline{U \ltimes C_0(U)}$.

Proof. We may assume that $a \in (\underline{U \ltimes C_0(U)})_{\mathrm{fin}}$. Moreover, we may assume that a is of the form $a_1 \otimes P_\Xi$ just as in the proof of Lemma 4.34.

Set $\widetilde{\not{\partial}}_0 := \mathrm{id} \otimes_1 \not{\partial}_L$. We prove the statement in the following way: (1) $(1+\widetilde{\not{\partial}}_0^2)^{-1} a \otimes_1 \mathrm{id}$ is compact; and (2) the difference $\left(1+\widetilde{\not{\partial}}_R^{\,2}\right)^{-1} a \otimes_1 \mathrm{id} - \left(1+\widetilde{\not{\partial}}_0^2\right)^{-1} a \otimes_1 \mathrm{id}$ is also compact.

(1) Firstly we note that $\left(1+\widetilde{\not{\partial}}_0^2\right)^{-1} a \otimes_1 \mathrm{id} = a \otimes_1 (1+\not{\partial}_L^2)^{-1}$. Notice that an element of

$$\mathcal{K}(\underline{L^2(U)}) \otimes \mathcal{K}_{\underline{U \ltimes_\tau \mathbb{C}}}(S_U \otimes (\underline{U \ltimes_\tau \mathbb{C}}))$$

gives a $\underline{U \ltimes_\tau \mathbb{C}}$-compact operator on $\underline{L^2(U)} \otimes S_U \otimes (\underline{U \ltimes_\tau \mathbb{C}})$. Since $a \in \mathcal{K}(\underline{L^2(U)})$, it suffices to check that $(1+\not{\partial}_L^2)^{-1}$ is $\underline{U \ltimes_\tau \mathbb{C}}$-compact.

Let us prove it by the spectral decomposition of $\not{\partial}_L^2$. Just like the computation in [36, Theorem 3.20], $\not{\partial}_L^2 = 2[N \otimes \mathrm{id} - i^{-1}\mathrm{id} \otimes (d\rho^*(d)\circ)]$, where $d\rho^*(d)\circ$ means the operator $k \mapsto d\rho^*(d) \circ k$. Thanks to the negative energy condition and the construction of S_U, we obtain the spectral decomposition. *In the following, the eigenvalue of a unit vector v is denoted by* $|v|$. There exists a complete orthonomal system consisting of eigenvectors $\{s_n \otimes \phi_m\}$ of $S_U \otimes \widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)^*$ such that

$$\begin{aligned} 2N(s_n) &= |s_n| s_n \\ \tfrac{2d\rho^*(d)}{i}\phi_m &= |\phi_m|\phi_m. \end{aligned}$$

Therefore, we obtain the formula on the spectral decomposition

$$\not{\partial}_L^2 = \sum_{n,m} (|s_n| + |\phi_m|) P_{s_n} \otimes (P_{\phi_m} \circ),$$

where P_{s_n} is the rank one projection onto $\mathbb{C}s_n$ on S_U, and P_{ϕ_m} is the rank one projection onto $\mathbb{C}\phi_m$ on $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*$. $P_{\phi_m}\circ$ is given by the composition (or equivalently the left multiplication). Recall that $|s_n| \to \infty$ and $|\phi_m| \to \infty$ as $n, m \to \infty$, and recall that $P_{s_n} \perp P_{s_{n'}}$ and $P_{\phi_m} \perp P_{\phi_{m'}}$ for $n \neq n'$ and $m \neq m'$.[11] These facts guarantee that the infinite sum

$$(1+\not{\partial}_L^2)^{-1} = \sum_{n,m} \frac{1}{|s_n| + |\phi_m| + 1} P_{s_n} \otimes (P_{\phi_m} \circ)$$

converges in norm. Since each operator appearing in the infinite sum is compact, the infinite sum is also compact.

(2) Let us compute the difference $(1+\widetilde{\not{\partial}}_R^{\,2})^{-1} a \otimes_1 \mathrm{id} - (1+\widetilde{\not{\partial}}_0^2)^{-1} a \otimes_1 \mathrm{id}$ when $a \in \underline{U \ltimes C_0(U)}$ is the image of the Schatten form $\phi \otimes \psi^* \in \mathcal{K}(L^2(U_N)) \cong U_N \ltimes C_0(U_N)$ corresponding to $\phi, \psi \in L^2(U_N)$ for some N. We may write this element as $a = (\phi \otimes \psi^*) \otimes P_\Xi$ as an element of $\mathcal{K}(\underline{L^2(U)})$, by taking into account Remarks 4.23 (3).

$$\begin{aligned} &\left[\left(1+\widetilde{\not{\partial}}_R^{\,2}\right)^{-1} \left[(\phi \otimes \psi^*) \otimes P_\Xi\right] \otimes_1 \mathrm{id} - \left(1+\widetilde{\not{\partial}}_0^2\right)^{-1} \left[(\phi \otimes \psi^*) \otimes P_\Xi\right] \otimes_1 \mathrm{id}\right](f \otimes s \otimes v) \\ &= \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)^{-1} \left[\left[(\phi \otimes \psi^*) \otimes P_\Xi\right] \otimes_1 \mathrm{id} - \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)\left(1+\widetilde{\not{\partial}}_0^2\right)^{-1} \left[(\phi \otimes \psi^*) \otimes P_\Xi\right] \otimes_1 \mathrm{id}\right](f \otimes s \otimes v) \\ &= \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)^{-1} \\ &\quad \times \left[\left[(\phi \otimes \psi^*) \otimes P_\Xi\right] \otimes_1 \mathrm{id}\,(f \otimes s \otimes v) - \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)\left(1+\widetilde{\not{\partial}}_0^2\right)^{-1} \left[(\phi \otimes \psi^*) \otimes P_\Xi\right] \otimes_1 \mathrm{id}\,(f \otimes s \otimes v)\right] \\ &= \langle \psi \otimes \Xi, f\rangle \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)^{-1} \left[(\phi \otimes \Xi) \otimes s \otimes v - \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)\left(1+\widetilde{\not{\partial}}_0^2\right)^{-1} ((\phi \otimes \Xi) \otimes s \otimes v)\right] \\ &=: \left(1+\widetilde{\not{\partial}}_R^{\,2}\right)^{-1} \mathrm{Rem}, \end{aligned}$$

where, in the fourth equality, we have used the definition of P_Ξ: $[(\phi \otimes \psi^*) \otimes P_\Xi](f) = [\phi \otimes \Xi] \otimes [\psi \otimes \Xi]^*(f) = \phi \otimes \Xi \langle \psi \otimes \Xi, f\rangle$.

Since $(1+\widetilde{\not{\partial}}_R^{\,2})^{-1}$ is a bounded operator, it is enough to check that the remainder term Rem is compact. For this aim, we prepare a formula by a simple calculation.

$$\begin{aligned} \widetilde{\not{\partial}}_R^2 &= D^2 \otimes_2 \mathrm{id} + 2\sum n \left(dR_{z_n} \otimes \mathrm{id} \otimes [d\rho^*(\overline{z_n})\circ] + dR_{\overline{z_n}} \otimes \mathrm{id} \otimes [d\rho^*(z_n)\circ]\right) + \mathrm{id} \otimes_1 \not{\partial}_L^2 \\ &=: \partial_1 + \partial_2 + \widetilde{\not{\partial}}_0^{\,2}. \end{aligned}$$

[11] For two projections p and q on a Hilbert space, $p \perp q$ means that the ranges of p and that of q are orthogonal.

This formula simplifies the remainder term:

$$\mathrm{Rem} = -\langle \psi \otimes \Xi, f\rangle \left((\partial_1+\partial_2)(1+\widetilde{\not\partial}_0^2)^{-1}\Big((\phi\otimes\Xi)\otimes s\otimes v\Big)\right).$$

For the ∂_1-part, we notice that

$$D^2 \otimes_2 \mathrm{id} \circ \left(1+\widetilde{\not\partial}_0^2\right)^{-1} = \sum_n n\left[(dR_{x_n})^2+(dR_{y_n})^2\right]\otimes(\mathrm{id}+\not\partial_L^2)^{-1}.$$

Note that the infinite sum $\sum_n n\left[(dR_{x_n})^2+(dR_{y_n})^2\right]\phi\otimes\Xi$ converges, thanks to the condition (4.1). Then the operator $\partial_1\circ(\mathrm{id}+\not\partial_L^2)^{-1}$ is of the following form:

$$[\text{rank one operator on } \underline{L^2(U)}]\otimes \text{compact operator on } S_U\otimes(\underline{U\ltimes_\tau\mathbb{C}}).$$

Therefore it is a compact operator.

For the ∂_2-part, we must prepare an estimate which looks like the "Rellich type estimates for closed manifolds": $[\mathrm{id}\otimes d\rho^*(\overline{z_n})]\circ(1+\widetilde{\not\partial}_0^2)^{-1}$ *is* $\underline{U\ltimes_\tau\mathbb{C}}$*-compact operator on* $S_U\otimes\underline{U\ltimes_\tau\mathbb{C}}$*. Its norm is bounded by* 1. In order to prove it, it is sufficient to check that the operator $[\mathrm{id}\otimes d\rho^*(\overline{z_n})]\circ(1+\not\partial_L^2)^{-1}$ on the Hilbert space $S_U\otimes\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*$, is compact (see the argument for (1)). Put $T_n := [\mathrm{id}\otimes d\rho^*(\overline{z_n})]\circ(1+\not\partial_0^2)^{-1}$, and consider $T_n^*T_n$:

$$T_n^*T_n = (1+\not\partial_0^2)^{-1}\left\{\mathrm{id}\otimes\left[d\rho^*(\overline{z_n})\right]^* d\rho^*(\overline{z_n})\right\}(1+\not\partial_0^2)^{-1}.$$

As easily calculated, this is a diagonal operator given as follows:

$$s\otimes\overline{z_1}^{m_1}\overline{z_2}^{m_2}\overline{z_3}^{m_3}\cdots \mapsto \frac{m_n+1}{(1+|s|+\sum km_k)^2}s\otimes\overline{z_1}^{m_1}\overline{z_2}^{m_2}\overline{z_3}^{m_3}\cdots$$

for an eigenvector s of N belonging to the eigenvalue $|s|$. For any positive number ε, there are at most finitely many combinations $(|s|, m_1, m_2, \ldots)$ satisfying that $\frac{m_n+1}{(1+|s|+\sum km_k)^2} > \varepsilon$. This means that $T_n^*T_n$ is compact. Since the maximum of $\frac{m_n+1}{(1+|s|+\sum km_k)^2}$ is 1, the operator norm is also 1. Consequently, T_n is a compact operator whose norm is 1.

This estimate guarantees that the composition

$$\begin{aligned} &dR_{z_n}\otimes\mathrm{id}\otimes d\rho(\overline{z_n})\circ(1+\widetilde{\not\partial}_0^2)^{-1}\circ a\otimes_1\mathrm{id}\\ &\quad=\left(dR_{z_n}(\phi\otimes\Xi)\otimes[\psi\otimes\Xi]^*\right)\otimes[\mathrm{id}\otimes d\rho(\overline{z_n})\circ(1+\not\partial_0^2)^{-1}]\end{aligned}$$

is compact. Moreover, since $dR_{z_n}(\Xi)$ is sufficiently small if n is large enough, the infinite sum converges in norm, thanks to the key condition (4.1). Consequently, the infinite sum is a compact operator.

The other part $dR_{z_n}\otimes\cdots$ can be dealt in the same way, and we omit the proof. □

4.4. The Mishchenko line bundle, the assembly map for U and the analytic index

The Mishchenko line bundle for finite-dimensional manifold is defined by a cut-off function as in Lemma 2.8. Recall that a cut-off function on a locally compact Hausdorff space equipped with a proper cocompact group action is a compactly supported continuous function whose integral on each orbit is 1. In the infinite-dimensional situation, however, we cannot follow this definition, because there are no non-trivial compactly supported functions on non-locally compact space. Thus we need to find some substituting concept of it for our problem.

In Lemma 3.6, we have described the K-theory element induced by the cut-off function under the isomorphism $G\ltimes C_0(G)\cong\mathcal{K}(L^2(G))$. Although the original definition of the Mishchenko line bundle is no longer valid for infinite-dimensional situation, the answer of this lemma, the Mishchenko line bundle corresponds to the rank one projection, still makes sense even in our situation. Then, we adopt this alternative description as the definition.

Definition 4.36. Let $\underline{[c]}\in KK(\mathbb{C},\underline{U\ltimes C_0(U)})$ be the K-theory element represented by the rank one projection P_Ξ onto $\mathbb{C}\Xi$, where $\Xi=\Xi_{\sigma_1}\otimes\Xi_{\sigma_2}\otimes\cdots$. In the language of Kasparov modules, $\underline{[c]}$ is represented by $(P_\Xi*(\underline{U\ltimes C_0(U)}),0)$.

We have defined the both of the "value of the descent homomorphism" and the "Mishenko line bundle". Thus we can define the "value of the assembly map" as follows. Notice that we have defined **neither** the homomorphism "μ_τ^U" **nor** the index element "$[\not\partial_R]$".

Definition 4.37. The "value of the assembly map" is defined by the Kasparov product

$$\underline{[c]} \otimes_{\underline{U \ltimes C_0(U)}} \underline{j_\tau^U([\not{\partial}_R])}$$

of two *KK*-elements $\underline{[c]}$ and $\underline{j_\tau^U([\not{\partial}_R])}$. This product is denoted by $\underline{\mu_\tau^U([\not{\partial}_R])}$.

This is the infinite-dimensional analogue of the assembly map. Hence we hope that it coincides with a certain analytic index, like Fact 1.3. The next task is to introduce an infinite-dimensional analogue of the analytic index. For this aim, we prepare an infinite-dimensional version of Lemma 3.10.

Lemma 4.38. $\underline{U \ltimes_{-\tau} \mathbb{C}} = \mathcal{K}(\widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+))$ *has a* $\underline{U \ltimes_\tau \mathbb{C}} = \mathcal{K}(\widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)^*)$*-Hilbert module structure given by*

$$f \cdot b := {}^t b \circ f$$
$$\left(f_1 \middle| f_2\right) := {}^t(f_2 f_1^*)$$

for $f, f_1, f_2 \in U \ltimes_{-\tau} \mathbb{C}$ *and* $b \in U \ltimes_\tau \mathbb{C}$.

In finite-dimensional cases, the index is defined by a certain completion of (a dense submodule of) the index element. The dense submodule is given by the set of compactly supported smooth sections. There is no infinite-dimensional analogue of it. Thus we must choose a nice dense submodule in another way; We adopt an algebraic way to choose it. Recall that $\underline{L^2(U, -\tau)}$ has a dense subspace consisting of finite energy vectors:

$$\underline{L^2(U, -\tau)_{\mathrm{fin}}} = \widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)_{\mathrm{fin}} \otimes^{\mathrm{alg}} \widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)^*_{\mathrm{fin}}.$$

This subspace has a pre-Hilbert $\underline{U \ltimes_\tau \mathbb{C}_{\mathrm{fin}}}$-module structure given by

$$(f \otimes \phi) \cdot b := {}^t b(f) \otimes \phi$$
$$\left(f_1 \otimes \phi_1 \middle| f_2 \otimes \phi_2\right) := f_1^* \otimes f_2 \left(\phi_1 \middle| \phi_2\right),$$

where $f, f_1, f_2 \in \widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)_{\mathrm{fin}}$ and $\phi, \phi_1, \phi_2 \in \widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)^*_{\mathrm{fin}}$.[12] The operator $\not{\partial}_R$ on $\underline{U \ltimes_{-\tau} \mathbb{C}} \otimes S_U$, can be defined in the same way of $\not{\partial}_L$. The completion of this pair gives the analytic index.

Definition 4.39. We define the analytic index of $[\not{\partial}_R]$, as an unbounded Kasparov $(\mathbb{C}, \underline{U \ltimes_\tau \mathbb{C}})$-module, by the following:

- The module is defined by the Hilbert $\underline{U \ltimes_\tau \mathbb{C}}$-module completion of $\underline{L^2(U, -\tau)_{\mathrm{fin}}} \otimes^{\mathrm{alg}} S_{U,\mathrm{fin}}$ with respect to the operations defined above.
- The operator is the closed operator extension of $\not{\partial}_R$.

This Kasparov module (and the corresponding *KK*-element) is denoted by $\underline{\mathrm{ind}_{U \ltimes_\tau \mathbb{C}}([\not{\partial}_R])}$.

The following is clear from the construction of our Hilbert module.

Proposition 4.40. *The analytic index* $\underline{\mathrm{ind}_{U \ltimes_\tau \mathbb{C}}([\not{\partial}_R])}$ *is given by*

$$((\underline{U \ltimes_{-\tau} \mathbb{C}}) \otimes S_U, \not{\partial}_R).$$

Let us prove the main theorem of this paper, which is an infinite-dimensional analogue of a part of Fact 1.3.

Theorem 4.41. *The analytic index* $\underline{\mathrm{ind}_{U \ltimes_\tau \mathbb{C}}([\not{\partial}_R])}$ *is a Kasparov product of* $\underline{[c]}$ *and* $\underline{j_\tau^U([\not{\partial}_R])}$.

Proof. Firstly, we note that the tensor product module $[P_\Xi * \underline{U \ltimes C_0(U)}] \otimes_{\underline{U \ltimes C_0(U)}} [\underline{L^2(U)} \otimes S_U \otimes \underline{U \ltimes_\tau \mathbb{C}}]$ is isomorphic to $S_U \otimes \underline{U \ltimes_\tau \mathbb{C}}$. It can be proved just like Proposition 3.8. Then, the transpose map

$$\underline{U \ltimes_{-\tau} \mathbb{C}} = \mathcal{K}(\widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)) \to \mathcal{K}(\widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)^*) = \underline{U \ltimes_\tau \mathbb{C}}$$

gives an isomorphism as $(U \ltimes_\tau \mathbb{C})$-Hilbert modules between the modules of $\underline{\mu_\tau^U([\not{\partial}_R])}$ and $\underline{\mathrm{ind}_{U \ltimes_\tau \mathbb{C}}([\not{\partial}_R])}$ just like Lemma 3.10.

Under this isomorphism, one can prove that $\not{\partial}_R$ satisfies the conditions to be a Kasparov product, just like Proposition 3.8 again, except that an infinite sum appears. This infinite sum can be dealt with in a similar way of Proposition 4.26. □

[12] Note that $\langle \bullet, \bullet \rangle$ is the bilinear pairing between $\widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)_{\mathrm{fin}}$ and $\widehat{S}((\mathrm{Lie}(U) \otimes \mathbb{C})_+)^*_{\mathrm{fin}}$, $(\bullet|\bullet)$ is an inner product which is anti-linear in the first variable, and $\bullet^*$ is the Riesz dual.

5. Notations

The following is a summary of the notations of infinite-dimensional objects.

Symbol	Brief definition or explanation	Address of correct definition
ρ	PER of U at level τ on $\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)$.	Definition 4.14
$\underline{L^2(U,-\tau)_{\mathrm{fin}}}$	$\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)_{\mathrm{fin}}\otimes^{\mathrm{alg}}\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*_{\mathrm{fin}}$.	Definition 4.15
$\underline{L^2(U,-\tau)}$	$\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)\otimes\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*$.	Definition 4.15
S_U, $S_{U,\mathrm{fin}}$	Exterior algebra of $(\mathrm{Lie}(U)\otimes\mathbb{C})_-$.	Definition 4.17
$\not{\partial}_R$	Algebraic Dirac operator acting on $\underline{L^2(U,-\tau)_{\mathrm{fin}}}\otimes S_{U,\mathrm{fin}}$.	Definition 4.18
$\underline{U\ltimes_\tau\mathbb{C}}$	$\mathcal{K}(\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+)^*)$.	Definition 4.20
$\underline{U\ltimes_{-\tau}\mathbb{C}}$	$\mathcal{K}(\widehat{S}((\mathrm{Lie}(U)\otimes\mathbb{C})_+))$.	Definition 4.20
$\underline{L^2_\infty(U)}$	$\varinjlim^{\mathrm{alg}} L^2_\infty(U_N)$.	Definition 4.22
$\underline{L^2_\infty(U)}$	$\varinjlim L^2(U_N)$.	Definition 4.22
Ξ_σ	Fourier transform of the characteristic function of the disk of center $(0,0)$ and radius σ on $\mathbb{R}^2$.	Definition 4.22
Ξ	$\Xi_{\sigma_{n+1}}\otimes\Xi_{\sigma_{n+2}}\otimes\cdots$. n can be any natural number including 0.	Remarks 4.23
D	"Dirac operator without potential" acting on $\underline{L^2(U)}\otimes S_U$.	Definition 4.24
$\underline{U\ltimes C_0(U)}$	$\varinjlim U_N\ltimes C_0(U_N)\cong\mathcal{K}(\underline{L^2(U)})$.	Definition 4.27
$\not{\partial}_L$	"Dirac operator" acting on $S_U\otimes(U\ltimes_\tau\mathbb{C})$ given by $\sum\sqrt{n}\Big(\gamma(z_n)\otimes d\rho^*(\overline{z_n})\circ+\gamma(\overline{z_n})\otimes d\rho^*(z_n)\circ\Big)$.	Definition 4.29
$\widetilde{\not{\partial}}_R$	$D\otimes_2\mathrm{id}+\mathrm{id}\otimes_1\not{\partial}_L$.	Theorem 4.31
$\underline{j^U_\tau([\not{\partial}_R])}$	$(\underline{L^2(U)}\otimes S_U\otimes\underline{U\ltimes_\tau\mathbb{C}},\widetilde{\not{\partial}}_R)$.	Definition 4.32
$\underline{[c]}$	Rank one projection onto $\mathbb{C}\Xi$ in $\mathcal{K}(\underline{L^2(U)})\cong\underline{U\ltimes C_0(U)}$.	Definition 4.36
$\underline{\mu^U_\tau([\not{\partial}_R])}$	$\underline{[c]}\otimes_{\underline{U\ltimes C_0(U)}}\underline{j^U_\tau([\not{\partial}_R])}$.	Definition 4.37
$\underline{\mathrm{ind}_{U\ltimes_\tau\mathbb{C}([\not{\partial}_R])}}$	$((\underline{U\ltimes_{-\tau}\mathbb{C}})\otimes S_U,\not{\partial}_R)$.	Definition 4.39

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References

[1] A. Alekseev, A. Malkin, E. Meinrenken, Lie group valued moment maps, J. Differential Geom. 48 (3) (1998) 445–495.
[2] A. Arai, Hilbert Spaces and Quantum Mechanics (Japanese), Kyoritsu Shuppan, 1997.
[3] M. Atiyah, G. Segal, The index of elliptic operators: II, Ann. of Math. (2) 87 (1968) 531–545.
[4] M. Atiyah, I. Singer, The index of elliptic operators: I, Ann. of Math. (2) 87 (1968) 484–530.
[5] M. Atiyah, I. Singer, The index of elliptic operators: III, Ann. of Math. (2) 87 (1968) 546–604.
[6] S. Baaj, P. Julg, Théorie bivariante de Kasparov et opérateurs non-bornés dans les C^*-modules hilbertiens, C. R. Acad. Sci., Paris I 296 (21) (1983) 875–878.
[7] B. Blackader, K-Theory for operator algebras, in: Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.
[8] A. Connes, Non-commutative differential geometry, Publ. Math. Inst. Hautes Études Sci. (62) (1985) 257–360.
[9] A. Connes, Noncommutative Geometry, Academic Press, Inc., San Diego, CA, 1994.
[10] D. Dumitrascu, J. Trout, On C^*-algebras and K-theory for infinite-dimensional Fredholm manifolds, Topol. Appl. 153 (14) (2006) 2528–2550.
[11] D. Freed, M.J. Hopkins, C. Teleman, Loop groups and twisted K-theory II, J. Amer. Math. Soc. 26 (3) (2013) 595–644.

[12] S. Gong, J. Wu, G. Yu, The Novikov conjecture, the group of volume preserving diffeomorphisms and Hilbert-Hadamard spaces, preprint, arXiv:1811.02086.
[13] N. Higson, G. Kasparov, E-Theory and KK-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (1) (2001) 23–74.
[14] N. Higson, G. Kasparov, J. Trout, A bott periodicity theorem for infinite-dimensional Euclidean space, Adv. Math. 135 (1) (1998) 1–40.
[15] A. Jaffe, A. Lesniewski, K. Osterwalder, Quantum K-theory I. The Chern character, Comm. Math. Phys. 118 (1) (1988) 1–14.
[16] A. Jaffe, A. Lesniewski, J. Weitsman, Index of a Family of Dirac operators on loop spaces, Comm. Math. Phys. 112 (1) (1987) 75–88.
[17] K. Jensen, K. Thomsen, Elements of KK-theory, in: Mathematics: Theory and Applications, Birkhäuser Boston, Inc., Boston, MA, 1991.
[18] V. Kac, I. Todorov, Superconformal current algebras and their unitary representations, Comm. Math. Phys. 102 (2) (1985) 337–347.
[19] G. Kasparov, An index for invariant elliptic operators, K-theory, and representations of Lie groups, Sov. Math. Dokl. 27 (1) (1983) 105–109.
[20] G. Kasparov, Equivariant KK-theory and the Novikov conjecture, Invent. Math. 91 (1) (1988) 147–201.
[21] G. Kasparov, Elliptic and transversally elliptic index theory from the viewpoint of KK- theory, J. Noncommut. Geom. 10 (4) (2016) 1303–1378.
[22] Y. Kazama, H. Suzuki, Characterization of $N = 2$ superconformal models generated by the coset space method, Phys. Lett. B 216 (1–2) (1989) 112–116.
[23] A.A. Kirillov, Lectures on the orbit methods, in: Graduate Studies in Mathematics, vol. 64, American Mathematical Society, Providence, RI, 2004.
[24] D. Kucerovsky, The KK-products of unbounded modules, K-Theory 11 (1) (1997) 17–34.
[25] G. Landweber, Dirac Operators on Loop Spaces Thesis (Ph.D.), Harvard University, 1999, p. 56.
[26] G. Landweber, Multiplets of representations and Kostant's Dirac operator for equal rank loop groups, Duke Math. J. 110 (1) (2001) 121–160.
[27] Y. Loizides, E. Meinrenken, Y. Song, Spinor modules for Hamiltonian loop group spaces, preprint, arXiv:1706.07493.
[28] D. McDuff, The moment map for circle actions on symplectic manifolds, J. Geom. Phys. 5 (2) (1988) 149–160.
[29] E. Meinrenken, Twisted K-homology and group-valued moment maps, Int. Math. Res. Not. IMRN (20) (2012) 4563–4618.
[30] G. Murphy, C^*-Algebras and Operator Theory, Academic Press, Inc., Boston, MA, 1990.
[31] J.P. Pier, Amenable locally compact groups, in: A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, 1984.
[32] A. Pressery, G. Segal, Loop groups, in: Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.
[33] Y. Song, Dirac operators on quasi-Hamiltonian G-spaces, J. Geom. Phys. 106 (2016) 70–86.
[34] D. Takata, An infinite-dimensional index theorem and the Higson–Kasparov-Trout algebra, preprint, arXiv:1811.06811.
[35] D. Takata, A Loop Group Equivariant Analytic Index Theory for Infinitedimensional Manifolds (Ph.D. thesis) available as: https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/232217/2/drigk04334.pdf.
[36] D. Takata, An analytic LT-equivariant index and noncommutative geometry, J. Noncommut. Geom. 13 (2) (2019) 553–586.
[37] A. Wasserman, Kac–Moody and Virasoro algebras, lecture notes, available as arXiv:1004.1287.
[38] D. Williams, Crossed products of C^*-algebras, in: Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007.
[39] E. Witten, The index of the Dirac operator in loop space, in: Elliptic Curves and Modular Forms in Algebraic Topology (Princeton, NJ, 1986) 161-181, in: Lecture Notes in Math., vol. 1326, Springer, Berlin, 1988.