Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 368–376
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On m-accretive Schrödinger-type operators with singular potentials on Riemannian manifolds

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Received 4 August 2006; accepted 22 November 2007
Available online 4 December 2007

Abstract

We consider a Schrödinger-type differential expression $H_V = \nabla^*\nabla + V$, where ∇ is a Hermitian connection on a Hermitian vector bundle E over a complete Riemannian manifold (M, g) with metric g and positive smooth measure $d\mu$, and V is a locally integrable section of the bundle of endomorphisms of E. We give a sufficient condition for m-accretivity of a realization of H_V in $L^2(E)$.

PACS: 58J50

Keywords: m-accretive operator; Riemannian manifold; Schrödinger operator; Singular potential

1. Introduction and the main result

1.1. The setting

Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric g and $\dim M = n$. We will assume that M is connected and complete. Moreover, we will assume that we are given a positive smooth measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \ldots, x^n$ there exists a strictly positive C^∞ density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \ldots dx^n$.

Let E be a Hermitian vector bundle over M. We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u, v) = \int_M \langle u(x), v(x)\rangle \,\mathrm{d}\mu(x). \tag{1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the fiberwise inner product in E_x.

In what follows, by $C^\infty(E)$ we denote the space of smooth sections of E and by $C_c^\infty(E)$ the space of smooth compactly supported sections of E. For $E = M \times \mathbb{C}$, we will use the notation $L^2(M)$, $C^\infty(M)$ and $C_c^\infty(M)$.

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.11.008

Let

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$$

be a Hermitian connection on E. We consider a Schrödinger-type differential expression $H_V = \nabla^*\nabla + V$, where

$$\nabla^* : C^\infty(T^*M \otimes E) \to C^\infty(E)$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1), and V is a linear bundle endomorphism of E, i.e. for every $x \in M$,

$$V(x) : E_x \to E_x \tag{2}$$

is a linear operator.

In this paper, we define a realization of the expression H_V as an operator in $L^2(E)$ and show that this realization is m-accretive.

We make the following assumption on V.

Assumption A1. Assume that $V \in L^p_{\text{loc}}(\text{End}\, E)$, where

(1) $p = \frac{2n}{n+2}$ for $n \geq 3$,
(2) $p > 1$ for $n = 2$,
(3) $p = 1$ for $n = 1$.

In the sequel, we will use the following notation:

$$V_1(x) := \frac{V(x) + (V(x))^*}{2}, \qquad V_2(x) := \frac{V(x) - (V(x))^*}{2\mathrm{i}}, \quad x \in M, \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$ and $(V(x))^*$ denotes the adjoint of the linear operator (2) (in the sense of linear algebra).

By (3), for all $x \in M$, we have the following decomposition:

$$V(x) = V_1(x) + \mathrm{i}V_2(x).$$

1.2. Sobolev space $W^{1,2}(E)$

By $W^{1,2}(E)$ we will denote the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u, v)_1 := (u, v) + (\nabla u, \nabla v) \quad u, v \in C_c^\infty(E). \tag{4}$$

Remark 1. Since (M, g) is complete, by [5, Proposition 1.4] it follows that

$$W^{1,2}(E) = \{u \in L^2(E) : \nabla u \in L^2(T^*M \otimes E)\}.$$

1.3. A realization of H_V in $L^2(E)$

Let V be as in Assumption A1. We define an operator S in $L^2(E)$ by $Su = H_V u$ with the domain

$$\text{Dom}(S) = \{u \in W^{1,2}(E) : H_V u \in L^2(E)\}. \tag{5}$$

(In Remark 4 it is shown that for all $u \in W^{1,2}(E)$ we have $Vu \in L^1_{\text{loc}}(E)$, so that $H_V u$ in (5) can be understood in the distributional sense.)

We now state the main result.

Theorem 2. *Assume that (M, g) is a Riemannian manifold with positive smooth measure $d\mu$. Assume that (M, g) is complete. Let E be a Hermitian vector bundle over M and let ∇ be a Hermitian connection on E. Assume that V satisfies Assumption A1. Additionally, assume that for all $x \in M$,*

$$V_1(x) \geq 0, \quad \text{as an operator } E_x \to E_x,$$

where $V_1(x)$ is as in (3). Then S is an m-accretive operator.

Remark 3. Kato [8, Theorem I] proved m-accretivity of the operator $-\Delta + V$ in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set, $-\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure and $V \in L^p_{\text{loc}}(\Omega)$, with p as in Assumption A1, is a complex-valued function such that $\operatorname{Re} V \geq 0$. Kato's result was extended in [10] to operator $H_V = \nabla^*\nabla + V$, where (M, g) is a manifold of bounded geometry, E is a Hermitian vector bundle of bounded geometry over M, ∇ is a C^∞-bounded Hermitian connection on E, and V as in Assumption A1. The method of [10] uses the Kato inequality technique alone, and it works well in a bounded geometry setting. In this paper we eliminate the bounded geometry assumptions by using the Kato inequality technique together with the domination of semigroups technique and positivity preserving property of the resolvent of the scalar Laplacian.

Remark 4. Let $u \in W^{1,2}(E)$. We will show that $Vu \in L^1_{\text{loc}}(E)$. For $n \geq 3$, by the first part of [1, Theorem 2.21], we have the continuous embedding

$$W^{1,2}_{\text{loc}}(E) \subset L^{p'}_{\text{loc}}(E), \tag{6}$$

where $1/p' = 1/2 - 1/n$.

Let $p = \frac{2n}{n+2}$ be as in Assumption A1. Since $1/p + 1/p' = 1$, by Hölder's inequality it follows that $Vu \in L^1_{\text{loc}}(E)$.

For $n = 2$, by the first part of [1, Theorem 2.21], we get continuous embedding (6) for all $2 < p' < \infty$. By Assumption A1, for $n = 2$, we have $p > 1$. We may assume that $1 < p < 2$ (if $V \in L^t_{\text{loc}}(\operatorname{End} E)$ with $t \geq 2$, then $V \in L^p_{\text{loc}}(\operatorname{End} E)$ for all $1 < p < 2$). Given $1 < p < 2$, we can take $p' > 2$ such that $1/p + 1/p' = 1$. By Hölder's inequality we have $Vu \in L^1_{\text{loc}}(E)$.

For $n = 1$, it is well known (see, for example, the second part of [1, Theorem 2.21]) that (6) holds with $p' = \infty$. By Assumption A1 for $n = 1$, we have $p = 1$. Thus by Hölder's inequality we have $Vu \in L^1_{\text{loc}}(E)$.

2. Proof of Theorem 2

Throughout this section, we assume that all hypotheses of Theorem 2 are satisfied.

In the sequel, by $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$. For the duality pairing between $W^{-1,2}(E)$ and $W^{1,2}(E)$, will use the notation

$$(f, u)_{-1}, \quad f \in W^{-1,2}(E),\ u \in W^{1,2}(E). \tag{7}$$

We begin by introducing another realization of H_V.

2.1. *Maximal realization of H_V between $W^{1,2}(E)$ and $W^{-1,2}(E)$*

We define an operator $T\colon W^{1,2}(E) \to W^{-1,2}(E)$ by $Tu = H_V u$ with domain

$$\operatorname{Dom}(T) = \{u \in W^{1,2}(E) : H_V u \in W^{-1,2}(E)\}. \tag{8}$$

Remark 5. The condition $H_V u \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$ makes sense since $H_V u$ is a distributional section of E by Remark 4. Since $\nabla^*\nabla u \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$, it follows that the condition $H_V u \in W^{-1,2}(E)$ in (8) is equivalent to $Vu \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$.

Lemma 6. *The following inclusion holds: $C^\infty_c(E) \subset \operatorname{Dom}(T)$.*

Proof. Let $u \in C^\infty_c(E)$. Then $Vu \in L^p_{\text{comp}}(E)$, where p is as in Assumption A1. By Remark 4, it follows that $W^{1,2}_{\text{loc}}(E) \subset L^{p'}_{\text{loc}}(E)$, where $1/p + 1/p' = 1$. By duality, we have $L^p_{\text{loc}}(E) \subset W^{-1,2}_{\text{loc}}(E)$. Thus $Vu \in W^{-1,2}_{\text{comp}}(E) \subset W^{-1,2}(E)$, and hence $u \in \operatorname{Dom}(T)$. $\square$

2.2. Minimal realization of H_V between $W^{1,2}(E)$ and $W^{-1,2}(E)$

By T_0 we will denote the restriction of T with $\mathrm{Dom}(T_0) = C_c^\infty(E)$. Clearly, T_0 is a densely defined operator.

Remark 7. Since $\mathrm{Dom}(S)$, where S is as in (5), does not necessarily contain $C_c^\infty(E)$, there is no minimal realization of H_V in $L^2(E)$ (in the sense of Section 2.2).

2.3. Maximal and minimal realization of H_{V^*}

Let V^* be the adjoint of V as in (3). In what follows, we will denote by T' and T_0' the maximal and minimal realizations of H_{V^*} in the sense of Section 2.1 and Section 2.2 respectively.

The proof of the following lemma is given in [10, Lemma 2.4]. As we shall refer to its proof later, we included it in Section 4.

Lemma 8. *The following property holds: $T = (T_0')^*$, where $*$ denotes the adjoint of an operator.*

In what follows, we will adopt the terminology of Kato [8] and distinguish between monotone and accretive operators. Accretive operators act within the same Hilbert space, while monotone operators act from a Hilbert space into its adjoint space (anti-dual).

The proofs of the following two lemmas are direct consequences of the definition of T_0; for details see [10, Lemmas 2.5 and 2.6].

Lemma 9. *The operator T_0 is monotone, i.e.*

$$\mathrm{Re}(T_0 s, s)_{-1} \geq 0, \quad \textit{for all } s \in C_c^\infty(E), \tag{9}$$

where $(\cdot,\cdot)_{-1}$ is as in (7).

Lemma 10. *The operator $1 + T_0$ is coercive in the sense that*

$$\|(1+T_0)s\|_{-1} \geq \|s\|_1, \quad \textit{for all } s \in \mathrm{Dom}(T_0) = C_c^\infty(E), \tag{10}$$

where $\|\cdot\|_{-1}$ is the norm in $W^{-1,2}(E)$, and $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$.

Lemma 11. *The following hold:*

(1) *The operator T_0 is closable with closure T_0^{**}.*
(2) $\mathrm{Ran}(1 + T_0^{**})$ *is closed.*

Proof. Let T' be as in Section 2.3. Since $T_0' \subset T'$ (as operators), it follows that T' is densely defined. By Lemma 8 we know that $T' = T_0^*$. Thus T_0^{**} exists and equals $\overline{T_0}$. This proves property (1).

The proof of property (2) uses coerciveness of $1 + T_0$ (and hence of $1 + T_0^{**}$) and the definition of the closed range; for details see [10, Lemma 2.7]. $\square$

2.4. Scalar Laplacian on M

Let $d: C^\infty(M) \to \Omega^1(M)$ be the standard differential, and let $d^*: \Omega^1(M) \to C^\infty(M)$ be the formal adjoint of d with respect to the inner product (1) with $E = M \times \mathbb{C}$. The operator

$$\Delta_M := d^* d: C^\infty(M) \to C^\infty(M)$$

is called the scalar Laplacian.

2.5. Kato's inequality

We will use the following variant of Kato's inequality for the Bochner Laplacian (for the proof, see [2, Theorem 5.7]).

Lemma 12. *Assume that (M, g) is a Riemannian manifold. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E. Assume that $w \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla w \in L^1_{\mathrm{loc}}(E)$. Then*

$$\Delta_M |w| \leq \mathrm{Re}\langle \nabla^*\nabla w, \operatorname{sign} w\rangle, \tag{11}$$

where

$$\operatorname{sign} w(x) = \begin{cases} \dfrac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 13. The original version of Kato's inequality was proven in Kato [7].

We will also need some facts on domination of semigroups.

2.6. Domination of semigroups

With (M, g), E and ∇ as in hypotheses of Theorem 2, it is well known that $\nabla^*\nabla$ and Δ_M are essentially self-adjoint on $C_c^\infty(E)$ and $C_c^\infty(M)$ respectively; see, for example, [5, Theorem 3.5]. Let S_0 and K_0 denote the self-adjoint closures of $\nabla^*\nabla|_{C_c^\infty(E)}$ and $\Delta_M|_{C_c^\infty(M)}$. Since S_0 and K_0 are non-negative self-adjoint operators, it follows that $-S_0$ and $-K_0$ generate self-adjoint contraction semigroups $\{\mathrm{e}^{-tS_0}\}_{t\geq 0}$ in $L^2(E)$ and $\{\mathrm{e}^{-tK_0}\}_{t\geq 0}$ in $L^2(M)$ respectively; see, for example, [3, Theorem 4.6]. Combining Kato's inequality (11) and [6, Theorem 2.15], it follows that the semigroup e^{-tS_0} is dominated by e^{-tK_0}:

$$\left|\mathrm{e}^{-tS_0} f\right| \leq \mathrm{e}^{-tK_0}|f|, \quad \text{for all } f \in L^2(E), \tag{12}$$

where $|\cdot|$ denotes the fiberwise norm in E_x.

2.7. Quadratic form associated with K_0 and S_0

It is well known that the quadratic forms associated (in the sense of [9, Theorem VI.2.1]) with the self-adjoint closures K_0 of $\Delta_M|_{C_c^\infty(M)}$ in $L^2(M)$ and S_0 of $\nabla^*\nabla|_{C_c^\infty(E)}$ in $L^2(E)$ are

$$t_0(v) := \int_M |dv|^2 \,\mathrm{d}\mu \quad \text{and} \quad h_0(w) := \int_M |\nabla w|^2 \,\mathrm{d}\mu,$$

with the domains $\mathrm{D}(t_0) = W^{1,2}(M)$ and $\mathrm{D}(h_0) = W^{1,2}(E)$.

In what follows,

$$(\cdot,\cdot)_{t_0} := t_0(\cdot,\cdot) + (\cdot,\cdot)_{L^2(M)} \tag{13}$$

denotes the inner product corresponding to the quadratic form t_0.

The following lemma is a special case of [11, Proposition 2.4].

Lemma 14. *Assume that (M, g) is a complete Riemannian manifold. Let t_0 be the quadratic form associated with the self-adjoint closure K_0 of $\Delta_M|_{C_c^\infty(M)}$. Assume that $0 \leq f \in \mathrm{D}(t_0)$. Then there exists a sequence $0 \leq \phi_k \in C_c^\infty(M)$ such that*

$$\|\phi_k - f\|_{t_0} \to 0, \quad \text{as } k \to \infty,$$

where $\|\cdot\|_{t_0}$ is the norm corresponding to the inner product (13).

The following proposition is of key importance.

Proposition 15. $\mathrm{Ran}(1+T_0^{**}) = W^{-1,2}(E)$.

Proof. By Lemma 11 it suffices to show that if $u \in W^{1,2}(E)$ and

$$((1+T_0)s, u)_{-1} = 0, \quad \text{for all } s \in C_c^\infty(E), \tag{14}$$

then $u = 0$.

Using (14), integration by parts (see [2, Lemma 8.8]) and complex conjugation, we obtain

$$\begin{aligned} 0 &= (s,u)_{-1} + (\nabla^*\nabla s, u)_{-1} + (Vs,u)_{-1} \\ &= \overline{(u,s)_{-1}} + \overline{(\nabla^*\nabla u, s)_{-1}} + (Vs,u)_{-1}, \quad \text{for all } s \in C_c^\infty(E), \end{aligned} \tag{15}$$

where $(\cdot,\cdot)_{-1}$ is as in (7) and $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$.

By repeating the same arguments as in the proof of the first two equalities in (27) (with V^* replaced by V), we get

$$(Vs,u)_{-1} = \int \langle (Vs)(x), u(x)\rangle \,\mathrm{d}\mu(x) = \int \overline{\langle (V^*u)(x), s(x)\rangle}\,\mathrm{d}\mu(x). \tag{16}$$

Using (15) and (16) we get for all $s \in C_c^\infty(E)$,

$$0 = \overline{(u,s)_{-1}} + \overline{(\nabla^*\nabla u, s)_{-1}} + \int \overline{\langle (V^*u)(x), s(x)\rangle}\,\mathrm{d}\mu(x). \tag{17}$$

Therefore, the following distributional equality holds:

$$\nabla^*\nabla u + V^*u + u = 0. \tag{18}$$

From (18), we have $\nabla^*\nabla u = -V^*u - u \in L^1_{\mathrm{loc}}(E)$. Therefore, by Lemma 12, we get

$$\Delta_M |u| \le \mathrm{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle = \mathrm{Re}\langle -u - V_1 u + \mathrm{i}V_2 u, \operatorname{sign} u\rangle \tag{19}$$

$$= -|u| - \langle V_1 u, \operatorname{sign} u\rangle \le -|u|, \tag{20}$$

where $V_1 \ge 0$ and V_2 are linear self-adjoint bundle endomorphisms as in (3).

By (19) we get the following distributional inequality:

$$(\Delta_M + 1)|u| \le 0. \tag{21}$$

Let t_0 and h_0 be as in Section 2.7. Since $u \in W^{1,2}(E) = \mathrm{D}(h_0)$ and since the domination relation (12) holds, by abstract Proposition 2.12 in [6] it follows that $|u| \in \mathrm{D}(t_0)$.

Using integration by parts, by (21) we have

$$(\phi, |u|)_{t_0} \le 0, \quad \text{for all } 0 \le \phi \in C_c^\infty(M), \tag{22}$$

where $(\cdot,\cdot)_{t_0}$ is as in (13).

Let $f := (K_0+1)^{-1}|u|$, where K_0 is as in Section 2.6. Then $f \in \mathrm{Dom}(K_0)$, and

$$(|u|,|u|)_{L^2(M)} = ((K_0+1)f, |u|)_{L^2(M)} = (f,|u|)_{t_0}. \tag{23}$$

(The last equality in (23) holds since t_0 is the quadratic form associated with K_0; see [9, Theorem VI.2.1].)

It is well known that $(K_0+1)^{-1}$ is positivity preserving in $L^2(M)$; see, for instance, [4, Sec. 5.1]. Since $0 \le |u| \in L^2(M)$, it follows that $0 \le f \in \mathrm{Dom}(K_0) \subset \mathrm{D}(t_0)$. Hence, by Lemma 14 there exists a sequence $0 \le \phi_k \in C_c^\infty(M)$ such that $\|\phi_k\|_{t_0} \to \|f\|_{t_0}$, where $\|\cdot\|_{t_0}$ denotes the norm corresponding to the inner product $(\cdot,\cdot)_{t_0}$.

Thus, by (23) and (22) we have

$$(|u|,|u|)_{L^2(M)} = (f,|u|)_{t_0} = \lim_{k\to\infty} (\phi_k, |u|)_{t_0} \le 0.$$

Hence $u = 0$, and the proposition is proven. □

Corollary 16. *T_0^{**} is a maximal monotone operator (in the sense that it is monotone and has no proper monotone extension).*

Proof. By Lemma 11 we know that $T_0^{**} = \overline{T_0}$. Hence, from (9) we get

$$\operatorname{Re}(T_0^{**}u, u)_{-1} \geq 0, \quad \text{for all } u \in \operatorname{Dom}(T_0^{**}).$$

Thus, T_0^{**} is a monotone operator. By Proposition 15 and the remark after the equation (3.38) in [9, Sec. V.3.10] it follows that T_0^{**} has no proper monotone extension. □

Proposition 17. *The following hold:*

(1) $T = T_0^{**}$.
(2) *The operator T is maximal monotone.*

Proof. We first prove property (1). Since $T_0 \subset T$ (as operators), it follows that $T_0^{**} \subset T$ because T is closed by Lemma 8. By Proposition 15, $\operatorname{Ran}(1 + T_0^{**}) = W^{-1,2}(E)$. By the same proposition (with V replaced by V^*) it follows that $\operatorname{Ran}(1 + (T_0')^{**}) = W^{-1,2}(E)$, where T_0' is as in Section 2.3. Since $1 + T = 1 + (T_0')^*$ (see Lemma 8), it follows that $\operatorname{Ker}(1 + T) = \{0\}$. Hence T cannot be a proper extension of T_0^{**}. This shows that $T_0^{**} = T$.

Property (2) follows immediately from property (1) and Corollary 16. □

3. Proof of Theorem 2

First note that the following holds: $u \in \operatorname{Dom}(S)$ if and only if $u \in \operatorname{Dom}(T)$ and $Tu \in L^2(E)$ (in which case $Su = Tu$).

By Propositions 15 and 17, it follows that $\operatorname{Ran}(1 + T) = W^{-1,2}(E)$. Therefore $\operatorname{Ran}(1 + S) = L^2(E)$.

We will now show that S is accretive. Since T is (maximal) monotone by Proposition 17, it follows that

$$\operatorname{Re}(Su, u) = \operatorname{Re}(Tu, u)_{-1} \geq 0, \quad \text{for all } u \in \operatorname{Dom}(S), \tag{24}$$

where $(\cdot, \cdot)$ is as in (1) and $(\cdot, \cdot)_{-1}$ is as in (7).

The inequality (24) shows that S is accretive; see the definition in the equation (3.37) of Sec. V.3.10 in [9].

We will now show that S is a closed operator. Let $u_k \in \operatorname{Dom}(S)$ be a sequence such that $u_k \to u$ in $L^2(E)$ and $Su_k \to v$ in $L^2(E)$. We need to show that $u \in \operatorname{Dom}(S)$ and $v = Su$. Since $u_k \in \operatorname{Dom}(S)$, by (5) it follows that $u_k \in W^{1,2}(E)$ and $H_V u_k \in L^2(E)$. By (8) we have $u_k \in \operatorname{Dom}(T)$, and from the definitions of T and S and the assumption $Su_k \to v$ in $L^2(E)$ we get $Tu_k = Su_k \to v$ in $W^{-1,2}(E)$. Since $T = T_0^{**}$, from (10) (which also holds for $T_0^{**} = \overline{T_0}$) it follows that u_k is a Cauchy sequence with respect to the norm of $W^{1,2}(E)$; hence, u_k converges in the norm of $W^{1,2}(E)$ to some element $z \in W^{1,2}(E)$. In particular, we get $u_k \to z$ in $L^2(E)$. Since (by assumption) $u_k \to u$ in $L^2(E)$, it follows that $u = z \in W^{1,2}(E)$. Since T is a closed operator and since $u_k \to u$ in $W^{1,2}(E)$ and $Tu_k \to v$ in $W^{-1,2}(E)$, it follows that $u \in \operatorname{Dom}(T)$ and $v = Tu$. Thus, $u \in W^{1,2}(E)$ and $v = Tu = H_V u \in W^{-1,2}(E)$. But, by assumption, we know that $v \in L^2(E)$; hence, $H_V u = v \in L^2(E)$. Now by definition of S it follows that $u \in \operatorname{Dom}(S)$ and $v = Su$; hence, S is a closed operator.

Thus we have proved that S is accretive, closed, and $\operatorname{Ran}(1 + S) = L^2(E)$. By the remark after the equation (3.37) in [9, Sec. V.3.10], it follows that S is m-accretive. □

4. Proof of Lemma 8

We need to show that for any $u \in W^{1,2}(E)$ and $f \in W^{-1,2}(E)$, the equation $Tu = f$ is true if and only if

$$(T's, u)_{-1} = \overline{(f, s)_{-1}}, \quad \text{for all } s \in C_c^\infty(E), \tag{25}$$

where $(\cdot, \cdot)_{-1}$ is as in (7), and $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

1. Assume that $u \in W^{1,2}(E)$, $f \in W^{-1,2}(E)$, and $Tu = f$. Then $Vu \in W^{-1,2}(E)$. By Lemma 6, for all $s \in C_c^\infty(E)$, we have $V^*s \in W^{-1,2}_{\mathrm{comp}}(E)$. Since $s \in C_c^\infty(E)$, we have $V^*s \in L^p_{\mathrm{comp}}(E)$ with p as in Assumption A1. By the proof in Remark 4, we have $u \in W^{1,2}(E) \subset L^{p'}_{\mathrm{loc}}(E)$ (continuous embedding), where $1/p + 1/p' = 1$.

By Hölder's inequality, $L^{p'}_{\mathrm{loc}}(E)$ is in a continuous duality with $L^{p}_{\mathrm{comp}}(E)$ by the usual integration. Thus, for all $s \in C_c^\infty(E)$, we have (after approximating u by sections $u_j \in C_c^\infty(E)$ in $W^{1,2}$-norm in a neighborhood of $\operatorname{supp} s$)

$$\begin{aligned}(V^*s, u)_{-1} &= \lim_{j\to\infty} (V^*s, u_j)_{-1} = \lim_{j\to\infty} \int \langle (V^*s)(x), u_j(x)\rangle \, d\mu(x)\\ &= \int \langle (V^*s)(x), u(x)\rangle \, d\mu(x). \end{aligned} \tag{26}$$

The second equality in (26) holds since $V^*s \in L^1_{\mathrm{loc}}(E)$ by Remark 4 and $u_j \in C_c^\infty(E)$.

Therefore, we obtain

$$\begin{aligned}(V^*s, u)_{-1} &= \int \langle (V^*s)(x), u(x)\rangle \, d\mu(x) = \int \overline{\langle (Vu)(x), s(x)\rangle} \, d\mu(x)\\ &= \overline{(Vu, s)_{-1}}. \end{aligned} \tag{27}$$

The first equality in (27) follows from (26). The second equality in (27) holds by the definition of $(V(x))^*\colon E_x \to E_x$. The third equality in (27) holds for all $s \in C_c^\infty(E)$ since $Vu \in W^{-1,2}(E)$ and $Vu \in L^1_{\mathrm{loc}}(E)$ by Remark 4.

Using (27), we obtain

$$\begin{aligned}(T's, u)_{-1} &= (\nabla^*\nabla s + V^*s, u)_{-1} = (\nabla^*\nabla s, u)_{-1} + (V^*s, u)_{-1}\\ &= \overline{(\nabla^*\nabla u, s)_{-1}} + \overline{(Vu, s)_{-1}} = \overline{(Tu, s)_{-1}}, \end{aligned} \tag{28}$$

where V^* is the adjoint of V as in (3). In the third equality in (28) we also used integration by parts; see, for example, [2, Lemma 8.8].

2. Assume that $u \in W^{1,2}(E)$, $f \in W^{-1,2}(E)$, and (25) holds. Then the first two equalities in (27) hold (we do not know a priori that $Vu \in W^{-1,2}(E)$ so the third equality in (27) is not yet justified). Thus for all $s \in C_c^\infty(E)$,

$$\begin{aligned}\overline{(f, s)_{-1}} &= (T's, u)_{-1}\\ &= (\nabla^*\nabla s, u)_{-1} + (V^*s, u)_{-1} = \overline{(\nabla^*\nabla u, s)_{-1}} + \int \langle (V^*s)(x), u(x)\rangle \, d\mu(x), \end{aligned}$$

where the second equality follows as in (28), and the third equality follows from integration by parts and the first equality in (27).

Since $\nabla^*\nabla u \in W^{-1,2}(E)$ and $f \in W^{-1,2}(E)$, we obtain

$$\overline{(f - \nabla^*\nabla u, s)_{-1}} = \int \langle (V^*s)(x), u(x)\rangle \, d\mu(x), \quad \text{for all } s \in C_c^\infty(E). \tag{29}$$

Since $u \in W^{1,2}(E)$, from Remark 4 we know that $Vu \in L^1_{\mathrm{loc}}(E)$. By (29) we get $Vu \in W^{-1,2}(E)$ since $C_c^\infty(E)$ is dense in $W^{1,2}(E)$. Thus, as in (27),

$$\int \langle (V^*s)(x), u(x)\rangle \, d\mu(x) = \overline{(Vu, s)_{-1}}, \quad \text{for all } s \in C_c^\infty(E). \tag{30}$$

From (29) and (30), we obtain

$$(f - \nabla^*\nabla u, s)_{-1} = (Vu, s)_{-1}, \quad \text{for all } s \in C_c^\infty(E). \tag{31}$$

Therefore,

$$(f, s)_{-1} = (\nabla^*\nabla u, s)_{-1} + (Vu, s)_{-1} = (Tu, s)_{-1}, \quad \text{for all } s \in C_c^\infty(E).$$

This shows that $Tu = f$, and the lemma is proven. □

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