



# Lightlike hypersurfaces along spacelike submanifolds in Minkowski space–time



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## ABSTRACT

We consider the singularities of lightlike hypersurfaces along spacelike submanifolds in Lorentz–Minkowski space of general codimension. As an application of the theory of Legendrian singularities, we investigate the geometric meanings of the singularities of lightlike hypersurfaces in terms of the contact of spacelike submanifolds with lightcones.

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## 1. Introduction

Studies on the extrinsic differential geometry of submanifolds in Lorentz manifolds are of special interest in relativity theory. In particular, *lightlike hypersurfaces* provide good models for studying different horizon types [1–3]. A lightlike hypersurface is also called a *light sheet* in theoretical physics [4] and plays a principal role in the quantum theory of gravity. Here we investigate the singularities of lightlike hypersurfaces in Lorentz–Minkowski space. Although Lorentz–Minkowski space has no gravity, the singularities of lightlike hypersurfaces give a typical model of horizons and important information for the shape of horizons in general Lorentz manifolds. The lightlike hypersurfaces are constructed as ruled hypersurfaces along spacelike submanifolds with lightlike rulings.

Tools in the theory of singularities have proven to be a useful description of geometric properties of submanifolds immersed in different ambient spaces from both local and global viewpoints [5–11]. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with models of the ambient space can be described by analysis of the singularities of appropriate families of contact functions or, equivalently, of their associated Legendrian maps [12–14]. When working in Lorentz–Minkowski space, the properties associated with the contacts of a given submanifold with lightlike hyperplanes or lightcones have a special relevance. A framework constructed for studying spacelike submanifolds of codimension two in Lorentz–Minkowski space revealed a Lorentz invariant concerning their contacts with lightlike hyperplanes [15,10]. The geometry related to this framework is called *lightlike geometry* (or *lightlike flat geometry*) of spacelike submanifolds of codimension two. Using the invariants of lightlike geometry, the singularities of lightlike hypersurfaces along spacelike surfaces in Lorentz–Minkowski 4-space were investigated [8]. It is not difficult to generalize the result from that study [8] to the case of codimension two in general-dimensional Lorentz–Minkowski space [16]. However,

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Fig. 1. Lightlike surface.

the situation is rather complicated for the general codimensional case. The main difference from the Euclidean space case is that the fiber of the canal hypersurface of a spacelike submanifold is neither connected nor compact. Here we investigate singularities of lightlike hypersurfaces along general codimension spacelike submanifolds in Lorentz–Minkowski space. To avoid the above difficulty, we arbitrarily choose a timelike future-directed unit normal vector field along the spacelike submanifold, which always exists for an orientable manifold [17]. Then we construct the unit spherical normal bundle relative to this timelike unit normal vector field, which can be considered as a codimension-two spacelike canal submanifold of the ambient Lorentz–Minkowski space. Therefore, we can apply the idea of the lightlike geometry of spacelike submanifolds of Lorentz–Minkowski space of codimension two. We previously used this approach to construct a framework of the lightlike geometry of spacelike submanifolds with general codimension [17] and investigated local and global properties. Here we apply this framework and the theory of Legendrian singularities to investigate the singularities of lightlike hypersurfaces along spacelike submanifolds in Lorentz–Minkowski space of general codimension. Fig. 1 shows an image of the lightlike surface along an ellipse in the Euclidean plane canonically embedded in Lorentz–Minkowski 3-space. We can observe four swallowtail singularities (defined in Section 6) on the surface that correspond to the vertices of the ellipse. This means that there might be interesting geometric meanings of the singularities of lightlike surfaces.

In Section 3 we briefly review the framework of the lightlike geometry of spacelike submanifolds of general codimension [17]. The notion of lightlike hypersurfaces along spacelike submanifolds is introduced and the basic properties are investigated in Section 4. The notion of the family of distance squared functions is useful for studying lightlike hypersurfaces. The critical value set of a lightlike hypersurface along a spacelike submanifold is called the *lightlike focal set* of the submanifold. In Section 5 we show that the lightlike focal set of a spacelike submanifold is a point if and only if the lightlike hypersurface along the submanifold is a subset of a lightcone (Proposition 5.1). Therefore, a lightcone is a model hypersurface of lightlike hypersurfaces. The geometric meaning of the singularities of a lightlike hypersurface is described by the theory of contact of submanifolds with model hypersurfaces. As an application of the theory of Legendrian singularities, we show that two lightlike hypersurfaces are locally diffeomorphic if and only if the types of contact of spacelike submanifolds with lightcones are the same in the sense of Montaldi [13] under some generic conditions (Theorem 5.5). In Section 6 we describe the case for codimension two as a special case. We also investigate spacelike curves in Lorentz–Minkowski 4-space as the simplest case of a higher codimension in Section 7. In Section 8 we consider the case of submanifolds lying in a spacelike hyperplane or in hyperbolic space. In this case, lightlike focal sets correspond to the focal sets in the Euclidean sense or hyperbolic and de Sitter focal sets [18].

## 2. Basic notations for Lorentz–Minkowski space

In this section we introduce some basic notions for Lorentz–Minkowski  $n + 1$ -space [19].

Let  $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R} \ (i = 0, 1, \dots, n)\}$  be an  $n + 1$ -dimensional Cartesian space. For any  $\mathbf{x} = (x_0, x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$ , the *pseudo scalar product* of  $\mathbf{x}$  and  $\mathbf{y}$  is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i.$$

We call  $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$  *Lorentz–Minkowski  $n + 1$ -space*. We write  $\mathbb{R}_1^{n+1}$  instead of  $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ . We say that a non-zero vector  $\mathbf{x} \in \mathbb{R}_1^{n+1}$  is *spacelike*, *lightlike* or *timelike* if  $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ ,  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  or  $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ , respectively. The norm of the vector  $\mathbf{x} \in \mathbb{R}_1^{n+1}$  is defined as  $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$ . The *signature* of a vector  $\mathbf{x} \in \mathbb{R}_1^{n+1} \setminus \{\mathbf{0}\}$  is defined as

$$\text{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} : \text{spacelike}, \\ 0 & \mathbf{x} : \text{lightlike}, \\ -1 & \mathbf{x} : \text{timelike}. \end{cases}$$

The canonical projection  $\pi : \mathbb{R}_1^{n+1} \rightarrow \mathbb{R}^n$  is defined as  $\pi(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n)$ . Here we identify  $\{\mathbf{0}\} \times \mathbb{R}^n$  with  $\mathbb{R}^n$  and it is considered as a Euclidean  $n$ -space whose scalar product is induced from the pseudo scalar product  $\langle \cdot, \cdot \rangle$ . For a vector  $\mathbf{v} \in \mathbb{R}_1^{n+1}$  and a real number  $c$ , we define a *hyperplane with pseudo normal*  $\mathbf{v}$  as

$$HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

We call  $HP(\mathbf{v}, c)$  a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if  $\mathbf{v}$  is timelike, spacelike or lightlike, respectively. We note that  $\{\mathbf{0}\} \times \mathbb{R}^n$  is a spacelike hypersurface in  $\mathbb{R}_1^{n+1}$ .

We define *hyperbolic  $n$ -space* as

$$H^n(-1) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}$$

and *de Sitter  $n$ -space* as

$$S_1^n = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}.$$

We define

$$LC_{\mathbf{a}} = \{\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\},$$

which is called the *lightcone with vertex*  $\mathbf{a}$ , and denote  $LC^* = LC_{\mathbf{0}} \setminus \{\mathbf{0}\}$ . If  $\mathbf{x} = (x_0, x_1, \dots, x_n)$  is a lightlike vector, then  $x_0 \neq 0$ . Therefore, we have

$$\tilde{\mathbf{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in S_+^{n-1} = \{\mathbf{x} = (x_0, x_1, \dots, x_n) \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_0 = 1\}.$$

We call  $S_+^{n-1}$  the *lightcone* (or *spacelike*) *unit  $n-1$ -sphere*.

For any  $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^n \in \mathbb{R}_1^{n+1}$ , we define the vector  $\mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \dots \wedge \mathbf{x}^n$  as

$$\mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \dots \wedge \mathbf{x}^n = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \dots & \mathbf{e}_n \\ x_0^1 & x_1^1 & \dots & x_n^1 \\ x_0^2 & x_1^2 & \dots & x_n^2 \\ \vdots & \vdots & \dots & \vdots \\ x_0^n & x_1^n & \dots & x_n^n \end{vmatrix},$$

where  $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$  is the canonical basis of  $\mathbb{R}_1^{n+1}$  and  $\mathbf{x}^i = (x_0^i, x_1^i, \dots, x_n^i)$ . We can easily show that

$$\langle \mathbf{x}, \mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \dots \wedge \mathbf{x}^n \rangle = \det(\mathbf{x}, \mathbf{x}^1, \dots, \mathbf{x}^n),$$

so that  $\mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \dots \wedge \mathbf{x}^n$  is pseudo orthogonal to any  $\mathbf{x}^i$  ( $i = 1, \dots, n$ ).

### 3. Differential geometry on spacelike submanifolds

In this section we introduce the basic geometric framework for studying spacelike submanifolds in Minkowski  $n+1$ -space analogous to the case of codimension two [10]. Let  $\mathbb{R}_1^{n+1}$  be an oriented and time-oriented space. We choose  $\mathbf{e}_0 = (1, 0, \dots, 0)$  as the future timelike vector field. Let  $\mathbf{X} : U \rightarrow \mathbb{R}_1^{n+1}$  be a spacelike embedding of codimension  $k$ , where  $U \subset \mathbb{R}^s$  ( $s+k=n+1$ ) is an open subset. We also write  $M = \mathbf{X}(U)$  and identify  $M$  and  $U$  through the embedding  $\mathbf{X}$ . We say that  $\mathbf{X}$  is *spacelike* if the tangent space  $T_p M$  of  $M$  at  $p$  is a spacelike subspace (i.e., consists of spacelike vectors) for any point  $p \in M$ . For any  $p = \mathbf{X}(u) \in M \subset \mathbb{R}_1^{n+1}$  we have

$$T_p M = \langle \mathbf{X}_{u_1}(u), \dots, \mathbf{X}_{u_s}(u) \rangle_{\mathbb{R}}.$$

Let  $N_p(M)$  be the pseudo-normal space of  $M$  at  $p$  in  $\mathbb{R}_1^{n+1}$ . Since  $T_p M$  is a spacelike subspace of  $T_p \mathbb{R}_1^{n+1}$ ,  $N_p(M)$  is a  $k$ -dimensional Lorentzian subspace of  $T_p \mathbb{R}_1^{n+1}$  [19]. On the pseudo-normal space  $N_p(M)$ , we have two types of pseudosphere:

$$N_p(M; -1) = \{\mathbf{v} \in N_p(M) \mid \langle \mathbf{v}, \mathbf{v} \rangle = -1\}$$

$$N_p(M; 1) = \{\mathbf{v} \in N_p(M) \mid \langle \mathbf{v}, \mathbf{v} \rangle = 1\},$$

so that we have two unit spherical normal bundles over  $M$ :

$$N(M; -1) = \bigcup_{p \in M} N_p(M; -1) \quad \text{and} \quad N(M; 1) = \bigcup_{p \in M} N_p(M; 1).$$

Then we have the Whitney sum decomposition

$$T\mathbb{R}_1^{n+1}|M = TM \oplus N(M).$$

Since  $M = \mathbf{X}(U)$  is spacelike,  $\mathbf{e}_0$  is a transversal future-directed timelike vector field along  $M$ . For any  $\mathbf{v} \in T_p \mathbb{R}_1^{n+1}|M$ , we have  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ , where  $\mathbf{v}_1 \in T_p M$  and  $\mathbf{v}_2 \in N_p(M)$ . If  $\mathbf{v}$  is timelike, then  $\mathbf{v}_2$  is timelike. Let  $\pi_{N(M)} : T\mathbb{R}_1^{n+1}|M \rightarrow N(M)$

be the canonical projection. Then  $\pi_{N(M)}(\mathbf{e}_0)$  is a future-directed timelike normal vector field along  $M$ . Thus, we always have a future-directed unit timelike normal vector field along  $M$  (even globally). We now arbitrarily choose a future-directed unit timelike normal vector field  $\mathbf{n}^T(u) \in N_p(M; -1)$ , where  $p = \mathbf{X}(u)$ . Therefore, we have the pseudo-orthonormal complement  $(\langle \mathbf{n}^T(u) \rangle_{\mathbb{R}})^{\perp}$  in  $N_p(M)$ , which is a  $k-1$ -dimensional subspace of  $N_p(M)$ . We can also choose a pseudo-normal section  $\mathbf{n}^S(u) \in (\langle \mathbf{n}^T(u) \rangle_{\mathbb{R}})^{\perp} \cap N(M; 1)$ , at least locally, and then we have  $\langle \mathbf{n}^S, \mathbf{n}^S \rangle = 1$  and  $\langle \mathbf{n}^S, \mathbf{n}^T \rangle = 0$ . We define a  $k-1$ -dimensional spacelike unit sphere in  $N_p(M)$  as

$$N_1(M)_p[\mathbf{n}^T] = \{\xi \in N_p(M; 1) \mid \langle \xi, \mathbf{n}^T(p) \rangle = 0\}.$$

Then we have a spacelike unit  $k-2$ -spherical bundle over  $M$  with respect to  $\mathbf{n}^T$  defined as

$$N_1(M)[\mathbf{n}^T] = \bigcup_{p \in M} N_1(M)_p[\mathbf{n}^T].$$

Since we have  $T_{(p,\xi)}N_1(M)[\mathbf{n}^T] = T_pM \times T_{\xi}N_1(M)_p[\mathbf{n}^T]$ , we have the canonical Riemannian metric on  $N_1(M)[\mathbf{n}^T]$ . We denote the Riemannian metric on  $N_1(M)[\mathbf{n}^T]$  by  $(G_{ij}(p, \xi))_{1 \leq i,j \leq n-1}$ .

For any future-directed unit normal  $\mathbf{n}^T$  along  $M$ , we arbitrarily choose (at least locally) the unit spacelike normal vector field  $\mathbf{n}^S$  with  $\mathbf{n}^S(u) \in N_1(M)_p[\mathbf{n}^T]$ , where  $p = \mathbf{X}(u)$ . We call  $(\mathbf{n}^T, \mathbf{n}^S)$  a future directed pair along  $M$ . Clearly, the vectors  $\mathbf{n}^T(u) \pm \mathbf{n}^S(u)$  are lightlike. Here we choose  $\mathbf{n}^T + \mathbf{n}^S$  as a lightlike normal vector field along  $M$ . We define a mapping

$$\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S) : U \longrightarrow LC^*$$

by  $\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S)(u) = \mathbf{n}^T(u) + \mathbf{n}^S(u)$  and call it the lightcone Gauss image of  $M = \mathbf{X}(U)$  with respect to  $(\mathbf{n}^T, \mathbf{n}^S)$ . Under the identification of  $M$  and  $U$  through  $\mathbf{X}$ , we have the linear mapping provided by the derivative of the lightcone Gauss image  $\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S)$  at each point  $p \in M$ ,

$$d_p\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S) : T_pM \longrightarrow T_p\mathbb{R}_1^{n+1} = T_pM \oplus N_p(M).$$

Consider the orthogonal projections  $\pi^t : T_pM \oplus N_p(M) \rightarrow T_p(M)$ . We define

$$d_p\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S)^t = \pi^t \circ d_p(\mathbf{n}^T + \mathbf{n}^S).$$

We call the linear transformations  $S_p(\mathbf{n}^T, \mathbf{n}^S) = -d_p\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S)^t$  the  $(\mathbf{n}^T, \mathbf{n}^S)$ -shape operator of  $M = \mathbf{X}(U)$  at  $p = \mathbf{X}(u)$ . Let  $\{\kappa_i(\mathbf{n}^T, \mathbf{n}^S)(p)\}_{i=1}^s$  be the eigenvalues of  $S_p(\mathbf{n}^T, \mathbf{n}^S)$ , which are called the lightcone principal curvatures with respect to  $(\mathbf{n}^T, \mathbf{n}^S)$  at  $p = \mathbf{X}(u)$ . Then the lightcone Gauss–Kronecker curvature with respect to  $(\mathbf{n}^T, \mathbf{n}^S)$  at  $p = \mathbf{X}(u)$  is defined as

$$K_{\ell}(\mathbf{n}^T, \mathbf{n}^S)(p) = \det S_p(\mathbf{n}^T, \mathbf{n}^S).$$

We say that a point  $p = \mathbf{X}(u)$  is an  $(\mathbf{n}^T, \mathbf{n}^S)$ -umbilical point if

$$S_p(\mathbf{n}^T, \mathbf{n}^S) = \kappa(\mathbf{n}^T, \mathbf{n}^S)(p)1_{T_pM}.$$

We say that  $M = \mathbf{X}(U)$  is totally  $(\mathbf{n}^T, \mathbf{n}^S)$ -umbilical if all points on  $M$  are  $(\mathbf{n}^T, \mathbf{n}^S)$ -umbilical. Moreover,  $M = \mathbf{X}(U)$  is said to be totally lightcone umbilical if it is totally  $(\mathbf{n}^T, \mathbf{n}^S)$ -umbilical for any future-directed pair  $(\mathbf{n}^T, \mathbf{n}^S)$ .

We now deduce the lightcone Weingarten formula. Since  $\mathbf{X}_{u_i} (i = 1, \dots, s)$  are spacelike vectors, we have a Riemannian metric (the lightcone first fundamental form) on  $M = \mathbf{X}(U)$  defined by  $ds^2 = \sum_{i=1}^s g_{ij} du_i du_j$ , where  $g_{ij}(u) = \langle \mathbf{X}_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$  for any  $u \in U$ . We also have a lightcone second fundamental invariant with respect to the normal vector field  $(\mathbf{n}^T, \mathbf{n}^S)$  defined by  $h_{ij}(\mathbf{n}^T, \mathbf{n}^S)(u) = \langle -(\mathbf{n}^T + \mathbf{n}^S)_{u_i}(u), \mathbf{X}_{u_j}(u) \rangle$  for any  $u \in U$ . Using similar arguments to those in the proof of [10, Proposition 3.2], we obtain the following proposition.

**Proposition 3.1.** We choose a pseudo-orthonormal frame  $\{\mathbf{n}^T, \mathbf{n}_1^S, \dots, \mathbf{n}_{k-1}^S\}$  of  $N(M)$  with  $\mathbf{n}_{k-1}^S = \mathbf{n}^S$ . Then we have the following lightcone Weingarten formula with respect to  $(\mathbf{n}^T, \mathbf{n}^S)$ :

- (a)  $\mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S)_{u_i} = \langle \mathbf{n}_{u_i}^S, \mathbf{n}^T \rangle (\mathbf{n}^T - \mathbf{n}^S) + \sum_{\ell=1}^{k-2} \langle (\mathbf{n}^T + \mathbf{n}^S)_{u_i}, \mathbf{n}_{\ell}^S \rangle \mathbf{n}_{\ell}^S - \sum_{j=1}^s h_{ij}^j(\mathbf{n}^T, \mathbf{n}^S) \mathbf{X}_{u_j}$
- (b)  $\pi^t \circ \mathbb{L}\mathbb{G}(\mathbf{n}^T, \mathbf{n}^S)_{u_i} = -\sum_{j=1}^s h_{ij}^j(\mathbf{n}^T, \mathbf{n}^S) \mathbf{X}_{u_j},$

where  $\left(h_{ij}^j(\mathbf{n}^T, \mathbf{n}^S)\right) = (h_{ik}(\mathbf{n}^T, \mathbf{n}^S)) (g^{kj})$  and  $(g^{kj}) = (g_{kj})^{-1}$ .

As a consequence of the above proposition, we can explicitly express the lightcone curvature as

$$K_{\ell}(\mathbf{n}^T, \mathbf{n}^S) = \frac{\det(h_{ij}(\mathbf{n}^T, \mathbf{n}^S))}{\det(g_{\alpha\beta})}.$$

Since  $\langle -(\mathbf{n}^T + \mathbf{n}^S)(u), \mathbf{X}_{u_j}(u) \rangle = 0$ , we have  $h_{ij}(\mathbf{n}^T, \mathbf{n}^S)(u) = \langle \mathbf{n}^T(u) + \mathbf{n}^S(u), \mathbf{X}_{u_i u_j}(u) \rangle$ . Therefore, the lightcone second fundamental invariant at point  $p_0 = \mathbf{X}(u_0)$  depends only on the values  $\mathbf{n}^T(u_0) + \mathbf{n}^S(u_0)$  and  $\mathbf{X}_{u_i u_j}(u_0)$ . Thus, the lightcone curvatures also depend only on  $\mathbf{n}^T(u_0) + \mathbf{n}^S(u_0)$ ,  $\mathbf{X}_{u_i}(u_0)$  and  $\mathbf{X}_{u_i u_j}(u_0)$ , independent of the derivation of the vector fields

$\mathbf{n}^T$  and  $\mathbf{n}^S$ .  $\kappa_i(\mathbf{n}_0^T, \mathbf{n}_0^S)(p_0)$  ( $i = 1, \dots, s$ ) and  $K_\ell(\mathbf{n}_0^T, \mathbf{n}_0^S)(u_0)$  denote the lightcone curvatures at  $p_0 = \mathbf{X}(u_0)$  with respect to  $(\mathbf{n}_0^T, \mathbf{n}_0^S) = (\mathbf{n}^T(u_0), \mathbf{n}^S(u_0))$ . We might also say that a point  $p_0 = \mathbf{X}(u_0)$  is  $(\mathbf{n}_0^T, \mathbf{n}_0^S)$ -umbilical because the lightcone  $(\mathbf{n}^T, \mathbf{n}^S)$ -shape operator at  $p_0$  depends only on the normal vectors  $(\mathbf{n}_0^T, \mathbf{n}_0^S)$ . Therefore, we denote  $h_{ij}(\mathbf{n}^T, \xi)(u_0) = h_{ij}(\mathbf{n}^T, \mathbf{n}^S)(u_0)$  and  $K_\ell(\mathbf{n}^T, \xi)(p_0) = K_\ell(\mathbf{n}_0^T, \mathbf{n}_0^S)(p_0)$ , where  $\xi = \mathbf{n}^S(u_0)$  for some local extension  $\mathbf{n}^T(u)$  of  $\xi$ . Analogously, we say that a point  $p_0 = \mathbf{X}(u_0)$  is an  $(\mathbf{n}_0^T, \mathbf{n}_0^S)$ -parabolic point of  $\mathbf{X} : U \rightarrow \mathbb{R}_1^{n+1}$  if  $K_\ell(\mathbf{n}_0^T, \mathbf{n}_0^S)(u_0) = 0$ . We also say that a point  $p_0 = \mathbf{X}(u_0)$  is a  $(\mathbf{n}_0^T, \mathbf{n}_0^S)$ -flat point if it is an  $(\mathbf{n}_0^T, \mathbf{n}_0^S)$ -umbilical point and  $K_\ell(\mathbf{n}_0^T, \mathbf{n}_0^S)(u_0) = 0$ .

**Definition 3.2.** We define the map  $\mathbb{L}\mathbb{G}(\mathbf{n}^T) : N_1(M)[\mathbf{n}^T] \rightarrow LC^*$  as  $\mathbb{L}\mathbb{G}(\mathbf{n}^T)(u, \xi) = \mathbf{n}^T(u) + \xi$ , which we call the *lightcone Gauss image* of  $N_1(M)[\mathbf{n}^T]$ .

This map leads us to the notion of curvature. Let  $T_{(p, \xi)}N_1(M)[\mathbf{n}^T]$  be the tangent space of  $N_1(M)[\mathbf{n}^T]$  at  $(p, \xi)$ . Under the canonical identification  $(\mathbb{L}\mathbb{G}(\mathbf{n}^T))^*T\mathbb{R}_1^{n+1})_{(p, \xi)} = T_{(\mathbf{n}^T(p) + \xi)}\mathbb{R}_1^{n+1} \cong T_p\mathbb{R}_1^{n+1}$ , we have

$$T_{(p, \xi)}N_1(M)[\mathbf{n}^T] = T_pM \oplus T_\xi S^{k-2} \subset T_pM \oplus N_p(M) = T_p\mathbb{R}_1^{n+1},$$

where  $T_\xi S^{k-2} \subset T_\xi N_p(M) \cong N_p(M)$  and  $p = \mathbf{X}(u)$ . Let

$$\Pi^t : \mathbb{L}\mathbb{G}(\mathbf{n}^T)^*T\mathbb{R}_1^{n+1} = TN_1(M)[\mathbf{n}^T] \oplus \mathbb{R}^{k+1} \rightarrow TN_1(M)[\mathbf{n}^T]$$

be the canonical projection. Then we have a linear transformation

$$S_\ell(\mathbf{n}^T)_{(p, \xi)} = -\Pi^t_{\mathbb{L}\mathbb{G}(\mathbf{n}^T)(p, \xi)} \circ d_{(p, \xi)}\mathbb{L}\mathbb{G}(\mathbf{n}^T) : T_{(p, \xi)}N_1(M)[\mathbf{n}^T] \rightarrow T_{(p, \xi)}N_1(M)[\mathbf{n}^T],$$

which is called the *lightcone shape operator* of  $N_1(M)[\mathbf{n}^T]$  at  $(p, \xi)$ . Let  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$  be the eigenvalues of  $S_\ell(\mathbf{n}^T)_{(p, \xi)}$  ( $i = 1, \dots, n-1$ ). Here,  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$  ( $i = 1, \dots, s$ ) denotes the eigenvalues belonging to the eigenvectors on  $T_pM$  and  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$  ( $i = s+1, \dots, n-1$ ) as the eigenvalues belonging to the eigenvectors on the tangent space of the fiber of  $N_1(M)[\mathbf{n}^T]$ . We previously showed that  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi) = -1$  ( $i = s+1, \dots, n-1$ ) [17]. We call  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$  ( $i = 1, \dots, s$ ) the *lightcone principal curvatures* of  $M$  with respect to  $(\mathbf{n}^T, \xi)$  at  $p \in M$ . The *lightcone Lipschitz–Killing curvature* of  $N_1(M)[\mathbf{n}^T]$  at  $(p, \xi)$  is defined as  $K_\ell(\mathbf{n}^T)(p, \xi) = \det S_\ell(\mathbf{n}^T)_{(p, \xi)}$ .

**Definition 3.3.** We define a mapping  $\mathbb{D}\mathbb{G}(\mathbf{n}^T) : N_1(M)[\mathbf{n}^T] \rightarrow S_1^n$  as  $\mathbb{D}\mathbb{G}(\mathbf{n}^T)(p, \xi) = \xi$ , which is called the *de Sitter Gauss image* of  $N_1(M)[\mathbf{n}^T]$ .

In a similar way to the above case, we can define the *de Sitter shape operator*  $S_d(\mathbf{n}^T)_{(p, \xi)}$ . The *de Sitter principal curvatures* of  $M$  with respect to  $(\mathbf{n}^T, \xi)$  at  $p \in M$  are defined as the eigenvalues of  $S_d(\mathbf{n}^T)_{(p, \xi)}$  belonging to the eigenvectors on  $T_pM$ , which are denoted by  $\kappa_d(\mathbf{n}^T)_i(p, \xi)$  ( $i = 1, \dots, s$ ).

**Definition 3.4.** We define a mapping  $\mathbb{H}\mathbb{G}(\mathbf{n}^T) : M \rightarrow H^n(-1)$  as  $\mathbb{H}\mathbb{G}(\mathbf{n}^T)(p) = \mathbf{n}^T(p)$ . We call this the *hyperbolic Gauss image* of  $M$  with respect to  $\mathbf{n}^T$ .

We define the *hyperbolic shape operator*  $S_h(\mathbf{n}^T)_p$  with respect to  $\mathbf{n}^T$  as  $S_h(\mathbf{n}^T) = -\pi^t \circ d\mathbb{H}\mathbb{G}(\mathbf{n}^T)(p)$ , where  $\pi^t : T_p\mathbb{R}_1^{n+1} = T_pM \oplus N_p(M) \rightarrow TM$  is the orthogonal projection under the identification  $T_{\mathbf{n}^T(p)}\mathbb{R}_1^{n+1} \cong T_p\mathbb{R}_1^{n+1}$ . We also define the *hyperbolic principal curvatures*  $\kappa_h(\mathbf{n}^T)_i(p)$  ( $i = 1, \dots, s$ ) of  $M$  as the eigenvalues of  $S_h(\mathbf{n}^T)$ . By assertion (b) of Proposition 3.1, we have the following corollary.

**Corollary 3.5.** Using the above notation, we have the following assertions:

- (1) The lightcone principal curvatures  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$  ( $i = 1, \dots, s$ ) are the eigenvalues of the matrix  $(h_i^j(\mathbf{n}^T(p), \mathbf{n}^S(p)))$ , where  $\mathbf{n}^S$  is the local section of  $N_1(M)[\mathbf{n}^T]$  with  $\mathbf{n}^T(p) = \xi$ .
- (2) We have the following relation:

$$\kappa_\ell(\mathbf{n}^T)_i(p, \xi) = \kappa_h(\mathbf{n}^T)_i(p) + \kappa_d(\mathbf{n}^T)_i(p, \xi), \quad (i = 1, \dots, s).$$

#### 4. Lightlike hypersurfaces

In this section we define a lightlike hypersurface along a spacelike submanifold, which is the main subject of our paper.

**Definition 4.1.** We define the hypersurface

$$\mathbb{L}\mathbb{H}_M(\mathbf{n}^T) : N_1(M)[\mathbf{n}^T] \times \mathbb{R} \rightarrow \mathbb{R}_1^{n+1}$$

as

$$\mathbb{L}\mathbb{H}_M((p, \xi), t) = \mathbf{X}(u) + t(\mathbf{n}^T + \xi)(u) = \mathbf{X}(u) + t\mathbb{L}\mathbb{G}(\mathbf{n}^T)(u, \xi),$$

where  $p = \mathbf{X}(u)$  is called the *lightlike hypersurface* along  $M$  relative to  $\mathbf{n}^T$ .

In general, a hypersurface  $H \subset \mathbb{R}_1^{n+1}$  is called a *lightlike hypersurface* if it is tangential to the lightcone at any regular point. We note that  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$  is a lightlike hypersurface.

We introduce the notion of Lorentzian distance-squared functions on spacelike submanifolds, which is useful for studying singularities of lightlike hypersurfaces.

**Definition 4.2.** We define a family of functions  $G : M \times \mathbb{R}_1^{n+1} \rightarrow \mathbb{R}$  on a spacelike submanifold  $M = \mathbf{X}(U)$  as

$$G(p, \lambda) = G(u, \lambda) = \langle \mathbf{X}(u) - \lambda, \mathbf{X}(u) - \lambda \rangle,$$

where  $p = \mathbf{X}(u)$ . We call  $G$  the *Lorentzian distance-squared function* on the spacelike submanifold  $M$ .

For any fixed  $\lambda_0 \in \mathbb{R}_1^{n+1}$ , we write  $g_{\lambda_0}(p) = G(p, \lambda_0)$  and have the following proposition.

**Proposition 4.3.** Let  $M$  be a spacelike submanifold and let  $G : M \times (\mathbb{R}_1^{n+1} \setminus M) \rightarrow \mathbb{R}$  be the Lorentzian distance-squared function on  $M$ . Suppose that  $p_0 \neq \lambda_0$ . Then we have the following.

(1)  $g_{\lambda_0}(p_0) = \partial g_{\lambda_0} / \partial u_i(p_0) = 0$  ( $i = 1, \dots, s$ ) if and only if there exist  $\xi_0 \in N_1(M)_{p_0}[\mathbf{n}^T]$  and  $\mu \in \mathbb{R} \setminus \{0\}$  such that

$$p_0 - \lambda_0 = \mu \mathbb{L}G(\mathbf{n}^T)(p_0, \xi_0).$$

(2)  $g_{\lambda_0}(p_0) = \partial g_{\lambda_0} / \partial u_i(p_0) = \det \mathcal{H}(g_{\lambda_0})(p_0) = 0$  ( $i = 1, \dots, s$ ) if and only if there exist  $\xi \in N_1(M)_{p_0}[\mathbf{n}^T]$  and  $\mu \in \mathbb{R} \setminus \{0\}$  such that

$$p_0 - \lambda_0 = \mu \mathbb{L}G(\mathbf{n}^T)(p_0, \xi_0)$$

and  $1/\mu$  is one of the non-zero lightcone principal curvatures  $\kappa_i(\mathbf{n}^T)(p_0, \xi_0)$  ( $i = 1, \dots, s$ ).

Here,  $\mathcal{H}(g_{\lambda_0})(p_0)$  is the Hessian matrix of  $g_{\lambda_0}$  at  $p_0$ .

**Proof.** (1) Let  $p = \mathbf{X}(u)$ . The condition  $g_{\lambda_0}(p) = \langle \mathbf{X}(u) - \lambda_0, \mathbf{X}(u) - \lambda_0 \rangle = 0$  means that  $\mathbf{X}(u) - \lambda_0 \in LC^*$ . We observe that  $\partial g / \partial u_i(p) = 2 \langle \mathbf{X}_{u_i}(u), \mathbf{X}(u) - \lambda_0 \rangle = 0$  if and only if  $\mathbf{X}(u) - \lambda_0 \in N_p M$ . Hence,  $g_{\lambda_0}(p_0) = \partial g_{\lambda_0} / \partial u_i(p_0) = 0$  ( $i = 1, \dots, s$ ) if and only if  $p_0 - \lambda_0 \in N_p M \cap LC^*$ . Then we denote  $v = \mathbf{X}(u) - \lambda_0 \in LC^*$ . If  $\langle \mathbf{n}^T(u), v \rangle = 0$ , then  $\mathbf{n}^T(u)$  has to be lightlike or spacelike. This contradicts the fact that  $\mathbf{n}^T(u)$  is a timelike unit vector, so that  $\langle \mathbf{n}^T(u), v \rangle \neq 0$ . We set

$$\xi_0 = \frac{-1}{\langle \mathbf{n}^T(u), v \rangle} v - \mathbf{n}^T(u).$$

Then we have

$$\langle \xi_0, \xi_0 \rangle = -2 \frac{-1}{\langle \mathbf{n}^T(u), v \rangle} \langle \mathbf{n}^T(u), v \rangle - 1 = 1$$

$$\langle \xi_0, \mathbf{n}^T(u) \rangle = \frac{-1}{\langle \mathbf{n}^T(u), v \rangle} \langle \mathbf{n}^T(u), v \rangle + 1 = 0.$$

This means that  $\xi_0 \in N_1(M)_p(M)$ . Since  $-v = \langle \mathbf{n}^T(u), v \rangle (\mathbf{n}^T(u) + \xi_0)$ , we have  $p_0 - \lambda_0 = \mu \mathbb{L}G(\mathbf{n}^T)(p_0, \xi_0)$ , where  $p_0 = \mathbf{X}(u)$  and  $\mu = -\langle \mathbf{n}^T(u), v \rangle$ . The converse assertion is trivial by definition.

(2) A straightforward calculation yields

$$\frac{\partial^2 g}{\partial u_i \partial u_j} = 2 \{ \langle \mathbf{X}_{u_i u_j}, \mathbf{X} - \lambda_0 \rangle + \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle \}.$$

Under the conditions  $p_0 - \lambda_0 = \mu (\mathbf{n}^T(u) + \xi_0)$  and  $p_0 = \mathbf{X}(u)$ , we have

$$\frac{\partial^2 g}{\partial u_i \partial u_j}(u) = 2 \{ \langle \mathbf{X}_{u_i u_j}(u), \mu (\mathbf{n}^T(u) + \xi_0) \rangle + g_{ij}(u) \}.$$

Therefore,

$$\left( \frac{\partial^2 g}{\partial u_i \partial u_j}(u) \right) (g^{k\ell}(u)) = (2 \{ -\mu h_j^i(\mathbf{n}^T(u), \mathbf{n}^S(u)) + \delta_j^i \}),$$

where  $\mathbf{n}^S$  is the local section of  $N_1(M)[\mathbf{n}^T]$  with  $\mathbf{n}^S(u) = \xi_0$ . It follows that  $\det \mathcal{H}(g)(p_0) = 0$  if and only if  $1/\mu$  is an eigenvalue of  $(h_j^i(\mathbf{n}^T(u), \mathbf{n}^S(u)))$ , which is equal to one of the lightcone principal curvatures  $\kappa_i(\mathbf{n}^T)(p_0, \xi_0)$  ( $i = 1, \dots, s$ ) by Corollary 3.5.  $\square$

To understand the geometric meaning of the assertions of Proposition 4.3, we briefly review the theory of Legendrian singularities [12,14]. Let  $\pi : PT^*(\mathbb{R}^{n+1}) \rightarrow \mathbb{R}^{n+1}$  be the projective cotangent bundle with its canonical contact structure. We review the geometric properties of this bundle. Consider the tangent bundle  $\tau : TPT^*(\mathbb{R}^{n+1}) \rightarrow PT^*(\mathbb{R}^{n+1})$  and the differential map  $d\pi : TPT^*(\mathbb{R}^{n+1}) \rightarrow T\mathbb{R}^{n+1}$  of  $\pi$ . For any  $X \in TPT^*(\mathbb{R}^{n+1})$ , there exists an element  $\alpha \in T^*\mathbb{R}_1^{n+1}$  such that



$\tau(X) = [\alpha]$ . For an element  $V \in T_x \mathbb{R}^{n+1}$ , the property  $\alpha(V) = 0$  does not depend on the choice of representative of the class  $[\alpha]$ . Thus, we can define the canonical contact structure on  $PT^*(\mathbb{R}^{n+1})$  as

$$K = \{X \in TPT^*(\mathbb{R}^{n+1}) \mid \tau(X)(d\pi(X)) = 0\}.$$

We have the trivialization  $PT^*(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times P^n(\mathbb{R})^*$  and we call

$$((v_0, v_1, \dots, v_n), [\xi_0 : \xi_1 : \dots : \xi_n])$$

the *homogeneous coordinates* of  $PT^*(\mathbb{R}^{n+1})$ , where  $[\xi_0 : \xi_1 : \dots : \xi_n]$  are the homogeneous coordinates of the dual projective space  $P^n(\mathbb{R})^*$ .

It is easy to show that  $X \in K_{(x, [\xi])}$  if and only if  $\sum_{i=0}^n \mu_i \xi_i = 0$ , where  $d\tilde{\pi}(X) = \sum_{i=0}^n \mu_i \partial/\partial v_i$ . An immersion  $i : L \rightarrow PT^*(\mathbb{R}^{n+1})$  is said to be a *Legendrian immersion* if  $\dim L = n$  and  $di_q(T_q L) \subset K_{i(q)}$  for any  $q \in L$ . The map  $\pi \circ i$  is also called the *Legendrian map* and the set  $W(i) = \text{image } \pi \circ i$ , the *wave front set* of  $i$ . Moreover,  $i$  (or the image of  $i$ ) is called the *Legendrian lift* of  $W(i)$ .

Let  $F : (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \rightarrow (\mathbb{R}, \mathbf{0})$  be a function germ. We say that  $F$  is a *Morse family of hypersurfaces* if the map germ

$$\Delta^* F = \left( F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \rightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is submersive, where  $(q, x) = (q_1, \dots, q_k, x_0, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0})$ . In this case we have a smooth  $n$ -dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ  $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \rightarrow PT^*\mathbb{R}^{n+1}$  defined as

$$\mathcal{L}_F(q, x) = \left( x, \left[ \frac{\partial F}{\partial x_0}(q, x) : \dots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion. We call  $F$  a *generating family* of  $\mathcal{L}_F$  and the wave front set is given by  $W(\mathcal{L}_F) = \pi_n(\Sigma_*(F))$ , where  $\pi_n : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the canonical projection. In the theory of unfoldings of function germs, the wave front set  $W(\mathcal{L}_F)$  is called a *discriminant set* of  $F$ , which we also denote by  $\mathcal{D}_F$ . Therefore, Proposition 4.3 asserts that the discriminant set of the Lorentzian distance-squared function  $G$  is given by

$$\mathcal{D}_G = \{ \lambda \mid \lambda = X(p) + t(\mathbf{n}^T \pm \xi)(p), p \in M, \xi \in N_1(M)_p[\mathbf{n}^T], t \in \mathbb{R} \},$$

which is the image of the lightlike hypersurface along  $M$  relative to  $\mathbf{n}^T$ .

By assertion (2) of Proposition 4.3, a singular point of the lightlike hypersurface is a point  $\lambda_0 = p_0 + t_0(\mathbf{n}^T + \xi_0)(p_0)$  for  $t_0 = 1/\kappa_i(\mathbf{n}^T)(p_0, \xi_0)$  ( $i = 1, \dots, s$ ). Then we have the following corollary.

**Corollary 4.4.** *The critical value of  $\mathbb{LH}_M(\mathbf{n}^T)$  is the point at which  $\kappa_i(\mathbf{n}^T)(p, \xi) \neq 0$  and*

$$\lambda = p + \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi)} \mathbb{LG}(\mathbf{n}^T)(p, \xi).$$

**Definition 4.5.** We define the mapping  $\mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i} : N_1(M)[\mathbf{n}^T] \rightarrow \mathbb{R}_1^{n+1}$  as

$$\mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) = p + \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi)} \mathbb{LG}(\mathbf{n}^T)(p, \xi).$$

We also define

$$\mathbb{LF}_M(\mathbf{n}^T) = \bigcup \{ \mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) \mid (p, \xi) \in N_1(M)[\mathbf{n}^T] \text{ s.t. } \kappa_\ell(\mathbf{n}^T)_i(p, \xi) \neq 0, i = 1, \dots, s \}.$$

We call  $\mathbb{LF}_M(\mathbf{n}^T)$  the *lightlike focal set* of  $M = X(U)$  relative to  $\mathbf{n}^T$ .

By definition, the lightlike focal set of  $M = X(U)$  relative to  $\mathbf{n}^T$  is the critical value set of the lightlike hypersurface  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$  along  $M$  relative to  $\mathbf{n}^T$ .

We can show that the image of the lightlike hypersurface along  $M$  is independent of the choice of the future-directed timelike normal vector field  $\mathbf{n}^T$  as a corollary of Proposition 4.3.

**Corollary 4.6.** *Let  $\mathbf{n}^T$  and  $\bar{\mathbf{n}}^T$  be future-directed timelike unit normal fields along  $M$ . Then we have*

$$\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R}) = \mathbb{LH}_M(\bar{\mathbf{n}}^T)(N_1(M)[\bar{\mathbf{n}}^T] \times \mathbb{R}) \quad \text{and} \quad \mathbb{LF}_M(\mathbf{n}^T) = \mathbb{LF}_M(\bar{\mathbf{n}}^T).$$

**Proof.** By Proposition 4.3, the images of the lightlike hypersurface along  $M$  relative to both  $\mathbf{n}^T$  and  $\overline{\mathbf{n}}^T$  are the discriminant sets of the Lorentzian distance-squared function  $G$  on  $M$ . Moreover, the focal set of  $M$  is the critical value set of the lightlike hypersurface along  $M$  relative to  $\mathbf{n}^T$ . Since  $G$  is independent of the choice of  $\mathbf{n}^T$ , the assertion follows.  $\square$

We have the following proposition.

**Proposition 4.7.** Let  $G$  be the Lorentzian distance-squared function on  $M$ . For any point  $(u, \lambda) \in G^{-1}(0)$ , the germ of  $G$  at  $(u, \lambda)$  is a Morse family of hypersurfaces.

**Proof.** We denote

$$\mathbf{X}(u) = (X_0(u), X_1(u), \dots, X_n(u)) \quad \text{and} \quad \lambda = (\lambda_0, \lambda_1, \dots, \lambda_n).$$

By definition, we have

$$G(u, \lambda) = -(X_0(u) - \lambda_0)^2 + (X_1(u) - \lambda_1)^2 + \dots + (X_n(u) - \lambda_n)^2.$$

We now prove that the mapping

$$\Delta^*G = \left( G, \frac{\partial G}{\partial u_1}, \dots, \frac{\partial G}{\partial u_s} \right)$$

is non-singular at  $(u, \lambda) \in G^{-1}(0)$ . The Jacobian matrix of  $\Delta^*G$  is given by

$$\begin{pmatrix} \mathbf{A} & 2(X_0 - \lambda_0) & -2(X_1 - \lambda_1) & \cdots & -2(X_n - \lambda_n) \\ 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_s} & -2X_{1u_s} & \cdots & -2X_{nu_s} \end{pmatrix},$$

where  $\mathbf{A}$  is the matrix

$$\begin{pmatrix} 2\langle \mathbf{X} - \lambda, \mathbf{X}_{u_1} \rangle & \cdots & 2\langle \mathbf{X} - \lambda, \mathbf{X}_{u_s} \rangle \\ 2(\langle \mathbf{X}_{u_1}, \mathbf{X}_{u_1} \rangle + \langle \mathbf{X} - \lambda, \mathbf{X}_{u_1u_1} \rangle) & \cdots & 2(\langle \mathbf{X}_{u_1}, \mathbf{X}_{u_s} \rangle + \langle \mathbf{X} - \lambda, \mathbf{X}_{u_1u_s} \rangle) \\ \vdots & \ddots & \vdots \\ 2(\langle \mathbf{X}_{u_s}, \mathbf{X}_{u_1} \rangle + \langle \mathbf{X} - \lambda, \mathbf{X}_{u_su_1} \rangle) & \cdots & 2(\langle \mathbf{X}_{u_s}, \mathbf{X}_{u_s} \rangle + \langle \mathbf{X} - \lambda, \mathbf{X}_{u_su_s} \rangle) \end{pmatrix}.$$

Since  $\mathbf{X}$  is an immersion, the rank of the matrix

$$\begin{pmatrix} 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_s} & -2X_{1u_s} & \cdots & -2X_{nu_s} \end{pmatrix}$$

is equal to  $s$ . Moreover,  $\mathbf{X} - \lambda$  is lightlike, so that it is linearly independent with respect to tangent vectors  $\mathbf{X}_{u_1}, \dots, \mathbf{X}_{u_s}$ . This means that the rank of the matrix

$$\begin{pmatrix} 2(X_0 - \lambda_0) & -2(X_1 - \lambda_1) & \cdots & -2(X_n - \lambda_n) \\ 2X_{0u_1} & -2X_{1u_1} & \cdots & -2X_{nu_1} \\ \vdots & \vdots & \ddots & \vdots \\ 2X_{0u_s} & -2X_{1u_s} & \cdots & -2X_{nu_s} \end{pmatrix}$$

is equal to  $s + 1$ . Therefore, the Jacobi matrix of  $\Delta^*G$  is non-singular at  $(u, \lambda) \in G^{-1}(0)$ .  $\square$

Since  $G$  is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathcal{L}_G : \Sigma_*(G) \longrightarrow PT^*(\mathbb{R}_1^{n+1})$$

defined as

$$\mathcal{L}_G(u, \lambda) = (\lambda, [(X_0(u) - \lambda_0) : (\lambda_1 - X_1(u)) : \cdots : (\lambda_n - X_n(u))]),$$

where

$$\Sigma_*(G) = \{(u, \lambda) \mid \lambda = \mathbb{LH}_M(\mathbf{n}^T)(p, \xi, t) \mid ((p, \xi), t) \in N_1(M)[\mathbf{n}^T] \times \mathbb{R}\}.$$

We observe that  $G$  is a generating family of the Legendrian immersion  $\mathcal{L}_G$  whose wave front is  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$ . Therefore, we say that the Lorentzian distance-squared function  $G$  on  $M$  gives a Lorentz-Minkowski canonical generating family for the Legendrian lift of  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$ .



## 5. Contact with lightcones

In this section we consider the geometric meaning of the singularities of lightlike hypersurfaces in terms of the theory of contact of submanifolds with model hypersurfaces [13]. We begin with the following basic observations.

**Proposition 5.1.** *Let  $\lambda_0 \in \mathbb{R}_1^{n+1}$  and let  $M$  be a spacelike submanifold without points satisfying  $K_\ell(\mathbf{n}^T)(p, \xi) = 0$ . Then  $M \subset LC_{\lambda_0}$  if and only if  $\{\lambda_0\} = \mathbb{LF}_M(\mathbf{n}^T)$  is the lightcone focal set. In this case we have  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T]) \subset LC_{\lambda_0}$  and  $M = \mathbf{X}(U)$  is totally lightcone umbilical.*

**Proof.** By Proposition 3.1,  $K_\ell(\mathbf{n}^T)(p_0, \xi_0) \neq 0$  if and only if

$$\{(\mathbf{n}^T + \mathbf{n}^S), (\mathbf{n}^T + \mathbf{n}^S)_{u_1}, \dots, (\mathbf{n}^T + \mathbf{n}^S)_{u_s}\}$$

is linearly independent for  $p_0 = \mathbf{X}(u_0) \in M$  and  $\xi_0 = \mathbf{n}^S(u_0)$ , where  $\mathbf{n}^S : U \rightarrow N_1(M)[\mathbf{n}^T]$  is a local section. By definition,  $M \subset LC_{\lambda_0}$  if and only if  $g_{\lambda_0}(u) \equiv 0$  for any  $u \in U$ , where  $g_{\lambda_0}(u) = G(u, \lambda_0)$  is the Lorentzian distance-squared function on  $M$ . It follows from Proposition 4.3 that there exist a smooth function  $\mu : U \times N_1(M)[\mathbf{n}^T] \rightarrow \mathbb{R}$  and section  $\mathbf{n}^S : U \rightarrow N_1(M)[\mathbf{n}^T]$  such that

$$\mathbf{X}(u) = \lambda_0 + \mu(u, \mathbf{n}^S(u))(\mathbf{n}^T(u) \pm \mathbf{n}^S(u)).$$

In fact, we have  $\mu(u, \mathbf{n}^S(u)) = -1/\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$   $i = 1, \dots, s$ , where  $p = \mathbf{X}(u)$  and  $\xi = \mathbf{n}^S(u)$ . It follows that  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi) = \tilde{\kappa}_\ell(\mathbf{n}^T)_j(p, \xi)$ , so that  $M = \mathbf{X}(U)$  is totally lightcone umbilical. Therefore, we have

$$\mathbb{LH}_M(\mathbf{n}^T)(u, \mathbf{n}^S(u), t) = \lambda_0 + (t + \mu(u, \mathbf{n}^S(u)))(\mathbf{n}^T(u) \pm \mathbf{n}^S(u)).$$

Hence,  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T]) \subset LC_{\lambda_0}$ . By Corollary 4.4, the critical value set of  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T])$  is the lightcone focal set  $\mathbb{LF}_M(\mathbf{n}^T)$ . However, this is equal to  $\{\lambda_0\}$  according to the previous arguments.

For the converse assertion, suppose that  $\{\lambda_0\} = \mathbb{LF}_M(\mathbf{n}^T)$ . Then we have

$$\lambda_0 = \mathbf{X}(u) + \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(\mathbf{X}(u), \xi)} \mathbb{LG}(\mathbf{n}^T)(u, \xi)$$

for any  $i = 1, \dots, s$  and  $(p, \xi) \in N_1(M)[\mathbf{n}^T]$ , where  $p = \mathbf{X}(u)$ . Thus,

$$\kappa_\ell(\mathbf{n}^T)_i(\mathbf{X}(u), \xi) = \kappa_\ell(\mathbf{n}^T)_j(\mathbf{X}(u), \xi)$$

for any  $i, j = 1, \dots, s$ , so that  $M$  is totally lightcone umbilical. Since  $\mathbb{LG}(\mathbf{n}^T)(u, \xi)$  is lightlike, we have  $\mathbf{X}(u) \in LC_{\lambda_0}$ . This completes the proof.  $\square$

According to the above proposition, the lightcone is regarded as a model lightlike hypersurface in  $\mathbb{R}_1^{n+1}$ . We now consider the contact between spacelike submanifolds and lightcones in terms of Montaldi's theory. We review the theory of contact for submanifolds [13]. Let  $X_i$  and  $Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$  and  $\dim Y_1 = \dim Y_2$ . We say that the *contact* of  $X_1$  and  $Y_1$  at  $y_1$  is of the same type as the *contact* of  $X_2$  and  $Y_2$  at  $y_2$  if there is a diffeomorphism germ  $\Phi : (\mathbb{R}^n, y_1) \rightarrow (\mathbb{R}^n, y_2)$  such that  $\Phi(X_1) = X_2$  and  $\Phi(Y_1) = Y_2$ . In this case we write  $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ . Since this definition of contact is local, we can replace  $\mathbb{R}^n$  by an arbitrary  $n$ -manifold. Montaldi gives the following characterization of contact using  $\mathcal{K}$ -equivalence [13].

**Theorem 5.2.** *Let  $X_i$  and  $Y_i$  ( $i = 1, 2$ ) be submanifolds of  $\mathbb{R}^n$  with  $\dim X_1 = \dim X_2$  and  $\dim Y_1 = \dim Y_2$ . Let  $g_i : (X_i, x_i) \rightarrow (\mathbb{R}^n, y_i)$  be immersion germs and  $f_i : (\mathbb{R}^n, y_i) \rightarrow (\mathbb{R}^p, 0)$  be submersion germs with  $(Y_i, y_i) = (f_i^{-1}(0), y_i)$ . Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

*if and only if  $f_1 \circ g_1$  and  $f_2 \circ g_2$  are  $\mathcal{K}$ -equivalent.*

We now return to our review of the theory of Legendrian singularities and introduce a natural equivalence relation among Legendrian submanifold germs. Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$  be Morse families of hypersurfaces. Then we say that  $\mathcal{L}_F(\Sigma_*(F))$  and  $\mathcal{L}_G(\Sigma_*(G))$  are *Legendrian equivalent* if there exists a contact diffeomorphism germ  $H : (PT^*\mathbb{R}^n, z) \rightarrow (PT^*\mathbb{R}^n, z')$  such that  $H$  preserves fibers of  $\pi$  and that  $H(\mathcal{L}_F(\Sigma_*(F))) = \mathcal{L}_G(\Sigma_*(G))$ , where  $z = \mathcal{L}_F(0)$ ,  $z' = \mathcal{L}_G(0)$ . Using the Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs in the ordinary way [12, Part III]. We can interpret the Legendrian equivalence using the notion of generating families. We denote by  $\mathcal{E}_n$  the local ring of function germs  $(\mathbb{R}^n, \mathbf{0}) \rightarrow \mathbb{R}$  with the unique maximal ideal  $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$ . Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$  be function germs. We say that  $F$  and  $G$  are *P- $\mathcal{K}$ -equivalent* if there exists a diffeomorphism germ  $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$  of the form  $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$  for  $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$  such that  $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$ . Here,  $\Psi^* : \mathcal{E}_{k+n} \rightarrow \mathcal{E}_{k+n}$  is the pull back  $\mathbb{R}$ -algebra isomorphism defined as  $\Psi^*(h) = h \circ \Psi$ . We say that  $F$  is an *infinitesimally  $\mathcal{K}$ -versal deformation* of  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$  if [20]

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \dots, \frac{\partial F}{\partial x_n} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\varepsilon_k}.$$

The main result in the theory of Legendrian singularities [12, Section 20.8] [14, Theorem 2] is the following theorem.

**Theorem 5.3.** *Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$  be Morse families of hypersurfaces. Then we have the following assertions:*

- (1)  $\mathcal{L}_F(\Sigma_*(F))$  and  $\mathcal{L}_G(\Sigma_*(G))$  are Legendrian equivalent if and only if  $F$  and  $G$  are  $P$ - $\mathcal{K}$ -equivalent.
- (2)  $\mathcal{L}_F(\Sigma_*(F))$  is Legendrian stable if and only if  $F$  is an infinitesimally  $\mathcal{K}$ -versal deformation of  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ .

Since  $F$  and  $G$  are function germs on the common space germ  $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ , we do not need the notion of stably  $P$ - $\mathcal{K}$ -equivalences in this situation [14, p. 27]. For any map germ  $f : (\mathbb{R}^p, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$ , we define the local ring of  $f$  as  $Q_*(f) = \varepsilon_n / (f^*(\mathfrak{M}_p)\varepsilon_n + \mathfrak{M}_n^{r+1})$ . We have the following classification result of Legendrian stable germs [8, Proposition A.4], which is key for our purpose in this section.

**Proposition 5.4.** *Let  $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$  be Morse families of hypersurfaces with  $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$  and  $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ . Suppose that  $\mathcal{L}_F$  and  $\mathcal{L}_G$  are Legendrian stable. Then the following conditions are equivalent:*

- (1)  $(W(\mathcal{L}_F), \mathbf{0})$  and  $(W(\mathcal{L}_G), \mathbf{0})$  are diffeomorphic as set germs.
- (2)  $\mathcal{L}_F(\Sigma_*(F))$  and  $\mathcal{L}_G(\Sigma_*(G))$  are Legendrian equivalent.
- (3)  $Q_{n+1}(f)$  and  $Q_{n+1}(g)$  are isomorphic as  $\mathbb{R}$ -algebras.

Returning to lightlike hypersurfaces, we now consider the function

$$\mathcal{G} : \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}$$

defined as  $\mathcal{G}(\mathbf{x}, \boldsymbol{\lambda}) = \langle \mathbf{x} - \boldsymbol{\lambda}, \mathbf{x} - \boldsymbol{\lambda} \rangle$ . Given  $\boldsymbol{\lambda}_0 \in \mathbb{R}_1^{n+1}$ , we denote  $\mathfrak{g}_{\boldsymbol{\lambda}_0}(\mathbf{x}) = \mathcal{G}(\mathbf{x}, \boldsymbol{\lambda}_0)$ , so we have  $\mathfrak{g}_{\boldsymbol{\lambda}_0}^{-1}(0) = LC_{\boldsymbol{\lambda}_0}$ . For any  $p_0 = \mathbf{X}(u_0) \in M$ ,  $t_0 \in \mathbb{R}$  and  $\boldsymbol{\xi}_0 \in N_1(M)_p[\mathbf{n}^T]$  we consider the point  $\boldsymbol{\lambda}_0 = \mathbf{X}(u_0) + t_0(\mathbf{n}^T(u_0) + \boldsymbol{\xi}_0)$ . Then we have

$$\mathfrak{g}_{\boldsymbol{\lambda}_0} \circ \mathbf{X}(u_0) = \mathcal{G} \circ (\mathbf{X} \times 1_{\mathbb{R}^{n+1}})(u_0, \boldsymbol{\lambda}_0) = G(p_0, \boldsymbol{\lambda}_0) = 0,$$

where  $t_0 = 1/\kappa_\ell(\mathbf{n}^T)_i(p_0, \boldsymbol{\xi}_0)$  ( $i = 1, \dots, s$ ). We also have the relations

$$\frac{\partial \mathfrak{g}_{\boldsymbol{\lambda}_0} \circ \mathbf{X}}{\partial u_i}(u_0) = \frac{\partial G}{\partial u_i}(p_0, \boldsymbol{\lambda}_0) = 0, \quad i = 1, \dots, s.$$

These imply that the lightcone  $\mathfrak{g}_{\boldsymbol{\lambda}_0}^{-1}(0) = LC_{\boldsymbol{\lambda}_0}$  is tangential to  $M = \mathbf{X}(U)$  at  $p_0 = \mathbf{X}(u_0)$ . In this case, we call  $LC_{\boldsymbol{\lambda}_0}$  a *tangent lightcone* of  $M = \mathbf{X}(U)$  at  $p_0 = \mathbf{X}(u_0)$ .

We now describe the contacts of spacelike submanifolds with lightcones. We denote by  $Q^\sigma(\mathbf{X}, u_0)$  the local ring of the function germ  $\tilde{g}_{\boldsymbol{\lambda}_0}^\sigma : (U, u_0) \longrightarrow \mathbb{R}$ , where  $\boldsymbol{\lambda}_0 = \mathbb{L}C_M(u_0, \boldsymbol{\xi}_0, t_0)$ . Note that we can explicitly write the local ring as

$$Q_{n+1}(\mathbf{X}, u_0) = \frac{C_{u_0}^\infty(U)}{\langle \langle \mathbf{X}(u) - \boldsymbol{\lambda}_0, \mathbf{X}(u) - \boldsymbol{\lambda}_0 \rangle \rangle_{C_{u_0}^\infty(U)} + \mathfrak{M}_{u_0}(U)^{n+2}},$$

where  $C_{u_0}^\infty(U)$  is the local ring of function germs at  $u_0$ .

Let  $\mathbb{L}H_{M_i}(\mathbf{n}_i^T) : (N_1(M_i)[\mathbf{n}_i^T] \times \mathbb{R}, (p_i, \boldsymbol{\xi}_i, t_i)) \longrightarrow (\mathbb{R}_1^{n+1}, \boldsymbol{\lambda}_i)$  ( $i = 1, 2$ ) be two lightlike hypersurface germs of spacelike submanifold germs  $\mathbf{X}_i : (U, u^i) \longrightarrow (\mathbb{R}_1^{n+1}, p_i)$ . Let  $G_i : (U \times \mathbb{R}_1^{n+1}, (u^i, \boldsymbol{\lambda}_i^\sigma)) \longrightarrow \mathbb{R}$  be the Lorentzian distance-squared function germ of  $\mathbf{X}_i$ . Then we have the following theorem.

**Theorem 5.5.** *Let  $\mathbf{X}_i : (U, u^i) \longrightarrow (\mathbb{R}_1^{n+1}, p_i)$  ( $i = 1, 2$ ) be spacelike surface germs such that the corresponding Legendrian submanifold germs  $\mathcal{L}_{G_i}(\Sigma_*(G_i))$  are Legendrian stable. Then the following conditions are equivalent:*

- (1)  $(\mathbb{L}H_{M_1}(N_1(M_1)[\mathbf{n}_1^T] \times \mathbb{R}), \boldsymbol{\lambda}_1)$  and  $(\mathbb{L}H_{M_2}(N_1(M_2)[\mathbf{n}_2^T] \times \mathbb{R}), \boldsymbol{\lambda}_2)$  are diffeomorphic.
- (2)  $(\mathcal{L}_{G_1}(\Sigma_*(G_1)), (u^1, \boldsymbol{\lambda}_1))$  and  $(\mathcal{L}_{G_2}(\Sigma_*(G_2)), (u^2, \boldsymbol{\lambda}_2))$  are Legendrian equivalent.
- (3)  $G_1$  and  $G_2$  are  $P$ - $\mathcal{K}$ -equivalent.
- (4)  $g_{1, \boldsymbol{\lambda}_1}$  and  $g_{2, \boldsymbol{\lambda}_2}$  are  $\mathcal{K}$ -equivalent.
- (5)  $K(\mathbf{X}_1(U), LC_{\boldsymbol{\lambda}_1}, p_1) = K(\mathbf{X}_2(U), LC_{\boldsymbol{\lambda}_2}, p_2)$ .
- (6)  $Q_{n+1}(\mathbf{X}_1, u^1)$  and  $Q_{n+1}(\mathbf{X}_2, u^2)$  are isomorphic as  $\mathbb{R}$ -algebras.

**Proof.** By Proposition 5.4, conditions (1), (2) and (6) are equivalent. This is also equivalent to the condition that the two generating families  $G_1$  and  $G_2$  are  $P$ - $\mathcal{K}$ -equivalent by Theorem 5.3. If we denote  $g_{i, \boldsymbol{\lambda}_i}(u) = G_i(u, \boldsymbol{\lambda}_i)$ , then we have  $g_{i, \boldsymbol{\lambda}_i}(u) = \mathfrak{g}_{\boldsymbol{\lambda}_i} \circ \mathbf{X}_i(u)$ . By Theorem 5.2,  $K(\mathbf{X}_1(U), LC_{\boldsymbol{\lambda}_1}, p_1) = K(\mathbf{X}_2(U), LC_{\boldsymbol{\lambda}_2}, p_2)$  if and only if  $\tilde{g}_{1, \boldsymbol{\lambda}_1}$  and  $\tilde{g}_{2, \boldsymbol{\lambda}_2}$  are  $\mathcal{K}$ -equivalent. This means that (4) and (5) are equivalent. By definition, (3) implies (4). From the uniqueness of the infinitesimally  $\mathcal{K}$ -versal deformation of  $g_{i, \boldsymbol{\lambda}_i}$  [20], we have that condition (4) implies (3). This completes the proof.  $\square$

For a spacelike embedding germ  $\mathbf{X} : (U, u_0) \longrightarrow (\mathbb{R}_1^{n+1}, p_0)$ , we consider a set germ  $(\mathbf{X}^{-1}(LC_{\lambda_0}), u_0)$ , which is called the *tangent lightcone indicatrix germ of  $\mathbf{X}$* , where  $\lambda_0 = \mathbb{LH}_M(p_0, \xi_0, t_0)$  and  $t_0 = -1/\kappa_\ell(\mathbf{n}^T)_i(p_0, \xi_0)$  ( $i = 1, \dots, s$ ). We have the following corollary of Theorem 5.5.

**Corollary 5.6.** *Under the assumptions of Theorem 5.5, if the lightlike hypersurface germs  $(\mathbb{LH}_{M_1}(N_1(M_1)[\mathbf{n}_1^T] \times \mathbb{R}), \lambda_1)$  and  $(\mathbb{LH}_{M_2}(N_1(M_2)[\mathbf{n}_2^T] \times \mathbb{R}), \lambda_2)$  are diffeomorphic as set germs, then tangent lightcone indicatrix germs*

$$(\mathbf{X}_1^{-1}(LC_{\lambda_1}), u^1) \quad \text{and} \quad (\mathbf{X}_2^{-1}(LC_{\lambda_2}), u^2)$$

*are diffeomorphic as set germs.*

**Proof.** Note that the tangent lightcone indicatrix germ of  $\mathbf{X}_i$  is the zero-level set of  $g_{i,\lambda_i}$ . Since  $\mathcal{K}$ -equivalence among function germs preserves their zero-level sets, the assertion follows from Theorem 5.5.  $\square$

We now consider generic properties of lightlike hypersurfaces along spacelike submanifolds. Let  $\text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1})$  be the space of spacelike embeddings with the Whitney  $C^\infty$ -topology for an open set  $U \subset \mathbb{R}_1^{n+1}$ . We consider the function  $g : \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}$  again. We claim that  $g_\lambda$  is a submersion at  $\mathbf{x} \neq \lambda$  for any  $\lambda \in \mathbb{R}_1^{n+1}$ . For any  $\mathbf{X} \in \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1})$ , we have  $G = g \circ (\mathbf{X} \times 1_{\mathbb{R}_1^{n+1}})$ . We have the  $r$ -jet extension  $j_1^r G : U \times \mathbb{R}_1^{n+1} \longrightarrow J^r(U, \mathbb{R})$ , defined as  $j_1^r G(u, \lambda) = j^r g_\lambda(u)$ , where  $J^k(U, \mathbb{R})$  is the  $k$ -jet space of functions on  $U$ . We consider the trivialization  $J^r(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^r(s, 1)$ . For any submanifold  $Q \subset J^r(s, 1)$ , we denote  $\tilde{Q} = U \times \mathbb{R} \times Q$ . As an application of [21, Lemma 6], the set

$$T_Q = \{\mathbf{X} \in \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1}) \mid j_1^r G \text{ is transversal to } \tilde{Q}\}$$

is a residual set of  $\text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1})$ . Moreover, if  $Q$  is a closed subset, then  $T_Q$  is open. It is known that there exists a semi-algebraic set  $W^r(s, 1) \subset J^k(s, 1)$  and a stratification  $\mathcal{A}^r(s, 1)$  of  $J^k(s, 1) \setminus W^r(s, 1)$  such that  $\lim_{k \rightarrow \infty} \text{cod } W^r(s, 1) = +\infty$  [22]. The stratification  $\mathcal{A}^r(s, 1)$  is called the *canonical stratification*. We define the stratification  $\mathcal{A}^r(U, \mathbb{R})$  of  $J^r(U, \mathbb{R}) \setminus W^r(U, \mathbb{R})$  as

$$U \times (\mathbb{R} \setminus \{0\}) \times (J^r(s, 1) \setminus W^r(s, 1)), \quad U \times \{0\} \times \mathcal{A}^r(s, 1),$$

where  $W^r(U, \mathbb{R}) = U \times \mathbb{R} \times W^r(s, 1)$ . Wan showed that if  $j_1^r G(U \times \mathbb{R}_1^{n+1}) \cap W^r(U, \mathbb{R}) = \emptyset$  and  $j_1^r G$  is transversal to  $\mathcal{A}^r(U, \mathbb{R})$ , then the map  $\pi|_{G^{-1}(0)} : G^{-1}(0) \longrightarrow \mathbb{R}$  is an MT-stable map germ at each point, where  $\pi : U \times \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}_1^{n+1}$  is the canonical projection [23]. Here, a map germ is said to be *MT-stable* if the jet extension is transversal to the canonical stratification of the jet space of sufficiently higher order [22, 24]. The main result of the theory of topological stability of Mather is that MT-stability implies topological stability. By Proposition 4.3, the lightlike hypersurface  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$  is the discriminant set of  $G$ , which is equal to the critical value set of  $\pi|_{G^{-1}(0)}$ . Since  $\text{cod } W^r(U, \mathbb{R}) > s + n + 1$  for sufficiently large  $k$ , the set

$$\mathcal{O}_1 = \{\mathbf{X} \in \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1}) \mid j_1^r G(U \times \mathbb{R}_1^{n+1}) \cap W^r(U, \mathbb{R}) = \emptyset\}$$

is a residual set. It follows that the set

$$\mathcal{O} = \{\mathbf{X} \in \mathcal{O}_1 \mid j_1^r G \text{ is transversal to } \mathcal{A}^r(U, \mathbb{R})\}$$

is a residual set. Therefore, we have the following theorem.

**Theorem 5.7.** *There exists a residual set  $\mathcal{O} \subset \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1})$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the germ of the lightlike hypersurface  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$  at any point is a germ of the critical value set of an MT-stable map germ.*

When  $n \leq 5$ , by the classification results of the  $\mathcal{K}$ -equivalence among function germs, the canonical stratification  $\mathcal{A}^k(s, 1)$  is given by the finite collection of the  $\mathcal{K}$ -orbits. Moreover, if  $j_1^r G$  is transversal to the  $\mathcal{K}$ -orbit of  $j^r g_{\lambda_0}(u_0)$  for sufficiently large  $r$ , then  $G$  is an infinitesimally  $\mathcal{K}$ -versal deformation of  $g_\lambda$  at  $(u_0, \lambda_0)$  [20]. By Theorem 5.3, we have the following theorem.

**Theorem 5.8.** *Suppose that  $n \leq 5$ . Then there exists a residual set  $\mathcal{O} \subset \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^{n+1})$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the germ of the lightlike hypersurface  $\mathbb{LH}_M(\mathbf{n}^T)(N_1(M)[\mathbf{n}^T] \times \mathbb{R})$  at any point is the germ of the wave front set of a stable Legendrian submanifold germ  $\mathcal{L}_G(\Sigma_*(G))$ .*

## 6. Spacelike submanifolds of codimension two

For  $s = n - 1$ ,  $N_1(M)[\mathbf{n}^T]$  is a double covering of  $M = \mathbf{X}(U)$ . We can construct a spacelike unit normal section  $\mathbf{n}^S(u) \in N_p(M)$  as

$$\mathbf{n}^S(u) = \frac{\mathbf{n}^T(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(u)}{\|\mathbf{n}^T(u) \wedge \mathbf{X}_{u_1}(u) \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}(u)\|}.$$

Then  $\sigma^\pm(u) = (\mathbf{X}(u), \pm \mathbf{n}^S(u))$  are sections of  $N_1(M)[\mathbf{n}^T]$ . Clearly, the vectors  $\mathbf{n}^T(u) \pm \mathbf{n}^S(u)$  are lightlike. We previously showed that  $\mathbf{n}^T(u) \pm \mathbf{n}^S(u)$  and  $\tilde{\mathbf{n}}^T(u) \pm \tilde{\mathbf{n}}^S(u)$  are respectively parallel for any two future-directed unit timelike normal

sections  $\mathbf{n}^T(u), \tilde{\mathbf{n}}^T(u) \in N_p(M)$  [10]. Therefore,  $\widetilde{\mathbf{n}^T(u) \pm \mathbf{n}^S(u)} = \tilde{\mathbf{n}}^T(u) \pm \tilde{\mathbf{n}}^S(u)$ . It follows that we have the mapping  $\widetilde{\mathbb{L}\mathbb{G}^\pm} : U \longrightarrow S_+^{n-1}$  defined as  $\widetilde{\mathbb{L}\mathbb{G}^\pm}(u) = \mathbf{n}^T(u) \pm \mathbf{n}^S(u)$ . We call  $\widetilde{\mathbb{L}\mathbb{G}^\pm}$  a *normalized lightcone Gauss map* of  $M = \mathbf{X}(U)$ , which is independent of the choice of  $\mathbf{n}^T$ . Since  $N_p(M)[\mathbf{n}^T]$  is a spacelike line in  $N_p(M)$ , we have  $\xi = \mathbf{n}^S(u)$  or  $\xi = -\mathbf{n}^S(u)$  for any  $(\mathbf{X}(u), \xi) \in N_1(M)[\mathbf{n}^T]$ . We previously defined the *normalized lightcone shape operator*  $\widetilde{S}_{\ell,p}^\pm : T_pM \rightarrow T_pM$  by taking the derivative of the normalized lightcone Gauss map  $\widetilde{\mathbb{L}\mathbb{G}^\pm}$  [10]. The *normalized principal curvatures*  $\{\tilde{\kappa}_{\ell,i}^\pm(p)\}_{i=1}^{n-1}$  are defined as the eigenvalues of the normalized lightcone shape operator  $\widetilde{S}_{\ell,p}^\pm$ . We showed that

$$\tilde{\kappa}_{\ell,i}^\pm(p) = \frac{1}{\ell_0^\pm(p)} \kappa_i(\mathbf{n}^T, \pm \mathbf{n}^S)(p),$$

where  $\mathbf{n}^T(u) \pm \mathbf{n}^S(u) = (\ell_0^\pm(p), \ell_1^\pm(p), \dots, \ell_n^\pm(p))$  and  $p = \mathbf{X}(u)$ . We define the *normalized lightlike hypersurface* along  $M = \mathbf{X}(U)$  as the images of the maps  $\widetilde{\mathbb{L}\mathbb{H}_M^\pm} : U \times \mathbb{R} \longrightarrow \mathbb{R}_1^{n+1}$  defined as  $\widetilde{\mathbb{L}\mathbb{H}_M^\pm}(u, t) = \mathbf{X}(u) + t\widetilde{\mathbb{L}\mathbb{G}^\pm}(u)$ . Since

$$\widetilde{\mathbb{L}\mathbb{G}^\pm}(u) = \frac{1}{\ell_0^\pm(p)} (\mathbf{n}^T(u) \pm \mathbf{n}^S(u)) = \frac{1}{\ell_0^\pm(p)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(p, \mathbf{n}^S(u)),$$

we have

$$\mathbb{L}\mathbb{H}_M(N_1(M)[\mathbf{n}^T] \times \mathbb{R}) = \widetilde{\mathbb{L}\mathbb{H}_M^+}(U \times \mathbb{R}) \cup \widetilde{\mathbb{L}\mathbb{H}_M^-}(U \times \mathbb{R}).$$

In this case the singular value of  $\widetilde{\mathbb{L}\mathbb{H}_M^\pm}(U \times \mathbb{R})$  is the point at which  $\tilde{\kappa}_{\ell,i}^\pm(p) \neq 0$  and

$$\lambda^\pm = \mathbf{X}(u) + \frac{1}{\tilde{\kappa}_{\ell,i}^\pm(p)} \widetilde{\mathbb{L}\mathbb{G}^\pm}(u) = \mathbf{X}(u) + \frac{1}{\kappa_i(\mathbf{n}^T, \pm \mathbf{n}^S)(p)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(p, \pm \mathbf{n}^S(u)).$$

Therefore, we have the mapping  $\widetilde{\mathbb{L}\mathbb{E}_{\tilde{\kappa}_{\ell,i}^\pm}} : U \longrightarrow \mathbb{R}_1^{n+1}$ , defined as

$$\widetilde{\mathbb{L}\mathbb{E}_{\tilde{\kappa}_{\ell,i}^\pm}}(u) = \mathbf{X}(u) + \frac{1}{\tilde{\kappa}_{\ell,i}^\pm(u)} \widetilde{\mathbb{L}\mathbb{G}^\pm}(u).$$

Then we define

$$\widetilde{\mathbb{L}\mathbb{E}_M^\pm} = \bigcup \left\{ \widetilde{\mathbb{L}\mathbb{E}_{\tilde{\kappa}_{\ell,i}^\pm}}(u) \mid u \in U \text{ such that } \tilde{\kappa}_{\ell,i}^\pm(u) \neq 0, i = 1, \dots, n-1 \right\}.$$

From the above arguments, we know that  $\widetilde{\mathbb{L}\mathbb{E}_M^\pm}$  is nothing but the lightlike focal set of  $M = \mathbf{X}(U)$ . However, we call this the *lightlike evolute* of  $M = \mathbf{X}(U)$  when  $\text{codim}M = 2$ .

We previously investigated the lightlike hypersurface  $\widetilde{\mathbb{L}\mathbb{H}_M^\pm}(U \times \mathbb{R})$  for a spacelike surface  $M = \mathbf{X}(U)$  in  $\mathbb{R}_1^4$  under a slightly different formulation [8]. Using the classification of stable Legendrian mappings [14], we have the following proposition [8].

**Proposition 6.1.** *There exists an open dense subset  $\mathcal{O} \subset \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^4)$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the germ of the normalized lightlike hypersurfaces  $\widetilde{\mathbb{L}\mathbb{H}_M^\pm}(U \times \mathbb{R})$  at any point is diffeomorphic to the image of one of the map germs  $A_k$  ( $1 \leq k \leq 4$ ) or  $D_4^\pm$ , where the  $A_k$ ,  $D_4^\pm$ -map germs  $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$  are given by*

$$\begin{aligned} (A_1) \quad & f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0) \\ (A_2) \quad & f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3) \\ (A_3) \quad & f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3) \\ (A_4) \quad & f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_1, u_2) \\ (D_4^+) \quad & f(u_1, u_2, u_3) = (2(u_1^2 + u_2^2) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3) \\ (D_4^-) \quad & f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3). \end{aligned}$$

Using the generic normal forms [14] of generating families (i.e. Lorentzian distance-squared functions) of  $\mathcal{L}_G(\Sigma_*(G))$  and Corollary 5.6, we have the following corollary.

**Corollary 6.2.** *There exists an open dense subset  $\mathcal{O} \subset \text{Emb}_{\text{sp}}(U, \mathbb{R}_1^4)$  such that for any  $\mathbf{X} \in \mathcal{O}$ , the germ of the corresponding tangent lightcone indicatrix at any point  $(x_0, y_0) \in U$  is diffeomorphic to one of the following germs:*

- (1)  $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^2 = 0\}$  (ordinary cusp)
- (2)  $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^4 \pm y^2 = 0\}$  (tachnode or point)
- (3)  $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^5 + y^2 = 0\}$  (rhamphoid cusp)
- (4)  $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 - xy^2 = 0\}$  (three lines)
- (5)  $\{(x, y) \in (\mathbb{R}^2, 0) \mid x^3 + y^3 = 0\}$  (line).

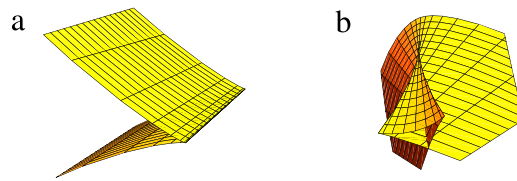


Fig. 2. (a) Cuspidal edge and (b) swallowtail.

We previously investigated the lightlike surface along a spacelike curve in  $\mathbb{R}_1^3$  [5]. We now provide a brief review of the results. Let  $\gamma : I \rightarrow \mathbb{R}_1^3$  be a unit speed spacelike curve with  $\|\gamma''(s)\| \neq 0$ , where  $I$  is an open interval. Then we define  $\mathbf{t}(s) = \gamma'(s)$  and call  $\mathbf{t}(s)$  a unit tangent vector of  $\gamma$  at  $s$ . The curvature of  $\gamma$  at  $s$  is defined as  $\kappa(s) = \sqrt{|\langle \gamma''(s), \gamma''(s) \rangle|}$ . If  $\kappa(s) \neq 0$ , then the unit principal normal vector  $\mathbf{n}(s)$  of the curve  $\gamma$  at  $s$  is given by  $\gamma''(s) = \kappa(s)\mathbf{n}(s)$ . We denote  $\delta(\gamma(s)) = \text{sign}(\langle \mathbf{n}(s), \mathbf{n}(s) \rangle)$ . The unit vector  $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$  is called a unit binormal vector of the curve  $\gamma$  at  $s$ . Since  $\mathbf{t}(s)$  is spacelike, we have  $\langle \mathbf{b}(s), \mathbf{b}(s) \rangle = -\delta(\gamma(s))$  and  $\text{sign}(\langle \gamma'(s), \gamma'(s) \rangle) = 1$ . Then the following Frenet–Serret-type formulae hold:

$$\begin{cases} \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \\ \mathbf{n}'(s) = -\delta(\gamma(s))\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \\ \mathbf{b}'(s) = \tau(s)\mathbf{n}(s), \end{cases}$$

where  $\tau(s)$  is the torsion of the curve  $\gamma$  at  $s$ . In this case we distinguish two cases as follows:

Case (1) If  $\delta(\gamma) = -1$ , then  $\mathbf{n}$  is timelike, so we choose  $\mathbf{n}^T = \mathbf{n}$ . We now consider the lightlike surface  $\mathbb{LH}_C^\pm[\mathbf{n}](I \times \mathbb{R})$  along  $C = \gamma(I)$ , defined as

$$\mathbb{LH}_C^\pm[\mathbf{n}](s, t) = \gamma(s) + t(\mathbf{n} \pm \mathbf{b})(s).$$

By the Frenet–Serret-type formulae, we have

$$\mathbf{n}'(s) \pm \mathbf{b}'(s) = -\delta(\gamma(s))\kappa(s)\mathbf{t}(s) + \tau(s)(\mathbf{b}(s) \pm \mathbf{n}(s)),$$

so that  $\kappa_\ell^\pm(s) = \delta(\gamma(s))\kappa(s) = -\kappa(s)$ .

Case (2) If  $\delta(\gamma) = 1$ , then  $\mathbf{n}$  is spacelike, so we choose  $\mathbf{n}^T = \mathbf{b}$ . We now consider the lightlike surface  $\mathbb{LH}_C^\pm[\mathbf{b}](I \times \mathbb{R})$  along  $C = \gamma(I)$ , defined as

$$\mathbb{LH}_C^\pm[\mathbf{b}](s, t) = \gamma(s) + t(\mathbf{b} \pm \mathbf{n})(s).$$

We also have

$$\mathbf{b}'(s) \pm \mathbf{n}'(s) = \mp\delta(\gamma(s))\kappa(s)\mathbf{t}(s) + \tau(s)(\mathbf{b}(s) \pm \mathbf{n}(s)),$$

so that  $\kappa_\ell^\pm(s) = \pm\delta(\gamma(s))\kappa(s) = \pm\kappa(s)$ .

Conversely, we have

$$\gamma(s) + t(\mathbf{b} \pm \mathbf{n})(s) = \gamma(s) \pm t(\mathbf{n} \pm \mathbf{b})(s).$$

If we define a diffeomorphism  $\Psi^\pm : I \times \mathbb{R} \rightarrow I \times \mathbb{R}$  as  $\Psi^\pm(s, t) = (s, \pm t)$ , then we have  $\mathbb{LH}_C^\pm[\mathbf{b}] = \mathbb{LH}_C^\pm[\mathbf{n}] \circ \Psi^\pm$ . Therefore, we have  $\mathbb{LH}_C^\pm[\mathbf{b}](I \times \mathbb{R}) = \mathbb{LH}_C^\pm[\mathbf{n}](I \times \mathbb{R})$ . The assertions of [5, Theorem B] can be interpreted as the following theorem under the framework in this paper.

**Theorem 6.3.** (1) Suppose that  $\delta(\gamma) = -1$ . Then we have the following:

(a) The lightlike surface  $\mathbb{LH}_C^\pm[\mathbf{n}](I \times \mathbb{R})$  is locally diffeomorphic to the cuspidal edge at  $\mathbb{LH}_C^\pm[\mathbf{n}](s_0, t_0)$  if and only if  $t_0 = \frac{-1}{\kappa(s_0)} = \frac{-1}{\kappa_\ell^\pm(s_0)}$ . Moreover, the lightlike evolute  $\mathbb{LE}_C^\pm$  is the critical locus of the cuspidal edge.

(b) The lightlike surface  $\mathbb{LH}_C^\pm[\mathbf{n}](I \times \mathbb{R})$  is locally diffeomorphic to the swallowtail at  $\mathbb{LH}_C^\pm[\mathbf{n}](s_0, t_0)$  if and only if  $t_0 = \frac{-1}{\kappa(s_0)} = \frac{-1}{\kappa_\ell^\pm(s_0)}$ ,  $(\kappa' - \tau\kappa)(s) = 0$  and  $(\kappa' - \tau\kappa)'(s) \neq 0$ .

(2) Suppose that  $\delta(\gamma) = 1$ . Then we have the following:

(a) The lightlike surface  $\mathbb{LH}_C^\pm[\mathbf{b}](I \times \mathbb{R})$  is locally diffeomorphic to the cuspidal edge at  $\mathbb{LH}_C^\pm[\mathbf{b}](s_0, t_0)$  if and only if  $t_0 = \frac{\pm 1}{\kappa(s_0)} = \frac{1}{\kappa_\ell^\pm(s_0)}$ . Moreover, the lightlike evolute  $\mathbb{LE}_C^\pm$  is the critical locus of the cuspidal edge.

(b) The lightlike surface  $\mathbb{LH}_C^\pm[\mathbf{b}](I \times \mathbb{R})$  is locally diffeomorphic to the swallowtail at  $\mathbb{LH}_C^\pm[\mathbf{b}](s_0, t_0)$  if and only if  $t_0 = \pm \frac{1}{\kappa(s_0)} = \frac{1}{\kappa_\ell^\pm(s_0)}$ ,  $(\kappa' - \tau\kappa)(s) = 0$  and  $(\kappa' - \tau\kappa)'(s) \neq 0$ .

Here, the cuspidal edge is the set germ  $CE = \{(u_1, u_2^2, u_2^3) | (u_1, u_2) \in \mathbb{R}^2\}$  and the swallowtail is the set germ  $SW = \{(3u_1^4 + u_1^2u_2, 4u_1^3 + 2u_1u_2, u_2) | (u_1, u_2) \in \mathbb{R}^2\}$  (Fig. 2).

## 7. Spacelike curves in Minkowski 4-space

In Section 6 we investigated spacelike submanifolds of codimension two and we described a classification of the singularities of lightlike hypersurfaces in  $\mathbb{R}_1^4$ . In this section we consider the higher codimensional case in  $\mathbb{R}_1^4$ , that is, spacelike curves in Minkowski 4-space, as a special case of the previous results.

Let  $\gamma : I \rightarrow \mathbb{R}_1^4$  be a spacelike curve with  $\|\gamma''(s)\| \neq 0$ . In this case we write  $C = \gamma(I)$  instead of  $M = \gamma(I)$ . Since  $\|\gamma'(s)\| > 0$ , we can reparameterize this by the arc length  $s$ . Thus, we have the unit tangent vector  $\mathbf{t}(s) = \gamma'(s)$  of  $\gamma(s)$ . We also have two unit normal vectors  $\mathbf{n}_1(s) = \frac{\gamma''(s)}{\kappa_1(s)}$  and  $\mathbf{n}_2(s) = \frac{\mathbf{n}'_1(s) + \delta_1 \kappa_1(s) \mathbf{t}(s)}{\|\mathbf{n}'_1(s) + \delta_1 \kappa_1(s) \mathbf{t}(s)\|}$  under the conditions  $\kappa_1(s) = \|\gamma''(s)\| \neq 0$  and  $\kappa_2(s) = \|\mathbf{n}'_1(s) + \delta_1 \kappa_1(s) \mathbf{t}(s)\| \neq 0$ , where  $\delta_i = \text{sign}(\mathbf{n}_i(s))$  and  $\text{sign}(\mathbf{n}_i(s))$  is the sign of  $\mathbf{n}_i(s)$  ( $i = 1, 2, 3$ ). Then we have another unit normal vector field  $\mathbf{n}_3(s)$ , defined as  $\mathbf{n}_3(s) = \mathbf{t}(s) \wedge \mathbf{n}_1(s) \wedge \mathbf{n}_2(s)$ . Therefore, we can construct a pseudo-orthogonal frame  $\{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)\}$ , which satisfies the *Frenet–Serret-type formulae*:

$$\begin{cases} \mathbf{t}'(s) = \kappa_1(s) \mathbf{n}_1(s), \\ \mathbf{n}'_1(s) = -\delta_1 \kappa_1(s) \mathbf{t}(s) + \kappa_2(s) \mathbf{n}_2(s), \\ \mathbf{n}'_2(s) = \delta_3 \kappa_2(s) \mathbf{n}_1(s) + \kappa_3(s) \mathbf{n}_3(s), \\ \mathbf{n}'_3(s) = \delta_1 \kappa_3(s) \mathbf{n}_2(s), \end{cases}$$

where  $\kappa_2(s) = \delta_2 \langle \mathbf{n}'_1(s), \mathbf{n}_2(s) \rangle$  and  $\kappa_3(s) = \delta_3 \langle \mathbf{n}'_2(s), \mathbf{n}_3(s) \rangle$ .

Since  $\mathbf{t}(s)$  is spacelike, we distinguish the following three cases:

Case 1:  $\mathbf{n}_1(s)$  is timelike, that is,  $\delta_1 = -1$  and  $\delta_2 = \delta_3 = 1$ .

Case 2:  $\mathbf{n}_2(s)$  is timelike, that is,  $\delta_2 = -1$  and  $\delta_1 = \delta_3 = 1$ .

Case 3:  $\mathbf{n}_3(s)$  is timelike, that is,  $\delta_3 = -1$  and  $\delta_1 = \delta_2 = 1$ .

We consider the lightlike hypersurface along  $C$  and calculate the Lorentzian distance-squared function on  $C$ , which is useful for studying the singularities of lightlike hypersurfaces in each case.

### 7.1. Case 1

Suppose that  $\mathbf{n}_1(s)$  is timelike. In this case we adopt  $\mathbf{n}^T(s) = \mathbf{n}_1(s)$  and denote  $\mathbf{b}_1(s) = \mathbf{n}_2(s)$ ,  $\mathbf{b}_2(s) = \mathbf{n}_3(s)$ . Then we have the pseudo-orthogonal frame

$$\{\mathbf{t}(s), \mathbf{n}^T(s), \mathbf{b}_1(s), \mathbf{b}_2(s)\}$$

for  $\delta_1 = -1$  and  $\delta_2 = \delta_3 = 1$ , which satisfies the following Frenet–Serret-type formulae:

$$\begin{cases} \mathbf{t}'(s) = \kappa_1(s) \mathbf{n}^T(s) \\ \mathbf{n}^{T'}(s) = \kappa_1(s) \mathbf{t}(s) + \kappa_2(s) \mathbf{b}_1(s) \\ \mathbf{b}'_1(s) = \kappa_2(s) \mathbf{n}^T(s) + \kappa_3(s) \mathbf{b}_2(s) \\ \mathbf{b}'_2(s) = -\kappa_3(s) \mathbf{b}_1(s). \end{cases}$$

Since  $N_1(C)[\mathbf{n}^T]$  is parameterized by

$$N_1(C)[\mathbf{n}^T] = \{(\gamma(s), \xi) \in \gamma^* T\mathbb{R}_1^4 \mid \xi = \cos \theta \mathbf{b}_1(s) + \sin \theta \mathbf{b}_2(s) \in N_{\gamma(s)}(C), s \in I\},$$

the lightcone Gauss image of  $N_1(C)_p[\mathbf{n}^T]$  is given by

$$\mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta) = \mathbf{n}^T(s) + \cos \theta \mathbf{b}_1(s) + \sin \theta \mathbf{b}_2(s).$$

Then we have the following lightlike hypersurface along  $C$ :

$$\mathbb{L}\mathbb{H}_C((s, \theta), t) = \gamma(s) + t(\mathbf{n}^T(s) + \cos \theta \mathbf{b}_1(s) + \sin \theta \mathbf{b}_2(s)) = \gamma(s) + t\mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta).$$

Note that the image of this lightlike hypersurface along  $C$  is independent of the choice of the future-directed timelike normal vector field  $\mathbf{n}^T$  by Corollary 4.6.

Now we investigate the Lorentzian distance-squared functions  $G : I \times \mathbb{R}_1^4 \rightarrow \mathbb{R}$  on a spacelike curve  $C = \gamma(I)$ , defined as

$$G(p, \lambda) = G(s, \lambda) = \langle \gamma(s) - \lambda, \gamma(s) - \lambda \rangle,$$

where  $p = \gamma(s)$ . For any fixed  $\lambda_0 \in \mathbb{R}_1^4$ , we write  $g(p) = g_{\lambda_0}(p) = G(p, \lambda_0)$ .

By Proposition 4.3, the discriminant set of the Lorentzian distance-squared function  $G$  is given by

$$\mathcal{D}_G = \mathbb{L}\mathbb{H}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R}) = \{\lambda = \gamma(s) + t\mathbb{L}\mathbb{G}(s, \theta) \mid \theta \in [0, 2\pi), s \in I, t \in \mathbb{R}\},$$

which is the image of the lightlike hypersurface along  $C$ . We also calculate  $g''(p) = 2\langle \gamma''(s), \gamma(s) - \lambda_0 \rangle + 2\langle \gamma'(s), \gamma'(s) \rangle = 2(-\mu\kappa_1 + 1)$ . Then  $g''(p) = 0$  if and only if  $\mu = 1/\kappa_1(s)$ . This means that a singular point of the lightlike hypersurface is a point  $\lambda_0 = \gamma(s_0) + t_0\mathbb{L}\mathbb{G}(\theta_0, s_0)$  for  $t_0 = 1/\kappa_1(s_0)$ . Therefore, the lightlike focal surface is

$$\mathbb{L}\mathbb{F}_C = \left\{ \lambda = \gamma(s) - \frac{1}{\kappa_1(s)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta) \mid s \in I, \theta \in [0, 2\pi) \right\}.$$

Moreover, if we calculate the third, fourth and fifth derivatives of  $g(s)$ , we have the following proposition.

**Proposition 7.1.** Let  $C$  be a spacelike curve and let  $G : C \times (\mathbb{R}_1^4 \setminus C) \rightarrow \mathbb{R}$  be the Lorentzian distance-squared function on  $C$ . Suppose that  $p_0 = \gamma(s_0) \neq \lambda_0$ . Then we have the following:

(1)  $g(p_0) = g'(p_0) = 0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  and  $\mu \in \mathbb{R} \setminus \{0\}$  such that

$$\gamma(s_0) - \lambda_0 = \mu \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0).$$

(2)  $g(p_0) = g'(p_0) = g''(p_0) = 0$  if and only if there exists  $\theta_0 \in [0, 2\pi)$  such that

$$\gamma(s_0) - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0).$$

(3)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = 0$  if and only if there exists  $\theta_0 \in [0, 2\pi)$  such that

$$\gamma(s_0) - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0)$$

and  $\kappa'_1(s_0) - \cos \theta_0 \kappa_1(s_0) \kappa_2(s_0) = 0$ , so that we can write  $\theta_0 = \theta(s_0)$ .

(4)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = 0$  if and only if there exists  $\theta_0 = \theta(s_0) \in [0, 2\pi)$  such that

$$\gamma(s_0) - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta(s_0)),$$

$\kappa'_1(s_0) - \cos \theta(s_0) \kappa_1(s_0) \kappa_2(s_0) = 0$  and  $(2\kappa'_1(s_0) \kappa_2(s_0) + \kappa_1(s_0) \kappa'_2(s_0)) \cos \theta(s_0) - \kappa''_1(s_0) - \kappa_1(s_0) \kappa_2^2(s_0) + \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) \sin \theta(s_0) = 0$ .

(5)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = g^{(5)}(p_0) = 0$  if and only if there exists  $\theta_0 = \theta(s_0) \in [0, 2\pi)$  such that

$$\gamma(s_0) - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta(s_0)),$$

$\kappa'_1(s_0) - \cos \theta(s_0) \kappa_1(s_0) \kappa_2(s_0) = 0$ ,  $(2\kappa'_1(s_0) \kappa_2(s_0) + \kappa_1(s_0) \kappa'_2(s_0)) \cos \theta(s_0) - \kappa''_1(s_0) - \kappa_1(s_0) \kappa_2^2(s_0) + \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) \sin \theta(s_0) = 0$  and  $((2\kappa'_1(s_0) \kappa_2(s_0) + \kappa_1(s_0) \kappa'_2(s_0)) \cos \theta(s_0) - \kappa''_1(s_0) - \kappa_1(s_0) \kappa_2^2(s_0) + \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) \sin \theta(s_0))' = 0$ .

Taking into account the above proposition, we denote  $\rho_1(s, \theta) = \kappa'_1(s) - \cos \theta \kappa_1(s) \kappa_2(s)$  and  $\eta_1(s, \theta) = (2\kappa'_1(s) \kappa_2(s) + \kappa_1(s) \kappa'_2(s)) \cos \theta - \kappa''_1(s) - \kappa_1(s) \kappa_2^2(s) + \kappa_1(s) \kappa_2(s) \kappa_3(s) \sin \theta$ , which might be important invariants of  $C = \gamma(I)$ . Then we can show that  $\rho_1(s, \theta) = \eta_1(s, \theta) = 0$  if and only if  $\rho_1(s, \theta) = \sigma_1(s) = 0$ , where

$$\sigma_1(s) = \left[ \kappa_1 \kappa_2 (\kappa'_1 + \kappa_1 \kappa_2^2) - \kappa'_1 (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \mp \kappa_1 \kappa_2 \kappa_3 \sqrt{(\kappa_1 \kappa_2)^2 - (\kappa'_1)^2} \right] (s).$$

## 7.2. Case 2

Suppose that  $\mathbf{n}_2(s)$  is timelike. Then we adopt  $\mathbf{n}^T(s) = \mathbf{n}_2(s)$  and denote  $\mathbf{b}_1(s) = \mathbf{n}_1(s)$ ,  $\mathbf{b}_2(s) = \mathbf{n}_3(s)$ . We have a pseudo-orthogonal frame  $\{\mathbf{t}(s), \mathbf{n}^T(s), \mathbf{b}_1(s), \mathbf{b}_2(s)\}$ ,  $\delta_2 = -1$  and  $\delta_1 = \delta_3 = 1$ , which satisfies the following Frenet–Serret-type formulae:

$$\begin{cases} \mathbf{t}'(s) = \kappa_1(s) \mathbf{b}_1(s) \\ \mathbf{b}'_1(s) = -\kappa_1(s) \mathbf{t}(s) + \kappa_2(s) \mathbf{n}^T(s) \\ \mathbf{n}^{T'}(s) = \kappa_2(s) \mathbf{b}_1(s) + \kappa_3(s) \mathbf{b}_2(s) \\ \mathbf{b}'_2(s) = \kappa_3(s) \mathbf{n}^T(s). \end{cases}$$

Here,  $N_1(C)[\mathbf{n}^T]$  is parameterized by

$$N_1(C)[\mathbf{n}^T] = \{(\gamma(s), \xi) \in \gamma^* T\mathbb{R}_1^4 \mid \xi = \cos \theta \mathbf{b}_1(s) + \sin \theta \mathbf{b}_2(s) \in N_{\gamma(s)}(C), s \in I\},$$

so that we have the following lightlike hypersurface along  $C = \gamma(I)$ :

$$\mathbb{LH}_C((s, \theta), t) = \gamma(s) + t \mathbb{L}G(\mathbf{n}^T)(s, \theta).$$

We consider the Lorentzian distance-squared function  $G : C \times \mathbb{R}_1^4 \rightarrow \mathbb{R}$  on a spacelike curve  $C = \gamma(I)$ . Using similar notation to Case 1, we have the following proposition.

**Proposition 7.2.** Let  $C$  be a spacelike curve and let  $G : C \times (\mathbb{R}_1^4 \setminus C) \rightarrow \mathbb{R}$  be the Lorentzian distance-squared function on  $C$ . Suppose that  $p_0 \neq \lambda_0$ . Then we have the following:



(1)  $g(p_0) = g'(p_0) = 0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  and  $\mu \in \mathbb{R} \setminus \{0\}$  such that

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = \mu \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0).$$

(2)  $g(p_0) = g'(p_0) = g''(p_0) = 0$  if and only if there exists  $\theta_0 \in [0, 2\pi)$  such that

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0).$$

(3)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = 0$  if and only if there exists  $\theta_0 \in [0, 2\pi)$  such that

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0)$$

and  $\kappa'_1(s_0) \cos \theta_0 - \kappa_1(s_0) \kappa_2(s_0) = 0$ , so that we can write  $\theta_0 = \theta(s_0)$ .

(4)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = 0$  if and only if there exists  $\theta_0 = \theta(s_0) \in [0, 2\pi)$  such that

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0) \cos \theta(s_0)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta(s_0)),$$

$\kappa'_1(s_0) \cos \theta(s_0) - \kappa_1(s_0) \kappa_2(s_0) = 0$  and  $(\kappa''_1(s_0) + \kappa_1(s_0) \kappa_2^2(s_0)) \cos \theta(s_0) - 2\kappa'_1(s_0) \kappa_2(s_0) - \kappa_1(s_0) \kappa'_2(s_0) + \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) \sin \theta(s_0) = 0$ .

(5)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = g^{(5)}(p_0) = 0$  if and only if there exists  $\theta_0 = \theta(s_0) \in [0, 2\pi)$  such that

$$\boldsymbol{\gamma}(s_0) - \boldsymbol{\lambda}_0 = -\frac{1}{\kappa_1(s_0) \cos \theta(s_0)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta(s_0)),$$

$\kappa'_1(s_0) \cos \theta(s_0) - \kappa_1(s_0) \kappa_2(s_0) = 0$ ,  $(\kappa''_1(s_0) + \kappa_1(s_0) \kappa_2^2(s_0)) \cos \theta(s_0) - 2\kappa'_1(s_0) \kappa_2(s_0) - \kappa_1(s_0) \kappa'_2(s_0) + \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) \sin \theta(s_0) = 0$  and  $((\kappa''_1(s_0) + \kappa_1(s_0) \kappa_2^2(s_0)) \cos \theta(s_0) - 2\kappa'_1(s_0) \kappa_2(s_0) - \kappa_1(s_0) \kappa'_2(s_0) + \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) \sin \theta(s_0))' = 0$ .

The above proposition states that the discriminant set of the Lorentzian distance-squared function  $G$  is given by

$$\mathcal{D}_G = \mathbb{L}\mathbb{H}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R}) = \{\boldsymbol{\lambda} = \boldsymbol{\gamma}(s) + t \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta) \mid s \in I, \theta \in [0, 2\pi), t \in \mathbb{R}\}.$$

Moreover, the lightlike focal surface is

$$\mathbb{L}\mathbb{F}_C = \left\{ \boldsymbol{\lambda} = \boldsymbol{\gamma}(s) - \frac{1}{\kappa_1(s) \cos \theta} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta) \mid s \in I, \theta \in [0, 2\pi) \right\}.$$

Here, we also denote  $\rho_2(s, \theta) = \kappa'_1(s) \cos \theta - \kappa_1(s) \kappa_2(s)$  and

$$\eta_2(s, \theta) = (\kappa''_1(s) + \kappa_1(s) \kappa_2^2(s)) \cos \theta - 2\kappa'_1(s) \kappa_2(s) - \kappa_1(s) \kappa'_2(s) + \kappa_1(s) \kappa_2(s) \kappa_3(s) \sin \theta.$$

We can also show that  $\rho_2(s, \theta) = \eta_2(s, \theta) = 0$  if and only if  $\rho_2(s) = \sigma_2(s) = 0$ , where

$$\sigma_2(s) = \left[ \kappa_1 \kappa_2 (\kappa''_1 + \kappa_1 \kappa_2^2) - \kappa'_1 (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \pm \kappa_1 \kappa_2 \kappa_3 \sqrt{-(\kappa_1 \kappa_2)^2 + (\kappa'_1)^2} \right] (s).$$

### 7.3. Case 3

Suppose that  $\mathbf{n}_3(s)$  is timelike. Then we adopt  $\mathbf{n}^T(s) = \mathbf{n}_3(s)$  and denote  $\mathbf{b}_1(s) = \mathbf{n}_1(s)$ ,  $\mathbf{b}_2(s) = \mathbf{n}_2(s)$ . We have a pseudo-orthogonal frame  $\{\mathbf{t}(s), \mathbf{n}^T(s), \mathbf{b}_1(s), \mathbf{b}_2(s)\}$  and  $\delta_3 = -1$  and  $\delta_1 = \delta_2 = 1$ , which satisfies the following Frenet–Serret-type formulae:

$$\begin{cases} \mathbf{t}'(s) = \kappa_1(s) \mathbf{b}_1(s) \\ \mathbf{b}_1'(s) = -\kappa_1(s) \mathbf{t}(s) + \kappa_2(s) \mathbf{b}_2(s) \\ \mathbf{b}_2'(s) = -\kappa_2(s) \mathbf{b}_1(s) + \kappa_3(s) \mathbf{n}^T(s) \\ \mathbf{n}^{T'}(s) = \kappa_3(s) \mathbf{b}_2(s). \end{cases}$$

Here,  $N_1(C)[\mathbf{n}^T]$  is parameterized by

$$N_1(C)[\mathbf{n}^T] = \{(\boldsymbol{\gamma}(s), \boldsymbol{\xi}) \in \boldsymbol{\gamma}^* T\mathbb{R}_1^4 \mid \boldsymbol{\xi} = \cos \theta \mathbf{b}_1(s) + \sin \theta \mathbf{b}_2(s) \in N_{\boldsymbol{\gamma}(s)}(C), s \in I\}$$

so that we have the following lightlike hypersurface along  $C$ :

$$\mathbb{L}\mathbb{H}_C((s, \theta), t) = \boldsymbol{\gamma}(s) + t \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta).$$

We investigate the Lorentzian distance-squared function on a spacelike curve  $C = \boldsymbol{\gamma}(I)$ . According to calculations similar to those for cases 1 and 2, we have the following proposition.

**Proposition 7.3.** Let  $C$  be a spacelike curve and let  $G : C \times (\mathbb{R}^4 \setminus C) \rightarrow \mathbb{R}$  be the Lorentzian distance-squared function on  $C = \gamma(I)$ . Suppose that  $p_0 \neq \lambda_0$ . Then we have the following:

(1)  $g(p_0) = g'(p_0) = 0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  and  $\mu \in \mathbb{R} \setminus \{0\}$  such that

$$\gamma(s_0) - \lambda_0 = \mu \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0).$$

(2)  $g(p_0) = g'(p_0) = g''(p_0) = 0$  if and only if there exists  $\theta_0 \in [0, \pi)$  such that

$$\gamma(s_0) - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0).$$

(3)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = 0$  if and only if there exists  $\theta_0 \in [0, \pi)$  such that

$$\gamma(s_0) - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0)$$

and  $\kappa'_1(s_0) \cos \theta_0 + \kappa_1(s_0) \kappa_2(s_0) \sin \theta_0 = 0$ , so we can write  $\theta_0 = \theta(s_0)$ .

(4)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = 0$  if and only if there exists  $\theta_0 = \theta(s_0) \in [0, 2\pi)$  such that

$$\gamma(s_0) - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta(s_0)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta(s_0)),$$

$\kappa'_1(s_0) \cos \theta(s_0) + \kappa_1(s_0) \kappa_2(s_0) \sin \theta(s_0) = 0$  and  $(2\kappa'_1(s_0) \kappa_2(s_0) + \kappa_1(s_0) \kappa'_2(s_0)) \sin \theta(s_0) + (\kappa''_1(s_0) - \kappa_1(s_0) \kappa_2^2(s_0)) \cos \theta(s_0) - \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0) = 0$ .

(5)  $g(p_0) = g'(p_0) = g''(p_0) = g'''(p_0) = g^{(4)}(p_0) = g^{(5)}(p_0) = 0$  if and only if there exists  $\theta_0 = \theta(s_0) \in [0, 2\pi)$  such that

$$\gamma(s_0) - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta(s_0)} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta(s_0)),$$

$\kappa'_1(s_0) \cos \theta(s_0) + \kappa_1(s_0) \kappa_2(s_0) \sin \theta(s_0) = 0$  and  $((2\kappa'_1(s_0) \kappa_2(s_0) + \kappa_1(s_0) \kappa'_2(s_0)) \sin \theta(s_0) + (\kappa''_1(s_0) - \kappa_1(s_0) \kappa_2^2(s_0)) \cos \theta(s_0) - \kappa_1(s_0) \kappa_2(s_0) \kappa_3(s_0))' = 0$ .

The above proposition states that the discriminant set of the Lorentzian distance-squared function  $G$  is given by

$$\mathcal{D}_G = \mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R}) = \{\lambda = \gamma(s) + t \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta) \mid s \in I, \theta \in [0, 2\pi), t \in \mathbb{R}\}.$$

Moreover, the lightlike focal surface is

$$\mathbb{LF}_C = \left\{ \lambda = \gamma(s) - \frac{1}{\kappa_1(s) \cos \theta} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s, \theta) \mid s \in I, \theta \in [0, 2\pi) \right\}.$$

Here, we also denote  $\rho_3(s, \theta) = \kappa'_1(s) \cos \theta + \kappa_1(s) \kappa_2(s) \sin \theta$  and

$$\eta_3(s, \theta) = (2\kappa'_1(s) \kappa_2(s) + \kappa_1(s) \kappa'_2(s)) \sin \theta + (\kappa''_1(s) - \kappa_1(s) \kappa_2^2(s)) \cos \theta - \kappa_1(s) \kappa_2(s) \kappa_3(s).$$

We can also show that  $\rho_3(s, \theta) = \eta_3(s, \theta) = 0$  if and only if  $\rho_3(s, \theta) = \sigma_3(s) = 0$ , where

$$\sigma_3(s) = \left[ \kappa_1 \kappa_2 (\kappa''_1 - \kappa_1 \kappa_2^2) - \kappa'_1 (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \mp \kappa_1 \kappa_2 \kappa_3 \sqrt{(\kappa_1 \kappa_2)^2 + (\kappa'_1)^2} \right] (s).$$

We can unify the invariants  $\sigma_i(s)$  ( $i = 1, 2, 3$ ) as follows:

$$\sigma(s) = \left[ \kappa_1 \kappa_2 (\kappa''_1 - \kappa_1 \kappa_2^2) - \kappa'_1 (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) \mp \delta_2 \kappa_1 \kappa_2 \kappa_3 \sqrt{\delta_1 (\kappa_1 \kappa_2)^2 + \delta_2 (\kappa'_1)^2} \right] (s).$$

#### 7.4. Classifications of singularities

Using the results for the three cases in Section 7.3, we classify the singularities of the lightlike hypersurface along  $\gamma$  as an application of the unfolding theory of functions. For a function  $f(s)$ , we say that  $f$  has  $A_k$ -singularity at  $s_0$  if  $f^{(p)}(s_0) = 0$  for all  $1 \leq p \leq k$  and  $f^{(k+1)}(s_0) \neq 0$ . Let  $F$  be an  $r$ -parameter unfolding of  $f$  and let  $f$  have an  $A_k$ -singularity ( $k \geq 1$ ) at  $s_0$ . We denote the  $(k-1)$ -jet of the partial derivative  $\partial F / \partial x_i$  at  $s_0$  by

$$j^{(k-1)} \left( \frac{\partial F}{\partial x_i}(s, \mathbf{x}_0) \right) (s_0) = \sum_{j=1}^{k-1} \alpha_{ji} (s - s_0)^j, \quad (i = 1, \dots, r).$$

If the rank of the  $k \times r$  matrix  $(\alpha_{0i}, \alpha_{ji})$  is  $k$  ( $k \leq r$ ), then  $F$  is called a *versal unfolding* of  $f$ , where  $\alpha_{0i} = \partial F / \partial x_i(s_0, \mathbf{x}_0)$ .

Inspired by the propositions in the previous subsections, we define the following set:

$$D_F^\ell = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = \cdots = \frac{\partial^\ell F}{\partial s^\ell}(s, \mathbf{x}) = 0 \right\},$$

which is called a *discriminant set of order  $\ell$* . Of course,  $D_F^1 = D_F$  and  $D_F^2$  is the set of singular points of  $D_F$ . Therefore, we have the following proposition.

**Proposition 7.4.** *For all the cases, we have*

$$D_G = D_G^1 = \text{LHC}(N_1(C)[\mathbf{n}^T] \times \mathbb{R}), \quad D_G^2 = \text{LFC} \quad \text{and} \quad D_G^3 \text{ is the critical value set of } \text{LFC}.$$

To understand the geometric properties of the discriminant set of order  $\ell$ , we introduce an equivalence relation among the unfoldings of functions. Let  $F$  and  $G$  be  $r$ -parameter unfoldings of  $f(s)$  and  $g(s)$ , respectively. We say that  $F$  and  $G$  are  $P$ - $\mathcal{R}$ -equivalent if there exists a diffeomorphism germ  $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow (\mathbb{R} \times \mathbb{R}^r, (s'_0, \mathbf{x}'_0))$  of the form  $\Phi(s, \mathbf{x}) = (\Phi_1(s, \mathbf{x}), \phi(\mathbf{x}))$  such that  $G \circ \Phi = F$ . Straightforward calculations yield the following proposition.

**Proposition 7.5.** *Let  $F$  and  $G$  be  $r$ -parameter unfoldings of  $f(s)$  and  $g(s)$ , respectively. If  $F$  and  $G$  are  $P$ - $\mathcal{R}$ -equivalent by a diffeomorphism germ  $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow (\mathbb{R} \times \mathbb{R}^r, (s'_0, \mathbf{x}'_0))$  of the form  $\Phi(s, \mathbf{x}) = (\Phi_1(s, \mathbf{x}), \phi(\mathbf{x}))$ , then  $\phi(D_F^\ell) = D_G^\ell$  are set germs.*

We have the following classification theorem of versal unfoldings [25, p. 149].

**Theorem 7.6.** *Let  $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow \mathbb{R}$  be an  $r$ -parameter unfolding of  $f$  with  $A_k$ -singularity at  $s_0$ . Suppose  $F$  is a versal unfolding of  $f$ . Then  $F$  is  $P$ - $\mathcal{R}$ -equivalent to one of the following unfoldings:*

- (a)  $k = 1 : \pm s^2 + x_1$
- (b)  $k = 2 : s^3 + x_1 + sx_2$
- (c)  $k = 3 : \pm s^4 + x_1 + sx_2 + s^2x_3$
- (d)  $k = 4 : s^5 + x_1 + sx_2 + s^2x_3 + s^3x_4$ .

We have the following classification result as a corollary of the above theorem.

**Corollary 7.7.** *Let  $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow \mathbb{R}$  be an  $r$ -parameter unfolding of  $f$  with  $A_k$ -singularity at  $s_0$ . Suppose  $F$  is a versal unfolding of  $f$ . Then we have the following assertions:*

- (a) *If  $k = 1$ , then  $D_F$  is diffeomorphic to  $\{0\} \times \mathbb{R}^{r-1}$  and  $D_F^2 = \emptyset$ .*
- (b) *If  $k = 2$ , then  $D_F$  is diffeomorphic to  $C(2, 3) \times \mathbb{R}^{r-2}$ ,  $D_F^2$  is diffeomorphic to  $\{0\} \times \mathbb{R}^{r-2}$  and  $D_F^3 = \emptyset$ .*
- (c) *If  $k = 3$ , then  $D_F$  is diffeomorphic to  $SW \times \mathbb{R}^{r-3}$ ,  $D_F^2$  is diffeomorphic to  $C(2, 3, 4) \times \mathbb{R}^{r-3}$ ,  $D_F^3$  is diffeomorphic to  $\{0\} \times \mathbb{R}^{r-3}$  and  $D_F^4 = \emptyset$ .*
- (d) *If  $k = 4$ , then  $D_F$  is locally diffeomorphic to  $BF \times \mathbb{R}^{r-4}$ ,  $D_F^2$  is diffeomorphic to  $C(BF) \times \mathbb{R}^{r-4}$ ,  $D_F^3$  is diffeomorphic to  $C(2, 3, 4, 5) \times \mathbb{R}^{r-4}$ ,  $D_F^4$  is diffeomorphic to  $\{0\} \times \mathbb{R}^{r-4}$  and  $D_F^5 = \emptyset$ .*

Note that all the diffeomorphisms in the above assertions are diffeomorphism germs.

We call  $C(2, 3) = \{(x_1, x_2) \mid x_1 = u^2, x_2 = u^3\}$  a  $(2, 3)$ -cusp,  $C(2, 3, 4) = \{(x_1, x_2, x_3) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4\}$  a  $(2, 3, 4)$ -cusp,  $C(2, 3, 4, 5) = \{(x_1, x_2, x_3, x_4) \mid x_1 = u^2, x_2 = u^3, x_3 = u^4, x_4 = u^5\}$  a  $(2, 3, 4, 5)$ -cusp,  $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$  a swallowtail,  $BF = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$  a butterfly, and  $C(BF) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 6u^5 + u^3v, x_2 = 25u^4 + 9u^2v, x_3 = 10u^3 + 3uv, x_4 = v\}$  a  $c$ -butterfly (i.e., the critical value set of the butterfly). We have the following key proposition for  $G$ .

**Proposition 7.8.** *If  $g(s)$  has  $A_k$ -singularity ( $k = 1, 2, 3, 4$ ) at  $p_0$ , then  $G$  is a versal unfolding of  $g$ .*

**Proof.** We denote  $\mathbf{y}(s) = (X_0(s), X_1(s), X_2(s), X_3(s))$  and  $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ .

By definition, we have

$$G(s, \boldsymbol{\lambda}) = -(X_0(s) - \lambda_0)^2 + (X_1(s) - \lambda_1)^2 + (X_2(s) - \lambda_2)^2 + (X_3(s) - \lambda_3)^2.$$

Thus, we have

$$\frac{\partial G}{\partial \lambda_i}(s, \boldsymbol{\lambda}) = 2(X_i(s) - \lambda_i), \quad \text{and} \quad \frac{\partial^2 G}{\partial s \partial \lambda_i}(s, \boldsymbol{\lambda}) = 2X'_i(s) \quad \text{for } i = 0, 1, 2, 3.$$

For a fixed  $\boldsymbol{\lambda}_0 = (\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{03})$ , the 3-jet of  $\partial G / \partial \lambda_i(s, \boldsymbol{\lambda}_0)$  ( $i = 0, 1, 2, 3$ ) at  $s_0$  is

$$j^{(3)} \frac{\partial G}{\partial \lambda_i}(s, \boldsymbol{\lambda}_0)(s_0) = 2X'_i(s_0)(s - s_0) - X''_i(s_0)(s - s_0)^2 - \frac{1}{3}X'''_i(s_0)(s - s_0)^3, \quad (i = 0, 1, 2, 3).$$

It is enough to show that the rank of the following matrix is four:

$$B = \begin{pmatrix} 2(X_0(s) - \lambda_0) & 2(X_1(s) - \lambda_1) & 2(X_2(s) - \lambda_2) & 2(X_3(s) - \lambda_3) \\ 2X_0'(s_0) & 2X_1'(s_0) & 2X_2'(s_0) & 2X_3'(s_0) \\ 2X_0''(s_0) & 2X_1''(s_0) & 2X_2''(s_0) & 2X_3''(s_0) \\ 2X_0'''(s_0) & 2X_1'''(s_0) & 2X_2'''(s_0) & 2X_3'''(s_0) \end{pmatrix}.$$

In fact,  $B = 2^t(\gamma(s) - \lambda, \gamma'(s), \gamma''(s), \gamma'''(s)) = 2^t(\gamma(s) - \lambda, \mathbf{t}(s), \mathbf{t}'(s), \mathbf{t}''(s))$  and  $\gamma(s) - \lambda, \mathbf{t}(s), \mathbf{t}'(s)$  and  $\mathbf{t}''(s)$  are linearly independent from each other in Cases 1, 2 and 3, respectively. This completes the proof.  $\square$

Finally, we can apply Corollary 7.7 to our condition. Then we have the following theorem.

**Theorem 7.9.** Let  $\gamma : I \longrightarrow +\mathbb{R}_1^4$  be a spacelike curve with  $\kappa_1(s) \neq 0$  and  $\kappa_2(s) \neq 0$ .

(A) For the lightlike hypersurfaces  $\mathbb{LH}_C((s, \theta), \mathbf{t})$  of  $C = \gamma(I)$  in Case 1, we have the following assertions:

- (1) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $C(2, 3) \times \mathbb{R}^2$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0)$$

and  $\rho_1(s_0, \theta_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is non-singular.

- (2) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $SW \times \mathbb{R}$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0),$$

$\rho_1(s_0, \theta_0) = 0$  and  $\sigma_1(s_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is locally diffeomorphic to  $C(2, 3, 4) \times \mathbb{R}$  and the critical value set of  $\mathbb{LF}_C$  is a regular curve.

- (3) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $BF$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = \frac{1}{\kappa_1(s_0)} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0),$$

$\rho_1(s_0, \theta_0) = 0$ ,  $\sigma_1(s_0) = 0$  and  $\sigma_1'(s_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is locally diffeomorphic to  $C(BF) \times \mathbb{R}$  and the critical value set is locally diffeomorphic to the  $C(2, 3, 4, 5)$ -cusp.

(B) For the lightlike hypersurfaces  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  of  $C = \gamma(I)$  in Case 2, we have the following assertions:

- (1) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $C(2, 3) \times \mathbb{R}^2$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0)$$

and  $\rho_2(s_0, \theta_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is non-singular.

- (2) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $SW \times \mathbb{R}$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0),$$

$\rho_2(s_0, \theta_0) = 0$  and  $\sigma(s_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is locally diffeomorphic to  $C(2, 3, 4) \times \mathbb{R}$  and the critical value set of  $\mathbb{LF}_C$  is a regular curve.

- (3) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $BF$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}G(\mathbf{n}^T)(s_0, \theta_0),$$

$\rho_2(s_0, \theta_0) = 0$ ,  $\sigma_2(s_0) = 0$  and  $\sigma_2'(s_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is locally diffeomorphic to  $C(BF) \times \mathbb{R}$  and the critical value set of  $\mathbb{LF}_C$  is locally diffeomorphic to the  $C(2, 3, 4, 5)$ -cusp.

(C) For the lightlike hypersurfaces  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  of  $C = \gamma(I)$  in Case 3, we have the following assertions:

- (1) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $C(2, 3) \times \mathbb{R}^2$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0)$$

and  $\rho_3(s_0, \theta_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is non-singular.

- (2) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $SW \times \mathbb{R}$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0),$$

$\rho_3(s_0, \theta_0) = 0$  and  $\sigma_3(s_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is locally diffeomorphic to  $C(2, 3, 4) \times \mathbb{R}$  and the critical value set of  $\mathbb{LF}_C$  is a regular curve.

- (3) The lightlike hypersurface  $\mathbb{LH}_C(N_1(C)[\mathbf{n}^T] \times \mathbb{R})$  is locally diffeomorphic to  $BF$  at  $\lambda_0$  if and only if there exist  $\theta_0 \in [0, 2\pi)$  such that

$$p_0 - \lambda_0 = -\frac{1}{\kappa_1(s_0) \cos \theta_0} \mathbb{L}\mathbb{G}(\mathbf{n}^T)(s_0, \theta_0),$$

$\rho_3(s_0, \theta_0) = 0$ ,  $\sigma_3(s_0) = 0$  and  $\sigma'_3(s_0) \neq 0$ . In this case, the lightlike focal set  $\mathbb{LF}_C$  is locally diffeomorphic to  $C(BF) \times \mathbb{R}$  and the critical value set of  $\mathbb{LF}_C$  is locally diffeomorphic to the  $C(2, 3, 4, 5)$ -cusp.

## 8. Submanifolds in Euclidean space or hyperbolic space

In this section we consider submanifolds in Euclidean space and hyperbolic space as special cases of the previous results.

### 8.1. Submanifolds in Euclidean space

Let  $\mathbb{R}_0^n$  be the Euclidean space given by  $x_0 = 0$  for  $\mathbf{x} = (x_0, x_1, \dots, x_n)$ . Consider an embedding  $\mathbf{X} : U \rightarrow \mathbb{R}_0^n$ , where  $U \subset \mathbb{R}^s$  is an open set. In this case we can adopt  $\mathbf{n}^T = \mathbf{e}_0 = (1, 0, \dots, 0)$  as a future-directed timelike unit normal vector field along  $M = \mathbf{X}(U)$  in  $\mathbb{R}_1^{n+1}$ . In this case,  $N_1(M)[\mathbf{n}^T] = N_1(M)[\mathbf{e}_0]$  is the unit normal bundle  $N_1^e(M)$  of  $M$  in  $\mathbb{R}_0^n$  in the Euclidean sense. Therefore, the lightcone Gauss map  $\mathbb{L}\mathbb{G}(\mathbf{n}^T)$  is given by  $\mathbb{L}\mathbb{G}(\mathbf{n}^T)(p, \xi) = \mathbf{e}_0 + \xi = \mathbf{e}_0 + \mathbb{G}(p, \xi)$ , where  $\mathbb{G} : N_1^e(M) \rightarrow S^{n-1}$  is the Gauss map of the unit normal bundle  $N_1^e(M)$  defined by  $\mathbb{G}(p, \xi) = \xi$  [26]. Since  $\mathbf{e}_0$  is a constant vector, we have  $d_{(p, \xi)} \mathbb{L}\mathbb{G}(\mathbf{n}^T) = d_{(p, \xi)} \mathbb{G}$ , so

$$\kappa_i(\mathbf{n}^T)(p, \xi) = \kappa_i(\mathbf{e}_0)(p, \xi) = \kappa_i(p, \xi),$$

where  $\kappa_i(p, \xi)$  ( $i = 1, \dots, s$ ) are the eigenvalues of  $-d_{(p, \xi)} \mathbb{G}$  belonging to the eigenvectors on  $T_p M$ , which are the principal curvatures of  $M$  with respect to  $\xi$  in the Euclidean sense.

The intersection of a lightcone with  $\mathbb{R}_0^n$  is a hypersphere in  $\mathbb{R}_0^n$ , so that the contact of a submanifold in  $\mathbb{R}_0^n$  with lightcones is equivalent to the contact with hyperspheres in  $\mathbb{R}_0^n$ . We define the projection  $\pi : \mathbb{R}_1^{n+1} \rightarrow \mathbb{R}_0^n$  as  $\pi(x_0, x_1, \dots, x_n) = (0, x_1, \dots, x_n)$ . Then we have

$$\pi \circ \mathbb{LF}_{\kappa_i(\mathbf{n}^T)(p, \xi)}(p, \xi, t) = \mathbf{X}(u) + \frac{1}{\kappa_i(p, \xi)} \mathbb{G}(p, \xi).$$

Therefore,  $\pi \circ \mathbb{LF}_M$  is the focal set of  $M = \mathbf{X}(U)$  in the Euclidean sense [27]. If  $s = n - 1$ ,  $\pi \circ \mathbb{LF}_M$  is called the *evolute* of  $M$  in  $\mathbb{R}_0^n$ .

Note that if  $\mathbf{n}^T = \mathbf{v}$  is a constant timelike unit vector, the spacelike submanifold  $M$  is a submanifold in the spacelike hyperplane  $HP(\mathbf{v}, c)$ . Since  $HP(\mathbf{v}, c)$  is isometric to the Euclidean space  $\mathbb{R}_0^n$ , all results for the case  $\mathbf{n} = \mathbf{e}_0$  hold in this case.

### 8.2. Submanifolds in hyperbolic space

Let  $\mathbf{X} : U \rightarrow H^n(-1)$  be an immersion into the hyperbolic space. Then we adopt  $\mathbf{n}^T(u) = \mathbf{X}(u)$ . In this case,  $N_1(M)[\mathbf{n}^T]$  is the unit normal bundle  $N_1^h(M)$  of  $M = \mathbf{X}(U)$  in  $H^n(-1)$ . Therefore, the lightcone Gauss image  $\mathbb{L}\mathbb{G}(\mathbf{n}^T)$  is given by  $\mathbb{L}\mathbb{G}(\mathbf{n}^T)(u, \xi) = \mathbf{X}(u) + \xi = \mathbb{L}(u, \xi)$ , where  $\mathbb{L} : N_1^h(M) \rightarrow S_+^{n-1}$  is the hyperbolic Gauss indicatrix of the unit normal bundle  $N_1^h(M)$  [28]. Since we identify  $M$  with  $U$  through  $\mathbf{X}$ ,  $d\mathbf{X}(u)$  can be regarded as  $1_{T_p M}$  for  $p = \mathbf{X}(u)$ . Therefore, we have  $\kappa_h(\mathbf{n}^T)_i(p) = -1$  and we denote  $\kappa_d(\mathbf{n}^T)_i(p, \xi) = \kappa_d(\xi)_i(p)$  ( $i = 1, \dots, s$ ), which we call the *de Sitter principal curvatures* of

$M$  at  $p = \mathbf{X}(u)$  with respect to  $\xi$  [6,29]. By Corollary 3.5, we have  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi) = -1 + \kappa_d(\xi)_i(p)$ . The lightlike hypersurface along  $M$  is given by  $\mathbb{LH}_M(p, \xi, t) = \mathbf{X}(u) + t(\mathbf{X}(u) + \xi)$ , where  $p = \mathbf{X}(u)$ . Since  $\langle \mathbb{LH}_M(p, \xi, t), \mathbb{LH}_M(p, \xi, t) \rangle = -1 - 2t$ ,

$$\mathbb{LH}_M(p, \xi, t) \text{ is } \begin{cases} \text{timelike} & \text{if and only if } t > -\frac{1}{2} \\ \text{lightlike} & \text{if and only if } t = -\frac{1}{2} \\ \text{spacelike} & \text{if and only if } t < -\frac{1}{2}. \end{cases}$$

We now define the mapping

$$\Phi : \mathbb{R}_1^{n+1} \setminus LC_0 \longrightarrow H^n(-1) \cup S_1^n$$

as  $\Phi(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ . We have  $\mathbb{R}_1^{n+1} \setminus LC_0 = T \cup S$ , where  $S = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle > 0\}$  and  $T = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle < 0\}$ . We define  $\Phi^S = \Phi|_S : S \longrightarrow S_1^n$  and  $\Phi^T = \Phi|_T : T \longrightarrow H^n(-1)$ .

We distinguish two cases as follows:

Case (1)  $t > -\frac{1}{2}$ , so that  $\mathbb{LH}_M(p, \xi, t)$  is timelike. Thus, we have  $\mathbb{LH}_M(p, \xi, t) \in T$ . It follows that we have the mapping  $\mathbb{LH}_M^T : N_1(M)[\mathbf{n}^T] \times \{t \in \mathbb{R} \mid 2t + 1 > 0\} \rightarrow T$ , defined as

$$\mathbb{LH}_M^T(p, \xi, t) = \Phi^T \circ \mathbb{LH}_M(p, \xi, t) = \frac{1}{\sqrt{2t+1}}((t+1)\mathbf{X}(u) + t\xi).$$

Case (2)  $t < -\frac{1}{2}$ , so that  $\mathbb{LH}_M(p, \xi, t)$  is spacelike. Thus, we have  $\mathbb{LH}_M(p, \xi, t) \in S$ . It follows that we have the mapping  $\mathbb{LH}_M^S : N_1(M)[\mathbf{n}^T] \times \{t \in \mathbb{R} \mid 2t + 1 < 0\} \rightarrow S$ , defined as

$$\mathbb{LH}_M^S(p, \xi, t) = \Phi^S \circ \mathbb{LH}_M(p, \xi, t) = \frac{1}{\sqrt{-2t-1}}((t+1)\mathbf{X}(u) + t\xi).$$

Then we have the following proposition.

**Proposition 8.1.** *Using the above notation, we have the following:*

- (1) For  $t_0 > -\frac{1}{2}$ ,  $(p_0, \xi_0, t_0)$  is a singular point of  $\mathbb{LH}_M$  if and only if it is a singular point of  $\mathbb{LH}_M^T$ .
- (2) For  $t_0 < -\frac{1}{2}$ ,  $(p_0, \xi_0, t_0)$  is a singular point of  $\mathbb{LH}_M$  if and only if it is a singular point of  $\mathbb{LH}_M^S$ .

**Proof.** (1) Let  $(p_0, \xi_0, t_0)$  be a regular point of  $\mathbb{LH}_M$ . Then the tangent hyperplane at  $(p_0, \xi_0, t_0)$  of  $\mathbb{LH}_M(N_1(M)[\mathbf{n}^T] \times \{t \in \mathbb{R} \mid 2t + 1 > 0\})$  is a lightlike hyperplane on which there are no timelike vectors. Since  $\mathbb{LH}_M(t_0, \xi_0, t_0)$  is timelike, it is transversal to the tangent hyperplane. Moreover,  $\mathbb{LH}_M(t_0, \xi_0, t_0)$  is directed to the fiber direction of the projection  $\Phi^T$ , so that  $\mathbb{LH}_M^T = \Phi^T \circ \mathbb{LH}_M$  is regular at  $(p_0, \xi_0, t_0)$ . The converse assertion is trivial.

(2) For  $t_0 > -\frac{1}{2}$ , we assume that  $(p_0, \xi_0, t_0)$  is a regular point of  $\mathbb{LH}_M$ . For any  $\mathbf{v} \in T_{(p_0, \xi_0)}N_1(M)[\mathbf{n}^T]$ , we take the directional derivative of the relation  $\langle \mathbb{LH}_M, \mathbb{LH}_M \rangle = -1 - 2t$  with respect to  $\mathbf{v}$  at  $(p_0, \xi_0, t_0)$ . Then we have

$$0 = D_{\mathbf{v}}(-1 - 2t)|_{t=t_0} = 2\langle \mathbb{LH}_M, D_{\mathbf{v}}\mathbb{LH}_M \rangle|_{(p_0, \xi_0, t_0)}.$$

Therefore,  $\mathbb{LH}_M(p_0, \xi_0, t_0)$  and  $D_{\mathbf{v}}\mathbb{LH}_M(p_0, \xi_0, t_0)$  are pseudo-orthogonal. Moreover,

$$\frac{\partial \mathbb{LH}_M}{\partial t}(p_0, \xi_0, t_0) = p_0 + \xi_0$$

is lightlike. Since  $\mathbb{LH}_M(p_0, \xi_0, t_0)$  is spacelike, it is transversal to the tangent hyperplane of  $\mathbb{LH}_M(N_1(M)[\mathbf{n}^T] \times \{t \in \mathbb{R} \mid 2t + 1 < 0\})$  at  $(p_0, \xi_0, t_0)$ . Thus,  $\mathbb{LH}_M^S = \Phi^S \circ \mathbb{LH}_M$  is regular at  $(p_0, \xi_0, t_0)$ . The converse assertion is trivial. This completes the proof.  $\square$

By Corollary 4.4, the singular point of  $\mathbb{LH}_M$  is  $\left(p, \xi, \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi)}\right)$  ( $i = 1, \dots, s$ ), so we have the following corollary.

**Corollary 8.2.** *We have the following assertions:*

- (1) If  $(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) > 0$ , then the critical value of  $\mathbb{LH}_M^T$  is

$$\Phi^T \circ \mathbb{LH}_M^T(p, \xi) = \frac{|\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 1|}{\sqrt{(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi)}} \left( p + \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 1} \xi \right)$$

for  $i = 1, \dots, s$ .

(2) If  $(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) < 0$ , then the critical value of  $\mathbb{LH}_M^S$  is

$$\Phi^S \circ \mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) = \frac{-(\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 1)}{\sqrt{-(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 - 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi)}} \left( p + \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 1} \xi \right)$$

for  $i = 1, \dots, s$ .

**Proof.** Suppose that  $(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) > 0$ . Since  $t = 1/\kappa_\ell(\mathbf{n}^T)_i(p, \xi)$ , we have

$$\Phi^T \circ \mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) = \sqrt{\frac{\kappa_\ell(\mathbf{n}^T)_i(p, \xi)}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 2}} \frac{\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi)} \left( p + \frac{1}{\kappa_\ell(\mathbf{n}^T)_i(p, \xi) + 1} \xi \right).$$

For convenience, we denote  $\kappa = \kappa_\ell(\mathbf{n}^T)_i(p, \xi)$ . If  $\kappa < 0$ , then  $\kappa + 2 < 0$ , so  $\kappa + 1 < -1$ . Therefore, we have

$$\sqrt{\frac{\kappa}{\kappa + 2}} \frac{\kappa + 1}{\kappa} = -\sqrt{\frac{\kappa}{(\kappa + 2)\kappa^2}} (\kappa + 1) = \frac{-(\kappa + 1)}{\sqrt{\kappa^2 + 2\kappa}}.$$

If  $\kappa > 0$ , then  $\kappa + 2 > 0$ , so we have

$$\sqrt{\frac{\kappa}{\kappa + 2}} \frac{\kappa + 1}{\kappa} = \sqrt{\frac{\kappa}{(\kappa + 2)\kappa^2}} (\kappa + 1) = \frac{(\kappa + 1)}{\sqrt{\kappa^2 + 2\kappa}}.$$

Thus, we have formula (1).

Suppose that  $(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) < 0$ . We also denote  $\kappa = \kappa_\ell(\mathbf{n}^T)_i(p, \xi)$ . Then we have

$$\Phi^S \circ \mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) = \sqrt{\frac{\kappa}{-\kappa - 2}} \frac{\kappa + 1}{\kappa} \left( p + \frac{1}{\kappa + 1} \xi \right).$$

Since  $-2 < \kappa < 0$ , we have

$$\sqrt{\frac{\kappa}{-\kappa - 2}} \frac{\kappa + 1}{\kappa} = -\sqrt{\frac{\kappa}{(-\kappa - 2)\kappa^2}} (\kappa + 1) = \frac{-(\kappa + 1)}{\sqrt{-\kappa^2 - 2\kappa}}.$$

Thus, we have formula (2). This completes the proof.  $\square$

Since  $\kappa_\ell(\mathbf{n}^T)_i(p, \xi) = \kappa_d(\xi)_i(p) - 1$ , we have

$$(\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) = (\kappa_d(\xi)_i(p))^2 - 1,$$

so that

$$\Phi^T \circ \mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) = \frac{|\kappa_d(\xi)_i(p)|}{\sqrt{(\kappa_d(\xi)_i(p))^2 - 1}} \left( p + \frac{1}{\kappa_d(\xi)_i(p)} \xi \right)$$

and

$$\Phi^S \circ \mathbb{LF}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) = \frac{-\kappa_d(\xi)_i(p)}{\sqrt{-(\kappa_d(\xi)_i(p))^2 - 1}} \left( p + \frac{1}{\kappa_d(\xi)_i(p)} \xi \right).$$

We now introduce the notion of focal sets of submanifolds in hyperbolic space. We define the *total focal set* of a submanifold  $M = \mathbf{X}(U) \subset H^n(-1)$  as

$$\text{TF}_M = \bigcup \left\{ \frac{\pm \kappa_d(\xi)_i(p)}{\sqrt{|(\kappa_d(\xi)_i(p))^2 - 1|}} \left( p + \frac{1}{\kappa_d(\xi)_i(p)} \xi \right) \mid \kappa_d(\xi)_i(p) \neq \pm 1, i = 1, \dots, s \right\}.$$

We have the following decomposition of the total focal set:

$$\text{TF}_M = \text{HIF}_M \cup \text{SF}_M,$$

where

$$\text{HIF}_M = \bigcup \left\{ \frac{\pm \kappa_d(\xi)_i(p)}{\sqrt{(\kappa_d(\xi)_i(p))^2 - 1}} \left( p + \frac{1}{\kappa_d(\xi)_i(p)} \xi \right) \mid (\kappa_d(\xi)_i(p))^2 > 1, i = 1, \dots, s \right\}$$

and

$$\text{SF}_M = \bigcup \left\{ \frac{\pm \kappa_d(\xi)_i(p)}{\sqrt{1 - (\kappa_d(\xi)_i(p))^2}} \left( p + \frac{1}{\kappa_d(\xi)_i(p)} \xi \right) \mid (\kappa_d(\xi)_i(p))^2 < 1, i = 1, \dots, s \right\}.$$



We call  $\mathbb{H}\mathbb{F}_M$  the *hyperbolic focal set* and  $\mathbb{S}\mathbb{F}_M$  the *de Sitter focal set*. We denote

$$\mathbb{L}\mathbb{F}_M^T = \bigcup \{ \mathbb{L}\mathbb{F}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) \mid (\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) > 0 \}$$

and

$$\mathbb{L}\mathbb{F}_M^S = \bigcup \{ \mathbb{L}\mathbb{F}_{\kappa_\ell(\mathbf{n}^T)_i}(p, \xi) \mid (\kappa_\ell(\mathbf{n}^T)_i(p, \xi))^2 + 2\kappa_\ell(\mathbf{n}^T)_i(p, \xi) < 0 \}.$$

We call  $\mathbb{L}\mathbb{F}_M^T$  and  $\mathbb{L}\mathbb{F}_M^S$  the *timelike* and *spacelike* parts, respectively, of the focal set of  $M$ . According to the previous arguments, we have the following proposition.

**Proposition 8.3.** *Let  $M = X(U)$  be a submanifold in the hyperbolic space  $H^n(-1)$ . Then we have*

$$\Phi^T(\mathbb{L}\mathbb{F}_M^T) \subset \mathbb{H}\mathbb{F}_M \quad \text{and} \quad \Phi^S(\mathbb{L}\mathbb{F}_M^S) \subset \mathbb{S}\mathbb{F}_M.$$

We previously introduced the notion of evolutes of a hypersurface in hyperbolic space and investigated the singularities of evolutes [18]. If  $M$  is a hypersurface in hyperbolic space, then  $M$  is a spacelike submanifold in  $\mathbb{R}_1^{n+1}$  of codimension two and  $N_1^h$  is a double covering of  $M$ . In this case, the above definition of the focal sets is the same as our definitions of evolutes [18]. Therefore, we denote  $\mathbb{L}\mathbb{E}_M^T$ ,  $\mathbb{L}\mathbb{E}_M^S$ ,  $\mathbb{H}\mathbb{E}_M$  and  $\mathbb{S}\mathbb{E}_M$  instead of  $\mathbb{L}\mathbb{F}_M^T$ ,  $\mathbb{L}\mathbb{F}_M^S$ ,  $\mathbb{H}\mathbb{F}_M$  and  $\mathbb{S}\mathbb{F}_M$ , respectively. Then we have the following corollary of Proposition 8.3.

**Corollary 8.4.** *Let  $M = X(U)$  be a hypersurface in the hyperbolic space  $H^n(-1)$ . Then we have*

$$\Phi^T(\mathbb{L}\mathbb{E}_M^T) \subset \mathbb{H}\mathbb{E}_M \quad \text{and} \quad \Phi^S(\mathbb{L}\mathbb{E}_M^S) \subset \mathbb{S}\mathbb{E}_M.$$

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