Stable quasimaps to GIT quotients

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Ionuţ Ciocan-Fontanine [a,b], Bumsig Kim [b,*], Davesh Maulik [c]

[a] School of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis MN, 55455, United States
[b] School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul, 130-722, Republic of Korea
[c] Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, United States

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ABSTRACT

We construct new compactifications with good properties of moduli spaces of maps from nonsingular marked curves to a large class of GIT quotients. This generalizes from a unified perspective many particular examples considered earlier in the literature.

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1. Introduction

The aim of this paper is to introduce new virtually smooth modular compactifications of spaces of maps from curves to a large class of targets.

Let C be a nonsingular complex projective curve, let X be a nonsingular complex projective variety, and let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class. The standard way of putting a scheme structure on the set $\mathrm{Map}_\beta(C, X)$ of algebraic maps from C to X of class β is by viewing it as an open subset in the Hilbert scheme of the product $C \times X$, via the identification of a map with its graph. The restriction of the universal family on the Hilbert scheme gives a universal map

$$C \times \mathrm{Map}_\beta(C, X) \overset{f}{\longrightarrow} X$$
$$\downarrow \pi$$
$$\mathrm{Map}_\beta(C, X).$$

The obstruction theory of the Hilbert scheme restricts to the obstruction theory

$$\left(R^\bullet \pi_* f^* T_X \right)^\vee \tag{1.0.1}$$

of the space of maps. While (1.0.1) is a perfect obstruction theory, and therefore $\mathrm{Map}_\beta(C, X)$ is virtually smooth, it is well-known that the obstruction theory of the Hilbert scheme fails to be perfect at points corresponding to subschemes which

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are not local complete intersections. Hence the compactification provided by the Hilbert scheme is not suitable to defining invariants of *X* via intersection theory. For a few special cases of varieties *X*, other compactifications have been used which do not suffer from this defect. Specifically,

(i) when *X* is a Grassmannian, the Quot schemes on *C* carry a perfect obstruction theory, see [1,2];

(ii) when *X* is a toric variety, the "toric compactifications" (or linear sigma models) of Givental and Morrison–Plesser also carry perfect obstruction theories;

(iii) when *X* is the Hilbert scheme of points in $\mathbb{C}^2$, Diaconescu's moduli spaces of ADHM sheaves, [3], give partial compactifications carrying perfect obstruction theories.

All these spaces have the common feature that the curve *C* is kept fixed and the map degenerates, so that points in the boundary correspond to *rational* maps $C -- \to X$.

In the case of a general *X*, Kontsevich approached the problem by insisting that in the compactification the boundary points still parametrize honest maps. This requires that the domain curve degenerates. It is then natural also to allow the domain curve to vary in moduli (and to consider markings). The approach leads to the virtually smooth moduli stacks of stable maps $\overline{M}_{g,k}(X, \beta)$ used to define the Gromov–Witten invariants of *X*. For stable map spaces, (1.0.1) becomes a relative perfect obstruction theory over the moduli Artin stack $\mathfrak{M}_{g,k}$ of (prestable) domain curves. In particular, the graph space $\overline{M}_{g,0}(X \times C, (\beta, 1))$ provides a virtually smooth compactification of $\mathrm{Map}_\beta(C, X)$.

The paper [4] introduced, in case *X* is a Grassmannian, a hybrid version of these compactifications, called the moduli of stable quotients. It allows both the domain curve to vary, and the map to acquire base points. Then [4] shows that after imposing an appropriate stability condition, the resulting moduli space is a proper Deligne–Mumford stack of finite type, carrying a relative perfect obstruction theory over $\mathfrak{M}_{g,k}$ and hence a virtual fundamental class.

The paper [5] treated the toric counterpart of the story, that is, the hybrid version of the compactifications in (ii) above when the domain curve is allowed to degenerate and to carry markings. The resulting objects, called *stable toric quasimaps*, are proven there to form a proper DM stack of finite type. Furthermore, it is observed that it is more natural to view the moduli spaces as stacks over (products of) *Picard stacks over* $\mathfrak{M}_{g,k}$, the corresponding relative obstruction theory being that of sections of line bundles on curves.

This suggests that the common feature of examples (i)–(iii) which should be exploited is that they are all GIT quotients. The obstruction theory that needs then to be studied is that of sections of fibrations associated to principal **G**-bundles over curves, with **G** a reductive complex algebraic group. Once this is recognized, the most natural way to view the problem is in a stack-theoretic context.

Let $X = W /\!\!/ \mathbf{G}$ be a (nonsingular, projective) GIT quotient obtained from a linearized **G**-action on a quasi-projective variety *W* for which the stable and semistable loci coincide. Let $\mathcal{A} = [W/\mathbf{G}]$ be the *stack quotient*. A map from a prestable curve *C* to $\mathcal{A}$ corresponds to a pair (P, u), with *P* a principal **G**-bundle on *C* and *u* a section of the induced fiber bundle $P \times_\mathbf{G} W \longrightarrow C$ with fiber *W*. It has a "homology" class $\beta \in \mathrm{Hom}_\mathbb{Z}(\mathrm{Pic}(\mathcal{A}), \mathbb{Z})$. We then have the moduli stack $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ parametrizing families of tuples

$$((C, p_1, \ldots, p_k), P, u)$$

with $(C, p_1, \ldots, p_k)$ a prestable *k*-pointed curve of genus *g*. While $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ contains the Deligne–Mumford stack

$$M_{g,k}(W /\!\!/ \mathbf{G}, \beta)$$

of maps from nonsingular pointed curves to $W /\!\!/ \mathbf{G}$ as an open substack, it is in general a nonseparated Artin stack of infinite type.

A sophisticated study of stacks of the type $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ was begun in [6], with the goal to produce an algebro-geometric version of the "Gromov–Witten gauge invariants" introduced in symplectic geometry via solutions to vortex equations (see [7–12]). The point of view we pursue in this paper is different in that we seek to obtain moduli stacks with good properties by imposing additional conditions on the maps we consider, as well as on the variety *W*. Namely, we will want our moduli stacks to be open substacks in $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ which are of finite type, Deligne–Mumford, proper, and to carry a perfect obstruction theory extending the natural such obstruction theory on the stack $M_{g,k}(W /\!\!/ \mathbf{G}, \beta)$. This will allow us to construct systems of numerical "invariants" (in fact, new Cohomological Field Theories) on the cohomology of the target varieties.

We highlight here the main points:

- If we restrict to $((C, p_1, \ldots, p_k), P, u)$ such that *u* sends the generic point of each irreducible component of *C* to the GIT-*stable locus* $W^s \subset W$, the resulting moduli stack is of finite type over $\mathfrak{M}_{g,k}$. We call these $((C, p_1, \ldots, p_k), P, u)$ *quasimaps to* $W /\!\!/ \mathbf{G}$. The closed points $y \in C$ with $u(y) \in W \setminus W^s$ are called the *base-points* of the quasimap (we assume throughout that there are no strictly semistable points in *W*).
- To ensure separatedness, we restrict to quasimaps whose base-points (if any) are away from the nodes and markings of *C*.
- To obtain a Deligne–Mumford stack of finite type, we require the quasimaps to satisfy a stability condition.
- To ensure properness we require *W* to be *affine*.
- To ensure that the natural obstruction theory is perfect we require that *W* has at most local complete intersection singularities.

The stability condition we consider initially generalizes the ones from [4,5] and we call the resulting objects stable quasimaps to $W /\!\!/ \mathbf{G}$. It is a simple fact that Kontsevich's stable maps to $W /\!\!/ \mathbf{G}$ are also obtained by imposing a stability condition of the same kind on quasimaps. Moreover, generalizing the observation of [13,14] for target $\mathbb{P}^n$ (extended in [15] to Grassmannians), we show that there is a one-parameter family of stability conditions producing open substacks of $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ with all the good properties we require, and which interpolate between stable maps and stable quasimaps. The virtual classes of these moduli spaces will potentially differ over loci in their boundaries. At the level of invariants, a correspondence between the stable quasimap and Gromov–Witten theories of $W /\!\!/ \mathbf{G}$ will then require a (generally nontrivial) transformation which records wall-crossing contributions. In this paper we restrict ourselves to setting up the foundations of quasimap theory, leaving the investigation of the correspondence to future work.

We now describe in more detail the contents and organization of the paper.

Section 2 is devoted to a description of the general set-up that will be used for the rest of the paper: we consider nonsingular GIT quotients obtained from the action of a reductive complex algebraic group $\mathbf{G}$ on an affine complex algebraic variety $W = \mathrm{Spec}(A)$, linearized by a character of $\mathbf{G}$. The quotient $W /\!\!/ \mathbf{G}$ is always projective over the "affine quotient" $\mathrm{Spec}(A^{\mathbf{G}})$ of W by $\mathbf{G}$, but we do not require the affine quotient to be a point. We also include here a quick discussion of the moduli stack $\mathfrak{B}un_{\mathbf{G}}$ parametrizing principal $\mathbf{G}$-bundles on (varying) prestable marked curves.

In Section 3 the notions of quasimaps and stable quasimaps to $W /\!\!/ \mathbf{G}$ are introduced. We then prove various boundedness results for families of quasimaps and stable quasimaps.

The main properties of the moduli stack $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ of stable quasimaps to $W /\!\!/ \mathbf{G}$ with fixed numerical data (g, k, β) are established in Section 4. While the fact that $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is an Artin stack locally of finite type follows quite easily by combining some results in [16,17], we give a straightforward direct construction which exhibits it as a stack with a schematic morphism to $\mathfrak{B}un_{\mathbf{G}}$, whose fibers are open subschemes in appropriate Hilbert schemes. Boundedness is used to show that the stack is of finite type, while stability implies that it is Deligne–Mumford. Finally, we prove that $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is proper over the affine quotient. Our argument uses crucially the assumption that W is affine.

The usefulness of the construction of the moduli stack described above is that it provides us immediately with a canonical obstruction theory for $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$, relative to the smooth stack $\mathfrak{B}un_{\mathbf{G}}$, given by the deformation theory of Hilbert schemes. When W is nonsingular it is very easy to see that this obstruction theory is perfect. By a more elaborate argument we prove that the obstruction theory remains perfect if lci singularities only are allowed in the unstable locus of W. It is also clear from the description of the obstruction theory that the result is optimal: as long as the stability condition allows base points, the obstruction theory will fail to be perfect if the singularities are worse than lci. In particular, this answers a question of Marian, Oprea, and Pandharipande in [4] by showing that the "moduli of stable quotients" to a projective subvariety $X \subset \mathbb{P}^n$ carries a natural virtual class only when X is a complete intersection. On the other hand, since the stability condition that leads to the usual Kontsevich stable maps forbids base points, the unstable locus in W plays no role and we obtain a perfect obstruction theory inducing the usual virtual class of $\overline{M}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ for all targets $W /\!\!/ \mathbf{G}$.

A short alternative construction is described in Section 5 for the case when $W = V$ is a vector space. It leads naturally to a simple description of the *absolute* perfect obstruction theory and the corresponding virtual class of Qmap is shown to be globally a refined top Chern class of a vector bundle on a smooth Deligne–Mumford stack. It is worth pointing out that in this case both the tangent bundle of $V /\!\!/ \mathbf{G}$ and the obstruction theory of $\mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$ may be presented using generalized "Euler sequences", which facilitate computations in many instances.

Section 6 introduces the descendent quasimap integrals, as well as the analogues in quasimap theory of the twisted Gromov–Witten invariants of Coates and Givental for proper targets $W /\!\!/ \mathbf{G}$. Furthermore, equivariant invariants for non-proper targets endowed with torus actions having proper fixed points loci are also discussed (Nakajima quiver varieties are a large class of particularly interesting examples of such targets).

Finally, Section 7 deals with two variants of our construction. The first is the variation of stability condition mentioned above. Namely, we introduce for each positive rational number ϵ the notion of ϵ-stable quasimaps. When ϵ is sufficiently close to zero, this coincides with the notion of stable quasimap, while when ϵ is sufficiently large it recovers stable maps to $W /\!\!/ \mathbf{G}$. The arguments from Sections 3 and 4 extend in a straightforward manner to prove that the moduli stacks of ϵ-stable quasimaps are Deligne–Mumford, of finite type, proper over the affine quotient, and carry natural perfect obstruction theories under the same condition on the singularities of W. For a fixed class β, the interval $(0, +\infty)$ over which the stability parameter ϵ ranges is divided into chambers by finitely many walls of the form $\epsilon = 1/m$ such that the moduli stack is constant in each chamber.

The second variant we discuss is that of stable quasimaps with one parametrized component. These are most interesting in genus zero and the corresponding moduli stacks are the analogues for quasimaps of the "graph spaces" $\overline{M}_{0,k}(X \times \mathbb{P}^1, (\beta, 1))$ used prominently by Givental in his proofs of mirror theorems. They are used to extend the definition of Givental's small I-functions, and of the big I-functions in [5], from toric targets to general GIT quotient targets $W /\!\!/ \mathbf{G}$. We expect there is a natural way to express the "mirror maps" transforming I into the J-function of $W /\!\!/ \mathbf{G}$ via wall-crossing for ϵ-stable quasimaps with one parametrized component.

2. Preliminaries

Throughout the paper we work over the base field $\mathbb{C}$. We fix a reductive (hence linearly reductive) complex algebraic group $\mathbf{G}$.

Recall that a principal **G**-bundle P over a base algebraic space B is an algebraic space P with a free (right) **G**-action and a **G**-equivariant map $\pi : P \longrightarrow B$ (where B has the trivial action) which is étale-locally trivial.

For a scheme W with (right) **G**-action, one can form the *mixed construction*

$$P \times_{\mathbf{G}} W := [(P \times W)/\mathbf{G}], \tag{2.0.1}$$

where the quotient is taken with respect to the diagonal action of **G**. It is an algebraic space with a morphism

$$\rho : P \times_{\mathbf{G}} W \longrightarrow B$$

which is (in the étale topology) a locally trivial fibration with fiber W.

2.1. Principal **G**-bundles on curves

By $\mathfrak{M}_{g,k}$ we denote the moduli stack of prestable k-pointed curves of genus g, and by $\overline{M}_{g,k}$ we denote the moduli stack of stable k-pointed curves of genus g. Recall that $\mathfrak{M}_{g,k}$ is a smooth Artin stack, locally of finite type, while $\overline{M}_{g,k}$ is a smooth Deligne–Mumford stack of finite type, which is proper. Usually $\mathfrak{C}_{g,k}$ and $\mathcal{C}_{g,k}$ will stand for the universal curves over $\mathfrak{M}_{g,k}$ and $\overline{M}_{g,k}$, respectively. The projection maps from the universal curves to their base will all be denoted π; this should not lead to confusion.

We will also consider the relative moduli stack

$$\mathfrak{Bun}_{\mathbf{G}} \xrightarrow{\phi} \mathfrak{M}_{g,k}$$

of principal **G**-bundles on the fibers of $\mathfrak{C}_{g,k}$. It is again an Artin stack, locally of finite type, as can be deduced immediately from [16, Prop. 2.18] and [17, Lemma C.5]. By [18], the morphism ϕ has smooth geometric fibers. In fact, ϕ is a smooth morphism, hence $\mathfrak{Bun}_{\mathbf{G}}$ is smooth. For convenience, we include a direct proof of these facts in the following proposition. See also the recent preprint [19].

Proposition 2.1.1. *The stack $\mathfrak{Bun}_{\mathbf{G}}$ is a smooth Artin stack, locally of finite type over $\mathbb{C}$.*

Proof. We check the requirements in the definition of an Artin stack.

1. The diagonal map Δ is representable, quasicompact, and separated:

(a) We first prove this for the case $\mathbf{G} = GL(r)$. Let T be a $\mathbb{C}$-scheme. Consider two vector bundles E_1, E_2 of rank r on a projective scheme Y/T with $\pi_{Y/T}$ the flat structural map $Y \rightarrow T$. We let $\text{Isom}_T(E_1, E_2)$ be the functor from the T-scheme category Sch$/T$ to the set category Sets sending T'/T to $\text{Hom}_{T'}(E_1|_{T'}, E_2|_{T'})$. Then the stack $\text{Isom}_T(E_1, E_2)$ over T is representable by a locally closed subscheme of the total space of the vector bundle associated to the locally free sheaf $(\pi_{Y/T})_*(E_1^\vee \otimes E_2(n))$ for some large enough integer n. Hence, $\text{Isom}_T(E_1, E_2)$ is a quasicompact separated T-scheme. Now consider two vector bundles F_1, F_2 on two families C_i/S of m-marked *stable* curves over a $\mathbb{C}$-scheme S, respectively. Notice that $\text{Isom}_S(C_1, C_2)$ is a quasi-projective scheme over S. The stack $\text{Isom}_S(F_1, F_2)$ over S is representable by a quasicompact separated T-scheme $\text{Isom}_T(\pi_1^* F_1, f^* F_2)$, where $T = \text{Isom}_S(C_1, C_2)$, $Y = C_1 \times_S T$, $\pi_1 : Y \rightarrow C_1$ is the first projection, and f is the evaluation map $Y \rightarrow C_2$. However, in general, C_i are families of prestable curves. In this case, by adding additional local markings, there is an étale covering $\{S_j\}$ of S such that $C_{ij} = C_i|_{S_j}$ is a projective scheme over S_j. Now glue $\text{Isom}_{S_j}(F_1|_{S_j}, F_2|_{S_j})$ to obtain an algebraic space over S, representing $\text{Isom}_S(F_1, F_2)$. This shows that Δ is representable, quasicompact, and separated.

(b) For a general reductive algebraic group $\mathbf{G} \subset GL(r)$, consider extensions $P_i' := P_i \times_{\mathbf{G}} GL(r)$, $i = 1, 2$, and the associated sections $s_i : C \rightarrow P_i'/\mathbf{G}$. Then $\text{Isom}_S(P_1, P_2)$ is the closed algebraic subspace of $\text{Isom}_S(P_1', P_2')$ parametrizing the isomorphisms compatible with s_i.

2. There is a smooth surjective morphism from a $\mathbb{C}$-scheme locally of finite type to $\mathfrak{Bun}_{\mathbf{G}}$:

Let S be a $\mathbb{C}$-scheme of finite type. For every pair of positive integers n, N and every family C/S of prestable curves with relatively ample line bundle $\mathcal{O}_C(1)$, consider an open subscheme $Z_{N,n}$ of Quot schemes parametrizing the locally free rank r quotients of $\mathcal{O}_C^{\oplus N}(n)$. Then when $\mathbf{G} = GL(r)$ the natural forgetful morphism from the disjoint union $Z := \coprod_{N,n} Z_{N,n}$ to the stack $\mathfrak{Bun}_{\mathbf{G}}(C/S)$ of principal G-bundles on C/S is surjective and smooth. For a general reductive algebraic group $\mathbf{G} \subset GL(r)$, we consider the natural morphism

$$\text{Hom}_Z(C \times_S Z, \mathcal{P}/\mathbf{G}) \rightarrow \mathfrak{Bun}_{\mathbf{G}}(C/S),$$

where $\mathcal{P}$ is the universal $GL(r)$-principal bundle on $C \times_S Z$. Since $GL(r)/\mathbf{G}$ is quasi-projective, and hence $\mathcal{P}/\mathbf{G}$ is quasi-projective over Z (see for example [20, Section 3.6.7]), we see that $\text{Hom}_Z(C \times_S Z, \mathcal{P}/\mathbf{G})$ is a quasi-projective scheme over Z. Now by this and the fact that $\mathfrak{M}_{g,k}$ is an Artin stack, we can build a scheme locally of finite type, smoothly covering $\mathfrak{Bun}_{\mathbf{G}}$.

3. We show the smoothness of $\mathfrak{Bun}_{\mathbf{G}}$. It is enough to show that $\mathfrak{Bun}_{\mathbf{G}}$ is formally smooth, i.e., the deformation problem for the corresponding moduli functor is unobstructed:

Let S_0 be the spectrum of a finitely generated $\mathbb{C}$-algebra A_0 and let $S = \text{Spec}(A)$, with A a square zero extension by a finite A_0-module M. Let C_0, C be families of prestable curves over S_0, respectively over S, with C an extension of C_0. Note that for any principal **G** bundle P' on a scheme Y, $\text{Aut}(P') = \text{Hom}_{\mathbf{G}}(P', \mathbf{G}) = \Gamma(Y, P' \times_{\mathbf{G}} \mathbf{G})$, where the **G**-action on **G** is the adjoint action. From this it is classical to derive that the obstruction to extending a principal **G**-bundle on C_0 to C lies in $H^2(C_0, P \times_{\mathbf{G}} \mathfrak{g}) \otimes_{A_0} M$. The fibers of C_0/S_0 are one-dimensional and S_0 is affine, hence the A_0-module $H^2(C_0, P \times_{\mathbf{G}} \mathfrak{g})$ vanishes. Since $\mathfrak{M}_{g,k}$ is a smooth stack over $\mathbb{C}$, we conclude the proof. $\square$

2.2. A class of GIT quotients

Let $W = \mathrm{Spec}(A)$ be an affine algebraic variety with an action by the reductive algebraic group $\mathbf{G}$. There are two natural quotients associated with this action: the *quotient stack* $[W/\mathbf{G}]$, and the *affine quotient* $W /_{\mathrm{aff}} \mathbf{G} = \mathrm{Spec}(A^{\mathbf{G}})$.

Let $\chi(\mathbf{G})$ be the character group of $\mathbf{G}$ and denote by $\mathrm{Pic}^{\mathbf{G}}(W)$ the group of isomorphism classes of $\mathbf{G}$-linearized line bundles on W. Any character $\xi \in \chi(\mathbf{G})$ determines a one-dimensional representation $\mathbb{C}_\xi$ of $\mathbf{G}$, hence a linearized line bundle

$$L_\xi = W \times \mathbb{C}_\xi$$

in $\mathrm{Pic}^{\mathbf{G}}(W)$.

Fix once and for all a character $\theta \in \chi(\mathbf{G})$. Since $\mathbf{G}$ is reductive, the graded algebra

$$S(L_\theta) := \bigoplus_{n \geq 0} \Gamma(W, L_\theta^{\otimes n})^{\mathbf{G}}$$

is finitely generated and we have the associated *GIT quotient*

$$W /\!\!/ \mathbf{G} := W /\!\!/_\theta \mathbf{G} := \mathrm{Proj}(S(L_\theta)),$$

which is a quasi-projective variety, with a projective morphism

$$W /\!\!/ \mathbf{G} \longrightarrow W /_{\mathrm{aff}} \mathbf{G}$$

to the affine quotient (see [21]). The GIT quotient is projective precisely when $W /_{\mathrm{aff}} \mathbf{G}$ is a point, i.e., when the only $\mathbf{G}$-invariant functions on W are the constant functions.

Let

$$W^s = W^s(\theta) \quad \text{and} \quad W^{ss} = W^{ss}(\theta)$$

be the open subsets of stable (respectively, semistable) points determined by the choice of linearization (again, see [21] for the definitions). From now on the following assumptions will be in force:

(i) $\emptyset \neq W^s = W^{ss}$.
(ii) W^s is nonsingular.
(iii) $\mathbf{G}$ acts freely on W^s.

It follows from Luna's slice theorem that $W^s \xrightarrow{\rho} W /\!\!/ \mathbf{G}$ is a principal $\mathbf{G}$-bundle in the étale topology. Hence $W /\!\!/ \mathbf{G}$ is a nonsingular variety, which coincides with $[W^s/\mathbf{G}]$, the stack quotient of the stable locus. In particular, it is naturally an open substack in $[W/\mathbf{G}]$.

Note that $W /\!\!/ \mathbf{G}$ comes with a (relative) polarization: the line bundle L_θ descends to a line bundle

$$\mathcal{O}_{W /\!\!/ \mathbf{G}}(\theta) = W^s \times_{\mathbf{G}} (L_\theta|_{W^s})$$

which is relatively ample over the affine quotient (the right-hand side is the mixed construction (2.0.1)). Replacing θ by a positive integer multiple gives the same quotient $W /\!\!/ \mathbf{G}$, but with polarization $\mathcal{O}_{W /\!\!/ \mathbf{G}}(m\theta) = \mathcal{O}_{W /\!\!/ \mathbf{G}}(\theta)^{\otimes m}$.

Remark 2.2.1. The assumption (iii) above can in fact be dropped, in which case one considers the smooth Deligne–Mumford open substack $[W^s/\mathbf{G}] \subset [W/G]$, with coarse moduli space $W /\!\!/ \mathbf{G}$, as the base of the principal $\mathbf{G}$-bundle. This leads to the theory of "orbifold" (or twisted) stable quasimaps to $[W^s/\mathbf{G}]$, which can be developed with the same methods we introduce here. For clarity, we decided to treat only the case of manifold targets in this paper. The orbifold theory will be discussed in a forthcoming paper, [22].

2.3. Maps from curves to $[W/\mathbf{G}]$

Let $(C, x_1, \ldots, x_k)$ be a prestable pointed curve, i.e., a connected projective curve (of some arithmetic genus g), with at most nodes as singularities, together with k distinct and nonsingular marked points on it. We will be interested in various moduli spaces of maps from such curves to the quotient stack $[W/\mathbf{G}]$. By the definition of quotient stacks, a map $[u] : C \longrightarrow [W/\mathbf{G}]$ corresponds to a pair $(P, \tilde{u})$, with

$$P \longrightarrow C$$

a principal $\mathbf{G}$-bundle on C and

$$\tilde{u} : P \longrightarrow W$$

a $\mathbf{G}$-equivariant morphism. Equivalently (and most often), we will consider the data (P, u), with

$$u : C \longrightarrow P \times_{\mathbf{G}} W$$

a section of the fiber bundle $\rho : P \times_{\mathbf{G}} W \longrightarrow C$. Then $[u] : C \rightarrow [W/\mathbf{G}]$ is obtained as the composite

$$C \xrightarrow{u} P \times_{\mathbf{G}} W \rightarrow [W/\mathbf{G}].$$

Note also that the composition

$$C \xrightarrow{[u]} [W/\mathbf{G}] \rightarrow W /_{\mathrm{aff}} \mathbf{G}$$

is always a constant map, since C is projective.

We denote by $\mathrm{Map}(C, [W/\mathbf{G}])$ the Artin stack parametrizing such pairs (P, u).

2.4. Induced line bundles and the degree of maps

Recall that $L \mapsto [L/\mathbf{G}]$ gives an identification $\mathrm{Pic}^{\mathbf{G}}(W) = \mathrm{Pic}([W/\mathbf{G}])$.

Let (C, P, u) be as above. For a $\mathbf{G}$-equivariant line bundle L on W, we have a cartesian diagram

$$\begin{array}{ccc} P \times_{\mathbf{G}} W & \longleftarrow & P \times_{\mathbf{G}} L \\ \downarrow & & \downarrow \\ [W/\mathbf{G}] & \longleftarrow & [L/\mathbf{G}]. \end{array}$$

Hence we get an induced line bundle $u^*(P \times_{\mathbf{G}} L) = [u]^*([L/\mathbf{G}])$ on C.

Definition 2.4.1. The *degree* β of $(P, u) \in \mathrm{Map}(C, [W/\mathbf{G}])$ is the homomorphism

$$\beta : \mathrm{Pic}^{\mathbf{G}}(W) \rightarrow \mathbb{Z}, \qquad \beta(L) = \deg_C(u^*(P \times_{\mathbf{G}} L)).$$

In fact, under the natural map $H_2^{\mathbf{G}}(W) \rightarrow \mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z})$, β is the image of the class of the equivariant cycle

$$\begin{array}{ccc} P & \longrightarrow & W \\ \downarrow & & \\ C & & \end{array}$$

given by $(C, P, \tilde{u})$. Here $H_2^{\mathbf{G}}(W)$ denotes the $\mathbf{G}$-equivariant homology. As a consequence we will also call β the (equivariant homology) *class* of (C, P, u).

Finally, we note that any $\mathbf{G}$-equivariant section $t \in \Gamma(W, L)^{\mathbf{G}}$ induces a section of $[L/\mathbf{G}]$, hence a section $u^*t = [u]^*t$ of $u^*(P \times_{\mathbf{G}} L)$.

2.5. Quotients of vector spaces

An important special case is when $W = V$ is a finite dimensional $\mathbb{C}$-vector space equipped with a *linear* action, i.e., $\mathbf{G}$ acts via a representation $\mathbf{G} \longrightarrow \mathrm{GL}(V)$. As an algebraic variety, V is an affine space and its Picard group is trivial. Hence there is an identification

$$\chi(\mathbf{G}) \cong \mathrm{Pic}^{\mathbf{G}}(V).$$

Let C be a nodal curve and let $(P, u) \in \mathrm{Map}(C, [V/\mathbf{G}])$ be as above. If ξ is a character with associated representation $\mathbb{C}_\xi$, then there is an isomorphism of line bundles on C

$$u^*(P \times_{\mathbf{G}} L_\xi) \cong P \times_{\mathbf{G}} \mathbb{C}_\xi. \tag{2.5.1}$$

Therefore in this case the degree

$$\beta = \beta_P \in \mathrm{Hom}_{\mathbb{Z}}(\chi(\mathbf{G}), \mathbb{Z})$$

is the usual degree of the principal $\mathbf{G}$-bundle on a curve.

Remark 2.5.1. Eq. (2.5.1) is a consequence of the following fact, which we will tacitly use from now on. Given $\mathbf{G}$-schemes W and E and a map (C, P, u) to $[W/\mathbf{G}]$, let

$$\rho : P \times_{\mathbf{G}} (W \times E) \longrightarrow P \times_{\mathbf{G}} W$$

be the map induced by the first projection $W \times E \rightarrow W$. Then there is a canonical isomorphism of C-schemes

$$P \times_{\mathbf{G}} E \cong C \times_{P \times_{\mathbf{G}} W} (P \times_{\mathbf{G}} (W \times E)),$$

where in the right-hand side we have the fiber product over the section $u : C \rightarrow P \times_{\mathbf{G}} W$ and the map ρ.

The following well-known fact will be used later in the paper. It can be found in many standard texts on Invariant Theory (see e.g. [23], page 94).

Proposition 2.5.2. *Let W be an affine algebraic variety over $\mathbb{C}$ and let $\mathbf{G}$ be a reductive group acting on W. There exists a finite dimensional vector space V with a linear $\mathbf{G}$-action, and a $\mathbf{G}$-equivariant closed embedding $W \hookrightarrow V$.*

3. Quasimaps and stable quasimaps to $W /\!\!/ G$

We keep the set-up from the previous section: we are given an affine variety W with an action of **G** linearized by a character $\theta \in \chi(\mathbf{G})$, satisfying the assumptions (i)–(iii) from Section 2.2.

3.1. Quasimaps and stable quasimaps

Fix integers $k, g \geq 0$ and a class $\beta \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic}^{\mathbf{G}}(W), \mathbb{Z})$.

Definition 3.1.1. A k-pointed, genus g *quasimap* of class β to $W /\!\!/ \mathbf{G}$ consists of the data

$$(C, p_1, \ldots, p_k, P, u),$$

where

- $(C, p_1, \ldots, p_k)$ is a connected, at most nodal, k-pointed projective curve of genus g,
- P is a principal **G**-bundle on C,
- u is a section of the induced fiber bundle $P \times_{\mathbf{G}} W$ with fiber W on C such that (P, u) is of class β,

and satisfying the following generic nondegeneracy condition:

there is a finite (possibly empty) set $B \subset C$ such that for every $p \in C \setminus B$ we have $u(p) \in W^s$.   (3.1.1)

In other words, a quasimap is a map to the quotient stack $[W/\mathbf{G}]$ such that u sends the generic point of each irreducible component of C to the stable locus W^s. The points in B will be called the *base points* of the quasimap.

Definition 3.1.2. The quasimap $(C, p_1, \ldots, p_k, P, u)$ is called *prestable* if the base points are disjoint from the nodes and markings on C.

It is called *stable* if it is prestable and the line bundle $\omega_C(\sum_{i=1}^{k} p_i) \otimes \mathcal{L}^\epsilon$ is ample for every rational number $\epsilon > 0$, where

$$\mathcal{L} := \mathcal{L}_\theta = u^*(P \times_{\mathbf{G}} L_\theta) \cong P \times_{\mathbf{G}} \mathbb{C}_\theta.$$

These definitions generalize and are motivated by two previous constructions in the literature.

Example 3.1.3. *Stable quotients:* Consider, for $r \leq n$, the vector space $V = \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n) = \operatorname{Mat}(n \times r)$ of $n \times r$ complex matrices, equipped with the action of $\mathbf{G} = GL(r)$ via right multiplication and with stability induced by the determinant character. In this case, the associated GIT quotient is the Grassmannian $G(r, n)$ and the definition of a stable quasimap to $V /\!\!/ \mathbf{G}$ simply recovers the definition of stable quotients first studied in [4]. It is shown there that quasimaps and stable maps to $G(r, n)$ give essentially identical numerical invariants.

Example 3.1.4. *Stable toric quasimaps:* Suppose we are given a complete nonsingular fan $\Sigma \subset \mathbb{R}^n$ generated by l one-dimensional integral rays inside an n-dimensional lattice. There is a smooth n-dimensional projective toric variety X_Σ associated to the fan, which can be expressed as a GIT quotient of $V = \mathbb{C}^l$ by the torus $\mathbf{G} = (\mathbb{C}^*)^{l-n}$. In this case, we obtain stable toric quasimaps studied in [5]. Already in this setting, the relation between stable quasimaps and stable maps is much more complicated, even at the numerical level.

Definition 3.1.5. An isomorphism between two quasimaps

$$(C, p_1, \ldots, p_k, P, u),$$

and

$$(C', p_1', \ldots, p_k', P', u'),$$

consists of an isomorphism $f : C \to C'$ of the underlying curves, along with an isomorphism $\sigma : P \to f^*P'$, such that the markings and the section are preserved:

$$f(p_j) = p_j', \qquad \sigma_W(u) = f^*(u'),$$

where $\sigma_W : P \times_{\mathbf{G}} W \to P' \times_{\mathbf{G}} W$ is the isomorphism of fiber bundles induced by σ.

Definition 3.1.6. A family of (stable) quasimaps to $W /\!\!/ \mathbf{G}$ over a base scheme S consists of the data

$$(\pi : \mathcal{C} \to S, \{p_i : S \to \mathcal{C}\}_{i=1,\ldots,k}, \mathcal{P}, u)$$

where

- $\pi : \mathcal{C} \to S$ is a flat family of curves over S, that is a flat proper morphism of relative dimension one,
- $p_i, i = 1, \ldots, k$ are sections of π,

- $\mathcal{P}$ is a principal **G**-bundle on $\mathcal{C}$,
- $u : \mathcal{C} \longrightarrow \mathcal{P} \times_{\mathbf{G}} W$ is a section

such that the restriction to every geometric fiber $\mathcal{C}_s$ of π is a (stable) k-pointed quasimap of genus g and class β. An isomorphism between two such families $(\mathcal{C} \to S, \dots)$ and $(\mathcal{C}' \to S, \dots)$ consists of an isomorphism of S-schemes $f : \mathcal{C} \to \mathcal{C}'$, and an isomorphism of **G**-bundles $\sigma : \mathcal{P} \to f^*\mathcal{P}'$, which preserve the markings and the section.

Stability of a quasimap $(C, p_1, \dots, p_k, P, u)$ implies immediately the following properties:

- Every rational component of the underlying curve C has at least two nodal or marked points and $\deg(\mathcal{L}) > 0$ on any such component with exactly two special points; in particular, this forces the inequality $2g - 2 + k \geq 0$.
- The automorphism group of a stable quasimap is finite and reduced. Indeed, it suffices to restrict to any rational component C' of C with exactly two special points q_1 and q_2. If the automorphism group $\mathbb{C}^* = \mathrm{Aut}(C', q_1, q_2)$ preserves the section u, this forces u to have no base points and $P|_{C'}$ to be trivial. However, this violates stability.

3.2. Boundedness

We prove in this subsection that quasimaps with fixed domain curve C and class β, as well as stable quasimaps with fixed numerical type (g, k, β) form bounded families.

We start with the following important observation.

Lemma 3.2.1. *If (C, P, u) is a quasimap, then $\beta(L_\theta) \geq 0$; and $\beta(L_\theta) = 0$ if and only if $\beta = 0$, if and only if the quasimap is a constant map to $W /\!\!/ \mathbf{G}$. Furthermore, the same holds for any subcurve C' and the induced quasimap.*

Proof. It is enough to prove the lemma when C is irreducible.

Recall that the equivariant line bundle L_θ on W descends to a line bundle $\mathcal{O}(\theta)$ on $W /\!\!/ \mathbf{G}$ which is relatively ample over the affine quotient. Pick $m > 0$ such that $\mathcal{O}(m\theta)$ is relatively very ample and let

$$W /\!\!/ \mathbf{G} \hookrightarrow \mathbb{P}_{A(W)^{\mathbf{G}}} \left((\Gamma(W, L_{m\theta})^\vee)^{\mathbf{G}} \right) \tag{3.2.1}$$

be the corresponding embedding over $\mathrm{Spec}(A(W)^{\mathbf{G}})$. If u has image contained in $P \times_{\mathbf{G}} W^s$, then it corresponds to a regular map $f : C \longrightarrow W /\!\!/ \mathbf{G}$ and $\beta(L_\theta) = \deg_C f^*\mathcal{O}(\theta)$. The claim is clear in this case. Assume now that there is a base-point $x \in C$ (so that $u(x)$ is unstable). Via composition with the inclusion (3.2.1), we view the quasimap as a *rational* map

$$C - - \rightarrow \mathbb{P}_{A(W)^{\mathbf{G}}} \left((\Gamma(W, L_{m\theta})^\vee)^{\mathbf{G}} \right)$$

with a base-point at x. It follows that every **G**-equivariant section t of $L_{m\theta}$ must satisfy $u^*t(x) = 0$. On the other hand, since the image of the generic point is stable, the line bundle $\mathcal{L}_{m\theta} = u^*(P \times_{\mathbf{G}} L_{m\theta})$ on C has a nonzero section of the form u^*t. Hence $m\beta(L_\theta) = \deg(\mathcal{L}_{m\theta}) > 0$. $\quad\square$

Definition 3.2.2. We will call elements

$$\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z})$$

which are realized as classes of quasimaps (possibly with disconnected domain curve) to $W /\!\!/_\theta \mathbf{G}$ (**G**-equivariant) L_θ-*effective classes*. These L_θ-effective classes form a semigroup, denoted $\mathrm{Eff}(W, \mathbf{G}, \theta)$.

Hence, if $\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)$, then either $\beta = 0$ or $\beta(L_\theta) > 0$.

An immediate consequence of Lemma 3.2.1 is a boundedness result for the underlying curves of stable quasimaps:

Corollary 3.2.3. *The number of irreducible components of the underlying curve of a stable quasimap to $V /\!\!/ \mathbf{G}$ is bounded in terms of g, k, and β only.*

Proof. See Corollary 3.1.5 in [5]. $\quad\square$

Next we bound quasimaps with given class β on a nodal curve.

Theorem 3.2.4. *Let $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z})$ and a nodal curve C be fixed. The family of quasimaps of class β from C to $W /\!\!/ \mathbf{G}$ is bounded.*

We begin by making some reductions.

First, by Proposition 2.5.2, it suffices to assume that $W = V$ is a vector space. Indeed, given a **G**-equivariant embedding $W \subset V$, every quasimap (C, P, u) to $W /\!\!/ \mathbf{G}$ is also a quasimap to $V /\!\!/ \mathbf{G}$, where we use the same linearization $\theta \in \chi(\mathbf{G})$ to define the stable and semistable loci on V. Note that the class of the quasimap to $V /\!\!/ \mathbf{G}$ is the image of β under the natural map

$$\mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(V), \mathbb{Z}) = \mathrm{Hom}(\chi(\mathbf{G}), \mathbb{Z}),$$

and is in fact just the degree of the underlying principal **G**-bundle P. Strictly speaking, we are in a slightly more general situation since the linearized action on V may fail to satisfy the assumptions in Section 2.2: there may be strictly semistable points in V, and points in $V^s \setminus W^s$ may have nontrivial stabilizers. However, the proof will only use that $V^s \neq \emptyset$, and that the representation **G** $\longrightarrow GL(V)$ has finite kernel. These follow respectively from the fact that W^s is nonempty, and from the freeness of the **G**-action on W^s (which implies the kernel is trivial).

Second, notice that the quasimaps (C, P, u) to $V /\!\!/ \mathbf{G}$ with fixed projective, connected nodal curve C and fixed principal **G**-bundle P on it, are parametrized by an open subset of the space of global sections of the induced *vector bundle*

$$\mathcal{V}_P = P \times_{\mathbf{G}} V$$

on C. Hence, after fixing C and a degree

$$\beta \in \mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(V), \mathbb{Z}) = \mathrm{Hom}(\chi(\mathbf{G}), \mathbb{Z}),$$

it suffices to bound the set S of principal **G**-bundles P on C admitting a quasimap (P, u) of class β to $V /\!\!/ \mathbf{G}$.

Finally, a principal **G**-bundle on a nodal curve C is given by a principal bundle $\widetilde{P}$ on the normalization $\widetilde{C}$, together with identifications of the fibers $\widetilde{P}_x$ and $\widetilde{P}_y$ for each pair of preimages of a node. For each node, these identifications are parametrized by the group **G**. It follows that we may assume that the curve C is irreducible and nonsingular.

Hence Theorem 3.2.4 is a consequence of the following result:

Theorem 3.2.5. *Let $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\chi(\mathbf{G}), \mathbb{Z})$ and a smooth projective curve C be fixed. Let V be a vector space with an action of* **G** *via a representation* **G** $\longrightarrow GL(V)$ *with finite kernel and let $\theta \in \chi(\mathbf{G})$ be a character such that $V^s(\mathbf{G}, \theta) \neq \emptyset$. Then the family of principal* **G**-*bundles P on C of degree β such that the vector bundle $\mathcal{V}_P = P \times_{\mathbf{G}} V$ admits a section u which sends the generic point of C to $V^s(\mathbf{G}, \theta)$ is bounded.*

Proof. We first prove the Theorem under the additional assumption that **G** is *connected*, using an argument similar to the one employed by Behrend [18] to show boundedness for semistable principal **G**-bundles of fixed degree on a curve. Essentially the same proof also appeared later in the paper [24] of Holla and Narasimhan. It is based on two general auxiliary results which we state below.

Let **T** be a maximal torus in **G** and let **B** be a Borel subgroup of **G** with $\mathbf{T} \subset \mathbf{B}$. Let $\mathbf{B}_r$ be the unipotent radical of **B**. Note that the composition $\mathbf{T} \subset \mathbf{B} \to \mathbf{B}/\mathbf{B}_r$ provides an isomorphism $\mathbf{T} \cong \mathbf{B}/\mathbf{B}_r$; and $\chi(\mathbf{T}) = \chi(\mathbf{B})$ since the unipotent radical has only the trivial character.

Lemma 3.2.6. *Every principal* **G**-*bundle on a smooth projective curve admits a reduction to a principal* **B**-*bundle.*

Proof. This is a corollary of Springer's result that any principal **G**-bundle on a projective smooth curve C is locally trivial in the Zariski topology of C (see [25, Section 2.11]). $\square$

Lemma 3.2.7. *Let S be a set of principal* **G**-*bundles on C with chosen* **B**-*reductions P' of P for each $P \in S$ and consider the set of associated* $\mathbf{T} = \mathbf{B}/\mathbf{B}_r$-*bundles*

$$R := \{\overline{P'} = P'/\mathbf{B}_r : P \in S\}. \tag{3.2.2}$$

If R is bounded, then S is also bounded. Furthermore, R is bounded if the set of degrees

$$\{d_{\overline{P'}} : \chi(\mathbf{T}) \longrightarrow \mathbb{Z} : \overline{P'} \in R\} \tag{3.2.3}$$

is a finite set. (Recall that $d_{\overline{P'}}$ is defined by $d_{\overline{P'}}(\xi) = \deg(\overline{P'} \times_{\mathbf{T}} \mathbb{C}_\xi)$.)

Proof. See [24, Proposition 3.1] and [24, Lemma 3.3]. $\square$

Now let S be the set of principal **G**-bundles in Theorem 3.2.5 and for each $P \in S$ pick a **B**-reduction P'. This is possible by Lemma 3.2.6. The following Lemma is the main step in the proof.

Lemma 3.2.8. *There is a $\mathbb{Q}$-basis $\{\theta_i\}$ of $\chi(\mathbf{T}) \otimes \mathbb{Q}$ such that:*

(1) *For each i, and each P', the line bundle*

$$P' \times_{\mathbf{B}} \mathbb{C}_{\theta_i} = P'/\mathbf{B}_r \times_{\mathbf{T}} \mathbb{C}_{\theta_i}$$

on C has nonnegative degree.
(2) *$\theta = \sum a_i \theta_i$ for some nonnegative rational numbers a_i, where θ is considered as a character of* **T**.

Proof. There is an open neighborhood A of θ in $\chi(\mathbf{T}) \otimes \mathbb{Q}$ such that

$$V^s(\mathbf{T}, \theta) \subset V^{ss}(\mathbf{T}, \xi) \tag{3.2.4}$$

for every $\xi \in A$. Here $V^s(\mathbf{T}, \theta)$ (resp. $V^{ss}(\mathbf{T}, \xi)$) denotes the θ-stable (resp. ξ-semistable) locus of V for the action of **T** linearized by the given character; and a point $p \in V$ is said to be ξ-semistable if it is so for some positive integer ℓ making

$\ell\xi$ an integral character in $\chi(\mathbf{T})$. The existence of A can be seen by Mumford's Numerical Criterion for (semi)stability, in the form Proposition 2.5 in [21], as follows.

Decompose V as a direct sum of one-dimensional $\mathbf{T}$-eigenspaces with (not necessarily distinct) weights δ_j. By the Numerical Criterion, a point $p = (p_1, \ldots, p_{\dim V}) \in V$ is θ-stable if and only if $(a_1 p_1, \ldots, a_{\dim V} p_{\dim V})$ is θ-stable for every $a_i \in \mathbb{C}^*$. Hence it is enough to check the inclusion (3.2.4) only at the points p with coordinates $p_j = 0$ or 1. For such a point p which is θ-stable, consider the cone $\sigma(p)$ in $(\chi(\mathbf{T}) \otimes \mathbb{Q})^\vee$ consisting of λ such that $\langle \delta_j, \lambda \rangle \geq 0$ for every j with $p_j = 1$. If we let A be the intersection of the (finitely many) dual open cones $\sigma(p)^\vee$, then (3.2.4) follows immediately from the Numerical Criterion.

Via the homomorphism $\mathbf{G} \longrightarrow GL(V)$, the image of the Borel subgroup $\mathbf{B}$ is contained in a Borel subgroup of $GL(V)$. It follows that we have a decomposition

$$V = \bigoplus_{l=1}^{\dim V} V_l / V_{l-1} \tag{3.2.5}$$

such that V_l is a $\mathbf{B}$-invariant l-dimensional subspace of V. The two sides of (3.2.5) are isomorphic as $\mathbf{T}$-representation spaces. Hence from the $\mathbf{B}$-equivariant map

$$P' \to P' \times_{\mathbf{B}} \mathbf{G} = P \to V$$
$$p' \mapsto (p', e) \mapsto u(p', e)$$

given by a quasimap u we obtain a $\mathbf{T}$-equivariant map

$$P'/\mathbf{B}_r \to \bigoplus_{l=1}^{\dim V} V_l / V_{l-1}. \tag{3.2.6}$$

Via composition with (3.2.6), any $\mathbf{T}$-equivariant map

$$s : \bigoplus_{l=1}^{\dim V} V_l / V_{l-1} = V \to \mathbb{C}_\xi$$

induces a section $[s]$ of $P'/\mathbf{B}_r \times_{\mathbf{T}} \mathbb{C}_\xi$. Since $V^s(\mathbf{G}, \theta) \subset V^s(\mathbf{T}, \theta) \subset V^{ss}(\mathbf{T}, \xi)$ for $\xi \in A$, and u is a quasimap, this section is nonzero at the generic point of C. Hence $\deg(P'/\mathbf{B}_r \times_{\mathbf{T}} \mathbb{C}_\xi) \geq 0$ for every $\xi \in A$. It is clear that there is a basis $\{\theta_i\}$ of $\chi(\mathbf{T}) \otimes \mathbb{Q}$ contained in A and satisfying condition (2) of the Lemma. $\square$

The proof of Theorem 3.2.5 for connected $\mathbf{G}$ is now immediate. First, $\beta(\theta) \geq 0$ by Lemma 3.2.1. Next, by Lemma 3.2.8

$$\beta(\theta) = \deg(P \times_{\mathbf{G}} \mathbb{C}_\theta) = \deg(P' \times_{\mathbf{B}} \mathbb{C}_{\sum a_i \theta_i}) = \sum a_i \deg(P' \times_{\mathbf{B}} \mathbb{C}_{\theta_i}),$$

for every P. Since all a_i's are nonnegative, we see that

$$0 \leq \deg(P' \times_{\mathbf{B}} \mathbb{C}_{\theta_i}) = \deg(\overline{P'} \times_{\mathbf{T}} \mathbb{C}_{\theta_i})$$

is uniformly bounded above as P varies. Since the θ_i's form a basis of $\chi(\mathbf{T}) \otimes \mathbb{Q}$, it follows that the set of degrees in (3.2.3) is finite, so we conclude by Lemma 3.2.7.

We now consider the case of a general reductive group $\mathbf{G}$. Let $\mathbf{G}_\circ$ be the connected component of the identity $e \in \mathbf{G}$. It is a normal subgroup and $\mathbf{G}/\mathbf{G}_\circ$ is a finite group. Given a principal $\mathbf{G}$-bundle P on C we have a factorization

$$
\begin{array}{ccc}
P & \xrightarrow{\pi} & P/\mathbf{G}_\circ \\
\rho \searrow & & \swarrow q \\
& C &
\end{array}
$$

with $P \longrightarrow P/\mathbf{G}_\circ$ a principal $\mathbf{G}_\circ$-bundle and $P/\mathbf{G}_\circ \xrightarrow{q} C$ a principal $\mathbf{G}/\mathbf{G}_\circ$-bundle, so that $\widetilde{C} := P/\mathbf{G}_\circ$ is a nonsingular projective curve and q is a finite étale map of degree $m := |\mathbf{G}/\mathbf{G}_\circ|$. Note that there are only finitely many possibilities for $\widetilde{C}$ when P varies, since there are only finitely many principal $\mathbf{G}/\mathbf{G}_\circ$-bundles on C.

Let $\widetilde{\beta} : \chi(\mathbf{G}_\circ) \to \mathbb{Z}$ be the degree of the $\mathbf{G}_\circ$-principal bundle $P \longrightarrow \widetilde{C}$. Then $\beta(\theta) = m\widetilde{\beta}(\theta)$, where we view θ as a character of $\mathbf{G}_\circ$ by restriction, so $\widetilde{\beta}(\theta)$ is fixed, independent on the principal $\mathbf{G}$-bundle P of degree β.

Finally, any $\mathbf{G}$-equivariant map $P \longrightarrow V$ is also $\mathbf{G}_\circ$-equivariant; furthermore

$$V^{ss}(\mathbf{G}, \theta) = V^{ss}(\mathbf{G}_\circ, \theta) \quad \text{and} \quad V^s(\mathbf{G}, \theta) = V^s(\mathbf{G}_\circ, \theta)$$

by [26, Prop 1.15]. Hence if $P \times_{\mathbf{G}} V \longrightarrow C$ admits a section u as in the Theorem, so does $P \times_{\mathbf{G}_\circ} V \longrightarrow \widetilde{C}$. Noting that the proof in the case of connected group only used that $\beta(\theta)$ was fixed, we conclude from that case that the sub-family of P's with fixed $\widetilde{C}$ is bounded, so we are done by virtue of the finiteness of the set of $\widetilde{C}$'s. $\square$

For later use in Section 5 we record the following immediate consequence of Theorem 3.2.5.

Corollary 3.2.9. *Let*

$$(\mathcal{C} \longrightarrow S, p_1, \dots, p_k : S \longrightarrow \mathcal{C}, \mathcal{P}, u)$$

be a family of quasimaps of genus g and class β to $V /\!\!/ \mathbf{G}$ with S a scheme of finite type and let $\mathcal{O}_{\mathcal{C}}(1)$ be an S-relatively ample line bundle on $\mathcal{C}$. There exists an integer $m >> 0$, depending on k, g, and β, but <u>independent</u> on $\mathcal{P}$ and u, such that the restriction to every geometric fiber of the vector bundle $\mathcal{V}_{\mathcal{P}}(m) = (\mathcal{P} \times_{\mathbf{G}} V) \otimes \mathcal{O}_{\mathcal{C}}(m)$ on $\mathcal{C}$ has vanishing higher cohomology.

Remark 3.2.10. As we already observed, the proof of Theorem 3.2.5 only required that $\beta(L_\theta)$ is fixed. The same is true about the proof of Corollary 3.2.3. It follows from the results of this subsection that for fixed $d \in \mathbb{Z}_{\geq 0}$ there are only finitely many $\beta \in \text{Eff}(W, \mathbf{G}, \theta)$ with $\beta(L_\theta) = d$.

4. Moduli of stable quasimaps to $W /\!\!/ \mathbf{G}$

4.1. The moduli stack

Definition 4.1.1. Let k, g be fixed with $2g - 2 + k \geq 0$, and let β be a L_θ-effective class. The moduli stack

$$\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$$

of k-pointed quasimaps of genus g and class β to $W /\!\!/ \mathbf{G}$ is the stack parametrizing families as in Definition 3.1.6.

Strictly speaking, we only have a prestack, but it is straightforward to check the sheaf condition.
There are natural forgetful morphisms of stacks

$$\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \xrightarrow{\mu} \mathfrak{B}un_{\mathbf{G}} \quad \text{and} \quad \text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \xrightarrow{\nu} \mathfrak{M}_{g,k}.$$

Theorem 4.1.2. *The moduli stack $\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is a Deligne–Mumford stack of finite type, with a canonical relative obstruction theory over $\mathfrak{B}un_{\mathbf{G}}$. If W has only lci singularities, then the obstruction theory is perfect. If $W /\!\!/ \mathbf{G}$ is projective, then $\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is proper over $\text{Spec}(\mathbb{C})$. In general, it has a natural proper morphism*

$$\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \text{Spec}(A(W)^{\mathbf{G}}).$$

We will split the proof of the Theorem over several subsections below.

4.2. The moduli stack is Deligne–Mumford and of finite type

Consider the forgetful morphisms

$$\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \xrightarrow{\mu} \mathfrak{B}un_{\mathbf{G}} \quad \text{and} \quad \text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \xrightarrow{\nu} \mathfrak{M}_{g,k}.$$

By a standard argument it follows from Corollary 3.2.3 that the map ν factors through an open substack of finite type $\mathfrak{S} \subset \mathfrak{M}_{g,k}$. Further, by Theorem 3.2.5 there is an open substack of finite type $\mathfrak{B}un_{\mathbf{G},\beta} \subset \mathfrak{B}un_{\mathbf{G}}$, fitting in a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{B}un_{\mathbf{G},\beta} & \subset & \mathfrak{B}un_{\mathbf{G}} \\
\downarrow & & \downarrow \\
\mathfrak{S} & \subset & \mathfrak{M}_{g,k}
\end{array}
$$

and such that μ factors through $\mathfrak{B}un_{\mathbf{G},\beta}$. This is again standard for $\mathbf{G} = GL_n$: the stack $\mathfrak{S} \times_{\mathfrak{M}_{g,k}} \mathfrak{B}un_{\mathbf{G}}$ has a smooth surjective cover $\mathcal{U}$ which is an infinite disjoint union of quot schemes and Theorem 3.2.5 implies that μ factors through the image of only finitely many such quot schemes; we take $\mathfrak{B}un_{\mathbf{G},\beta}$ to be this image. The case general of a general group $\mathbf{G}$ is reduced to GL_n as in the proof of Proposition 2.1.1.

Claim: The morphism μ is representable and of finite type. More precisely, its fibers are open subschemes in appropriate Hilbert schemes.

Granting the claim, it follows that $\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is an Artin stack of finite type. By stability, the automorphism group of a stable quasimap is finite and reduced, so we get in fact a DM stack.

Proof of the Claim. Let $\mathfrak{C} \xrightarrow{\pi} \mathfrak{B}un_{\mathbf{G},\beta}$ be the universal curve, with universal sections $\mathfrak{p}_i$, $i = 1, \dots, k$, and let $\mathfrak{P}$ be the universal principal $\mathbf{G}$-bundle over $\mathfrak{C}$. If $\omega_{\mathfrak{C}}$ denotes the relative dualizing sheaf, then we have line bundles

$$\mathfrak{L} = \mathfrak{L}_\theta = \mathfrak{P} \times_{\mathbf{G}} \mathbb{C}_\theta \quad \text{and} \quad \omega_{\mathfrak{C}}\left(\sum \mathfrak{p}_i\right) \otimes \mathfrak{L}^{\otimes \epsilon}, \quad \epsilon \in \mathbb{Q}$$

on $\mathfrak{C}$. There is an open substack $\mathfrak{B}un^\circ_{\mathbf{G},\beta} \subset \mathfrak{B}un_{\mathbf{G},\beta}$ obtained by requiring that $\omega_\mathfrak{C}(\sum \mathfrak{p}_i) \otimes \mathcal{L}^{\otimes \epsilon}$ is π-relatively ample for some positive ϵ. We will continue to denote by $\mathfrak{C} \xrightarrow{\pi} \mathfrak{B}un^\circ_{\mathbf{G},\beta}$ and $\mathfrak{P}$ the universal curve and universal principal $\mathbf{G}$ bundle restricted to this open substack. Note that π is now a *projective* morphism since π is of finite type and $\mathfrak{B}un^\circ_{\mathbf{G},\beta}$ is quasicompact.

Forming the induced bundle $\mathfrak{P} \times_{\mathbf{G}} W$ with fiber W, we have a commutative diagram of projections

$$
\begin{array}{ccc}
\mathfrak{P} \times_{\mathbf{G}} W & \xrightarrow{\varrho} & \mathfrak{C} \\
q \searrow & & \swarrow \pi \\
& \mathfrak{B}un^\circ_{\mathbf{G},\beta} &
\end{array}
$$

The fiber of q over a $\mathbb{C}$-point $(C, p_1, \ldots, p_k, P)$ of $\mathfrak{B}un^\circ_{\mathbf{G},\beta}$ is $P \times_{\mathbf{G}} W$.

Let $\overline{W}$ be a projective $\mathbf{G}$-equivariant completion of W. For each scheme T with a morphism

$$T \longrightarrow \mathfrak{B}un^\circ_{\mathbf{G},\beta},$$

consider the pulled-back universal curve $\mathcal{C}_T \xrightarrow{\pi} T$ and universal bundle P_T on $\mathcal{C}_T$. A section u of

$$P_T \times_{\mathbf{G}} \overline{W} \xrightarrow{\varrho} \mathcal{C}_T$$

can be identified with its image inside $P_T \times_{\mathbf{G}} \overline{W}$. It is easily checked that the Hilbert polynomial of the image is determined by g and β, hence u corresponds to a point in the (relative) Hilbert scheme

$$\mathrm{Hilb}_{\beta,g}(P_T \times_{\mathbf{G}} \overline{W}/T) \longrightarrow T.$$

In the terminology of [27, Section 14], associating this Hilbert scheme to each $T \longrightarrow \mathfrak{B}un_{\mathbf{G},\beta}$ gives a *schematic local construction of finite type*, so we have an Artin stack with a schematic morphism of finite type

$$\mathfrak{H}ilb_{\beta,g}(\mathfrak{P} \times_{\mathbf{G}} \overline{W}/\mathfrak{B}un^\circ_{\mathbf{G},\beta}) \xrightarrow{\mu} \mathfrak{B}un^\circ_{\mathbf{G},\beta},$$

whose points parametrize closed subschemes Y (with Hilbert polynomials specified as above) in the fibers of the projection

$$q : \mathfrak{P} \times_{\mathbf{G}} \overline{W} \longrightarrow \mathfrak{B}un^\circ_{\mathbf{G},\beta}.$$

We now define a substack $\mathcal{H}$ of $\mathfrak{H}ilb_{\beta,g}(\mathfrak{P} \times_{\mathbf{G}} \overline{W}/\mathfrak{B}un^\circ_{\mathbf{G},\beta})$ by successively imposing the following requirements:

(1) the subscheme Y lies inside $P \times_{\mathbf{G}} W$
(2) Y is the image of a section $u : C \longrightarrow P \times_{\mathbf{G}} W$
(3) the section u maps the generic points of components of C, the nodes, and the markings into $P \times_{\mathbf{G}} W^s$
(4) stability holds, i.e., $\omega_C(\sum p_i) \otimes \mathcal{L}^{\otimes \epsilon}$ is ample for *every* $\epsilon > 0$.

Since each of these is an open condition, $\mathcal{H}$ is an open substack, which is obviously identified with $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$. The claim is proved.

Remark 4.2.1. Let $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ denote the moduli stack parametrizing maps of class β from prestable k-pointed curve of genus g to the quotient stack $[W/\mathbf{G}]$. Using the representability of the Hilbert functor in algebraic spaces, the argument above shows that the natural morphism

$$\mu : \mathfrak{M}_{g,k}([W/\mathbf{G}], \beta) \longrightarrow \mathfrak{B}un_{\mathbf{G}}$$

is representable and of finite type. Hence the stack $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ is algebraic and locally of finite type. (As for $\mathfrak{B}un_{\mathbf{G}}$, this conclusion also follows immediately from [16, Prop. 2.18] and [17, Lemma C.5].) Theorem 3.2.4 then shows that imposing the quasimap condition (Definition 3.1.1) determines an open substack of $\mathfrak{M}_{g,k}([W/\mathbf{G}], \beta)$ which is of finite type over $\mathfrak{M}_{g,k}$. Finally, imposing in addition a stability condition on quasimaps, such as the one in Definition 3.1.2, gives an open substack which is Deligne–Mumford and of finite type.

4.3. Properness

Proposition 4.3.1. *The stack* $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ *is proper over the affine quotient* $\mathrm{Spec}(A(W)^{\mathbf{G}})$.

Proof. Assume first for simplicity that $A(W)^{\mathbf{G}} = \mathbb{C}$, so that $W /\!\!/ \mathbf{G}$ is projective. The proof that $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is separated follows from the valuative criterion by an argument identical to the one for the toric case, given in Section 4.1 of [5] (in turn, that argument is a straightforward modification of the one for stable quotients in [4]). For the more general case here, one needs the statement that if two principal $\mathbf{G}$-bundles on a regular complex surface S agree outside finitely many points, then they agree on S. This follows from Hartogs' theorem. We should also point out that the "prestable" assumption that the base-points of stable quasimaps are away from nodes and markings is crucially used in the proof.

It remains to show completeness. For this we use the valuative criterion and standard semistable reduction techniques. Let $(\Delta, 0)$ be a smooth curve with a point 0 and let $\Delta^\circ = \Delta \setminus \{0\}$. Let $((C, p_i), P, u)$ be a Δ°-family of stable quasimaps. It is enough to show that, possibly after replacing $(\Delta, 0)$ by making a base-change ramified over 0, there is a stable quasimap extension of $((C, p_i), P, u)$ over Δ. Let B denote the base locus of this family with reduced scheme structure. As in [28,4,5], after possibly shrinking Δ and making an étale base change, we may regard B as additional sections y_j. Then $(C, p_i, y_j) \longrightarrow \Delta^\circ$ is a family of prestable curves. Further, we may assume that $C \longrightarrow \Delta^\circ$ has smooth irreducible fibers and that the total space is a smooth surface.

The given family of quasimaps $((C, p_i), P, u)$ gives a rational map

$$[u] : C -- \to W /\!\!/ G,$$

which is regular on $C \setminus B$. Since $W /\!\!/ G$ is projective, after shrinking Δ if necessary, it extends to a regular map, denoted by $[u_{reg}]$, on all of C. Stability of the quasimaps implies immediately that $((C, p_i, y_j), [u_{reg}])$ is a Δ°-family of *stable maps* to $W /\!\!/ G$. We may view it also as a family of quasimaps, and as such it has a class $\beta_{reg} \in \mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z})$; the image of β_{reg} in $\mathrm{Hom}(\mathrm{Pic}(W /\!\!/ \mathbf{G}), \mathbb{Z})$ is the element determined by the usual homology class of the stable map. In general β and β_{reg} are different. In fact, by Lemma 7.1.2 from Section 7 below, $\beta - \beta_{reg}$ is L_θ-effective and it is nonzero if the base locus is nonempty.

By the properness of the stable map functor, there is a family of stable maps

$$(\widehat{C}, p_i, y_j) \longrightarrow \Delta, \quad [\widehat{u}] : \widehat{C} \longrightarrow W /\!\!/ \mathbf{G}$$

extending $((C, p_i, y_j), [u_{reg}])$. The surface $\widehat{C}$ is normal and may have singularities only at nodes in the central fiber.

Pulling back via $[\widehat{u}]$ the principal $\mathbf{G}$-bundle $W^s \longrightarrow W /\!\!/ \mathbf{G}$, we obtain a pair

$$(\widehat{P}, \widehat{u}) \tag{4.3.1}$$

consisting of a principal $\mathbf{G}$-bundle $\widehat{P}$ on $\widehat{C}$ and an induced section $\widehat{u} : \widehat{C} \longrightarrow \widehat{P} \times_{\mathbf{G}} W$.

The central fiber $\widehat{C}_0$ of $\widehat{C}$ may contain "rational tails" with respect to the original set of markings. Specifically, consider all maximal trees $\Gamma_1, \ldots, \Gamma_N$ of rational curves in $\widehat{C}_0$ such that each Γ_l contains none of the markings p_i and meets the rest of the curve $(\widehat{C}_0 \setminus \Gamma_l)$ in a single point z_l. Each of these trees can be contracted by a succession of blow-downs of (-1)-curves.

Denote by $\bar{C}$ the surface obtained by contracting the trees $\Gamma_1, \ldots, \Gamma_N$. It is a Δ-family of prestable curves with markings p_i, while some of the additional sections y_j may come together at points z_l in the central fiber where contracted trees were attached. Note that each of $z_1, \ldots, z_N$ is a nonsingular point of $\bar{C}$. The pair $(\widehat{P}, \widehat{u})$ from (4.3.1) restricts to a pair (P_1, u_1) on the open subset

$$U_1 := \bar{C} \setminus \left((\cup_j y_j) \cup \{z_1, \ldots, z_N\} \right).$$

On the other hand, we have the original pair (P, u) on the open subset

$$U_2 := \bar{C} \setminus \bar{C}_0 \cong C.$$

By construction, (P_1, u_1) and (P, u) agree on the intersection $U_1 \cap U_2$, hence they glue together to give a pair $(\bar{P}^0, \bar{u}^0)$ defined on all of $\bar{C}$, except at the points z_l where rational tails were contracted, and at the intersections of the sections y_j with the central fiber. These points are all away from the nodes and markings in the central fiber.

Now, Lemma 4.3.2 below gives an extension $\bar{P}$ of $\bar{P}^0$ to $\bar{C}$. Furthermore, $\bar{u}^0$ is a section of $\bar{P} \times_G W$ defined over $\bar{C}$ except at finitely many nonsingular points in the central fiber. By Hartogs' theorem, we can extend the section as well to a section $\bar{u}$ defined on all of $\bar{C}$. This is the key place where W affine is important for the argument.

We have constructed a family of prestable quasimaps

$$((\bar{C}, p_i), \bar{P}, \bar{u}).$$

The limit quasimap will have base-points precisely at the points z_l where rational tails were contracted, and at the intersections of the sections y_j with the central fiber. This is because the rational tails in the stable map limit $((\widehat{C}_0, p_i, y_j), [\widehat{u}]|_{\widehat{C}_0})$ all carried either a nonzero part of β_{reg}, or some of the points y_j (or both), while each of the points y_j correspond to a nonzero part of the difference $\beta - \beta_{reg}$. (For a more precise analysis of the degrees carried by the base points, see the proof of Theorem 7.1.6 in Section 7.) It is now immediate to see that the limit is in fact stable. Indeed, we need to check that the line bundle $\mathcal{L}_\theta$ is nontrivial on rational components with only two special points (nodes or markings p_i) in the central fiber. This is clear if the quasimap has a base-point on the component. However, if such a component does not contain any of the points y_j and z_l, then by construction the restriction of the quasimap to it is a stable map to $W /\!\!/ \mathbf{G}$, hence non-constant, and $\mathcal{L}_\theta$ is the pullback of the polarization on $W /\!\!/ \mathbf{G}$. From the valuative criterion we conclude that $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is proper.

Assume now that we are in the general situation when we have a projective morphism

$$W /\!\!/ \mathbf{G} \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}}).$$

The morphism from $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ to the affine quotient is obtained as follows: First of all, it is well-known that

$$q : W \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}})$$

is a categorical quotient for the **G**-action on W. Therefore, for every quasimap

$$(C, P, \tilde{u} : P \longrightarrow W)$$

the **G**-equivariant morphism $\tilde{u}$ induces a map

$$C \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}}),$$

which must be constant, since C is projective and $\mathrm{Spec}(A(W)^{\mathbf{G}})$ is affine.

More generally, the universal family

$$\left(\mathcal{C} \longrightarrow \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta), \mathfrak{P}, \tilde{u} : \mathfrak{P} \longrightarrow W\right)$$

induces a map from the universal curve $\mathcal{C}$ to the affine quotient. By the previous paragraph, this map is constant on the fibers of the projection $\pi : \mathcal{C} \longrightarrow \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$. Hence it descends to our desired morphism

$$\eta : \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}}).$$

The properness of η now follows from the valuative criterion, by the relative version of argument given above in the absolute case. □

Lemma 4.3.2. *Let S be an algebraic surface (irreducible and reduced) and let P^0 be a principal **G**-bundle on S minus a finite set of nonsingular points. Then P^0 extends to a principal **G**-bundle on S.*

Proof. Replacing S by its nonsingular locus if necessary, we may assume that it is nonsingular. Let U denote the complement of a finite set over which P^0 is defined.

In the case $\mathbf{G} = GL(n)$, we can replace P^0 with a rank n vector bundle E^0 over U. It suffices to show that this extends to S. There exists a rank n coherent sheaf E on S extending E^0; by replacing E with its double-dual, we can assume E is reflexive. Since S is two-dimensional and regular, this forces E to be locally free, giving the extension of P^0.

In the case of general **G**, we choose some embedding $\mathbf{G} \subset GL(n)$. Let $Q^0 = P^0 \times_{\mathbf{G}} GL(n)$ be the associated $GL(n)$-bundle over U. By the last paragraph, Q^0 extends to a $GL(n)$-bundle Q over all of S. Reductions of Q to a principal **G**-bundle correspond to sections of the associated $GL(n)/\mathbf{G}$-bundle $Q/\mathbf{G}$ over S. Since **G** is reductive, it follows from Matsushima's criterion [29] that the homogeneous space $GL(n)/\mathbf{G}$ is affine. Moreover, by assumption, $Q/\mathbf{G}$ admits a section over U. Thus, by Hartogs' theorem, this section extends over S. □

Remark 4.3.3. Note that the statement of the lemma is false if **G** is not reductive: for any linear algebraic group with nontrivial unipotent radical, there exist principal **G**-bundles over $U = \mathbb{C}^2 \setminus \{0\}$ which do not extend to the entire plane. Indeed, for $\mathbf{G} = \mathbf{G}_a$, principal bundles over U are classified by $H^1(U, \mathcal{O}_U) \neq 0$. Since any unipotent **G** is built from $\mathbf{G}_a$ by taking successive extensions, the same calculation yields the statement for unipotent groups. Finally if **G** has nontrivial unipotent radical **U**, then the long exact sequence for group cohomology shows that the natural map $H^1(U, \mathbf{U}) \to H^1(U, \mathbf{G})$ is injective and any elements in its image cannot be extended to the plane.

4.4. Obstruction theory

Let

$$\mathcal{C} \xrightarrow{\pi} \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$$

be the universal curve, with universal principal **G**-bundle $\mathfrak{P}$ on it. ($\mathfrak{P}$ is the pull-back of the universal bundle on the universal curve over $\mathfrak{Bun}_{\mathbf{G}}$.) Let

$$\varrho : \mathfrak{P} \times_{\mathbf{G}} W \longrightarrow \mathcal{C}$$

be the induced bundle with fiber W and let

$$u : \mathcal{C} \longrightarrow \mathfrak{P} \times_{\mathbf{G}} W$$

be the universal section.

Recall from the construction in Section 4.2 that the fibers of the map

$$\mu : \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathfrak{Bun}_{\mathbf{G}}$$

are open in Hilbert schemes, with points corresponding to graphs of the sections u. It follows that there is a *canonical* relative obstruction theory for $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ over $\mathfrak{Bun}_{\mathbf{G}}$ given by

$$E_\mu^\bullet := (R^\bullet \pi_*(\mathcal{H}om(\mathbb{L}_u, \mathcal{O}_{\mathcal{C}})[1]))^\vee \tag{4.4.1}$$

where $\mathbb{L}_u$ is the relative cotangent complex of u.

Proposition 4.4.1. *If W is smooth, then the relative obstruction theory over $\mathfrak{Bun}_{\mathbf{G}}$ is perfect.*

Proof. When W is smooth the morphism u is an lci closed embedding, hence its cotangent complex is quasi-isomorphic to a single locally free sheaf in degree -1 (the conormal bundle of the section). Since π is projective of relative dimension 1, the proposition follows. □

4.5. Perfect obstruction theory when W has lci singularities

Note that since u is a section of ϱ, we have

$$\mathbb{L}_u \cong u^* \mathbb{L}_\varrho[1],$$

hence we can rewrite the relative obstruction theory as

$$\left(R^\bullet \pi_*(u^* \mathbb{R}T_\varrho)\right)^\vee,$$

with $\mathbb{R}T_\varrho$ the relative tangent complex of ϱ.

Assume now that W has lci singularities only (contained in $W \setminus W^s$). Choose a nonsingular **G**-variety V and a **G**-equivariant embedding $W \hookrightarrow V$ corresponding to an ideal sheaf $I \subset \mathcal{O}_V$. For example, we can take V to be a vector space, as in Proposition 2.5.2. The tangent complex of W is then the two-term complex of vector bundles

$$\mathbb{R}T_W = \left[T_V|_W \longrightarrow (I/I^2)^\vee\right]$$

up to quasi-isomorphism.

We have an induced complex of vector bundles

$$\left[\mathfrak{P} \times_\mathbf{G} T_V|_W \longrightarrow \mathfrak{P} \times_\mathbf{G} (I/I^2)^\vee\right]$$

on $\mathfrak{P} \times_\mathbf{G} W$ which is quasi-isomorphic to $\mathbb{R}T_\varrho$. Denote by

$$F^\bullet = [F^0 \longrightarrow F^1]$$

its pull-back to $\mathcal{C}$ via u.

Lemma 4.5.1. *Let C be a geometric fiber of $\mathcal{C} \xrightarrow{\pi} \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$. Then $H^1(F^\bullet|_C)$ is a torsion sheaf on C.*

Proof. Let $B \subset C$ be the set of base points of the stable quasimap. On $C \setminus B$ the complex $F^\bullet$ is quasi-isomorphic to the pull-back of the tangent bundle of the (smooth) stable locus W^s via $u|_{C \setminus B}$. $\square$

Theorem 4.5.2. *Under the assumption that W has at most lci singularities, the relative obstruction theory $\left(R^\bullet \pi_*(u^* \mathbb{R}T_\varrho)\right)^\vee$ is perfect.*

Proof. We need to check that $R^\bullet \pi_*(F^\bullet)$ is perfect, of amplitude in $[0, 1]$.

Using the Lemma, a spectral sequence computation shows that the second hypercohomology group

$$\mathbb{H}^2(F^\bullet|_C)$$

vanishes. The assertion now follows by standard arguments. $\square$

Remark 4.5.3. Since $\mathfrak{B}un_\mathbf{G}$ is a smooth Artin stack, it follows that there is an absolute perfect obstruction theory on $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$,

$$E^\bullet = Cone(f)[-1],$$

the shifted cone of the composite morphism

$$f : E^\bullet_\mu \longrightarrow \mathbb{L}_\mu \longrightarrow \mu^* \mathbb{L}_{\mathfrak{B}un_\mathbf{G}}[1].$$

Furthermore, since $\phi : \mathfrak{B}un_\mathbf{G} \longrightarrow \mathfrak{M}_{g,k}$ is a smooth map between smooth Artin stacks (see Section 2.1), we also have an induced perfect obstruction theory $E^\bullet_\nu$ relative to

$$\nu : \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathfrak{M}_{g,k},$$

fitting in a distinguished triangle

$$E^\bullet_\nu \longrightarrow E^\bullet_\mu \longrightarrow \mu^* \mathbb{L}_\phi[1].$$

One sees immediately that the absolute obstruction theories corresponding to $E^\bullet_\mu$ and $E^\bullet_\nu$ coincide. It is a well-known fact (see, e.g. [30, Proposition 3]) that in this case all three perfect obstruction theories determine the same virtual fundamental class $[\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)]^{vir}$.

Remark 4.5.4. The arguments in this section show that any open DM substack of the Artin stack of quasimaps to $W /\!\!/ \mathbf{G}$ obtained by imposing a stability condition has the canonical relative obstruction theory (4.4.1) over $\mathfrak{Bun}_\mathbf{G}$. For example, if the stability condition imposed is that the image of u is contained in the stable locus W^s (i.e., there are *no* base points) and that the line bundle

$$\omega_C \left(\sum_{i=1}^k p_i \right) \otimes \mathcal{L}_\theta^{\otimes 3}$$

is ample, then we obtain the Kontsevich moduli stack $\overline{M}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ of stable maps to $W /\!\!/ \mathbf{G}$. The proof of Proposition 4.4.1 shows that the obstruction theory is perfect in this case, irrespective of the nature of singularities of $W \setminus W^s$. The induced relative obstruction theory over $\mathfrak{M}_{g,k}$ is the usual obstruction theory of stable map spaces, so the virtual classes are again the same.

In fact, as we discuss in Section 7 below there is a one-parameter family of stability conditions interpolating between stable maps and stable quasimaps, giving DM moduli stacks with perfect obstruction theories relative to $\mathfrak{Bun}_\mathbf{G}$ given by (4.4.1).

In particular, all these Deligne–Mumford stacks, together with their canonical perfect obstruction theories, will agree over the open substacks parametrizing honest stable maps with *semistable* underlying curve.

Example 4.5.5. The typical context where the results of this subsection apply is a generalization of the case (suggested in [4]) of complete intersections in $\mathbb{P}^n$.

Namely, let V be a vector space with $\mathbf{G}$-action linearized by a character θ satisfying the assumptions in Section 2.2. Let E be a linear $\mathbf{G}$-representation and consider the $\mathbf{G}$-equivariant vector bundle $V \times E$ on V. It induces a vector bundle

$$\overline{E} := V^s \times_\mathbf{G} E$$

on $Y := V /\!\!/ \mathbf{G}$. Let $t \in \Gamma(V, V \times E)^\mathbf{G}$ be a regular $\mathbf{G}$-invariant section and let

$$W := Z(t) \subset V$$

be its zero locus, which we assume to be not entirely contained in the unstable locus in V. The section t descends to a section $\overline{t}$ of $\overline{E}$ on $Y = V /\!\!/ \mathbf{G}$, whose zero locus

$$X := Z(\overline{t}) \subset Y$$

is identified with $W /\!\!/ \mathbf{G}$. We assume that X is nonsingular (or, equivalently, that $W^s = W \cap V^s$ is nonsingular).

Since (after forgetting the $\mathbf{G}$-action) t is identified with a rank(E)-tuple of functions on V, we deduce that W is an affine complete intersection subscheme of V and therefore we have the moduli stacks $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ carrying canonical perfect obstruction theories.

In particular, if $Y = \mathbb{P}^n = \mathbb{C}^{n+1} /\!\!/ \mathbb{C}^*$, the above construction gives precisely the nonsingular complete intersections $X \subset \mathbb{P}^n$, with W the affine cone over X.

Note, however, that for an arbitrary nonsingular projective subvariety $X \subset \mathbb{P}^n$ the affine cone W may have a worse than lci singularity at the origin, and the obstruction theory of $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbb{C}^*, \beta)$ will *not* be perfect in general.

Marian, Oprea, and Pandharipande introduced a "moduli space of stable quotients on a projective subvariety $X \subset \mathbb{P}^n$" as the closed substack of $\mathrm{Qmap}_{g,k}(\mathbb{C}^{n+1} /\!\!/ \mathbb{C}^*, d)$ given by the ideal sheaf induced by the equations of X in $\mathbb{P}^n$ (see Section 10.1 of [4]). They also suggested there that this moduli space carries a natural perfect obstruction theory when X is a complete intersection and asked if this holds for arbitrary X. The above discussion answers their question. Indeed, it is easy to see that their moduli space is the disjoint union of $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbb{C}^*, \beta)$ over all curve classes β which are sent to d via the map

$$\mathrm{Hom}(\mathrm{Pic}^{\mathbb{C}^*}(W), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{Pic}^{\mathbb{C}^*}(V), \mathbb{Z}) \cong \mathbb{Z}.$$

4.6. Dependence on GIT presentation

Given a reductive group $\mathbf{G}$, a quasi-projective scheme W equipped with a $\mathbf{G}$-action, and a $\mathbf{G}$-linearized ample line bundle L, we can consider the pair of Artin stacks

$$\mathcal{A} = [W/\mathbf{G}], \qquad \mathcal{A}^{ss} = [W^{ss}/G],$$

where $[W/\mathbf{G}]$ denotes the stack quotient and W^{ss} denotes the locus of semistable point in W with respect to L. We say that the data $(\mathbf{G}, W, L)$ is a GIT presentation for the pair $(\mathcal{A}, \mathcal{A}^{ss})$.

One can ask to what extent the moduli stack of quasimaps depends on the choice of GIT presentation. It is easy to see that if one presentation satisfies the conditions stated at the beginning of this section, so does any other. Also, the space of curve classes is simply the dual of $\mathrm{Pic}(\mathcal{A})$ so is independent of presentation. The following proposition is then immediate from the results of this section.

Proposition 4.6.1. *The moduli stack* $\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ *and its perfect obstruction theory only depend on the underlying pair* $(\mathcal{A}, \mathcal{A}^{ss})$.

Notice that this is weaker than claiming that the theory of quasimaps depends on either the GIT quotient or the stack quotient alone. Given an Artin stack $\mathcal{A}$ with reductive stabilizers, an approach to defining Gromov–Witten invariants intrinsically to $\mathcal{A}$ has been initiated in [6]. It would be interesting to compare their construction with ours.

5. Quasimaps to quotients of vector spaces

In this section we discuss an alternative construction of the stable quasimaps moduli in the important case when $W = V$ is a vector space. This is a situation general enough to include many interesting instances of GIT quotient targets, such as toric varieties and flag manifolds of type *A*. One reason this approach could be potentially useful is that it comes automatically with an easy and concrete description of the virtual fundamental class: the moduli space is realized as the zero locus of a section of a vector bundle on an appropriate *nonsingular* Deligne–Mumford stack and the virtual class is simply given by the Fulton–MacPherson refined top Chern class of this vector bundle.

The construction presented here is a straightforward extension of the one used in [5] for the moduli of stable toric quasimaps. Note also that it recovers the moduli of stable quotients with its virtual class by an approach different than that of [4].

5.1. A generalized Euler sequence

Recall our setup. Let V be a finite dimensional $\mathbb{C}$-vector space equipped with a $\mathbf{G}$-action via a representation $\mathbf{G} \longrightarrow \text{GL}(V)$. We fix a linearization $\theta \in \chi(\mathbf{G}) = \text{Pic}^{\mathbf{G}}(V)$ satisfying

(i) $\emptyset \neq V^s = V^{ss}$.
(ii) $\mathbf{G}$ acts freely on V^s.

Let $\rho : V^s \longrightarrow V /\!\!/ \mathbf{G} = V^s/\mathbf{G}$ be the projection.

It is useful to note the following presentation of the tangent bundle of $V /\!\!/ \mathbf{G}$. Namely, the infinitesimal action of $\mathbf{G}$ on V^s induces an exact sequence of vector bundles

$$0 \longrightarrow V^s \times \mathfrak{g} \longrightarrow TV^s = V^s \times V \longrightarrow \rho^* T(V /\!\!/ \mathbf{G}) \longrightarrow 0$$

where $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$. This sequence is $\mathbf{G}$-equivariant with respect to the adjoint action on $\mathfrak{g}$, so descends to an exact sequence

$$0 \longrightarrow V^s \times_{\mathbf{G}} \mathfrak{g} \longrightarrow V^s \times_{\mathbf{G}} V \longrightarrow T_{V /\!\!/ \mathbf{G}} \longrightarrow 0. \tag{5.1.1}$$

This construction generalizes the Euler sequence presentation of the tangent bundle of projective space.

5.2. Quasimaps from curves to $V /\!\!/ \mathbf{G}$

To give a quasimap of class β to $V /\!\!/ \mathbf{G}$ on a curve C is the same as giving a principal $\mathbf{G}$-bundle P on C of degree $\beta_P = \beta$ and a global section

$$u \in \Gamma(C, \mathcal{V}_P)$$

of the induced vector bundle $\mathcal{V}_P := P \times_{\mathbf{G}} V$, satisfying the generic nondegeneracy condition (3.1.1).

Let k, g be fixed with $2g - 2 + k \geq 0$, and let β be a L_θ-effective class. We have the moduli stack

$$\text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$$

of k-pointed, genus g stable quasimaps of class β to $V /\!\!/ \mathbf{G}$.

Theorem 5.2.1. *The moduli space* $\text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$ *is a finite type Deligne–Mumford stack, proper over the affine quotient* $V /_{\text{aff}} \mathbf{G} = \text{Spec}(\text{Sym}(V^\vee)^{\mathbf{G}})$, *and it carries a perfect obstruction theory.*

Proof. Given the boundedness results we established in Section 3.2, the argument is almost identical to the one given in [5] for the special case of toric varieties.

There are forgetful morphisms of stacks making a commutative diagram

$$
\begin{array}{ccc}
\text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta) & = & \text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta) \\
\mu \downarrow & & \downarrow \nu \\
\mathfrak{Bun}_{\mathbf{G}} & \longrightarrow & \mathfrak{M}_{g,k}.
\end{array}
$$

By Corollary 3.2.3, the image of ν is contained in an open and closed substack of finite type $\mathfrak{S} \subset \mathfrak{M}_{g,k}$. By Corollary 3.2.3 and Theorem 3.2.5, the image of μ is contained in an open and closed substack of finite type $\mathfrak{B}un_{\mathbf{G},\beta} \subset \mathfrak{B}un_{\mathbf{G}}$, lying over $\mathfrak{S}$.

On the universal curve $\mathfrak{C} \xrightarrow{\pi} \mathfrak{B}un_{\mathbf{G},\beta}$ we have the universal principal bundle $\mathfrak{P}$ and the induced universal vector bundle $\mathcal{V}_{\mathfrak{P}} = \mathfrak{P} \times_{\mathbf{G}} V$. As in Section 4.2, after replacing $\mathfrak{B}un_{\mathbf{G},\beta}$ by an open substack, we may assume that π is a projective morphism. Let $\mathcal{O}_{\mathfrak{C}}(1)$ be a π-relatively very ample line bundle and fix a global section $0 \longrightarrow \mathcal{O}_{\mathfrak{C}} \longrightarrow \mathcal{O}_{\mathfrak{C}}(1)$. Define a sheaf $\mathcal{G}_m$ on $\mathfrak{C}$ by the exact sequence

$$0 \longrightarrow \mathcal{V}_{\mathfrak{P}} \longrightarrow \mathcal{V}_{\mathfrak{P}}(m) \longrightarrow \mathcal{G}_m \longrightarrow 0.$$

By Corollary 3.2.9 there is an $m >> 0$, which we will fix from now on, such that

$$\pi_*(\mathcal{V}_{\mathfrak{P}}(m)) \quad \text{and} \quad \pi_*(\mathcal{G}_m)$$

are vector bundles on $\mathfrak{B}un_{\mathbf{G},\beta}$.

Let $\mathfrak{X}$ be the total space of the vector bundle $\pi_*(\mathcal{V}_{\mathfrak{P}}(m))$. It is a smooth Artin stack of finite type, whose $\mathbb{C}$-points parametrize tuples

$$(C, p_1, \ldots, p_k, P, u \in \Gamma(C, \mathcal{V}_P(m))) . \tag{5.2.1}$$

Let $\mathfrak{X}^{\circ}$ be the open substack in $\mathfrak{X}$ determined by imposing the requirements

- u sends all but possibly finitely many points of C to V^s; here we identify the fibers of $\mathcal{V}_P(m)$ with V.[1]
- the line bundle $\omega_C(\sum_{i=1}^k p_i) \otimes \mathcal{L}^{\epsilon}$ is ample for every rational $\epsilon > 0$, where $\mathcal{L} = P \times_{\mathbf{G}} L_{\theta}$.

By stability, $\mathfrak{X}^{\circ}$ is a smooth Deligne–Mumford stack. It comes with a projection $p : \mathfrak{X}^{\circ} \longrightarrow \mathfrak{B}un_{\mathbf{G},\beta}$. The vector bundle

$$\mathcal{F} := p^*(\pi_*(\mathcal{G}_m)) \tag{5.2.2}$$

on $\mathfrak{X}^{\circ}$ has a tautological section s induced by the maps

$$\Gamma(\mathfrak{C}, \mathcal{V}_{\mathfrak{P}}(m)) \longrightarrow \Gamma(\mathfrak{C}, \mathcal{G}_m).$$

Its zero locus $Z(s)$ is the closed substack parametrizing tuples as in (5.2.1), but with $u \in \Gamma(C, \mathcal{V}_P) \subset \Gamma(C, \mathcal{V}_P(m))$.

Finally, $\mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$ is identified with the open substack of $Z(s)$ obtained by imposing the strong nondegeneracy condition that the base-points are away from nodes and markings, and is therefore a Deligne–Mumford stack of finite type.

Properness has been proved more generally in the previous section.

Finally, it is easy now to describe the relative and absolute obstruction theories. For this, we note first that the order in which the last two conditions were imposed in the construction of $\mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$ may be in principle reversed. Namely, there is an open substack $\mathcal{E} \subset \mathfrak{X}^{\circ}$ (which is necessarily DM and smooth) such that $\mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$ is cut out in $\mathcal{E}$ as the zero locus $Z(s)$ of the tautological section of the bundle (5.2.2) *restricted* to $\mathcal{E}$.

Let

$$\mathfrak{C} \xrightarrow{\pi} \mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$$

be the universal curve. The above description gives an absolute perfect obstruction theory

$$\left[\mathcal{F}^{\vee}\big|_{\mathrm{Qmap}} \xrightarrow{ds^{\vee}} \Omega^1_{\mathcal{E}}\big|_{\mathrm{Qmap}} \right]$$

on $\mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$, whose virtual class is the refined top Chern class of $\mathcal{F}^{\vee}\big|_{\mathrm{Qmap}}$.

As explained in [5, Section 5], the induced relative obstruction theory over $\mathfrak{B}un_{\mathbf{G}}$ is

$$\left[\mathcal{F}^{\vee}\big|_{\mathrm{Qmap}} \xrightarrow{ds^{\vee}} \Omega^1_{\mathcal{E}/\mathfrak{B}un_{\mathbf{G},\beta}}\Big|_{\mathrm{Qmap}} \right] \overset{\mathrm{qis}}{\sim} (R^{\bullet}\pi_*(\mathcal{V}_{\mathfrak{P}}))^{\vee}$$

and the relative obstruction theory over $\mathfrak{M}_{g,k}$ is

$$(R^{\bullet}\pi_*(\mathcal{Q}))^{\vee},$$

where $\mathcal{Q}$ is defined by the "Euler sequence" on $\mathfrak{C}$

$$0 \longrightarrow \mathfrak{P} \times_G \mathfrak{g} \longrightarrow \mathcal{V}_{\mathfrak{P}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

induced by (5.1.1). $\square$

---

[1] A priori, this identification only makes sense locally, in a trivialization $\{U_i, \varphi_i\}$ of $\mathcal{V}_P(m)$ on C. However, one checks that the local $V^s \times U_i$ glue together into a global open subscheme $\mathcal{V}_P(m)^s \subset \mathcal{V}_P(m)$ under the transition functions of $\mathcal{V}_P(m)$. This is so since the $\mathbf{G}$-invariant locus $V^{ss} = V^s$ is also invariant under the scaling action by the homotheties of V. In turn, this invariance follows easily from the definition of θ-semistable points and the fact that the group of homotheties is identified with the center of $GL(V)$, and therefore its action on V commutes with the action of $\mathbf{G}$.

Remark 5.2.2. If W is a **G**-invariant affine subvariety of V, then we have the induced GIT quotient $W /\!\!/ \mathbf{G}$ (using the same linearization θ). By restricting only to quasimaps $((C, p_i), P, u)$ for which

$$u(C) \subset P \times_{\mathbf{G}} W \subset P \times_{\mathbf{G}} V$$

we obtain a closed substack of $\mathrm{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)$. It is obvious that this substack is identified with

$$\coprod_{\widetilde{\beta} \mapsto \beta} \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \widetilde{\beta}),$$

with the disjoint union over all curve classes $\widetilde{\beta}$ which are mapped to β via the morphism

$$\mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(V), \mathbb{Z})$$

induced by the pull-back $\mathrm{Pic}^{\mathbf{G}}(V) \longrightarrow \mathrm{Pic}^{\mathbf{G}}(W)$. Hence Theorem 5.2.1 provides a different proof of the fact that $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \widetilde{\beta})$ is a Deligne–Mumford stack of finite type for these targets $W /\!\!/ \mathbf{G}$.

6. Quasimap invariants

6.1. Descendant invariants

Throughout this subsection and the next we assume that $W /\!\!/ \mathbf{G}$ is projective. Since the moduli spaces $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ are proper Deligne–Mumford stacks with perfect obstruction theories, a system of invariants is obtained by integrating natural cohomology classes against the virtual class. Precisely, we have the following canonical structures:

- Evaluation maps

$$ev_i : \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow W /\!\!/ \mathbf{G}, \quad i = 1, \dots, k$$

  at the marked points. These are well-defined since base-points cannot occur at markings.
- Cotangent line bundles

$$M_i := s_i^*(\omega_{\mathcal{C}/\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)}), \quad i = 1, \dots, k$$

  where $\mathcal{C}$ is the universal curve, $\omega_{\mathcal{C}/\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)}$ is the relative dualizing sheaf (which is a line bundle), and

$$s_i : \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathcal{C}$$

  are the universal sections. We denote

$$\psi_i := c_1(M_i).$$

Definition 6.1.1. The descendant quasimap invariants are

$$\langle \tau_{n_1}(\gamma_1), \dots, \tau_{n_k}(\gamma_k) \rangle_{g,k,\beta}^{quasi} := \int_{[\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)]^{vir}} \prod_{i=1}^{k} \psi_i^{n_i} ev_i^*(\gamma_i),$$

with $\gamma_1, \dots, \gamma_k \in H^*(W /\!\!/ \mathbf{G}, \mathbb{Q})$ and $n_1, \dots, n_k$ nonnegative integers.

As explained in [5], the quasimap invariants satisfy the analogue of the Splitting Axiom in Gromov–Witten theory. Furthermore, there is a natural map

$$f : \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \overline{M}_{g,k}$$

which forgets the principal **G**-bundle and the section, and contracts the unstable components of the underlying pointed curve. We may define *quasimap classes* in $H^*(\overline{M}_{g,k}, \mathbb{Q})$ by

$$f_* \left(\prod_{i=1}^{k} \psi_i^{n_i} ev_i^*(\gamma_i) \right),$$

which by the splitting property give rise to a *Cohomological Field Theory* on $H^*(W /\!\!/ \mathbf{G}, \mathbb{Q})$.

On the other hand, the universal curve over $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$ is *not* isomorphic to $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$, and in general there are no maps

$$\mathrm{Qmap}_{g,k+1}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$$

which forget one marking. Note that the corresponding maps for the Kontsevich moduli spaces of stable maps are crucially used in the proofs of the string and divisor equations for descendant Gromov–Witten invariants. Some examples showing that the exact analogue of the string and divisor equations fail for quasimap invariants of certain non-Fano toric varieties are

given in [5]. However, the quasimap and Gromov–Witten theories of Grassmannians are shown to coincide in [4]; the same was conjectured to hold for Fano toric varieties, see [5], and is proved in [31]. Furthermore, for general targets we expect that the two theories are related by "wall-crossing" formulas involving the ϵ-stable quasimaps described in Section 7.1 below (this will be addressed elsewhere). Consequently, the quasimap invariants should still satisfy some modified versions of the equations. At the moment it is not yet clear to us if reasonable general formulae can be written down.

6.2. Twisted invariants

In Gromov–Witten theory, twisted invariants have been introduced and studied by Coates and Givental, [32]. Their counterparts in quasimap theory are easily obtained.

Let

$$\mathcal{C} \xrightarrow{\pi} \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$$

be the universal curve, with universal principal **G**-bundle $\mathfrak{P}$ on it. Let

$$\tilde{u} : \mathfrak{P} \longrightarrow W$$

be the universal **G**-equivariant map and let

$$u : \mathcal{C} \longrightarrow \mathfrak{P} \times_{\mathbf{G}} W$$

be the induced universal section. Let E be any **G**-equivariant vector bundle on W (for example, we could take a linear **G**-representation E and, by slight abuse of notation, denote also by E the associated **G**-equivariant vector bundle $W \times E$) and let $\bar{E} = [E|_{W^s}/G]$ be the induced vector bundle on $W /\!\!/ \mathbf{G}$. Then $\mathfrak{P} \times_{\mathbf{G}} E$ is a vector bundle on $\mathfrak{P} \times_{\mathbf{G}} W$ and we consider the vector bundle

$$E_{g,k,\beta} := u^*(\mathfrak{P} \times_{\mathbf{G}} E)$$

on the universal curve.

Claim: $R^\bullet \pi_* E_{g,k,\beta} := [R^0 \pi_* E_{g,k,\beta}] - [R^1 \pi_* E_{g,k,\beta}]$ is an element in

$$K^\circ(\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)),$$

the K group of vector bundles on $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$.

Proof. Let $\mathcal{O}(1)$ be a π-relatively ample line bundle on $\mathcal{C}$. For $m >> 0$, we have a surjection

$$B \longrightarrow E_{g,k,\beta}(m) \longrightarrow 0,$$

with B a trivial vector bundle. The kernel, call it A, is also a vector bundle on $\mathcal{C}$, and there is an exact sequence

$$0 \longrightarrow A(-m) \longrightarrow B(-m) \longrightarrow E_{g,k,\beta} \longrightarrow 0.$$

Since

$$R^0 \pi_*(A(-m)) = R^0 \pi_*(B(-m)) = 0,$$

we have a complex of vector bundles

$$R^1 \pi_*(A(-m)) \longrightarrow R^1 \pi_*(B(-m))$$

whose cohomology is precisely $R^\bullet \pi_* E_{g,k,\beta}$.   □

Let $\mathbb{C}^*$ act trivially on $\mathrm{Qmap} = \mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)$, with equivariant parameter λ, so that

$$H^*_{\mathbb{C}^*}(\mathrm{Qmap}, \mathbb{Q}) \cong H^*(\mathrm{Qmap}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[\lambda].$$

Similarly, the rational Grothendieck group of $\mathbb{C}^*$-equivariant vector bundles is

$$K^\circ_{\mathbb{C}^*}(\mathrm{Qmap}) \cong K^\circ(\mathrm{Qmap}) \otimes \mathbb{Q}[\lambda, \lambda^{-1}].$$

Now fix an invertible multiplicative class c, that is, a homomorphism

$$c : K^\circ_{\mathbb{C}^*}(\mathrm{Qmap}) \longrightarrow U\left(H^*(\mathrm{Qmap}, \mathbb{Q}) \otimes \mathbb{Q}[\lambda, \lambda^{-1}]\right)$$

to the group of units in the localized $\mathbb{C}^*$-equivariant cohomology ring. In addition, fix a **G**-equivariant bundle E on W. Let $\mathbb{C}_\lambda$ denote the 1-dimensional representation of $\mathbb{C}^*$ with weight λ. Applying the construction described earlier in this subsection to the $\mathbf{G} \times \mathbb{C}^*$-equivariant bundle $E \otimes \mathbb{C}_\lambda$ (for the trivial action of $\mathbb{C}^*$ on W) yields an element

$$R^\bullet \pi_* E_{g,k,\beta}(\lambda) \in K^\circ(\mathrm{Qmap}) \otimes \mathbb{Q}[\lambda, \lambda^{-1}].$$

Definition 6.2.1. The (E, c)-twisted quasimap invariants are

$$\langle \tau_{n_1}(\gamma_1), \ldots, \tau_{n_k}(\gamma_k) \rangle_{g,k,\beta}^{quasi,(c,E)} := \int_{[\text{Qmap}]^{vir}} c(R^\bullet \pi_* E_{g,k,\beta}(\lambda)) \prod_{i=1}^k \psi_i^{n_i} ev_i^*(\gamma_i).$$

By definition, the twisted invariants lie in $\mathbb{Q}[\lambda, \lambda^{-1}]$.

We discuss next an important example of this construction. A typical choice of multiplicative class for twisting is the equivariant Euler class. For an ordinary bundle F of rank r on a space Y, with Chern roots $f_1, \ldots, f_r$, the Euler class of $F \otimes \mathbb{C}_\lambda$ is

$$e(F \otimes \mathbb{C}_\lambda) = \prod_{i=1}^r (f_i + \lambda) = \lambda^r + \lambda^{r-1} c_1(F) + \cdots + c_r(F),$$

i.e., a version of the Chern polynomial of F.

Suppose now that we are in the situation described in Example 4.5.5: V is a vector space, E is the bundle $V \times E$ coming from a linear **G**-representation, $W = Z(t)$ is the zero locus of a regular section $t \in \Gamma(V, E)^{\mathbf{G}}$, and $W \cap V^s$ is nonsingular. For equivariant curve classes $\beta \in \text{Hom}(\text{Pic}^{\mathbf{G}}(V), \mathbb{Z})$ and $\tilde{\beta} \in \text{Hom}(\text{Pic}^{\mathbf{G}}(W), \mathbb{Z})$ we write $\tilde{\beta} \mapsto \beta$ if $\tilde{\beta}$ is sent to β under the natural map between the duals of equivariant Picard groups. Put

$$\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) := \coprod_{\tilde{\beta} \mapsto \beta} \text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \tilde{\beta}).$$

There is an induced closed embedding of stacks

$$i : \text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta).$$

Proposition 6.2.2. *Assume that $R^1 \pi_* E_{g,k,\beta} = 0$. Then*

$$i_*[\text{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta)]^{vir} = e(R^\bullet \pi_* E_{g,k,\beta}) \cap [\text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)]^{vir}$$
$$= \left(e(R^\bullet \pi_* E_{g,k,\beta}(\lambda)) \cap [\text{Qmap}_{g,k}(V /\!\!/ \mathbf{G}, \beta)]^{vir} \right)\big|_{\lambda=0}.$$

Proof. The assumed vanishing implies that $R^0 \pi_* E_{g,k,\beta}$ is a vector bundle. The second equality is now immediate from the definitions, while the first follows from [30] by the same argument as the one given there for moduli of stable maps. $\square$

As we already mentioned in the proof, Proposition 6.2.2 holds for stable maps, however, the required vanishing is true only for $g = 0$. It is a very important unsolved problem in Gromov–Witten theory to find a useful expression for the push-forward of the virtual class in the proposition when $g \geq 1$. As observed in [4] (for the special case of complete intersections in $\mathbb{P}^n$), one can do slightly better in quasimap theory. We will say that a character $\eta \in \chi(\mathbf{G})$ is *positive* with respect to our given linearization θ if $V^s(\theta) = V^s(\eta) = V^{ss}(\eta)$ (in particular, the line bundle induced by η on $V /\!\!/ \mathbf{G}$ is ample), and we will say it is *semi-positive* if $V^s(\theta) \subset V^{ss}(\eta)$.

Proposition 6.2.3. *The vanishing $R^1 \pi_* E_{g,k,\beta} = 0$, hence the conclusion of Proposition 6.2.2, holds in the following cases:*

 (i) $g = 0$, k arbitrary, and $E = \oplus_{i=1}^r \mathbb{C}_{\eta_i}$, with each η_i semi-positive.
 (ii) $g = 1$, $k = 0$, and $E = \oplus_{i=1}^r \mathbb{C}_{\eta_i}$, with each η_i positive.

Proof. Let C be a geometric fiber of the universal curve, let P be the corresponding principal bundle, and let $u : C \longrightarrow P \times_{\mathbf{G}} V$ be the corresponding section. The restriction to any irreducible component C' of C of the vector bundle

$$u^*(P \times_{\mathbf{G}} (V \times E))$$

decomposes as the direct sum of line bundles

$$\oplus_i u^* \left(P|_{C'} \times_{\mathbf{G}} (V \times \mathbb{C}_{\eta_i}) \right) \cong \oplus_i \left(P|_{C'} \times_{\mathbf{G}} \mathbb{C}_{\eta_i} \right).$$

In case (i) each irreducible component C' of C is a rational curve. The semi-positivity assumption implies that each of the line bundles has nonnegative degree on C' by the argument of Lemma 3.2.1. We deduce that

$$H^1(C, u^*(P \times_{\mathbf{G}}(V \times E))) = 0,$$

from which the vanishing follows.

In case (ii) the underlying curve of a stable quasimap is either an irreducible elliptic curve, or a cycle of rational curves (this last case includes the cycle of length one, i.e., an irreducible nodal curve of arithmetic genus one). Since there are no special line bundles of positive degree on such curves, the vanishing follows from the positivity of the η_i's. $\square$

Remark 6.2.4. It is clear that in (i) it suffices to assume that E is a representation such that the bundle $V \times E$ is generated by **G**-equivariant global sections.

6.3. Invariants for noncompact $W/\!\!/\mathbf{G}$

Recall that when $W/\!\!/\mathbf{G}$ is not projective we have only a proper morphism to the affine quotient

$$\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta) \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}}).$$

As is the case in Gromov–Witten theory, even though the integrals against the virtual class are not well-defined due to the lack of properness of $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)$, there are many situations for which one can get an interesting theory of *equivariant* invariants via the virtual localization formula of [33]. This happens when there is a torus $\mathbf{S} \cong (\mathbb{C}^*)^m$ with an action on W, commuting with the given $\mathbf{G}$-action, and such that the fixed locus of the induced $\mathbf{S}$-action on $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)$ is proper.

The $\mathbf{S}$-action on $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)$, is given on $\mathbb{C}$-points by

$$s \cdot ((C, p_1, \ldots, p_k), P, u) = ((C, p_1, \ldots, p_k), P, s \circ u).$$

Here we view $s \in \mathbf{S}$ as an automorphism of W, giving rise to a C-automorphism of the fibration $P \times_{\mathbf{G}} W \longrightarrow C$, still denoted by s.

Note that the map

$$\mu : \mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta) \longrightarrow \mathfrak{Bun}_{\mathbf{G},\beta}$$

is $\mathbf{S}$-equivariant, where $\mathbf{S}$ acts trivially on $\mathfrak{Bun}_{\mathbf{G},\beta}$.

From its description in (4.4.1), it is clear that the μ-relative obstruction theory is $\mathbf{S}$-equivariant, hence the same holds for the absolute obstruction theory as well.

Furthermore, the stack $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)$ admits an $\mathbf{S}$-equivariant closed embedding into a smooth Deligne–Mumford stack. When $W = V$ is a vector space, this is shown already by the construction in Section 5; in general, one can essentially deduce it from this case using Proposition 2.5.2 (some care is need to handle the fact that V may have strictly semistable points and stable points with nontrivial stabilizer).

It follows from [33] that each component F of the fixed point locus $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)^{\mathbf{S}}$ has a virtual fundamental class and a virtual normal bundle, whose Euler class is invertible in $H^*_{\mathbf{S}}(F, \mathbb{Q}) \otimes \mathbb{Q}(\lambda_1, \ldots, \lambda_m)$. Here $\lambda_1, \ldots, \lambda_m$ denote the equivariant parameters, so that

$$H^*_{\mathbf{S}}(\mathrm{pt}, \mathbb{Q}) = H^*(B\mathbf{S}) \cong \mathbb{Q}[\lambda_1, \ldots, \lambda_m].$$

Finally, as mentioned above, we will make the following

Assumption 6.3.1. The $\mathbf{S}$-fixed closed substack $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)^{\mathbf{S}}$ is proper.

For example, this will always hold when the fixed point locus for the induced $\mathbf{S}$-action on the affine quotient $\mathrm{Spec}(A(W)^{\mathbf{G}})$ is proper — and therefore a finite set of points. This follows immediately from the fact that the map $\eta : \mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta) \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}})$ is $\mathbf{S}$-equivariant and proper.

We can then define invariants as sums of equivariant residues via the virtual localization formula: for equivariant cohomology classes $\gamma_1, \ldots, \gamma_k \in H^*_{\mathbf{S}}(W/\!\!/\mathbf{G}, \mathbb{Q})$ and nonnegative integers $n_1, \ldots, n_k$, the equivariant descendant quasimap invariant is

$$\langle \tau_{n_1}(\gamma_1), \ldots, \tau_{n_k}(\gamma_k) \rangle^{quasi}_{g,k,\beta} := \sum_F \int_{[F]^{vir}} \frac{i_F^* \left(\prod_{i=1}^{k} \psi_i^{n_i} ev_i^*(\gamma_i) \right)}{e(N_F^{vir})},$$

the sum over the connected components F of the fixed point locus $\mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta)^{\mathbf{S}}$, with equivariant embeddings

$$i_F : F \longrightarrow \mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta).$$

By the above definition, the quasimap invariants lie in $\mathbb{Q}(\lambda_1, \ldots, \lambda_m)$.

Similarly, we may also define twisted invariants as in Section 6.2, using invertible multiplicative classes of $\mathbf{S}$-equivariant vector bundles.

The main examples we have in mind of noncompact targets with a well-defined theory of quasimap invariants are found among quiver varieties.

Example 6.3.2 (*Nakajima Quiver Varieties*)**.** Let Γ be an oriented graph on a finite set of vertices S with edge set E, equipped with source and target maps

$$i, o : E \to S.$$

Suppose we are given two dimension vectors

$$\overrightarrow{v}, \overrightarrow{u} \in (\mathbb{Z}_{\geq 0})^S.$$

The Nakajima quiver variety associated to this data is defined as follows. For each vertex $s \in S$, we fix vector spaces V_s, U_s of dimension v_s and u_s respectively. Consider the associated affine space

$$\mathbb{H} = \bigoplus_{e \in E} \left(\mathrm{Hom}(V_{i(e)}, V_{o(e)}) \oplus \mathrm{Hom}(V_{o(e)}, V_{i(e)}) \right) \bigoplus_{s \in S} \left(\mathrm{Hom}(U_s, V_s) \oplus \mathrm{Hom}(V_s, U_s) \right).$$

Given an element $(A_e, B_e, i_s, j_s)_{e \in E, s \in S} \in \mathbb{H}$, and a vertex $s \in S$, we can associate the following endomorphism

$$\phi_s = \sum_{e, i(e)=s} [A_e, B_e] - \sum_{e, o(e)=s} [A_e, B_e] + i_s \circ j_s \in \mathrm{End}(V_s).$$

Let $\mathbf{G} = \prod_{s \in S} GL(V_s)$ act on $\mathbb{H}$ with polarization given by the determinant character. We are interested in the GIT quotient of the $\mathbf{G}$-invariant affine variety W defined by the equations $\phi_s = 0$:

$$\mathcal{M}(\overrightarrow{v}, \overrightarrow{u}) = \{ h \in \mathbb{H} \mid \phi_s = 0 \text{ for all } s \} /\!\!/ \mathbf{G}.$$

These quotients are typically noncompact. However, if one considers the $\mathbf{S} = \mathbb{C}^*$-action defined by scaling A_e and i_s, it is easy to see that it preserves W and $(\mathrm{Spec}\,\mathbb{C}[W]^{\mathbf{G}})^{\mathbb{C}^*}$ is compact (see, for instance [34]).

In particular, this gives us a rich source of examples where the noncompact invariants make sense.

For other applications of quasimap theory with noncompact targets the reader is referred to the papers [35,36].

7. Variants and applications

7.1. ϵ-stable quasimaps

As we have observed in Section 4 (see in particular Remarks 4.2.1 and 4.5.4) one may view the moduli spaces of stable quasimaps and stable maps with target $W /\!\!/ \mathbf{G}$ as two instances of the same construction: they are (relatively) proper, finite type, Deligne–Mumford substacks of the Artin stack, locally of finite type parametrizing prestable quasimaps to $W /\!\!/ \mathbf{G}$, obtained by imposing a stability condition. Furthermore, one may think loosely of passing from stable maps to stable quasimaps as a process in which, by changing the stability condition, rational tails are replaced with base points that keep track of their degrees (cf. the proof of Proposition 4.3.1). It is natural to try to do this sequentially, removing first rational tails of lowest degree, then those of next lowest degree and so on.

In the case of target $\mathbb{P}^n$, this loose interpretation is literally true, and the result of the procedure is the factorization of a natural morphism from the moduli of stable maps to the moduli of stable quasimaps into a sequence of blow-downs to intermediate moduli spaces. These intermediate spaces correspond to stability conditions depending on a rational parameter ϵ. This was first noticed and proved some time ago by Mustaţă et al. in [13,14].

More recently, using moduli of stable quotients, Toda ([15]) has extended the story to Grassmannian targets $G(r, n)$. In this case, there are typically no morphisms between the various ϵ-stable moduli spaces, so one obtains an instance of the general wall-crossing phenomenon when varying the stability condition.

In this subsection we treat the variation of stability for general targets $W /\!\!/ \mathbf{G}$. Recall that a quasimap is said to be prestable if the base points are away from nodes and markings.

Definition 7.1.1. The *length* $\ell(x)$ at a point $x \in C$ of a prestable quasimap $((C, p_i), P, u)$ to $W /\!\!/ \mathbf{G}$ is defined by

$$\ell(x) := \min \left\{ \left. \frac{(u^*s)_x}{m} \right| \; s \in H^0(W, L_{m\theta})^{\mathbf{G}}, \; u^*s \not\equiv 0, \; m > 0 \right\},$$

where $(u^*s)_x$ is the coefficient of the divisor (u^*s) at x.

The following properties follow easily:

- For every $x \in C$ we have

  $$\beta(L_\theta) \geq \ell(x) \geq 0$$

  and $\ell(x) > 0$ if and only if x is a base point of the quasimap. This is because the scheme-theoretic unstable locus W^{us} is the subscheme defined by the ideal $J_{W^{us}} \subset A(W)$ generated by $\{s \mid s \in H^0(W, L_{m\theta})^{\mathbf{G}}, \; m > 0\}$.
- If $H^0(W, L_\theta)^{\mathbf{G}}$ generates $\bigoplus_{m \geq 0} H^0(W, L_{m\theta})^{\mathbf{G}}$ as an algebra over $A(W)^{\mathbf{G}}$ (for example, if the relatively ample line bundle $\mathcal{O}(\theta)$ induced by L_θ on $W /\!\!/ \mathbf{G}$ is relatively very ample over the affine quotient) then we can also write

  $$\ell(x) = \min \left\{ (u^*s)_x \mid \; s \in H^0(W, L_{m\theta})^{\mathbf{G}}, \; u^*s \not\equiv 0, \; m > 0 \right\}. \tag{7.1.1}$$

Alternatively, consider the ideal sheaf $\mathcal{J}$ of the closed subscheme $P \times_{\mathbf{G}} W^{us}$ of $P \times_{\mathbf{G}} W$. Then it is clear that $\ell(x)$ from (7.1.1) satisfies

$$\ell(x) = \mathrm{length}_x(\mathrm{coker}(u^*\mathcal{J} \longrightarrow \mathcal{O}_C)), \tag{7.1.2}$$

which may be viewed as the order of contact of $u(C)$ with the unstable subscheme $P \times_{\mathbf{G}} W^{us}$ at $u(x)$.

Let $((C, p_i), P, u)$ be a prestable quasimap of class β. Let $B \subset C$ be the base locus. Let

$$[u] : C \setminus B \to W /\!\!/ \mathbf{G}$$

be the induced map. By the prestable condition and the projectivity of $W /\!\!/ \mathbf{G} \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}})$, $[u]$ extends to a regular map

$$[u_{reg}] : C \longrightarrow W /\!\!/ \mathbf{G}.$$

Let P_{reg} be the principal $\mathbf{G}$-bundle on C which is obtained as the pull-back of $W^s \longrightarrow W /\!\!/ \mathbf{G}$ via $[u_{reg}]$. Let $u_{reg} : C \longrightarrow P_{reg} \times_G W$ be the induced section. The data

$$((C, p_i), P_{reg}, u_{reg})$$

is also a quasimap to $W /\!\!/ \mathbf{G}$, so it has a class $\beta_{reg} \in \mathrm{Hom}(\mathrm{Pic}^{\mathbf{G}}(W), \mathbb{Z})$.

Lemma 7.1.2. *We have*

$$(\beta - \beta_{reg})(L_\theta) = \sum_{x \in B} \ell(x). \tag{7.1.3}$$

In particular, $\beta - \beta_{reg}$ is L_θ-effective.

Proof. It suffices to assume that L_θ descends to a relatively very ample line bundle on $W /\!\!/ \mathbf{G}$, since the equality (7.1.3) is invariant under replacing θ by a positive multiple. But in this case the equality is immediate. □

Fix a positive rational number ϵ.

Definition 7.1.3. A prestable quasimap $((C, p_i), P, u)$ is called ϵ-*stable* if

(1) $\omega_C(\sum p_i) \otimes \mathcal{L}_\theta^\epsilon$ is ample.
(2) $\epsilon \ell(x) \leq 1$ for every point x in C.

Remark 7.1.4. The notion of ϵ-stability depends obviously on θ. However, this dependence is very simple under rescaling: for a positive integer m, the quasimap $((C, p_i), P, u)$ is ϵ-stable with respect to θ if and only if it is $(\frac{\epsilon}{m})$-stable with respect to $m\theta$.

As a consequence, we may assume (and will assume from now on) that the polarization $\mathcal{O}(\theta)$ on $W /\!\!/ \mathbf{G}$ is relatively very ample over the affine quotient. Hence we can use (7.1.1) and (7.1.2) as (equivalent) definitions of length. In particular, $\ell(x) \in \mathbb{Z}_+$ for every point x of C.

Again, we list some immediate consequences of this definition.

- The underlying curve of an ϵ-stable quasimap may have rational components containing only one special point (i.e., rational tails). However, any such component C' must satisfy

$$\epsilon \deg(\mathcal{L}_\theta|_{C'}) > 1. \tag{7.1.4}$$

In addition, the degree of $\mathcal{L}_\theta$ must be positive on rational components containing exactly two special points.
- A prestable quasimap $((C, p_i), P, u)$ of some class β is a stable quasimap (as in Definition 3.1.2) if and only it is an ϵ-stable quasimap for some $\epsilon \leq 1/\beta(L_\theta)$.
- Assume that $(g, k) \neq (0, 0)$. Then a prestable quasimap is a stable map to $W /\!\!/ \mathbf{G}$ if and only if it is an ϵ-stable quasimap for some $\epsilon > 1$. In the case $(g, k) = (0, 0)$ the same is true but with $\epsilon > 2$.

Therefore, for the extremal values of ϵ we recover the notions of stable quasimaps and stable maps, respectively.

Proposition 7.1.5. *Let $\epsilon > 0$ be fixed. The automorphism group of an ϵ-stable quasimap is finite and reduced.*

Proof. Assume first $(g, k) \neq (0, 0)$. It suffices to check what happens on rational tails of the underlying curve C. Let C' be a rational tail and let $\beta_{C'}$ be the class of the induced quasimap on C'. Any automorphism of the quasimap must preserve the map $[u_{reg}]$, as well as the base points. If $[u_{reg}]|_{C'}$ is not a constant map we are done, so assume it is constant. Let B' be the base locus supported on C'. By Lemma 7.1.2 and (7.1.4) we get

$$\sum_{x \in B'} \ell(x) = \beta_{C'}(L_\theta) > \frac{1}{\epsilon}.$$

If B' is a single point x, then x violates condition (2) in Definition 7.1.3. Hence B' must contain at least two distinct points.

If $(g, k) = (0, 0)$, we have the additional case of a rational curve C with no special points. The same argument will work, using now that $\beta_C(L_\theta) > \frac{2}{\epsilon}$. □

Theorem 7.1.6. *The stack* $\mathrm{Qmap}_{g,k}^\epsilon(W /\!\!/ \mathbf{G}, \beta)$ *is a separated Deligne–Mumford stack of finite type, admitting a canonical obstruction theory. If* W *has at most lci singularities, then the obstruction theory is perfect. Furthermore, there is a natural morphism*

$$\mathrm{Qmap}_{g,k}^\epsilon(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathrm{Spec}(A(W)^{\mathbf{G}})$$

which is proper.

Proof. Most of the proof of Theorem 4.1.2 from Section 4 carries over unchanged. First, the proof of Corollary 3.2.3 works for any ϵ, so we obtain (see Remark 4.2.1) that the stack is algebraic and of finite type. Proposition 7.1.5 is then used to conclude that the stack is Deligne–Mumford. The natural obstruction theory is the μ-relative obstruction theory for the forgetful map

$$\mu : \mathrm{Qmap}_{g,k}^\epsilon(W /\!\!/ \mathbf{G}, \beta) \longrightarrow \mathfrak{Bun}_{\mathbf{G}},$$

which is given by (4.4.1). It is perfect if W has at worst lci singularities (cf. Remark 4.5.4).

In fact, the only part of the argument that needs modification is the proof of completeness in Proposition 4.3.1. Namely, when passing from $\widehat{C}$ to $\bar{C}$ as in that proof, we do so by contracting only some of the trees Γ_i in the central fiber $\widehat{C}_0$. We go briefly through the details.

Let $(\Delta, 0)$ be a smooth pointed curve and let $\Delta^\circ = \Delta \setminus \{0\}$. Let $((C, p_i), P, u)$ be a Δ°-family of ϵ-stable quasimaps. Let B denote the base locus of the family. We may regard B as additional sections y_j, and assume that $(C, p_i, y_j) \longrightarrow \Delta^\circ$ is a family of prestable pointed curves with irreducible fibers and nonsingular total space C. By shrinking Δ if necessary, we may also assume that the length of the quasimaps in the family along each section y_j is constant. Denote this length by $\ell(y_j)$. By assumption we have $\ell(y_j) \leq 1/\epsilon$.

Let $[u] : C \setminus B \longrightarrow W /\!\!/ G$ be the induced map and let $[u_{reg}]$ be its extension to C (shrink Δ, if needed). The proof of Proposition 7.1.5 shows that $((C, p_i, y_j), [u_{reg}])$ is a Δ°-family of stable maps to $W /\!\!/ \mathbf{G}$, of class β_{reg}. By Lemma 7.1.2 we have

$$\beta(L_\theta) - \beta_{reg}(L_\theta) = \sum_j \ell(y_j). \tag{7.1.5}$$

Let

$$(\widehat{C}, p_i, y_j) \longrightarrow \Delta, \qquad [\widehat{u}] : \widehat{C} \longrightarrow W /\!\!/ \mathbf{G}$$

be the unique family of stable maps extending $((C, p_i, y_j), [u_{reg}])$. For each subcurve D in the central fiber $\widehat{C}_0$ we define its total degree with respect to L_θ to be

$$\deg(D, L_\theta) := \beta_{reg}\big|_D (L_\theta) + \sum_{y_{j,0} \in D} \ell(y_j), \tag{7.1.6}$$

where $y_{j,0}$ denotes the intersection point of the section y_j with $\widehat{C}_0$.

Let Γ be a tree of rational curves in $\widehat{C}_0$ which contains none of the markings p_i and meets the rest of the curve $\overline{(\widehat{C}_0 \setminus \Gamma)}$ in a single point z. Note that such a tree always has positive total degree with respect to L_θ by stability of the map $[u_{reg}]$. We contract the tree Γ exactly when its total degree satisfies

$$\deg(\Gamma, L_\theta) \leq \frac{1}{\epsilon}. \tag{7.1.7}$$

Now the exact same reasoning as in the proof of Proposition 4.3.1 gives a family of prestable quasimaps

$$((\bar{C}, p_i), \bar{P}, \bar{u})$$

of class β, extending the original family over Δ°. We claim that the limit quasimap is also ϵ-stable. Let $\Gamma_1, \ldots, \Gamma_N$ be all trees satisfying (7.1.7) that have been contracted and let $z_1, \ldots, z_N$ be their respective attaching points in the special fiber $\bar{C}_0$. Then

$$\beta(L_\theta) = \beta_{reg}\bigg|_{\overline{\widehat{C}_0 \setminus \cup_l \Gamma_l}} (L_\theta) + \sum_{l=1}^N \ell(z_l) + \sum_{y_{j,0} \notin \cup_l \Gamma_l} \ell(y_{j,0}) \tag{7.1.8}$$

by Lemma 7.1.2 applied to the limit quasimap.

By a semicontinuity argument we have

$$\ell(y_j) \leq \ell(y_{j,0}), \quad \forall y_{j,0} \notin \cup_l \Gamma_l. \tag{7.1.9}$$

The length at an attachment point z_l is made out of contributions from two sources. First, it is clear that the restriction of the stable map $[u_{reg}]$ to Γ_l contributes $\beta_{reg}|_{\Gamma_l}(L_\theta)$ to $\ell(z_l)$. Second, each of the points $y_{j,0} \in \Gamma_l$ contributes at least $\ell(y_j)$ by

semicontinuity. We deduce

$$\deg(\Gamma_l, L_\theta) = \beta_{reg}\big|_{\Gamma_l}(L_\theta) + \sum_{y_{j,0} \in \Gamma_l} \ell(y_j) \le \ell(z_l), \quad l = 1, \ldots, N. \tag{7.1.10}$$

From (7.1.5), (7.1.8), (7.1.9) and (7.1.10),

$$\begin{aligned}
\beta(L_\theta) &= \beta_{reg}(L_\theta) + \sum_j \ell(y_j) \\
&= \beta_{reg}\Big|_{\overline{\widehat{C}_0 \setminus \cup_l \Gamma_l}}(L_\theta) + \sum_l \beta_{reg}\big|_{\Gamma_l}(L_\theta) + \sum_j \ell(y_j) \\
&\le \beta_{reg}\Big|_{\overline{\widehat{C}_0 \setminus \cup_l \Gamma_l}}(L_\theta) + \sum_{l=1}^N \ell(z_l) + \sum_{y_{j,0} \notin \cup_l \Gamma_l} \ell(y_{j,0}) = \beta(L_\theta).
\end{aligned}$$

It follows that all inequalities in (7.1.9) and (7.1.10) must be equalities. Hence, for every $1 \le l \le N$ and every $y_{j,0} \notin \cup_l \Gamma_l$

$$\ell(z_l) = \deg(\Gamma, L_\theta) \le \frac{1}{\epsilon} \quad \text{and} \quad \ell(y_{j,0}) = \ell(y_j) \le \frac{1}{\epsilon}$$

satisfy condition (2) in Definition 7.1.3. The ampleness condition (1) of Definition 7.1.3 holds by construction on the remaining rational tails in $\bar{C}$, while on rational components with exactly two special points it is verified in the same way as in the proof of Proposition 4.3.1. We conclude that the limit thus constructed is an ϵ-stable quasimap. $\quad\square$

Theorem 7.1.6 is a generalization of the corresponding results in [13,14] for projective spaces and [15] for Grassmannians. The chamber structure for the stability parameter also extends to general targets $W/\!\!/\mathbf{G}$. We restrict for brevity to describing it when $(g, k) \neq (0, 0), (0, 1)$ and leave the remaining cases to the reader. We have already noted that

$$\mathrm{Qmap}^\epsilon_{g,k}(W/\!\!/\mathbf{G}, \beta) = \mathrm{Qmap}_{g,k}(W/\!\!/\mathbf{G}, \beta) \quad \text{for } \epsilon \in \left(0, \frac{1}{\beta(L_\theta)}\right],$$

while

$$\mathrm{Qmap}^\epsilon_{g,k}(W/\!\!/\mathbf{G}, \beta) = \overline{M}_{g,k}(W/\!\!/\mathbf{G}, \beta) \quad \text{for } \epsilon \in (1, \infty).$$

It is equally easy to see that for every integer $1 \le m \le \beta(L_\theta) - 1$, the moduli space of ϵ-stable quasimaps to $W/\!\!/\mathbf{G}$ with fixed numerical data (g, k, β) stays constant when $\epsilon \in \left(\frac{1}{m+1}, \frac{1}{m}\right]$. The underlying curve of a quasimap parametrized by this constant moduli space is allowed only rational tails of total degree at least $m + 1$ (with respect to L_θ), while the order of contact of the quasimap with the unstable locus W^{us} at each base point must be at most m.

In contrast with Toda's results for Grassmannians in [15], it is not true in general that ϵ-quasimap invariants remain unchanged when crossing a wall $\epsilon = \frac{1}{m}$. Some aspects of this are discussed in the work [31], where it is proved for example that the equality still holds for toric Fano varieties, but fails in the absence of the Fano condition. The wall-crossing contributions in the non-Fano case (and genus zero) may be viewed as being responsible for Givental's mirror formulas, [37]. Similar interpretations may be given for K-nef complete intersections in toric varieties. We believe this is a general phenomenon and plan to study it elsewhere.

7.2. Quasimaps with one parametrized component

This version of quasimap moduli spaces is discussed in detail in [5] for the case of toric varieties. They are the quasimap analogues of the so-called *graph spaces* in Gromov–Witten theory and lead to extensions of the I-functions introduced by Givental to the large parameter space, i.e., to the entire Frobenius manifold given by the Gromov–Witten theory of a target space $W/\!\!/\mathbf{G}$.

The quasimaps with one parametrized $\mathbb{P}^1$ and the associated I-functions will play a significant role in the forthcoming work [31], which will deal with comparisons between quasimap and GW theories for certain classes of targets. We restrict here to giving the definitions and a few basic facts.

Fix integers $g, k \ge 0$, a GIT quotient $W/\!\!/\mathbf{G}$ with W affine and lci as in Section 4, and a curve class β. In addition, fix a nonsingular irreducible projective curve D.

Definition 7.2.1. A stable, k-pointed quasimap of genus g and class β to $W/\!\!/\mathbf{G}$ is specified by the data

$$((C, p_1, \ldots, p_k), P, u, \varphi),$$

where

- $(C, p_1, \ldots, p_k)$ is a connected, at most nodal, projective curve of genus g, and p_i are distinct nonsingular points of C,
- P is a principal $\mathbf{G}$-bundle on C,

- u is a section of the fiber bundle

$$P \times_{\mathbf{G}} W \to C,$$

- $\varphi : C \longrightarrow D$ is a regular map,

  subject to the conditions:

(1) (parametrized component) $\varphi_*[C] = [D]$. Equivalently, there is a distinguished component C_0 of C such that φ restricts to an isomorphism $C_0 \cong D$ and $\varphi(C \setminus C_0)$ is zero-dimensional (or empty, if $C = C_0$).
(2) (generic nondegeneracy) There is a finite (possibly empty) set of points $B \subset C$ such that $u(C \setminus B)$ is contained in the stable locus $P \times_{\mathbf{G}} W^s$.
(3) (prestable) The base locus B is disjoint from the nodes and markings on C.
(4) (stability) The line bundle

$$\omega_C(p_1 + \cdots + p_k) \otimes \mathcal{L}^\epsilon \otimes \varphi^*(\omega_D^{-1} \otimes \mathcal{M})$$

  is ample for every rational $\epsilon > 0$, where $\mathcal{L} := \mathcal{L}_\theta = P \times_{\mathbf{G}} \mathbb{C}_\theta$ and $\mathcal{M}$ is any ample line bundle on D. (Equivalently, $\omega_{\tilde{C}}(\sum p_i + \sum q_j) \otimes \mathcal{L}^\epsilon$ is ample, where $\tilde{C}$ is the closure of $C \setminus C_0$, p_i are the markings on $\tilde{C}$, and q_j are the nodes $\tilde{C} \cap C_0$.)
(5) the class of the quasimap $((C, p_1, \ldots, p_k), P, u)$ is β.

We denote by

$$\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta; D)$$

the stack parametrizing the stable quasimaps in Definition 7.2.1. Note that it is empty if $g < g(D)$. However, since stability imposes no condition on the distinguished component, the inequality $2g - 2 + k \geq 0$ is not required anymore. By the same arguments as in Section 4 we obtain the following.

Theorem 7.2.2. *If $W /\!\!/ \mathbf{G}$ is a proper, then $\mathrm{Qmap}_{g,k}(W /\!\!/ \mathbf{G}, \beta; D)$ is a proper Deligne–Mumford stack of finite type, with a perfect obstruction theory. In general, it is proper over the affine quotient $\mathrm{Spec}(A(W)^{\mathbf{G}})$.*

The most interesting case is when $g = 0$ and (necessarily) $D = \mathbb{P}^1$, to which we restrict from now on. The underlying curve for a point in $\mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$ is a k-pointed tree of rational curves with one parametrized component $C_0 \cong \mathbb{P}^1$, that is, a point in the stack $\widetilde{\mathbb{P}^1[k]}$, the Fulton–MacPherson space of (not necessarily stable) configurations of k distinct points on $\mathbb{P}^1$. This is a smooth Artin stack, locally of finite type (see Section 2.8 in [38]). As in Section 1, we have a smooth Artin stack

$$\mathfrak{Bun}_{\mathbf{G}} \longrightarrow \widetilde{\mathbb{P}^1[k]},$$

parametrizing principal $\mathbf{G}$-bundles on the fibers of the universal curve over $\widetilde{\mathbb{P}^1[k]}$.

Forgetting the appropriate data, we obtain morphisms of stacks

$$\mu : \mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1) \longrightarrow \mathfrak{Bun}_{\mathbf{G}}$$

and

$$\nu : \mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1) \longrightarrow \widetilde{\mathbb{P}^1[k]}.$$

The natural obstruction theory to consider is the μ-relative one, given again by (4.4.1).

As before, if $W = V$ is a vector space, one sees easily that we have an Euler sequence

$$0 \longrightarrow \mathfrak{P} \times_{\mathbf{G}} \mathfrak{g} \longrightarrow \mathfrak{P} \times_{\mathbf{G}} V \longrightarrow \mathscr{F} \longrightarrow 0$$

on the universal curve

$$\pi : \mathcal{C} \longrightarrow \mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$$

and that the μ-relative perfect obstruction theory is given by

$$(R^\bullet \pi_*(\mathfrak{P} \times_{\mathbf{G}} V))^\vee ,$$

while the ν-relative perfect obstruction theory is

$$(R^\bullet \pi_* \mathscr{F})^\vee .$$

Example 7.2.3. The unpointed genus zero quasimap moduli spaces compactify the Hom-schemes parametrizing maps from $\mathbb{P}^1$ to $W /\!\!/ \mathbf{G}$ and have been studied extensively earlier for several special cases of targets, though it was not widely recognized before that the important common feature of all these examples is that they are all GIT quotients.

- Let $W = \mathbb{C}^N$ and $\mathbf{G} = (\mathbb{C}^*)^r$. Then $W /\!\!/ \mathbf{G}$ is a toric variety and $\mathrm{Qmap}_{0,0}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$ is the "toric compactification" of $\mathrm{Map}_\beta(\mathbb{P}^1, W /\!\!/ \mathbf{G})$ introduced by Givental, and by Morrison and Plesser [37,39].

- Let $W = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $\mathbf{G} = GL(r, \mathbb{C})$ with the obvious action, linearized by the determinant. Then $W /\!\!/ \mathbf{G}$ is the Grassmannian $\mathrm{G}(r, n)$ of r-dimensional subspaces in $\mathbb{C}^n$ and $\mathrm{Qmap}_{0,0}(W /\!\!/ \mathbf{G}, d; \mathbb{P}^1)$ is the Quot scheme parametrizing degree d and rank $n - r$ quotients of $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ on $\mathbb{P}^1$, first studied in detail by Strømme, [40]. More generally, the "Laumon spaces", or "hyperquot schemes" on $\mathbb{P}^1$ of [41], see also [42,43] are quasimap moduli spaces to flag varieties of type A, viewed as GIT quotients of a vector space by a product of GL's, as described e.g. in [44].
- Consider the vector space

$$V := \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$$

  and let $\mathbf{G} = GL(n, \mathbb{C})$ act on the right by

$$(A, B, i, j) \cdot g = (g^{-1}Ag, g^{-1}Bg, g^{-1}i, jg),$$

  with linearization given by the determinant character.

  Let W be the $\mathbf{G}$-invariant nonsingular affine subvariety given by the ADHM equation

$$[A, B] - ij = 0.$$

  Then $W /\!\!/ \mathbf{G}$ is the moduli space of rank r torsion-free sheaves on $\mathbb{P}^2$, trivialized on the line at infinity (in particular, for $r = 1$ we get the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ of n points in the plane), see e.g. [45], and $\mathrm{Qmap}_{0,0}(W /\!\!/ \mathbf{G}, d; \mathbb{P}^1)$ is Diaconescu's moduli space of ADHM sheaves on $\mathbb{P}^1$, see [3].

  Note that W is the zero locus of a section of the equivariant $\mathbf{G}$-bundle $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \times V$, as discussed in Example 4.5.5. Note also that neither $V /\!\!/ \mathbf{G}$, nor $W /\!\!/ \mathbf{G}$ are proper; in fact, these examples are special cases of quiver variety targets, as discussed in Section 6.3.

Remark 7.2.4. It is immediate to extend Definition 7.2.1 and Theorem 7.2.2 to include all ϵ-stability conditions, as in Section 7.1. The same is true about the material on I-functions discussed in Section 7.3 below. We leave this to the reader.

7.3. The I-function of $W /\!\!/ \mathbf{G}$

One of the most important objects in genus zero Gromov–Witten theory is the J-function of a target variety. For smooth toric varieties Givental introduced in [37] their *small I-functions* via the unpointed genus zero parametrized quasimap spaces and proved that they are related to the small J-functions via "mirror transformations". In [5], the parametrized quasimap spaces with markings were used to define the big I-function of a smooth toric variety. Here we generalize these notions to all GIT quotients $W /\!\!/ \mathbf{G}$.

The moduli spaces $\mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$ come equipped with a natural $\mathbb{C}^*$-action, induced from the usual action on $\mathbb{P}^1$. The I-function will be defined via certain equivariant residues for this action, so we start by describing the $\mathbb{C}^*$-fixed loci in $\mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$.

A stable quasimap

$$((C, p_1, \ldots, p_k), P, u, \varphi)$$

is fixed by $\mathbb{C}^*$ if and only if it satisfies the following properties:

- $C_0 \cap \overline{(C \setminus C_0)} \subset \{0, \infty\}$; here C_0 is identified with $\mathbb{P}^1$ via φ.
- There are no markings on $C_0 \setminus \{0, \infty\}$.
- The curve class β is "concentrated at 0 or ∞". Precisely, this means the following: there are no base points on $C_0 \setminus \{0, \infty\}$ and the resulting map

$$C_0 \setminus \{0, \infty\} \longrightarrow W /\!\!/ \mathbf{G}$$

is constant.

The absolute perfect obstruction theory

$$\mathbb{E}^\bullet = [E^{-1} \longrightarrow E^0]$$

of $\mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$ is $\mathbb{C}^*$-equivariant, hence each component F of the fixed point locus has an induced perfect obstruction theory, given by the $\mathbb{C}^*$-fixed part $\mathbb{E}_F^{\bullet, f}$ of the complex $\mathbb{E}^\bullet|_F$, and a virtual normal bundle N_F^{vir} given by the moving part $\mathbb{E}_F^{\bullet, m}$.

In fact, for defining the I-function we are only interested in the component F_0 of the fixed point locus for which the curve class β is concentrated only at $0 \in C_0$, i.e., the component parametrizing $\mathbb{C}^*$-fixed quasimaps for which $C_0 \cap \overline{(C \setminus C_0)} = \{0\}$ and there is no base point or marking at ∞. The cases $k = 0$ and $k \geq 1$ exhibit different behavior.

Case $k \geq 1$: We have $F_0 \cong \mathrm{Qmap}_{0,k+1}(W /\!\!/ \mathbf{G}, \beta)$ and $\mathbb{E}_{F_0}^{\bullet, f}$ is the usual obstruction theory of $\mathrm{Qmap}_{0,k+1}(W /\!\!/ \mathbf{G}, \beta)$. The (virtual) codimension of F_0 in $\mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)$ is equal to 2. As explained in Lemma 7.2.7 of [5], the virtual normal bundle has

equivariant Euler class

$$e^{\mathbb{C}^*}(N_{F_0}^{vir}) = z(z - \psi),$$

where $\psi = \psi_{k+1}$ is the cotangent class at the last marking and z is the equivariant parameter (i.e., $H^*_{\mathbb{C}^*}(point) \cong \mathbb{C}[z]$).

Associating to each $\mathbb{C}^*$-fixed stable quasimap in F_0 the point in $W /\!\!/ \mathbf{G}$ which is the image of the constant map

$$C_0 \setminus \{0, \infty\} \longrightarrow W /\!\!/ \mathbf{G},$$

gives a morphism

$$F_0 \longrightarrow W /\!\!/ \mathbf{G} \tag{7.3.1}$$

which is clearly just the evaluation map

$$ev_{k+1} : \mathrm{Qmap}_{0,k+1}(W /\!\!/ \mathbf{G}, \beta) \longrightarrow W /\!\!/ \mathbf{G}$$

at the last marking.

Case $k = 0$: In this case the above identification of F_0 with a space of unparametrized quasimaps is not possible, as $\mathrm{Qmap}_{0,1}(W /\!\!/ \mathbf{G}, \beta)$ does not exist.

For toric varieties, when $\mathbf{G} = (\mathbb{C}^*)^n$ is a torus and W is a vector space, F_0 is identified with a certain nonsingular subvariety of $W /\!\!/ \mathbf{G}$ (depending on β). Its virtual class is the usual fundamental class and the virtual normal bundle can be explicitly computed using appropriate Euler sequences, see e.g. [37,5]. For now, we contend ourselves to remark that in the general case the set-theoretic "evaluation" map defined as in (7.3.1) can easily be seen to give a morphism

$$ev : F_0 \longrightarrow W /\!\!/ \mathbf{G}.$$

In the toric case, this is simply the embedding of F_0 in $W /\!\!/ \mathbf{G}$. For $W /\!\!/ \mathbf{G}$ a Grassmannian, the quasimap space with parametrized $\mathbb{P}^1$ and no markings is a Quot scheme on $\mathbb{P}^1$. The fixed locus F_0 is identified with a disjoint union of flag bundles over the Grassmannian, while the evaluation map is the projection. Again, the top Chern class of the virtual normal bundle, as well as its push-forward by the evaluation map, can be calculated via appropriate Euler sequences, see [46]. One can extend this description of F_0 and the calculation of the normal bundle in a straightforward manner to all quotients of a vector space by a general linear group.

We are now ready to define the *I*-function. Recall from Definition 3.2.2 that the curve classes of quasimaps form a semigroup $\mathrm{Eff}(W, \mathbf{G}, \theta)$. We let $N(W, \mathbf{G}, \theta)$ denote the Novikov ring of formal power series

$$N(W, \mathbf{G}, \theta) = \left\{ \sum_{\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)} a_\beta Q^\beta \,\middle|\, a_\beta \in \mathbb{C} \right\},$$

the Q-adic completion of the semigroup ring $\mathbb{C}[\mathrm{Eff}(W, \mathbf{G}, \theta)]$.

Definition 7.3.1. The (big) *I*-function of $W /\!\!/ \mathbf{G}$ is

$$\begin{aligned}
I_{W /\!\!/ \mathbf{G}}(\mathbf{t}) = {} & 1 + \frac{\mathbf{t}}{z} + \sum_{\beta \neq 0} Q^\beta ev_* \left(\frac{[F_0]^{vir}}{e^{\mathbb{C}^*}(N_{F_0}^{vir})} \right) \\
& + \sum_\beta Q^\beta \sum_{k \geq 1} \frac{1}{k!} (ev_{k+1})_* \left(\frac{[\mathrm{Qmap}_{0,k+1}(W /\!\!/ \mathbf{G}, \beta)]^{vir}}{z(z - \psi)} \prod_{j=1}^{k} ev_j^*(\mathbf{t}) \right).
\end{aligned}$$

It is a formal function of $\mathbf{t} \in H^*(W /\!\!/ \mathbf{G}, \mathbb{C})$, taking values in

$$H^*(W /\!\!/ \mathbf{G}, \mathbb{C}) \otimes_{\mathbb{C}} N(W, \mathbf{G}, \theta) \left\{\!\!\left\{ \frac{1}{z} \right\}\!\!\right\}.$$

The above definition is an exact analogue of Givental's big *J*-function of $W /\!\!/ \mathbf{G}$ in Gromov–Witten theory, with the graph spaces

$$G_{k,\beta}(W /\!\!/ \mathbf{G}) := \overline{M}_{0,k}(W /\!\!/ \mathbf{G} \times \mathbb{P}^1, (\beta, 1))$$

replaced by the quasimap spaces with one parametrized component. Each of the terms in the sums is the push-forward via the appropriate evaluation map of the residue at F_0 of the class

$$\left[\mathrm{Qmap}_{0,k}(W /\!\!/ \mathbf{G}, \beta; \mathbb{P}^1)\right]^{vir} \cap \prod_{j=1}^{k} ev_j^*(\mathbf{t}).$$

In fact, as mentioned in Remark 7.2.4, we may define by the same formula an ϵ-*I*-function for each $\epsilon > 0$, using the virtual classes of ϵ-stable parametrized quasimaps. When $\epsilon > 1$ this definition reproduces precisely the usual definition of the *J*-function.

The *small I*-function is defined by restriction to the so-called small parameter space:

$$I^{small}_{W/\!\!/\mathbf{G}}(\mathbf{t}^0, \mathbf{t}^2) = I_{W/\!\!/\mathbf{G}}(\mathbf{t}_{small}),$$

with

$$\mathbf{t}_{small} = \mathbf{t}^0 + \mathbf{t}^2 \in H^0(W/\!\!/\mathbf{G}, \mathbb{C}) \oplus H^2(W/\!\!/\mathbf{G}, \mathbb{C}).$$

By analogy with [37], we also define formally *Givental's small I*-function by the formula

$$
\begin{aligned}
I^{Givental}_{W/\!\!/\mathbf{G}}(\mathbf{t}^0, \mathbf{t}^2) &= e^{(\mathbf{t}^0+\mathbf{t}^2)/z} \left(1 + \sum_{\beta \neq 0} Q^\beta e^{\beta(\mathbf{t}^2)} I^{Givental}_\beta \right) \\
&:= e^{(\mathbf{t}^0+\mathbf{t}^2)/z} \left(1 + \sum_{\beta \neq 0} Q^\beta e^{\beta(\mathbf{t}^2)} ev_* \left(\frac{[F_0]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{F_0})} \right) \right).
\end{aligned}
\tag{7.3.2}
$$

(To avoid technicalities with the definition of $\beta(\mathbf{t}^2)$, assume here that we have an identification between effective **G**-equivariant curve classes and actual effective curve classes $\beta \in H_2(W/\!\!/\mathbf{G}, \mathbb{Z})$.) When quasimap integrals of $W/\!\!/\mathbf{G}$ satisfy both the string and divisor equations, the two small *I*-functions coincide, however, they will be different in general.

Acknowledgments

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