



Low-dimensional filiform Lie superalgebras



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ABSTRACT

The present work is regarding *filiform Lie superalgebras* which is an important type of nilpotent Lie superalgebras. In general, classifying nilpotent Lie superalgebras is at present an open and unsolved problem. Throughout the present work we contribute to the resolution of this wide problem by classifying filiform Lie superalgebras of low dimensions, in particular less or equal to 7. Furthermore we would establish a method that could be applied to obtain similar results for higher dimensions. Thus, this method would mainly consist in using infinitesimal deformations of the model filiform Lie superalgebra.

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1. Introduction

The concept of *filiform Lie algebras* was firstly introduced in [1] by Vergne. Moreover this type of nilpotent Lie algebras has important properties such as every filiform Lie algebra can be obtained by a deformation of the model filiform algebra L_n .

The present work is regarding *filiform Lie superalgebras*, a generalization of filiform Lie algebras and an important type of nilpotent Lie superalgebras. It has been proved that in the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$. Thus, throughout this work we give a complete classification (up to isomorphisms) of complex filiform Lie superalgebras of dimension less or equal to 7 by means of the above mentioned deformations. Furthermore our method could be used in higher dimensions.

We will therefore consider infinitesimal deformations of $L^{n,m}$ which are defined by even 2-cocycles in $Z_0^2(L^{n,m}, L^{n,m})$. These deformations have been almost totally determined along the papers [2–5]. In some particular cases before applying our method we had to obtain some of these deformations according to the instructions described in the above mentioned papers.

All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be \mathbb{C} -vector spaces of finite dimension. Moreover, we shall use the well-known convention that for the definition of a (super) Lie bracket in terms of a basis only the non-vanishing brackets in some ordering of the base are explicitly mentioned.

2. Preliminaries

A *superspace* is nothing but a vector space with a \mathbb{Z}_2 -grading: $V = V_0 \oplus V_1$. Elements of the space V_0 are usually called even, and elements of the space V_1 , odd; the indices 0 and 1 are modulo 2. A linear map $\phi : V \rightarrow W$ between two super

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vector spaces is called *even* iff $\phi(V_0) \subset W_0$ and $\phi(V_1) \subset W_1$ and is called *odd* iff $\phi(V_0) \subset W_1$ and $\phi(V_1) \subset W_0$. Thus, we will have $\text{Hom}(V, W) = \text{Hom}(V, W)_0 \oplus \text{Hom}(V, W)_1$ where the first summand is composed by all the even and the second summand by all the odd linear maps.

A Lie superalgebra (see [6,7]) is a superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with an even bilinear commutation operation (or “supercommutation”) $[\cdot, \cdot]$, which satisfies the conditions:

- $[X, Y] = -(-1)^{\alpha\beta}[Y, X] \quad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
- $(-1)^{\gamma\alpha}[X, [Y, Z]] + (-1)^{\alpha\beta}[Y, [Z, X]] + (-1)^{\beta\gamma}[Z, [X, Y]] = 0$
for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$. (Graded Jacobi identity).

Thus, \mathfrak{g}_0 is an ordinary Lie algebra, and \mathfrak{g}_1 is a module over \mathfrak{g}_0 ; the Lie superalgebra structure also contains the symmetric pairing $S^2 \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$.

The descending central sequence of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined by $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$ for all $k \geq 0$. If $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k , the Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the *nilindex* of \mathfrak{g} . Analogously as for Lie algebras, the dimension of each term of the descending central sequence is an invariant of the superalgebra. Likewise it can be noted that each term of this sequence has an even part and an odd part, that is $\mathcal{C}^k(\mathfrak{g}) = (\mathcal{C}^k(\mathfrak{g}))_0 \oplus (\mathcal{C}^k(\mathfrak{g}))_1$, whose dimensions are also invariants of the superalgebra.

There are also defined two other descending sequences called $\mathcal{C}^k(\mathfrak{g}_0)$ and $\mathcal{C}^k(\mathfrak{g}_1)$:

$$\mathcal{C}^0(\mathfrak{g}_i) = \mathfrak{g}_i, \quad \mathcal{C}^{k+1}(\mathfrak{g}_i) = [\mathfrak{g}_0, \mathcal{C}^k(\mathfrak{g}_i)], \quad k \geq 0, i \in \{0, 1\}.$$

If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a nilpotent Lie superalgebra, then \mathfrak{g} has super-nilindex or *s-nilindex* (p, q) , if the following conditions hold:

$$(\mathcal{C}^{p-1}(\mathfrak{g}_0)) \neq 0 \quad (\mathcal{C}^{q-1}(\mathfrak{g}_1)) \neq 0, \quad \mathcal{C}^p(\mathfrak{g}_0) = \mathcal{C}^q(\mathfrak{g}_1) = 0.$$

It can be noted that a module $A = A_0 \oplus A_1$ of the Lie superalgebra \mathfrak{g} is an even bilinear map $\mathfrak{g} \times A \rightarrow A$ satisfying

$$\forall X \in \mathfrak{g}_\alpha, \quad Y \in \mathfrak{g}_\beta \quad a \in A: \quad X(Ya) - (-1)^{\alpha\beta}Y(Xa) = [X, Y]a.$$

Lie superalgebra cohomology is defined in the following well-known way (see e.g. [6,8]): the superspace of *q-dimensional cocycles* of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with coefficients in the \mathfrak{g} -module $A = A_0 \oplus A_1$ is given by

$$C^q(\mathfrak{g}; A) = \bigoplus_{q_0+q_1=q} \text{Hom}(\wedge^{q_0} \mathfrak{g}_0 \otimes S^{q_1} \mathfrak{g}_1, A).$$

The above space is graded by $C^q(\mathfrak{g}; A) = C_0^q(\mathfrak{g}; A) \oplus C_1^q(\mathfrak{g}; A)$ with

$$C_p^q(\mathfrak{g}; A) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r \equiv p \pmod{2}}} \text{Hom}(\wedge^{q_0} \mathfrak{g}_0 \otimes S^{q_1} \mathfrak{g}_1, A_r).$$

Thus we have the *cohomology groups*

$$H_p^q(\mathfrak{g}; A) = Z_p^q(\mathfrak{g}; A)/B_p^q(\mathfrak{g}; A)$$

where, in particular, the elements of $Z_0^q(\mathfrak{g}; A)$ and $Z_1^q(\mathfrak{g}; A)$ are called *even q-cocycles* and *odd q-cocycles* respectively.

On the other hand and in complete analogy to Lie algebras [9–11] we denote by $\mathcal{N}^{n+1,m}$ the variety of nilpotent Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\dim \mathfrak{g}_0 = n+1$ and $\dim \mathfrak{g}_1 = m$. Thus, we will have the following definition

Definition 2.1 ([12]). Any nilpotent Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{N}^{n+1,m}$ with s-nilindex (n, m) is called *filiform*.

We denote by $\mathcal{F}^{n+1,m}$ the subset of $\mathcal{N}^{n+1,m}$ consisting of all the filiform Lie superalgebras.

Before studying this family of Lie superalgebras it is convenient to solve the problem of finding a suitable basis, a so-called *adapted basis*.

Theorem 2.1 ([12]). If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \in \mathcal{F}^{n+1,m}$, then there exists an adapted basis of \mathfrak{g} , namely $\{X_0, X_1, \dots, X_n, Y_1, \dots, Y_m\}$, with $\{X_0, X_1, \dots, X_n\}$ a basis of \mathfrak{g}_0 and $\{Y_1, \dots, Y_m\}$ a basis of \mathfrak{g}_1 , such that:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1, \\ [X_0, X_n] = 0, \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1, \\ [X_0, Y_m] = 0. \end{cases}$$

X_0 is called the *characteristic vector*.

From now on all the filiform superalgebras, that we are going to consider throughout the present paper, would be expressed in an adapted basis.

It can be observed that the simplest filiform Lie superalgebra, denoted by $L^{n,m}$, will be defined by the following brackets:

$$L^{n,m} : \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq n-1, \\ [X_0, Y_j] = Y_{j+1}, & 1 \leq j \leq m-1, \end{cases}$$

with $\{X_0, X_1, \dots, X_n, Y_1, \dots, Y_m\}$ a basis of $L^{n,m}$. We will note by μ_0 the law of $L^{n,m}$.

$L^{n,m}$ will be the most important filiform Lie superalgebra, in complete analogy to Lie algebras, since all the other filiform Lie superalgebras can be obtained from it by deformations. Thus, we are going to consider its infinitesimal deformations that will be given by the even 2-cocycles, $Z_0^2(L^{n,m}, L^{n,m})$.

Therefore, an infinitesimal deformation of $L^{n,m}$ will be an element of the following space

$$\begin{aligned} Z_0^2(L^{n,m}, L^{n,m}) &= Z^2(L^{n,m}, L^{n,m}) \cap \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_0, \mathfrak{g}_0) \\ &\quad \oplus Z^2(L^{n,m}, L^{n,m}) \cap \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_1, \mathfrak{g}_1) \\ &\quad \oplus Z^2(L^{n,m}, L^{n,m}) \cap \text{Hom}(S^2 \mathfrak{g}_1, \mathfrak{g}_0) \\ &= A \oplus B \oplus C \end{aligned}$$

where $\mathfrak{g}_0 = L_0^{n,m}$ and $\mathfrak{g}_1 = L_1^{n,m}$. The third component has been determined in [3] and [4], and the first and the second component has been determined in [5] and [2] respectively.

3. Method of classification

The main result that we are going to use is that every filiform Lie superalgebra can be expressed by

$$\mu_0 + \Psi$$

with μ_0 the law of the model filiform Lie superalgebra and Ψ a linear (infinitesimal) deformation that verifies $\Psi \circ \Psi = 0$, with

$$\Psi \circ \Psi(x, y, z) = \Psi(\Psi(x, y), z) + \Psi(\Psi(z, x), y) + \Psi(\Psi(y, z), x).$$

The above result is not only true for Lie algebras or superalgebras but it is also true for color Lie superalgebras. Color Lie superalgebras are a generalization of Lie superalgebras and likewise filiform color Lie superalgebras are nothing but a generalization of filiform Lie superalgebras. Indeed, Lie superalgebras are defined through antisymmetric or symmetric products, although for color Lie superalgebras the product is neither symmetric nor antisymmetric and it is defined by means of a commutation factor β . Furthermore and similarly to Lie algebras, the basic tool for defining color Lie superalgebras is a grading determined, in this case, by an abelian group G instead of \mathbb{Z}_2 which is associated with Lie superalgebras. Thus in [13, p. 14] can be found the following theorem.

Theorem 2 ([13]). (1) Any filiform (G, β) -color Lie superalgebra law μ is isomorphic to $\mu_0 + \varphi$ where μ_0 is the law of the model filiform (G, β) -color Lie superalgebra and φ is an infinitesimal deformation of μ_0 verifying that $\varphi(X_0, X) = 0$ for all $X \in L$, with X_0 the characteristic vector of the model one.

(2) Conversely, if φ is an infinitesimal deformation of a model filiform (G, β) -color Lie superalgebra law μ_0 with $\varphi(X_0, X) = 0$ for all $X \in L$, then the law $\mu_0 + \varphi$ is a filiform (G, β) -color Lie superalgebra law iff $\varphi \circ \varphi = 0$.

$$\varphi \circ \varphi(X, Y, Z) = \beta(k, g)\varphi(X, \varphi(Y, Z)) + \beta(h, k)\varphi(Z, \varphi(X, Y)) + \beta(g, h)\varphi(Y, \varphi(Z, X)) \text{ for all } X \in L_g, Y \in L_h \text{ and } Z \in L_k.$$

The proof of the theorem can be consulted in the above mentioned paper. Consequently, if we replace the generic abelian group G by \mathbb{Z}_2 and the commutator factor β by $\beta(i, j) = (-1)^{ij} \forall i, j \in \mathbb{Z}_2$, we will obtain that “any filiform Lie superalgebra is a linear deformation of the corresponding model filiform Lie superalgebra”. Thus, it can be expressed by $\mu_0 + \Psi$ with μ_0 the law of the model filiform Lie superalgebra and Ψ a linear (infinitesimal) deformation verifying $\Psi \circ \Psi = 0$.

We are therefore going to consider Ψ such that

$$\Psi = a\Psi_A + b\Psi_B + c\Psi_C$$

with $a, b, c \in \mathbb{C}$, $\Psi_A \in A$, $\Psi_B \in B$ and $\Psi_C \in C$, and we are going to impose the condition to be integrable, that is $\Psi \circ \Psi = 0$. Finally we will classify the family of laws $\mu_0 + \Psi$ which remains. Two important facts to be considered: first we will always have $c \neq 0$ otherwise we have Lie superalgebras that are actually Lie algebras (degenerate cases); second if $a = b = 0$ then Ψ is always integrable.

4. Case $n = 2$

In this section we are going to consider the model filiform Lie superalgebra $L^{2,m}$ for each m such that $1 \leq m \leq 4$.

4.1. $n = 2, m = 1$

In this case, as infinitesimal deformations of A we only have $\Psi_{1,2}$ defined by $\Psi_{1,2}(X_1, X_2) = X_2$, see [5]. It can be seen that $\mu_0 + \Psi_{1,2}$ would lead to a non-nilpotent Lie algebra and therefore to a non-nilpotent Lie superalgebra and thus it will not

be considered. Searching the cocycles of B and in spite of the fact that in [2] we cannot find explicitly our case, following the steps described in the paper it can be obtained the infinitesimal deformation of B to be considered, that is $\Psi_{1,1}^1$ defined by $\Psi_{1,1}^1(X_1, Y_1) = Y_1$. But this infinitesimal deformation would not be faithful with the structure of filiform Lie superalgebra. Thus, only remain to consider the subspace C , studied in [3], or $\varphi_{1,2}$ defined by $\varphi_{1,2}(Y_1, Y_1) = X_2$. Recall that all the cocycles belonging to C are integrable and so $\varphi_{1,2} \circ \varphi_{1,2} = 0$ which leads to the unique filiform Lie superalgebra with $n = 2$ and $m = 1$ whose law can be expressed in the adapted basis $\{X_0, X_1, X_2, Y_1\}$ by the following bracket products

$$\mu_0 + \varphi_{1,2} : \begin{cases} [X_0, X_1] = X_2 \\ (Y_1, Y_1) = X_2. \end{cases}$$

Remark 4.1. In order to differentiate between symmetric and skew-symmetric products we will note, from now on, the symmetric bracket products by $(\ , \)$.

4.2. $n = 2, m = 2$

From consulting the papers: [5, p. 1169], [2, p. 138] and [4, p. 853], we can extract the following basis of infinitesimal deformations:

- Basis of A : $\{\Psi_{1,2}\}$ with $\Psi_{1,2}(X_1, X_2) = X_2$.
- Basis of B : $\{\Psi_{1,1}^1, \Psi_{1,1}^2, \Psi_{2,1}^2\}$ defined by $\{\Psi_{1,1}^1(X_1, Y_1) = Y_1, \Psi_{1,1}^1(X_1, Y_2) = Y_2\}$, $\{\Psi_{1,1}^2(X_1, Y_1) = Y_2\}$ and $\{\Psi_{2,1}^2(X_1, Y_2) = -Y_2, \Psi_{2,1}^2(X_2, Y_1) = Y_2\}$.
- Basis of C : $\{\varphi_{1,1}, \varphi_{1,2}\}$ defined by $\{\varphi_{1,1}(Y_1, Y_1) = X_1, \varphi_{1,2}(Y_1, Y_2) = \frac{1}{2}X_2\}$, and $\{\varphi_{1,2}(Y_1, Y_1) = X_2\}$.

Note that Ψ_{\dots} and Ψ_{\dots} are always skew symmetric and φ_{\dots} is symmetric.

If we consider the set $\{\Psi_{1,2}, \Psi_{1,1}^1, \Psi_{2,1}^2\}$ it can be seen that neither of them is faithful with the filiform structure and analogously any linear combination of them neither. Next, we consider therefore

$$\Psi = c\Psi_{1,1}^2 + a\varphi_{1,1} + b\varphi_{1,2}$$

with a or b non-zero. Thus, more specifically we would have the following bracket products

$$\Psi : \begin{cases} [X_1, Y_1] = cY_2 \\ (Y_1, Y_1) = aX_1 + bX_2, \quad (Y_1, Y_2) = \frac{a}{2}X_2. \end{cases}$$

Applying the condition to be integrable, $\Psi \circ \Psi = 0$, leads to the equation $ac = 0$. After that if we consider $\mu_0 + \Psi$ with $a = 0$ (the only possibility in which could appear c)

$$\begin{cases} [X_0, X_1] = X_2 \\ [X_0, X_1] = Y_2, \quad [X_1, Y_1] = cY_2 \\ (Y_1, Y_1) = bX_2, \end{cases}$$

and we apply the isomorphism defined by the matrix

$$\begin{pmatrix} 1 & -c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{cases} X'_0 = X_0 \\ X'_1 = X_1 - cX_0 \\ X'_2 = X_2 \\ Y'_i = Y_i, \quad 1 \leq i \leq 3 \end{cases}$$

we would obtain a new superalgebra with adapted basis $\{X'_0, X'_1, X'_2, Y'_1, Y'_2\}$, that is $[X'_0, X'_1] = X'_2$ and $[X'_0, Y'_1] = Y'_2$, verifying $(Y'_1, Y'_1) = bX'_2$ and $[X'_1, Y'_1] = [X_1 - cX_0, Y_1] = cY_2 - cY_2 = 0$. Therefore, we can suppose $c = 0$ without loss of generality. Thus, $\mu_0 + \Psi = \mu_0 + a\varphi_{1,1} + b\varphi_{1,2}$ and $\{\varphi_{1,1}, \varphi_{1,2}\}$ is a basis of C . It is not difficult to see that we will only have two non-isomorphic FLS expressed in the adapted basis $\{X_0, X_1, X_2, Y_1, Y_2\}$ by

$$\begin{aligned} \mu_0 + \varphi_{1,1} : \quad & \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = X_1, \quad (Y_1, Y_2) = \frac{1}{2}X_2 \end{cases} & \mu_0 + \varphi_{1,2} : \quad & \begin{cases} [X_0, X_1] = X_2, \quad [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = X_2, \end{cases} \end{aligned}$$

Remark 4.2. It can be seen that the above isomorphism is faithful with the adapted basis. In fact it can be determined only by the change of the basis vectors $X'_0 = X_0$, $X'_1 = X_1 - cX_0$ and $Y'_1 = Y_1$ which can be regarded as generators of the adapted basis in the sense that all the rest can be obtained from them. Thus, $X'_2 = [X'_0, X'_1]$ and $Y'_2 = [X'_0, Y'_1]$ and so on. Therefore, from now on and for simplicity instead of giving the whole matrix of the corresponding isomorphism we will provide X'_0, X'_1 and Y'_1 and the rest would be completed by the brackets that define the adapted basis: $X_{i+1} := [X'_0, X'_i]$ and $Y_{j+1} := [X'_0, Y'_j]$.

Remark 4.3. It can be observed too that if we have a FLS as $\mu_0 + \Psi = \mu_0 + a\Psi_a + b\Psi_b + c\varphi_c$, with $\Psi_a \in A$, $\Psi_b \in B$, $\Psi_c \in C$, then the Lie part of μ_0 , that is the brackets between X_i , together to $a\Psi_a$ will be a filiform Lie algebra. Therefore we can consult the classification of these algebras in [14] and use only the cocycles of A that appear in the corresponding classification according to the dimension.

4.3. $n = 2, m = 3$

From consulting the papers: [14, p. 148], [2, p. 138] and [4, p. 858], and after eliminating the infinitesimal deformations of B either they do not verify the condition $\Psi(X_0, -) = 0$ or they are not faithful with the filiform structure, remain the following infinitesimal deformations:

- Basis of B : $\{\Psi_{1,1}^2, \Psi_{1,1}^3, \Psi_{2,1}^3\}$ defined by $\{\Psi_{1,1}^2(X_1, Y_1) = Y_2, \Psi_{1,1}^3(X_1, Y_2) = Y_3\}$, $\{\Psi_{1,1}^3(X_1, Y_1) = Y_3\}$ and $\{\Psi_{2,1}^3(X_1, Y_2) = -Y_3, \Psi_{2,1}^3(X_2, Y_1) = Y_3\}$.
- Basis of C : $\{\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,2}\}$ defined by $\{\varphi_{1,1}(Y_1, Y_1) = X_1, \varphi_{1,2}(Y_1, Y_2) = \frac{1}{2}X_2\}$, $\{\varphi_{1,2}(Y_1, Y_1) = X_2\}$ and $\{\varphi_{2,2}(Y_1, Y_3) = -X_2, \varphi_{2,2}(Y_2, Y_2) = X_2\}$.

We will therefore consider $\Psi = a\Psi_{1,1}^2 + b\Psi_{1,1}^3 + c\Psi_{2,1}^3 + d\varphi_{1,1} + e\varphi_{1,2} + f\varphi_{2,2}$.

$$\Psi : \begin{cases} [X_1, Y_1] = aY_2 + bY_3 & (Y_1, Y_1) = dX_1 + eX_2 \\ [X_1, Y_2] = (a - c)Y_3 & (Y_1, Y_2) = \frac{d}{2}X_2 \\ [X_2, Y_1] = cY_3 & (Y_1, Y_3) = -fX_2 \\ (Y_2, Y_2) = fX_2. \end{cases}$$

Applying the condition to be integrable, $\Psi \circ \Psi = 0$, leads to the equations

- (1) $af = 0$
- (2) $ad = 0$
- (3) $bd + ec = 0$
- (4) $fc = 0$.

If we consider $\mu_0 + \Psi = \mathfrak{g}$ then it can be seen that the dimension of the even part of the third term of the descending central sequence depends on whether $f = 0$ or not. Thus, $\dim((\mathcal{C}^3(\mathfrak{g}))_0) = 0$ if $f = 0$ and $\dim((\mathcal{C}^3(\mathfrak{g}))_0) = 1$ otherwise. This fact allows us to differentiate between two cases non-isomorphic.

Case 1. $f \neq 0$. In this case the above mentioned equations remain $a = c = 0$ and $bd = 0$. By using generic isomorphisms, or so called change of basis, faithful with the expression of the superalgebra in an adapted basis it can be seen that it is not possible to eliminate b . More specifically, if we apply the change of basis defined by $\{X'_0 = 0, X'_1 = X_1 - AX_0, Y'_1 = Y_1 + BY_2\}$ we would obtain $[X'_1, Y'_1] = -AY'_2 + bY'_3$. Therefore we can consider two subcases non-isomorphic:

Subcase 1.1. $b \neq 0$. We would have $d = 0$ and the family of laws that follows:

$$\begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ [X_1, Y_1] = bY_3, & (Y_1, Y_1) = eX_2 \\ (Y_1, Y_3) = -fX_2, & (Y_2, Y_2) = fX_2. \end{cases}$$

We can suppose without loss of generality that $f = 1$. In fact, by means of the change of scale defined by $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = \frac{1}{\sqrt{f}}Y_1\}$ we would always obtain $f' = 1$, that is $(Y_1, Y_3) = -X_2$ and $(Y_2, Y_2) = X_2$. Note that a change of scale is nothing but a change of basis where each generator is a multiple of itself. Likewise it can be supposed firstly $e = 0$, thanks to the change of basis defined by $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = Y_1 + \frac{e}{2}Y_3\}$ and secondly $b = 1$, by the change of scale $\{X'_0 = bX_0, X'_1 = bX_1, Y'_1 = Y_1\}$. Thus, we will have the following FLS

$$\mu_0 + \Psi_{1,1}^3 + \varphi_{2,2} : \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_1] = Y_2 \\ [X_0, Y_2] = Y_3, & [X_1, Y_1] = Y_3 \\ (Y_1, Y_3) = -X_2, & (Y_2, Y_2) = X_2. \end{cases}$$

Subcase 1.2. $b = 0$. Thanks to the change of scale defined by $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = \frac{1}{\sqrt{f}}Y_1\}$ it can be supposed $f = 1$ and afterwards and by means of the change of basis $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = Y_1 + \frac{e}{2}Y_3\}$ it can be seen that there is no loss of generality in supposing $e = 0$ and consequently it remains

$$\begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ (Y_1, Y_1) = dX_1, & (Y_1, Y_2) = \frac{d}{2}X_2 \\ (Y_1, Y_3) = -X_2, & (Y_2, Y_2) = X_2. \end{cases}$$

It is not difficult to see that depending on whether d vanishes or not we would have different dimensions of the first term of the descending central sequence and therefore we can distinguish between the cases $d = 0$ and $d \neq 0$. In the former we

would obtain directly the FLS $\mu_0 + \varphi_{2,2}$ while in the latter we would obtain the FLS $\mu_0 + \varphi_{1,1} + \varphi_{2,2}$ after applying a change of scale, that is

$$\begin{aligned}\mu_0 + \varphi_{2,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ (Y_1, Y_3) = -X_2, & (Y_2, Y_2) = X_2 \end{cases} \\ \mu_0 + \varphi_{1,1} + \varphi_{2,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ (Y_1, Y_1) = X_1, & (Y_1, Y_2) = \frac{1}{2}X_2 \\ (Y_1, Y_3) = -X_2, & (Y_2, Y_2) = \tilde{X}_2. \end{cases}\end{aligned}$$

Case 2. $f = 0$. In this case we would have the equations $ad = 0$ and $bd + ec = 0$, and moreover it can be applied the same reasoning as the one applied in the precedent subcase regarding the value of d , which in turn leads to the subcases that follow.

Subcase 2.1. $d = 0$. The above equations remain $ec = 0$, but considering the restriction of obtaining always non-degenerated lie superalgebras we would obtain $e \neq 0$ and consequently $c = 0$. Next, using the change of basis $\{X'_0 = X_0, X'_1 = X_1 - aX_0, Y'_1 = Y_1\}$ leads to $a = 0$ and then it can be applied the change of scale $\{X'_0 = eX_0, X'_1 = X_1, Y'_1 = Y_1\}$ which would allow us to suppose $e = 1$. By using generic changes of basis, or isomorphisms, faithful with the expression of the superalgebra in an adapted basis it can be seen that it is not possible to eliminate b . Therefore we can distinguish between the cases $b = 0$ and $b \neq 0$, in the former we would obtain directly the FLS $\mu_0 + \varphi_{1,2}$ while in the latter we would obtain the FLS $\mu_0 + \Psi_{1,1}^3 + \varphi_{1,2}$ after applying a change of scale.

$$\begin{aligned}\mu_0 + \varphi_{1,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ (Y_1, Y_1) = X_2 \end{cases} \\ \mu_0 + \Psi_{1,1}^3 + \varphi_{1,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ [X_1, Y_1] = Y_3, & (Y_1, Y_1) = X_2. \end{cases}\end{aligned}$$

Subcase 2.1. $d \neq 0$. From the equations we would obtain $a = 0$ and $b = \frac{-ec}{d}$ and we would have

$$\mu_0 + \Psi : \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ [X_1, Y_1] = \frac{-ec}{d}Y_3 & (Y_1, Y_1) = dX_1 + eX_2 \\ [X_1, Y_2] = -cY_3 & (Y_1, Y_2) = \frac{d}{2}X_2 \\ [X_2, Y_1] = cY_3. \end{cases}$$

Applying the change of basis defined by $\{X'_0 = X_0, X'_1 = dX_1 + eX_2, Y'_1 = Y_1\}$ leads to $e = 0$ and $d = 1$. Next, if we consider $\mu_0 + \Psi = \mathfrak{g}$ then it can be seen that the dimension of the odd part of the third term of the descending central sequence depends on whether $c = 0$ or not. Thus, $\dim((\mathcal{C}^3(\mathfrak{g}))_1) = 0$ if $c = 0$ and $\dim((\mathcal{C}^3(\mathfrak{g}))_1) = 1$ otherwise. This fact allows us to differentiate between two cases non-isomorphic $c = 0$ and $c \neq 0$, in the former we would obtain directly the FLS $\mu_0 + \varphi_{1,1}$ while in the latter we would obtain the FLS $\mu_0 + \Psi_{2,1}^3 + \varphi_{1,1}$ after applying a change of scale.

$$\begin{aligned}\mu_0 + \varphi_{1,1} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ (Y_1, Y_1) = X_1 & (Y_1, Y_2) = \frac{1}{2}X_2 \end{cases} \\ \mu_0 + \Psi_{2,1}^3 + \varphi_{1,1} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 2 \\ [X_1, Y_2] = -Y_3, & [X_2, Y_1] = Y_3 \\ (Y_1, Y_1) = X_1, & (Y_1, Y_2) = \frac{1}{2}X_2. \end{cases}\end{aligned}$$

4.4. $n = 2, m = 4$

Analogously as the above case, from consulting the papers [14,2,4] and after eliminating the infinitesimal deformations of B either they do not verify the condition $\Psi(X_0, -) = 0$ or they are not faithful with the filiform structure, remain the following infinitesimal deformations:

- Basis of B : $\{\Psi_{1,1}^2, \Psi_{1,1}^3, \Psi_{1,1}^4, \Psi_{2,1}^3, \Psi_{2,1}^4\}$ defined by $\{\Psi_{1,1}^2(X_1, Y_1) = Y_2, \Psi_{1,1}^1(X_1, Y_2) = Y_3, \Psi_{1,1}^1(X_1, Y_3) = Y_4\}, \{\Psi_{1,1}^3(X_1, Y_1) = Y_3, \Psi_{1,1}^3(X_1, Y_2) = Y_4\}, \{\Psi_{1,1}^4(X_1, Y_1) = Y_4\}$ and $\{\Psi_{2,1}^3(X_1, Y_2) = -Y_3, \Psi_{2,1}^3(X_1, Y_3) = -2Y_4, \Psi_{2,1}^3(X_2, Y_1) = Y_3, \Psi_{2,1}^3(X_2, Y_2) = Y_4\}$ and $\{\Psi_{2,1}^4(X_1, Y_2) = -Y_4, \Psi_{2,1}^4(X_2, Y_1) = Y_4\}$.
- Basis of C : $\{\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,2}, \varphi_{2,1}\}$ with $\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,2}$ already defined in the above case and $\varphi_{2,1}$ verifying $\{\varphi_{2,1}(Y_1, Y_3) = -X_1, \varphi_{2,1}(Y_1, Y_4) = \frac{-3}{2}X_2, \varphi_{2,1}(Y_2, Y_2) = X_1, \varphi_{2,1}(Y_2, Y_3) = \frac{1}{2}X_2\}$.

Therefore we will consider $\Psi = A\Psi_{1,1}^2 + B\Psi_{1,1}^3 + C\Psi_{1,1}^4 + D\Psi_{2,1}^3 + E\Psi_{2,1}^4 + a\varphi_{1,1} + b\varphi_{1,2} + c\varphi_{2,2} + d\varphi_{2,1}$.

$$\Psi : \begin{cases} [X_1, Y_1] = AY_2 + BY_3 + CY_4 & (Y_1, Y_1) = aX_1 + bX_2 \\ [X_1, Y_2] = (A - D)Y_3 + (B - E)Y_4 & (Y_1, Y_2) = \frac{a}{2}X_2 \\ [X_1, Y_3] = (A - 2D)Y_4 & (Y_1, Y_3) = -dX_1 - cX_2 \\ [X_2, Y_1] = DY_3 + EY_4 & (Y_1, Y_4) = \frac{-3d}{2}X_2 \\ [X_2, Y_2] = DY_4 & (Y_2, Y_2) = dX_1 + cX_2 \\ (Y_2, Y_3) = \frac{d}{2}X_2. \end{cases}$$

Applying the condition to be integrable, $\Psi \circ \Psi = 0$, leads to the equations

$$\begin{aligned} (1) D &= 0 & (4) Aa &= 0 & (7) Bd &= 0 \\ (2) Ad &= 0 & (5) Ba &= 0 & (8) Cd + Ec &= 0 \\ (3) Ac &= 0 & (6) Ca + Eb &= 0 & (9) Ed &= 0. \end{aligned}$$

If we consider $\mu_0 + \Psi = \mathfrak{g}$ then it can be seen that there is no loss of generality in supposing $A = 0$. In fact, if we had $A \neq 0$ and consequently $d = c = a = 0$ we would apply the change of basis $\{X'_0 = X_0, X'_1 = X_1 - AX_0, Y'_1 = Y_1\}$. Next, it can be observed that the dimension of the fourth term of the descending central sequence depends on whether $d = 0$ or not. Thus, $\dim((\mathcal{C}^4(\mathfrak{g}))) = 0$ if $d = 0$ and $\dim((\mathcal{C}^4(\mathfrak{g}))) = 1$ otherwise. This fact allows us to differentiate between two cases non-isomorphic.

Case 1. $d \neq 0$. In that case $B = E = 0 = C$ and thanks to the change of basis $\{X'_0 = X_0, X'_1 = dX_1 + cX_2, Y'_1 = Y_1\}$ it can be always supposed $d = 1$ and $c = 0$. Next, firstly, applying the change of basis $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = Y_1 + \frac{b}{3}Y_4\}$ leads to $b = 0$ and secondly, applying the change of basis $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = Y_1 + \frac{a}{2}Y_3\}$ leads to $a = 0$ obtaining the following FLS

$$\mu_0 + \varphi_{2,1} : \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ (Y_1, Y_3) = -X_1, & (Y_1, Y_4) = \frac{-3}{2}X_2 \\ (Y_2, Y_2) = X_1, & (Y_2, Y_3) = \frac{1}{2}X_2. \end{cases}$$

Case 2. $d = 0$. In this case we would have the following family of laws

$$\begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = BY_3 + CY_4 & (Y_1, Y_1) = aX_1 + bX_2 \\ [X_1, Y_2] = (B - E)Y_4 & (Y_1, Y_2) = \frac{a}{2}X_2 \\ [X_2, Y_1] = EY_4 & (Y_1, Y_3) = -cX_2 \\ (Y_2, Y_2) = cX_2 \end{cases}$$

with the equations $Ec = 0$, $Ca + Eb = 0$ and $Ba = 0$. It can be noted that the value of a determines the dimension of $\mathcal{C}^1(\mathfrak{g})$ and thus, we can divide the present case into two different subcases non-isomorphic.

Subcase 2.1. $a \neq 0$. We would have $B = 0$, $Ec = 0$ and $C = \frac{-Eb}{a}$. Furthermore, applying the change of basis $\{X'_0 = X_0, X'_1 = aX_1 + bX_2, Y'_1 = Y_1\}$ we would have $a = 1$ and $b = 0$, and consequently $C = 0$. Next, it can be noted that $\dim((\mathcal{C}^3(\mathfrak{g}))_0) = 0$ if $c = 0$ and $\dim((\mathcal{C}^3(\mathfrak{g}))_0) = 1$ otherwise, thus we will have the following subcases.

Subcase 2.1.1. $c \neq 0$. In this case we would have $E = 0$ and after applying the change of scale $\{X'_0 = \frac{1}{c}X_0, X'_1 = X_1, Y'_1 = Y_1\}$ we obtain the following FLS.

$$\mu_0 + \varphi_{1,1} + \varphi_{2,2} : \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ (Y_1, Y_1) = X_1, & (Y_1, Y_2) = \frac{1}{2}X_2 \\ (Y_1, Y_3) = -X_2, & (Y_2, Y_2) = X_2. \end{cases}$$

Subcase 2.1.2. $c = 0$. By using generic change of basis, or isomorphisms, faithful with the currently expression of the superalgebra it can be seen that it is not possible to eliminate E . Therefore we can distinguish between the cases $E = 0$ and $E \neq 0$, in the former we would obtain directly the FLS $\mu_0 + \varphi_{1,1}$ while in the latter we would obtain the FLS $\mu_0 + \Psi_{2,1}^4 + \varphi_{1,1}$ after applying a change of scale.

$$\begin{aligned} \mu_0 + \varphi_{1,1} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ (Y_1, Y_1) = X_1, & (Y_1, Y_2) = \frac{1}{2}X_2 \end{cases} \\ \mu_0 + \Psi_{2,1}^4 + \varphi_{1,1} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_2] = -Y_4, & (Y_1, Y_1) = X_1 \\ [X_2, Y_1] = Y_4, & (Y_1, Y_2) = \frac{1}{2}X_2. \end{cases} \end{aligned}$$

Subcase 2.2. $a = 0$. The equations that remain in this case would be $Eb = 0$ and $Ec = 0$. Imposing the condition of not obtaining Lie algebras, which would be degenerate or split cases, leads to $E = 0$. Next, it can be noted that $\dim((\mathcal{C}^2(\mathfrak{g}))_0) = 0$ if $c = 0$ and $\dim((\mathcal{C}^2(\mathfrak{g}))_0) = 1$ otherwise, thus we will have the following subcases non-isomorphic.

Subcase 2.2.1. $c \neq 0$. Firstly, thanks to the change of basis $\{X'_0 = X_0, X'_1 = X_1, Y'_1 = Y_1 + \frac{b}{2c}Y_3\}$ it can be supposed $b = 0$ without loss of generality. Secondly, the change of scale $\{X'_0 = \frac{1}{c}X_0, X'_1 = X_1, Y'_1 = Y_1\}$ leads to $c = 1$. Next, it can be noted that neither B nor C can be eliminated by means of generic changes of basis faithful with the current structure of the family of superalgebras. Thus, we will distinguish between four cases non-isomorphic: $B = C = 0$, $B \neq 0$ and $C = 0$, $B = 0$ and $C \neq 0$ and finally $B \neq 0 \neq C$; obtaining the FLS $\mu_0 + \varphi_{2,2}$, $\mu_0 + \Psi_{1,1}^3 + \varphi_{2,2}$, $\mu_0 + \Psi_{1,1}^4 + \varphi_{2,2}$ and $\mu_0 + \Psi_{1,1}^3 + \Psi_{1,1}^4 + \varphi_{2,2}$ respectively. Note that in the last three cases changes of scale have been required, for instance in the case $B \neq 0 \neq C$ it has been applied $\{X'_0 = \frac{c}{B}X_0, X'_1 = \frac{c^2}{B^2}X_1, Y'_1 = \frac{\sqrt{c}}{B}Y_1\}$.

$$\begin{aligned} \mu_0 + \varphi_{2,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ (Y_1, Y_3) = -X_2 & (Y_2, Y_2) = X_2 \end{cases} \\ \mu_0 + \Psi_{1,1}^3 + \varphi_{2,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = Y_3, & (Y_1, Y_3) = -X_2 \\ [X_1, Y_2] = Y_4, & (Y_2, Y_2) = X_2 \end{cases} \\ \mu_0 + \Psi_{1,1}^4 + \varphi_{2,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = Y_4, & (Y_1, Y_3) = -X_2 \\ (Y_2, Y_2) = X_2 \end{cases} \\ \mu_0 + \Psi_{1,1}^3 + \Psi_{1,1}^4 + \varphi_{2,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = Y_3 + Y_4, & (Y_1, Y_3) = -X_2 \\ [X_1, Y_2] = Y_4, & (Y_2, Y_2) = X_2. \end{cases} \end{aligned}$$

Subcase 2.2.2. $c = 0$. In order to obtain a non-degenerate case we have $b \neq 0$ and after applying the change of scale $\{X'_0 = bX_0, X'_1 = X_1, Y'_1 = Y_1\}$ it can be supposed $b = 1$. A similar reasoning as the one applied in the precedent case regarding B and C , would be applied at this point obtaining the four FLS that follow

$$\begin{aligned} \mu_0 + \varphi_{1,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ (Y_1, Y_1) = X_2 \end{cases} \\ \mu_0 + \Psi_{1,1}^3 + \varphi_{1,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = Y_3, & (Y_1, Y_1) = X_2 \\ [X_1, Y_2] = Y_4. \end{cases} \\ \mu_0 + \Psi_{1,1}^4 + \varphi_{1,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = Y_4, & (Y_1, Y_1) = X_2 \end{cases} \\ \mu_0 + \Psi_{1,1}^3 + \Psi_{1,1}^4 + \varphi_{1,2} : & \begin{cases} [X_0, X_1] = X_2, & [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, Y_1] = Y_3 + Y_4, & (Y_1, Y_1) = X_2 \\ [X_1, Y_2] = Y_4. \end{cases} \end{aligned}$$

5. Case $m = 1$

In this section we are going to consider the model filiform Lie superalgebra $L^{n,1}$ for each n such that $3 \leq n \leq 5$.

Thus, if $n = 3$ the reasoning is very similar to the case $n = 2$. Therefore we would only have the FLS that follows

$$\mu_0 + \varphi_{1,3} : \begin{cases} [X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 2 \\ (Y_1, Y_1) = X_3. \end{cases}$$

For the cases $n = 4$ and $n = 5$ we would have to add either $\varphi_{1,4}$ or $\varphi_{1,5}$ to the classification of filiform Lie algebras given in [14]. Thus if $n = 4$ we would have two FLS

$$\begin{aligned} \mu_0 + \varphi_{1,4} : & \begin{cases} [X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 3 \\ (Y_1, Y_1) = X_4 \end{cases} & \mu_0 + \Psi_{1,4} + \varphi_{1,4} : & \begin{cases} [X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 3 \\ [X_1, X_2] = X_4, \quad (Y_1, Y_1) = X_4. \end{cases} \end{aligned}$$

And finally for $n = 5$ we would have five FLS

$$\begin{aligned} \mu_0 + \varphi_{1,5} : & \begin{cases} [X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 4 \\ (Y_1, Y_1) = X_5 \end{cases} & \mu_0 + \Psi_{1,4} + \varphi_{1,5} : & \begin{cases} [X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 4 \\ [X_1, X_2] = X_4, \quad (Y_1, Y_1) = X_5 \\ [X_1, X_3] = X_5, \end{cases} \end{aligned}$$

$$\begin{aligned}
&\mu_0 + \Psi_{1,5} + \varphi_{1,5} : & \mu_0 + \Psi_{2,5} + \varphi_{1,5} : \\
&\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq 4 \\ [X_1, X_2] = X_5, & (Y_1, Y_1) = X_5 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq 4 \\ [X_1, X_4] = -X_5, & (Y_1, Y_1) = X_5 \\ [X_2, X_3] = X_5 \end{cases} \\
&\mu_0 + \Psi_{1,4} + \Psi_{2,5} + \varphi_{1,5} : \\
&\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq 4 \\ [X_1, X_2] = X_4, & [X_1, X_4] = -X_5 \\ [X_1, X_3] = X_5, & [X_2, X_3] = X_5 \\ (Y_1, Y_1) = X_5. \end{cases}
\end{aligned}$$

Throughout the next cases it would be possible to apply a similar reasoning to which have been used in precedent cases. Thus, we are going to give only the expressions of the FLS resulting as well as the invariants if necessary.

6. Case $m = 2$

6.1. $n = 3, m = 2$

In this case we would have the FLS that follow, with i such that $1 \leq i \leq 2$.

$$\begin{aligned}
&\mu_0 + \overline{\varphi}_{2,3} : & \mu_0 + \varphi_{1,2} : & \mu_0 + \Psi_{1,1}^2 + \varphi_{1,2} : \\
&\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = 2X_1 \\ (Y_1, Y_2) = X_2 \\ (Y_2, Y_2) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \end{cases} \\
&\mu_0 + \varphi_{1,3} : & \mu_0 + \Psi_{1,1}^2 + \varphi_{1,3} : \\
&\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \\ (Y_1, Y_1) = X_3. \end{cases}
\end{aligned}$$

It can be noted that in the first FLS $\mu_0 + \overline{\varphi}_{2,3}$ we would have $\dim(\mathcal{C}^3(\mathfrak{g})) = 1$ but for the rest we would have $\dim(\mathcal{C}^3(\mathfrak{g})) = 0$. The four last FLS can be differentiated by means of generic changes of basis.

6.2. $n = 4, m = 2$

In this case we would have the FLS that follow, with i such that $1 \leq i \leq 3$.

$$\begin{aligned}
&\mu_0 + \overline{\varphi}_{2,4} : & \mu_0 + \varphi_{1,3} : & \mu_0 + \Psi_{1,1}^2 + \varphi_{1,3} : \\
&\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = 2X_2 \\ (Y_1, Y_2) = X_3 \\ (Y_2, Y_2) = X_4 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = X_3 \\ (Y_1, Y_2) = \frac{1}{2}X_4 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \\ (Y_1, Y_1) = X_3 \\ (Y_1, Y_2) = \frac{1}{2}X_4 \end{cases} \\
&\mu_0 + \varphi_{1,4} : & \mu_0 + \Psi_{1,1}^2 + \varphi_{1,4} : & \mu_0 + \Psi_{1,4} + \varphi_{1,3} : \\
&\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ (Y_1, Y_1) = X_4 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \\ (Y_1, Y_1) = X_4 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \\ (Y_1, Y_1) = X_3 \\ (Y_1, Y_2) = \frac{1}{2}X_4 \end{cases} \\
&\mu_0 + \Psi_{1,4} + \Psi_{1,1}^2 + \varphi_{1,3} : & \mu_0 + \Psi_{1,4} + \varphi_{1,4} : & \mu_0 + \Psi_{1,4} + \Psi_{1,1}^2 + \varphi_{1,4} : \\
&\begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \\ [X_1, Y_1] = Y_2 \\ (Y_1, Y_1) = X_3 \\ (Y_1, Y_2) = \frac{1}{2}X_4 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \\ (Y_1, Y_1) = X_4 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_1] = Y_2 \\ [X_1, X_2] = X_4 \\ [X_1, Y_1] = Y_2 \\ (Y_1, Y_1) = X_4. \end{cases}
\end{aligned}$$

It can be observed that the nine FLS are pairwise non-isomorphic by means of generic changes of basis.

7. Case $m = 3$

The only case to consider will be $n = 3$ and $m = 3$. In this case we have fifteen filiform Lie superalgebras and two one-parametric families of filiform Lie superalgebras. All of them are pairwise non-isomorphic mainly thanks to the use of generic changes of basis. The expression of such FLS is as follows, with i and j such that $1 \leq i, j \leq 2$.

$$\begin{array}{lll}
 \mu_0 + \Psi_{2,1}^3 + \varphi_{1,1} + \frac{1}{2}\varphi_{2,3} : & \mu_0 + \varphi_{1,1} + \alpha\varphi_{2,3} : & \mu_0 + \Psi_{1,1}^3 + \varphi_{1,2} + \varphi_{2,3} : \\
 \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_2] = -Y_3 \\ [X_2, Y_1] = Y_3 \\ (Y_1, Y_1) = X_1 \\ (Y_1, Y_2) = \frac{1}{2}X_2 \\ (Y_2, Y_2) = \frac{1}{2}X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ (Y_1, Y_1) = X_1 \\ (Y_1, Y_2) = \frac{1}{2}X_2 \\ (Y_1, Y_3) = \left(\frac{1}{2} - \alpha\right)X_3 \\ (Y_2, Y_2) = \alpha X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_3 \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \\ (Y_1, Y_3) = -X_3 \\ (Y_2, Y_2) = X_3. \end{cases} \\
 \\
 \mu_0 + \varphi_{1,2} + \varphi_{2,3} : & \mu_0 + \Psi_{1,1}^2 + \Psi_{1,1}^3 + \varphi_{1,2} : & \mu_0 + \Psi_{1,1}^2 + \varphi_{1,2} : \\
 \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \\ (Y_1, Y_3) = -X_3 \\ (Y_2, Y_2) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_2 + Y_3 \\ [X_1, Y_2] = Y_3 \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_2 \\ [X_1, Y_2] = Y_3 \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3. \end{cases} \\
 \\
 \mu_0 + \Psi_{1,1}^3 + \varphi_{1,2} : & \mu_0 + \varphi_{1,2} : & \mu_0 + \Psi_{1,1}^2 + \Psi_{1,1}^3 + 2\Psi_{2,1}^3 + \varphi_{2,3} : \\
 \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_3 \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ (Y_1, Y_1) = X_2 \\ (Y_1, Y_2) = \frac{1}{2}X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_2 + Y_3 \\ [X_1, Y_2] = -Y_3 \\ [X_2, Y_1] = 2Y_3 \\ (Y_1, Y_3) = -X_3 \\ (Y_2, Y_2) = X_3. \end{cases} \\
 \\
 \mu_0 + \Psi_{1,1}^2 + 2\Psi_{2,1}^3 + \varphi_{2,3} : & \mu_0 + \Psi_{1,1}^3 + \varphi_{2,3} : & \mu_0 + \varphi_{2,3} : \\
 \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_2 \\ [X_1, Y_2] = -Y_3 \\ [X_2, Y_1] = 2Y_3 \\ (Y_1, Y_3) = -X_3 \\ (Y_2, Y_2) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_3 \\ (Y_1, Y_3) = -X_3 \\ (Y_2, Y_2) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ (Y_1, Y_3) = -X_3 \\ (Y_2, Y_2) = X_3. \end{cases} \\
 \\
 \mu_0 + \Psi_{1,1}^2 + \alpha\Psi_{2,1}^3 + \varphi_{1,3} : & \mu_0 + \Psi_{2,1}^3 + \varphi_{1,3} : & \mu_0 + \Psi_{1,1}^2 + \Psi_{1,1}^3 + \varphi_{1,3} : \\
 \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_2 \\ [X_1, Y_2] = (1 - \alpha)Y_3 \\ [X_2, Y_1] = \alpha Y_3 \\ (Y_1, Y_1) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_2] = -Y_3 \\ [X_2, Y_1] = Y_3 \\ (Y_1, Y_1) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_2 + Y_3 \\ [X_1, Y_2] = Y_3 \\ (Y_1, Y_1) = X_3. \end{cases} \\
 \\
 \mu_0 + \Psi_{1,1}^3 + \varphi_{1,3} : & \mu_0 + \varphi_{1,3} : & \\
 \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ [X_1, Y_1] = Y_3 \\ (Y_1, Y_1) = X_3 \end{cases} & \begin{cases} [X_0, X_i] = X_{i+1} \\ [X_0, Y_j] = Y_{j+1} \\ (Y_1, Y_1) = X_3. \end{cases} &
 \end{array}$$

8. Main result

Theorem 8.1 (Classification's Theorem). *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a non-degenerated filiform Lie superalgebra with $\dim(\mathfrak{g}_0) = n + 1$ and $\dim(\mathfrak{g}_1) = m$. If the total dimension is less or equal to seven then the law of \mathfrak{g} will be isomorphic to a law $\mu_{(n+1,m)}^i$ of the*

following list of laws. Note that the laws $\mu_{(n+1,m)}^i$ and $\mu_{(n+1,m)}^j$ are not isomorphic for $i \neq j$. Furthermore two laws of the same one-parametric family $\mu_{(n+1,m)}^i(\alpha)$ and $\mu_{(n+1,m)}^i(\alpha')$ with $\alpha \neq \alpha'$ are also non-isomorphic.

List of Laws

Pair of Dimensions: $n = 2, m = 1$.

$$\mu_{(3,1)}^1 : \mu_0 + \varphi_{1,2}.$$

Pair of Dimensions: $n = 3, m = 1$.

$$\mu_{(4,1)}^1 : \mu_0 + \varphi_{1,3}.$$

Pair of Dimensions: $n = 4, m = 1$.

$$\mu_{(5,1)}^1 : \mu_0 + \varphi_{1,4}$$

$$\mu_{(5,1)}^2 : \mu_0 + \Psi_{1,4} + \varphi_{1,4}.$$

Pair of Dimensions: $n = 5, m = 1$.

$$\mu_{(6,1)}^1 : \mu_0 + \varphi_{1,5}$$

$$\mu_{(6,1)}^2 : \mu_0 + \Psi_{1,4} + \varphi_{1,5}$$

$$\mu_{(6,1)}^3 : \mu_0 + \Psi_{1,5} + \varphi_{1,5}$$

$$\mu_{(6,1)}^4 : \mu_0 + \Psi_{2,5} + \varphi_{1,5}$$

$$\mu_{(6,1)}^5 : \mu_0 + \Psi_{1,4} + \Psi_{2,5} + \varphi_{1,5}.$$

Pair of Dimensions: $n = 2, m = 2$.

$$\mu_{(3,2)}^1 : \mu_0 + \varphi_{1,1}$$

$$\mu_{(3,2)}^2 : \mu_0 + \varphi_{1,2}.$$

Pair of Dimensions: $n = 3, m = 2$.

$$\mu_{(4,2)}^1 : \mu_0 + \overline{\varphi}_{2,3}$$

$$\mu_{(4,2)}^2 : \mu_0 + \varphi_{1,2}$$

$$\mu_{(4,2)}^3 : \mu_0 + \Psi_{1,1}^2 + \varphi_{1,2}$$

$$\mu_{(4,2)}^4 : \mu_0 + \varphi_{1,3}$$

$$\mu_{(4,2)}^5 : \mu_0 + \Psi_{1,1}^2 + \varphi_{1,3}.$$

Pair of Dimensions: $n = 4, m = 2$.

$$\mu_{(5,2)}^1 : \mu_0 + \overline{\varphi}_{2,4}$$

$$\mu_{(5,2)}^2 : \mu_0 + \varphi_{1,3}$$

$$\mu_{(5,2)}^3 : \mu_0 + \Psi_{1,1}^2 + \varphi_{1,3}$$

$$\mu_{(5,2)}^4 : \mu_0 + \varphi_{1,4}$$

$$\mu_{(5,2)}^5 : \mu_0 + \Psi_{1,1}^2 + \varphi_{1,4}$$

$$\mu_{(5,2)}^6 : \mu_0 + \Psi_{1,4} + \varphi_{1,3}$$

$$\mu_{(5,2)}^7 : \mu_0 + \Psi_{1,4} + \Psi_{1,1}^2 + \varphi_{1,3}$$

$$\mu_{(5,2)}^8 : \mu_0 + \Psi_{1,4} + \varphi_{1,4}$$

$$\mu_{(5,2)}^9 : \mu_0 + \Psi_{1,4} + \Psi_{1,1}^2 + \varphi_{1,4}.$$

Pair of Dimensions: $n = 2, m = 3$.

$$\mu_{(3,3)}^1 : \mu_0 + \Psi_{1,1}^3 + \varphi_{2,2}$$

$$\mu_{(3,3)}^2 : \mu_0 + \varphi_{2,2}$$

$$\mu_{(3,3)}^3 : \mu_0 + \varphi_{1,1} + \varphi_{2,2}$$

$$\mu_{(3,3)}^4 : \mu_0 + \varphi_{1,2}$$

$$\mu_{(3,3)}^5 : \mu_0 + \Psi_{1,1}^3 + \varphi_{1,2}$$

$$\mu_{(3,3)}^6 : \mu_0 + \varphi_{1,1}$$

$$\mu_{(3,3)}^7 : \mu_0 + \Psi_{2,1}^3 + \varphi_{1,1}.$$

Pair of Dimensions: $n = 3, m = 3$.

$$\mu_{(4,3)}^1 : \mu_0 + \Psi_{2,1}^3 + \varphi_{1,1} + \frac{1}{2}\varphi_{2,3}$$

$$\mu_{(4,3)}^2(\alpha) : \mu_0 + \varphi_{1,1} + \alpha\varphi_{2,3}$$

$$\mu_{(4,3)}^3 : \mu_0 + \Psi_{1,1}^3 + \varphi_{1,2} + \varphi_{2,3}$$

$$\mu_{(4,3)}^4 : \mu_0 + \varphi_{1,2} + \varphi_{2,3}$$

$$\begin{aligned}
\mu_{(4,3)}^5 &: \mu_0 + \psi_{1,1}^2 + \psi_{1,1}^3 + \varphi_{1,2} \\
\mu_{(4,3)}^6 &: \mu_0 + \psi_{1,1}^2 + \varphi_{1,2} \\
\mu_{(4,3)}^7 &: \mu_0 + \psi_{1,1}^3 + \varphi_{1,2} \\
\mu_{(4,3)}^8 &: \mu_0 + \varphi_{1,2} \\
\mu_{(4,3)}^9 &: \mu_0 + \psi_{1,1}^2 + \psi_{1,1}^3 + 2\psi_{2,1}^3 + \varphi_{2,3} \\
\mu_{(4,3)}^{10} &: \mu_0 + \psi_{1,1}^2 + 2\psi_{2,1}^3 + \varphi_{2,3} \\
\mu_{(4,3)}^{11} &: \mu_0 + \psi_{1,1}^3 + \varphi_{2,3} \\
\mu_{(4,3)}^{12} &: \mu_0 + \varphi_{2,3} \\
\mu_{(4,3)}^{13}(\alpha) &: \mu_0 + \psi_{1,1}^2 + \alpha\psi_{2,1}^3 + \varphi_{1,3} \\
\mu_{(4,3)}^{14} &: \mu_0 + \psi_{2,1}^3 + \varphi_{1,3} \\
\mu_{(4,3)}^{15} &: \mu_0 + \psi_{1,1}^2 + \psi_{1,1}^3 + \varphi_{1,3} \\
\mu_{(4,3)}^{16} &: \mu_0 + \psi_{1,1}^3 + \varphi_{1,3} \\
\mu_{(4,3)}^{17} &: \mu_0 + \varphi_{1,3}.
\end{aligned}$$

Pair of Dimensions: $n = 2, m = 4$.

$$\begin{aligned}
\mu_{(3,4)}^1 &: \mu_0 + \varphi_{2,1} \\
\mu_{(3,4)}^2 &: \mu_0 + \varphi_{1,1} + \varphi_{2,2} \\
\mu_{(3,4)}^3 &: \mu_0 + \varphi_{1,1} \\
\mu_{(3,4)}^4 &: \mu_0 + \psi_{2,1}^4 + \varphi_{1,1} \\
\mu_{(3,4)}^5 &: \mu_0 + \varphi_{2,2} \\
\mu_{(3,4)}^6 &: \mu_0 + \psi_{1,1}^3 + \varphi_{2,2} \\
\mu_{(3,4)}^7 &: \mu_0 + \psi_{1,1}^4 + \varphi_{2,2} \\
\mu_{(3,4)}^8 &: \mu_0 + \psi_{1,1}^3 + \psi_{1,1}^4 + \varphi_{2,2} \\
\mu_{(3,4)}^9 &: \mu_0 + \varphi_{1,2} \\
\mu_{(3,4)}^{10} &: \mu_0 + \psi_{1,1}^3 + \varphi_{1,2} \\
\mu_{(3,4)}^{11} &: \mu_0 + \psi_{1,1}^4 + \varphi_{1,2} \\
\mu_{(3,4)}^{12} &: \mu_0 + \psi_{1,1}^3 + \psi_{1,1}^4 + \varphi_{1,2}.
\end{aligned}$$

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